2-representations of finitary 2-categories

(joint work with Vanessa Miemietz)

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- natural (strict) axioms;

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- the category $\mathfrak{R}_{\Bbbk}$ of small categories equivalent to module categories of finite-dimensional associative $\Bbbk$-algebras;

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Example. The 2-category $\mathscr{C}_A$ was defined via its **defining** representation.

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Examples. $\mathscr{S}$ is fiat; $\mathscr{C}_A$ is fiat if and only if A is self-injective and weakly symmetric (i.e. the top and the socle of each indecomposable projective are isomorphic).

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Example 2. $(\mathbb{Z}_+, \diamond)$, where $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and

$$m \diamond n = \{i : \quad |m - n| \leq i \leq m + n; \quad i \equiv m + n \mod 2\}.$$

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$$\bigcup_{s \in a \diamond b} s \diamond c = \bigcup_{t \in b \diamond c} a \diamond t, \qquad \text{for all } a, b, c \in S$$

Example 1. Any semigroup is a multisemigroup.

Example 2. $(\mathbb{Z}_+, \diamond)$, where $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and

$$m \diamond n = \{i : \quad |m - n| \leq i \leq m + n; \quad i \equiv m + n \mod 2\}.$$

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- $a \sim_J b$ iff $S \diamond a \diamond S = S \diamond b \diamond S$

Multisemigroup of a fiat category

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$S(\mathscr{C}_{\mathfrak{sl}_2})' \overset{1:1}{\leftrightarrow} \mathbb{Z}_+$ (via highest weight) and $(S(\mathscr{C}_{\mathfrak{sl}_2})', \diamond) \cong (\mathbb{Z}_+, \diamond)$

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note: $\underbrace{A_j f \otimes_{\Bbbk} eA_i}_{F^*} \otimes_A \underbrace{A_i e \otimes_{\Bbbk} fA_j}_{F} \cong \dim(A_i e)\, A_j f \otimes_{\Bbbk} fA_j$ and $\dim(A_i e)$ is constant on a right cell!!!

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Definition. $G_{\mathcal{L}}$ is the *Duflo involution* in $\mathcal{L}$

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Definition. Two 2-representations of $\mathscr{C}$ are called **equivalent** if there is a finite sequence of 2-representations starting with the first one and ending with the second one such that every pair of neighbors in the sequence are elementary equivalent.

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Example. Works for both $\mathscr{C}$ (in type A) and $\mathscr{C}_A$.