2-categories, 2-representations and their applications

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LMS Northern regional Meeting and Workshop on
TRIANGULATIONS AND MUTATIONS
March 18, 2013, Newcastle, UK

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2-categories: over monoids, part 4: functoriality, part 1

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$$\begin{tikzcd}
r \arrow[d, "x"'] & & r' \arrow[d, "x'"'] & & rr' \arrow[d, "xx'"'] & & \\
s \arrow[d, "y"'] \arrow[r, phantom, "\circ_0"] & {} & s' \arrow[d, "y'"'] \arrow[r, phantom, "\mapsto"] & {} & ss' \arrow[d, "yy'"'] \arrow[r, phantom, "\overset{\circ_1}{\mapsto}"] & {} & rr' \arrow[d, "yy'xx'"] \\
t & & t' & & tt' & & tt'
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Conclusion 1: $(y \circ_0 y') \circ_1 (x \circ_0 x') = yy'xx'$.

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s \arrow[dd, "y"'] & \circ_1 & s' \arrow[dd, "y'"'] & \mapsto & & \circ_0 & & \mapsto & \\
 & & & & t & & t' & & tt' \\
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Conclusion 2: $(y \circ_1 x) \circ_0 (y' \circ_1 x') = yxy'x'$.

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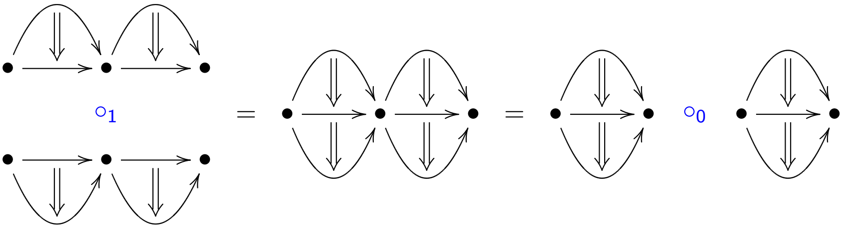

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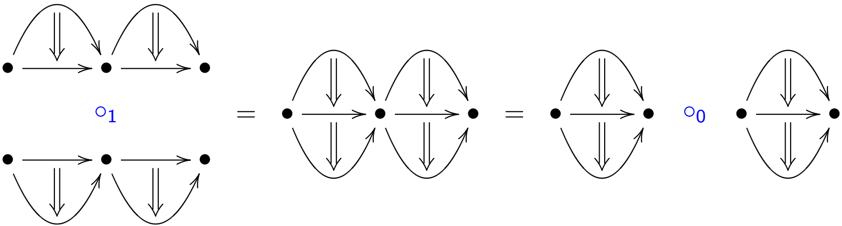

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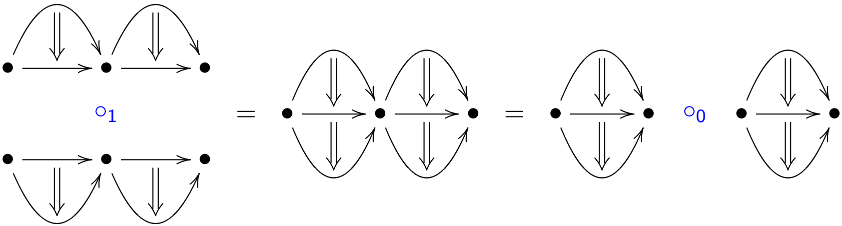

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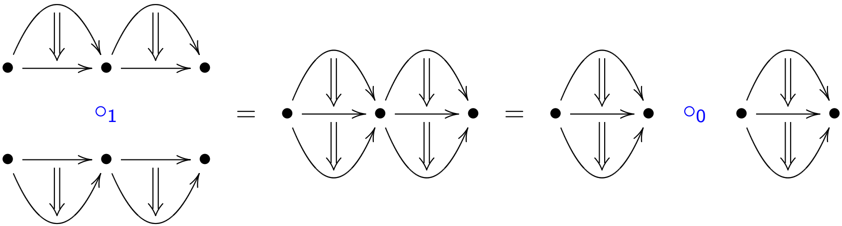

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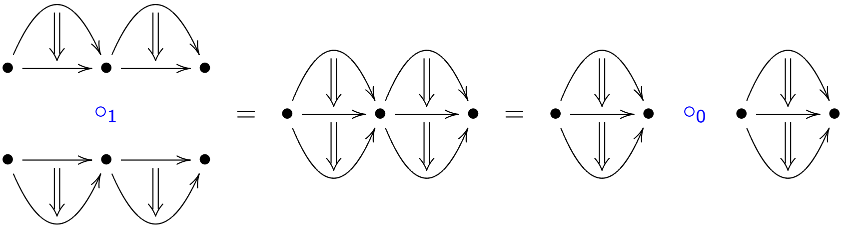

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$$
\begin{tikzcd}
\mathtt{j} \arrow[rr, "{\color{red}F\circ \_}"] & & \mathtt{k} \\
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\begin{tikzcd}
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\mathbf{C}^{\{i\}}\text{-mod} \arrow[ur, "\mathrm{Res}"] \arrow[rr, bend left=10, "E_{i+1}{:=}\mathrm{Ind}\circ\mathrm{Res}"] & & \mathbf{C}^{\{i+1\}}\text{-mod} \arrow[ll, bend left=10, "F_{i+1}{:=}\mathrm{Ind}\circ\mathrm{Res}"] \arrow[dl, "\mathrm{Res}"] \\
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