

Interaction of Ringel and Koszul dualities for quasi-hereditary algebras

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1. Notation for quasi-hereditary algebras

\mathbb{k} — algebraically closed field.

A — basic finite-dimensional \mathbb{k} -algebra.

$\{1, 2, \dots, n\}$ — indexing set for iso-classes of simple A -modules.

\leq — natural linear order on $\{1, 2, \dots, n\}$.

We assume that (A, \leq) is *quasi-hereditary*.

$L(\mathbf{i}), \mathbf{i} \in \Lambda$, — simple A -modules.

$P(\mathbf{i}), \mathbf{i} \in \Lambda$, — indecomposable projective A -modules.

$I(\mathbf{i}), \mathbf{i} \in \Lambda$, — indecomposable injective A -modules.

$\Delta(\mathbf{i}), \mathbf{i} \in \Lambda$, — standard A -modules.

$\nabla(\mathbf{i}), \mathbf{i} \in \Lambda$, — costandard A -modules.

$T(\mathbf{i}), \mathbf{i} \in \Lambda$, — indecomposable tilting A -modules.

$L = \bigoplus_{\mathbf{i}=1}^n L(\mathbf{i}), P = \bigoplus_{\mathbf{i}=1}^n P(\mathbf{i}),$ etc.

2. The Ringel dual of a quasi-hereditary algebra

$R(A) = \text{End}_A(T)$ — the *Ringel dual* of A .

$(R(A), \leq^{opp})$ — quasi-hereditary.

$(R(R(A)), \leq) \cong (A, \leq)$.

3. Standard Koszul quasi-hereditary algebras

Graded means \mathbb{Z} -graded.

Assume that $A = \bigoplus_{i \geq 0} A_i$, $\text{Rad}(A) = \bigoplus_{i > 0} A_i$ (i.e. A is *positively graded*).

Then L , P , I , T , Δ , and ∇ are gradeable.

Fix the following gradings:

Concentrate L in degree 0.

Fix grading on P such that $P \twoheadrightarrow L$ has degree 0.

Fix grading on Δ such that $\Delta \twoheadrightarrow L$ has degree 0.

Fix grading on I such that $L \hookrightarrow I$ has degree 0.

Fix grading on ∇ such that $L \hookrightarrow \nabla$ has degree 0.

Fix grading on T such that $\Delta \hookrightarrow T$ has degree 0.

This gives $R(A)$ a canonical *induced* grading.

$\langle 1 \rangle$ — the shift of grading (moves 0 to -1).

Definition. Let M be some fixed graded A -module. A complex,

$$\text{Com}(A) \ni \mathcal{C}^\bullet : \quad \dots \rightarrow \mathcal{C}^{i-1} \rightarrow \mathcal{C}^i \rightarrow \mathcal{C}^{i+1} \rightarrow \dots$$

is called *linear* (w.r.t. M) provided that $\mathcal{C}^i \in \text{Add}(M\langle i \rangle)$.

Definition. (Ágoston-Dlab-Lukács) A is called *standard Koszul* if Δ admits a linear projective resolution and ∇ admits a linear injective coresolution.

Recall that a positively graded algebra is called *Koszul* provided that L admits a linear projective resolution.

$E(A) = \text{Ext}_A^*(L, L)$ — the extension algebra of L .

Theorem. (Ágoston-Dlab-Lukács) If A is standard Koszul, then A is Koszul, moreover, $(E(A), \leq^{opp})$ is quasi-hereditary and standard Koszul.

Note that $E(E(A)) \cong A$ by Koszulity.

$E(A)$ — Koszul dual of A .

Question. Do Ringel and Koszul dualities commute?

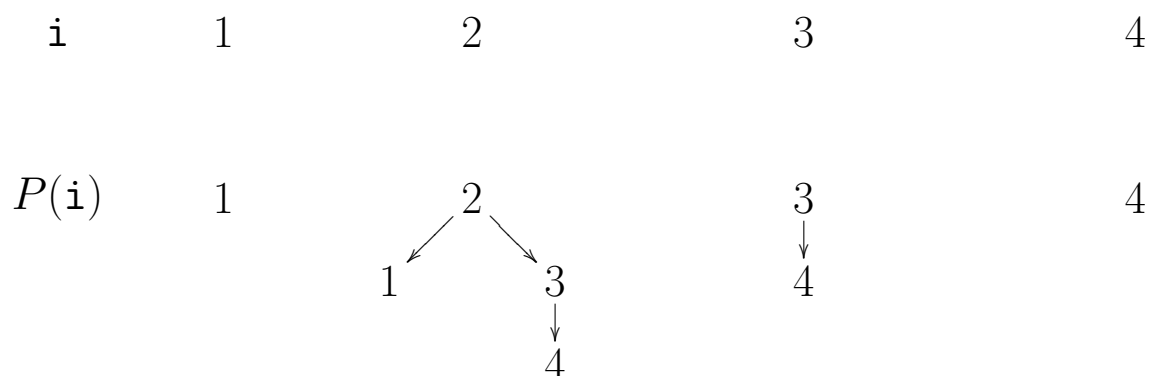
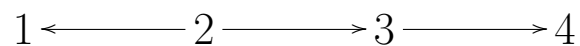
4. The answer is “no in general”, an example

