Simple supermodules for classical Lie superalgebras

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Classification of simple modules for semi-simple Lie algebras

"Full" answer: Only for $\mathfrak{sl}_2$, R. Block 1979, — reduces to description of equivalence classes of irreducible elements in a non-commutative Euclidean ring

Some partial answers:

- Finite dimensional modules: E. Cartan 1913
- Whittaker modules: B. Kostant 1978
- Weight modules with fin.-dim. weight spaces: O. Mathieu 2000

Some other classes of simple modules:

- Parabolically induced modules: V. Futorny, E. McDowell, O. Khomenko, D. Miličić, W. Soergel, C. Stroppel, V. M. and others 1980's - now
- Gelfand-Zetlin modules: Yu. Drozd, V. Futorny, S. Ovsienko 1989
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Classical Lie superalgebras

$\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$

$\mathfrak{g}_{\overline{0}}$ — finite dimensional reductive

$\mathfrak{g}_{\overline{1}}$ — finite dimensional and semi-simple over $\mathfrak{g}_{\overline{0}}$

Some examples:

- General linear Lie superalgebra $\mathfrak{gl}(m|n)$
- Queer Lie superalgebra $\mathfrak{q}(n)$
- Generalized Takiff Lie superalgebra $\mathfrak{g}_{\mathfrak{a},V}$ where $\mathfrak{g}_{\overline{0}} = \mathfrak{a}$, $\mathfrak{g}_{\overline{1}} = V \in \mathfrak{a}\text{-mod}$ and $[V, V] = 0$.

Main problem: Classification of simple $\mathfrak{g}$-supermodules

Reduction: Modulo classification of simple $\mathfrak{g}_{\overline{0}}$-modules

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"Full" answer:

- $\mathfrak{gl}(1,1)$ and $\mathfrak{q}(1)$ — exercise
- $\mathfrak{osp}(1,2)$: V. Bavula, F. van Oystaeyen 2000
- $\mathfrak{p}(2)$: V. Serganova 2002
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Special cases:

- Typical generic modules for basic: I. Penkov 1994
- Strongly typical modules for basic: M. Gorelik 2002
- Weight modules with fin.-dim. weight spaces for type I: D. Grantcharov 2003
- Regular strongly typical for $\mathfrak{q}_n$: A. Frisk, V. M. 2009
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Further reduction

L — simple $\mathfrak{g}$-supermodule

$\mathrm{Ann}_{U(\mathfrak{g})}(L)$ — the annihilator of L in $U(\mathfrak{g})$

$\mathrm{Ann}_{U(\mathfrak{g})}(L)$ is a primitive ideal of $U(\mathfrak{g})$

Theorem. (I. Musson 1992) There is a simple highest weight $\mathfrak{g}$-supermodule $\mathrm{L}(\lambda)$ such that $\mathrm{Ann}_{U(\mathfrak{g})}(L) = \mathrm{Ann}_{U(\mathfrak{g})}(\mathrm{L}(\lambda))$.

$\mathrm{L}(\lambda)$ is of finite length over $U(\mathfrak{g}_{\overline{0}})$

Take any μ such that $\mathrm{L}^{\mathfrak{g}_{\overline{0}}}(\mu)$ is a simple $\mathfrak{g}_{\overline{0}}$-submodule of $\mathrm{L}(\lambda)$

Note: μ is not uniquely defined

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Harish-Chandra bimodules

X — $\mathfrak{g}_{\overline{0}}$-module

Y — $\mathfrak{g}$-supermodule

$\mathcal{L}(X,Y)$ — the set of locally $\mathrm{ad}(\mathfrak{g}_{\overline{0}})$-finite linear maps from X to Y

$\mathcal{L}(X,Y)$ is a $U(\mathfrak{g})$-$U(\mathfrak{g}_{\overline{0}})$–bimodule (a *Harish-Chandra* bimodule)

$$\mathcal{L}(X,Y) \bigotimes_{U(\mathfrak{g}_{\overline{0}})} \_ : U(\mathfrak{g}_{\overline{0}})\text{-mod} \to U(\mathfrak{g})\text{-smod}$$

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The main conjecture

L — simple $\mathfrak{g}$-supermodule

$\mathcal{I} := \mathrm{Ann}_{U(\mathfrak{g})}(L)$

$\mathrm{L}(\lambda)$ — a simple highest weight module with $\mathcal{I} = \mathrm{Ann}_{U(\mathfrak{g})}(\mathrm{L}(\lambda))$

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Main conjecture. Tensoring with $\mathcal{L}$ induces a bijection between isomorphism classes of simple $U(\mathfrak{g}_{\overline{0}})$-modules with annihilator J and isomorphism classes of simple $U(\mathfrak{g})$-supermodules with annihilator $\mathcal{I}$.

