Interaction of Ringel and Koszul dualities for quasi-hereditary algebras

Volodymyr Mazorchuk

Uppsala University, SWEDEN

Joint work with Serge Ovsienko, Kyiv University.

1. Notation for quasi-hereditary algebras

$\Bbbk$ — algebraically closed field.

A — basic finite-dimensional $\Bbbk$-algebra.

$\{1, 2, \ldots, n\}$ — indexing set for iso-classes of simple A-modules.

$\leq$ — natural linear order on $\{1, 2, \ldots, n\}$.

We assume that $(A, \leq)$ is *quasi-hereditary*.

$L(\mathtt{i})$, $\mathtt{i} \in \Lambda$, — simple A-modules.

$P(\mathtt{i})$, $\mathtt{i} \in \Lambda$, — indecomposable projective A-modules.

$I(\mathtt{i})$, $\mathtt{i} \in \Lambda$, — indecomposable injective A-modules.

$\Delta(\mathtt{i})$, $\mathtt{i} \in \Lambda$, — standard A-modules.

$\nabla(\mathtt{i})$, $\mathtt{i} \in \Lambda$, — costandard A-modules.

$T(\mathtt{i})$, $\mathtt{i} \in \Lambda$, — indecomposable tilting A-modules.

$L = \oplus_{\mathtt{i}=1}^{n} L(\mathtt{i})$, $P = \oplus_{\mathtt{i}=1}^{n} P(\mathtt{i})$, etc.

2. The Ringel dual of a quasi-hereditary algebra

$R(A) = \mathrm{End}_A(T)$ — the *Ringel dual* of A.

$(R(A), \leq^{opp})$ — quasi-hereditary.

$(R(R(A)), \leq) \cong (A, \leq)$.

3. Standard Koszul quasi-hereditary algebras

Graded means $\mathbb{Z}$-*graded.*

Assume that $A = \oplus_{i\geq 0} A_i$, $\mathrm{Rad}(A) = \oplus_{i>0} A_i$ (i.e. A is *positively graded*).

Then L, P, I, T, Δ, and ∇ are gradeable.

Fix the following gradings:

Concentrate L in degree 0.

Fix grading on P such that $P \twoheadrightarrow L$ has degree 0.

Fix grading on Δ such that $\Delta \twoheadrightarrow L$ has degree 0.

Fix grading on I such that $L \hookrightarrow I$ has degree 0.

Fix grading on ∇ such that $L \hookrightarrow \nabla$ has degree 0.

Fix grading on T such that $\Delta \hookrightarrow T$ has degree 0.

This gives $R(A)$ a canonical *induced* grading.

$\langle 1 \rangle$ — the shift of grading (moves 0 to -1).

Definition. Let M be some fixed graded A-module. A complex,

$$\mathrm{Com}(A) \ni \mathcal{C}^\bullet : \quad \dots \to \mathcal{C}^{i-1} \to \mathcal{C}^i \to \mathcal{C}^{i+1} \to \dots$$

is called *linear* (w.r.t. M) provided that $\mathcal{C}^i \in \mathrm{Add}(M\langle i\rangle)$.

Definition. (Ágoston-Dlab-Lukács) A is called *standard Koszul* if Δ admits a linear projective resolution and ∇ admits a linear injective coresolution.

Recall that a positively graded algebra is called *Koszul* provided that L admits a linear projective resolution.

$E(A) = \mathrm{Ext}_A^*(L, L)$ — the extension algebra of L.

Theorem. (Ágoston-Dlab-Lukács) If A is standard Koszul, then A is Koszul, moreover, $(E(A), \leq^{opp})$ is quasi-hereditary and standard Koszul.

Note that $E(E(A)) \cong A$ by Koszulity.

$E(A)$ — Koszul dual of A.

Question. Do Ringel and Koszul dualities commute?

4. The answer is “no in general”, an example

It is natural to restrict ourselves to standard Koszul algebras.

Statement. The Ringel dual of a standard Koszul quasi-hereditary algebra does not have to be standard Koszul.

Reason: $R(A)$ does not have to be positively graded!

Example: A is the path algebra of

$$1 \longleftarrow 2 \longrightarrow 3 \longrightarrow 4$$

$$\begin{array}{ccccc} \mathtt{i} & 1 & 2 & 3 & 4 \\ \\ P(\mathtt{i}) & 1 & \begin{tikzcd}[column sep=small] & 2 \arrow[dl] \arrow[dr] & \\ 1 & & 3 \arrow[d] \\ & & 4 \end{tikzcd} & \begin{tikzcd} 3 \arrow[d] \\ 4 \end{tikzcd} & 4 \end{array}$$