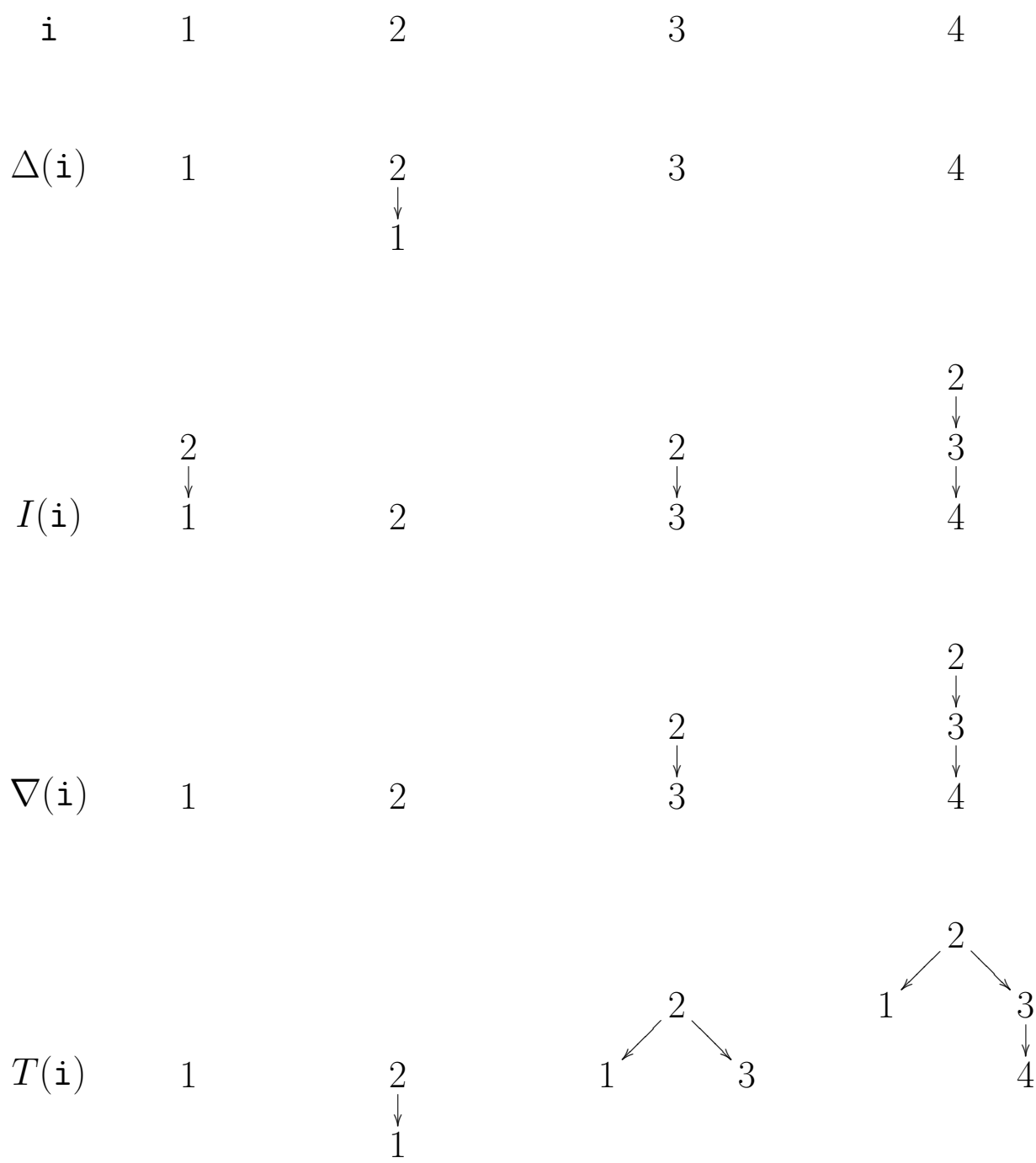
It is natural to restrict ourselves to standard Koszul algebras.