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The $\mathfrak{q}(2)$-example

Theorem. (V. M. 2010) The main conjecture is true for $\mathfrak{q}(2)$.

Root system: $\{\pm\alpha\}$

Alternatives: $\mu \in \{\lambda, \lambda - \alpha\}$ (depending on regularity, typicality etc.)

Bonus: Describes the rough structure of any simple $U(\mathfrak{q}(2))$-supermodule as a $U(\mathfrak{gl}(2))$-module

Very special feature: Every simple $U(\mathfrak{q}(2))$-supermodule is of finite length as a $U(\mathfrak{gl}(2))$-module

Rough structure: (O. Khomenko, V. M. 2004) Multiplicities of simple subquotients with "minimal possible" annihilators occurring in the module

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Rough structure conjecture

$\mathfrak{g}$ — classical Lie superalgebra

L — simple $\mathfrak{g}$-supermodule

$U(\mathfrak{g})$ is finite over $U(\mathfrak{g}_{\overline{0}})$

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$\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\overline{0}}}(L)$ does not have to be artinian (T. Stafford. 1985)

Rough structure conjecture. The rough structures of L and $\mathrm{L}(\lambda)$ "coincide" in the sense that under the bijection given by the main conjecture the multiplicities are preserved.

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Simple supermodules are submodules in induced modules

Lemma. Let L be a simple $\mathfrak{g}$-supermodule. Then there exists a simple $\mathfrak{g}_{\overline{0}}$-module N such that $L \subset \mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}}(N)$ or $L \subset \Pi\ \mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}}(N)$.

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Zorn's lemma implies that $\mathrm{Res}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}}(L)$ has a simple quotient, say N.

$\mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}} \cong \Pi^{\dim \mathfrak{g}_{\overline{1}}} \circ \mathrm{Coind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}}$

Adjunction: $\mathrm{Hom}_{\mathfrak{g}}(L, \mathrm{Coind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}}(N)) = \mathrm{Hom}_{\mathfrak{g}_{\overline{0}}}(\mathrm{Res}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}}(L), N) \neq 0.$

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$U(\mathfrak{g}_{\overline{0}})$ is noetherian, $\mathrm{Res}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}}(L)$ is noetherian

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Adjunction: $\mathrm{Hom}_{\mathfrak{g}}(L, \mathrm{Coind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}}(N)) = \mathrm{Hom}_{\mathfrak{g}_{\overline{0}}}(\mathrm{Res}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}}(L), N) \neq 0.$

Q.E.D.

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Simple supermodules are quotients of induced modules

Dual statement: Each simple supermodule is a quotient of an induced module.

Question: Is this true?

Idea: Same proof as above works?

Need: If L is a simple $\mathfrak{g}$-supermodule, then $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}}(L)$ has a simple submodule.

Note: This is obviously true if $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}}(L)$ has finite length.

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Main result

Theorem: Let $\mathfrak{a}$ be a finite dimensional reductive Lie algebra, V a simple $\mathfrak{a}$-module and E a simple finite dimensional $\mathfrak{a}$-module. Then $E \otimes V$ has a well-defined *socle*, that is there exists a unique submodule N of $E \otimes V$ which has the following properties:

1. N has finite length;
2. N is semi-simple;
3. any non-zero submodule of $E \otimes V$ intersects N in a non-zero way.

Corollary 1: Every simple $\mathfrak{g}$ supermodule has a well-defined socle (as a $\mathfrak{g}_{\overline{0}}$-module).