$\mathtt{i}$	1	2	3	4
$\Delta(\mathtt{i})$	1	$\begin{smallmatrix} 2 \\ \downarrow \\ 1 \end{smallmatrix}$	3	4
$I(\mathtt{i})$	$\begin{smallmatrix} 2 \\ \downarrow \\ 1 \end{smallmatrix}$	2	$\begin{smallmatrix} 2 \\ \downarrow \\ 3 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow \\ 3 \\ \downarrow \\ 4 \end{smallmatrix}$
$\nabla(\mathtt{i})$	1	2	$\begin{smallmatrix} 2 \\ \downarrow \\ 3 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow \\ 3 \\ \downarrow \\ 4 \end{smallmatrix}$
$T(\mathtt{i})$	1	$\begin{smallmatrix} 2 \\ \downarrow \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} & 2 & \\ \swarrow & & \searrow \\ 1 & & 3 \end{smallmatrix}$	$\begin{smallmatrix} & 2 & \\ \swarrow & & \searrow \\ 1 & & 3 \\ & & \downarrow \\ & & 4 \end{smallmatrix}$

There is $0 \neq f \in \mathrm{Hom}_A(T(1), T(3))$ of degree 0.

5. Making the induced grading on $R(A)$ positive

Theorem. Let A be a positively graded quasi-hereditary algebra. Then the following conditions are equivalent.

1. A is standard Koszul and the induced grading on $R(A)$ is positive.
2. Δ admits a linear coresolution by tilting modules, and ∇ admits a linear resolution by tilting modules.

In the example above Δ admits a linear coresolution by tilting modules, but the resolution of ∇ by tilting modules is *not* linear.

Definition. The algebra A satisfying the equivalent conditions above will be called an *SCT-algebra* (abbreviating standard-co-standard-tilting).

Proposition. A is an SCT-algebra if and only if A^{opp} is an SCT-algebra if and only if $R(A)$ is an SCT-algebra.

Conjecture A. Let A be an SCT-algebra, the the algebras $R(A)$, $E(A)$, $R(E(A))$ and $E(R(A))$ are SCT as well and $R(E(A)) \cong E(R(A))$ as quasi-hereditary algebras.

6. The main result

Definition. The SCT-algebra A is called *balanced* provided that $E(R(A))$ is an SCT-algebra.

Main Theorem. Let A be balanced, then the algebras $R(A)$, $E(A)$, $R(E(A))$ and $E(R(A))$ are balanced as well and $R(E(A)) \cong E(R(A))$ as quasi-hereditary algebras.

Examples of balanced algebras: Associative algebras of the blocks of the BGG category $\mathcal{O}$.

Conjecture B. Each SCT-algebra is balanced.

Conjecture B implies Conjecture A.

7. The category of linear complexes of tilting modules

Need: a realization for $E(R(A))$ using A-modules.

Denote by $\mathcal{T}$ the category of linear complexes of tilting A-modules, i.e. $\mathcal{C}^\bullet \in \mathcal{T}$ if and only if $\mathcal{C}^i \in \mathrm{Add}(T\langle i\rangle)$ for all $i \in \mathbb{Z}$.

Proposition. Let A be balanced. Then the category $\mathcal{T}$ is equivalent to the category of locally finite-dimensional graded $E(R(A))^{opp}$-modules.

Idea of the proof: Using the graded Ringel duality we can reformulate the statement in the following way: the category of linear complexes of projective $R(A)$-modules is equivalent to the category of locally finite-dimensional graded $E(R(A))^{opp}$-modules.

A is balanced, hence SCT, hence $R(A)$ is SCT as well. In particular, $R(A)$ is Koszul, and hence $E(R(A))^{opp} \cong R(A)^!$, the quadratic dual of $R(A)$ (i.e. if $R(A) = \oplus_{i\in\mathbb{Z}} R_i$, then $R(A)^! = R_0[R_1^*]/(\mu^*(R_1^*))$, where μ is the multiplication in R).

Now everything follows from the main result of [R. Martinez Villa, M. Saorin, Koszul equivalences and dualities. Pacific J. Math. 214 (2004), no. 2, 359–378]. **Q.E.D.**

Note: $R(A)$ is standard Koszul, hence $(E(R(A))^{opp}, \leq)$ is quasi-hereditary.

What are structural $(E(R(A))^{opp}, \leq)$-modules inside $\mathcal{T}$?

Obvious: simple $(E(R(A))^{opp}, \leq)$-modules are indecomposable tilting A-modules.

Easy: standard $(E(R(A))^{opp}, \leq)$-modules are linear tilting coresolutions of standard A-modules.

Easy: costandard $(E(R(A))^{opp}, \leq)$-modules are linear tilting resolutions of costandard A-modules.

Proposition. Let A be balanced. Then L is isomorphic in $D^b(A)$ to a linear complex of tilting A-modules. The corresponding complex is the characteristic tilting module in $\mathcal{T}$.

Final remark: For balanced A it is now not difficult to show that $\mathcal{T}$ is equivalent to the category of locally finite-dimensional $R(E(A))^{opp}$-modules, implying $E(R(A)) \cong R(E(A))$.