Statement. The Ringel dual of a standard Koszul quasi-hereditary algebra does not have to be standard Koszul.

Reason: $R(A)$ does not have to be positively graded!

Example: A is the path algebra of





There is $0 \neq f \in \text{Hom}_A(T(1), T(3))$ of degree 0.

5. Making the induced grading on $R(A)$ positive

Theorem. Let A be a positively graded quasi-hereditary algebra. Then the following conditions are equivalent.

1. A is standard Koszul and the induced grading on $R(A)$ is positive.
2. Δ admits a linear coresolution by tilting modules, and ∇ admits a linear resolution by tilting modules.

In the example above Δ admits a linear coresolution by tilting modules, but the resolution of ∇ by tilting modules is *not* linear.

Definition. The algebra A satisfying the equivalent conditions above will be called an *SCT-algebra* (abbreviating standard-co-standard-tilting).

Proposition. A is an SCT-algebra if and only if A^{opp} is an SCT-algebra if and only if $R(A)$ is an SCT-algebra.

Conjecture A. Let A be an SCT-algebra, then the algebras $R(A)$, $E(A)$, $R(E(A))$ and $E(R(A))$ are SCT as well and $R(E(A)) \cong E(R(A))$ as quasi-hereditary algebras.

6. The main result

Definition. The SCT-algebra A is called *balanced* provided that $E(R(A))$ is an SCT-algebra.

Main Theorem. Let A be balanced, then the algebras $R(A)$, $E(A)$, $R(E(A))$ and $E(R(A))$ are balanced as well and $R(E(A)) \cong E(R(A))$ as quasi-hereditary algebras.

Examples of balanced algebras: Associative algebras of the blocks of the BGG category \mathcal{O} .

Conjecture B. Each SCT-algebra is balanced.

Conjecture B implies Conjecture A.

7. The category of linear complexes of tilting modules

Need: a realization for $E(R(A))$ using A -modules.

Denote by \mathcal{T} the category of linear complexes of tilting A -modules, i.e. $\mathcal{C}^\bullet \in \mathcal{T}$ if and only if $\mathcal{C}^i \in \text{Add}(T\langle i \rangle)$ for all $i \in \mathbb{Z}$.

Proposition. Let A be balanced. Then the category \mathcal{T} is equivalent to the category of locally finite-dimensional graded $E(R(A))^{opp}$ -modules.

Idea of the proof: Using the graded Ringel duality we can reformulate the statement in the following way: the category of linear complexes of projective $R(A)$ -modules is equivalent to the category of locally finite-dimensional graded $E(R(A))^{opp}$ -modules.

A is balanced, hence SCT, hence $R(A)$ is SCT as well. In particular, $R(A)$ is Koszul, and hence $E(R(A))^{opp} \cong R(A)^\dagger$, the quadratic dual of $R(A)$ (i.e. if $R(A) = \bigoplus_{i \in \mathbb{Z}} R_i$, then $R(A)^\dagger = R_0[R_1^*]/(\mu^*(R_1^*))$, where μ is the multiplication in R).

Now everything follows from the main result of [R. Martinez Villa, M. Saorin, Koszul equivalences and dualities. Pacific J. Math. 214 (2004), no. 2, 359–378]. **Q.E.D.**

Note: $R(A)$ is standard Koszul, hence $(E(R(A))^{opp}, \leq)$ is quasi-hereditary.

What are structural $(E(R(A))^{opp}, \leq)$ -modules inside \mathcal{T} ?

Obvious: simple $(E(R(A))^{opp}, \leq)$ -modules are indecomposable tilting A -modules.

Easy: standard $(E(R(A))^{opp}, \leq)$ -modules are linear tilting coresolutions of standard A -modules.

Easy: costandard $(E(R(A))^{opp}, \leq)$ -modules are linear tilting resolutions of costandard A -modules.

Proposition. Let A be balanced. Then L is isomorphic in $D^b(A)$ to a linear complex of tilting A -modules. The corresponding complex is the characteristic tilting module in \mathcal{T} .

Final remark: For balanced A it is now not difficult to show that \mathcal{T} is equivalent to the category of locally finite-dimensional $R(E(A))^{opp}$ -modules, implying $E(R(A)) \cong R(E(A))$.