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Idea of the proof: projective functors

$\mathfrak{a}$ — be a finite dimensional reductive Lie algebra

$\mathcal{M}$ — the full subcategory in $\mathfrak{a}$-Mod consisting of modules on which the action of $Z(\mathfrak{a})$ is locally finite

$E \otimes \_ : \mathcal{M} \to \mathcal{M}$ — a projective functor (in the sense of I. Bernstein and S. Gelfand 1980)

Indecomposable projective functors are classified (I. Bernstein and S. Gelfand 1980)

The tensor category of projective functors is generated by:

1. Jantzen's translation functors (equivalences of categories);
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enough to prove the claim for indecomposable projective functors

induction reduces the claim to one of the three types of projective functors described above

for equivalences of categories the claim is obvious

for the translation to a wall the claim follows from (A. Beilinson and V. Ginzburg 1999)

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Main idea: Exploit the 2-categorical structure on the tensor category (2-category) of projective functors

the endomorphism algebra of the translation θ out of a wall is known (I. Bernstein and S. Gelfand 1980)

this endomorphism algebra is commutative, has simple socle, and $Z(\mathfrak{a})$ surjects onto it (this is the algebra of certain invariants in a certain coinvariant algebra), it is related to the endomorphism algebra of a certain projective in the BGG category $\mathcal{O}$

by noetherianity, we have at least one simple quotient of $\theta\, V$

applying the socle endomorphism of θ produces a simple submodule in $\theta\, V$

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Rough structure of supermodules: setup

$\mathfrak{a}$ — reductive finite dimensional Lie algebra **of type** A

V — simple $\mathfrak{a}$-module

$J := \mathrm{Ann}_{U(\mathfrak{a})}(V)$

λ — a weight such that $J = \mathrm{Ann}_{U(\mathfrak{a})}(L(\lambda))$

λ' — the most singular weight with comparable annihilator appearing in $\mathcal{JH}(E \otimes L(\lambda))$ where E is finite dimensional

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V' — the corresponding simple (sub)quotient of $E \otimes V$

Note: V is a quotient of $E^* \otimes V'$

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$\mathrm{Coker}(E \otimes V')$ — full subcategory of $\mathfrak{a}$-mod consisting of modules with presentation $X_1 \to X_0 \to M \to 0$ with $X_1, X_0 \in \mathrm{add}(E \otimes V')$ for some finite dimensional E

Proposition. V' is projective in $\mathrm{Coker}(E \otimes V')$ (compare with R. Irving and B. Shelton 1988)

Theorem. (V.M. and C. Stroppel 2008) $\mathrm{Coker}(E \otimes V')$ does not depend on V' (if J' is fixed), up to equivalence.

Corollary. The rough structure conjecture is true if $\mathfrak{g}_{\overline{0}}$ is of type A.

Consequently: Enough to describe the rough structure for highest weight supermodules.

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Rough structure of supermodules: the $\mathfrak{q}(2)$ example

α — the positive root

$L(0)$ — trivial supermodule

$L(\lambda)_{\overline{0}} \cong L(\lambda)_{\overline{1}}$ if $\lambda \neq 0$

Atypical $\lambda \neq 0$**:** $L(\lambda)_{\overline{0}} = \mathrm{L}^{\mathfrak{g}_{\overline{0}}}(\lambda)$

Regular typical $\lambda \neq 0$**:** $L(\lambda)_{\overline{0}} = \mathrm{L}^{\mathfrak{g}_{\overline{0}}}(\lambda) \oplus \mathrm{L}^{\mathfrak{g}_{\overline{0}}}(\lambda - \alpha)$

Singular typical $\lambda \neq 0$**:** $L(\lambda)_{\overline{0}}$ is indecomposable, $\mathrm{L}^{\mathfrak{g}_{\overline{0}}}(\lambda - \alpha) \hookrightarrow L(\lambda)_{\overline{0}} \twoheadrightarrow \mathrm{L}^{\mathfrak{g}_{\overline{0}}}(\lambda - \alpha)$, this sequence has one-dimensional homology (i.e. the **fine** structure is different from the rough structure)

Note Taking e.g. a simple dense $\mathfrak{g}$-supermodule with the same annihilator as $L(\lambda)$, the corresponding sequence will be **exact**, that is in this case the fine structure coincides with the rough structure.

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THANK YOU!!!