Koszul duality between generalized Takiff Lie algebras and superalgebras

(joint work with Jacob Greenstein)

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Lie Superalgebras
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Beilinson-Ginzburg-Soergel setup

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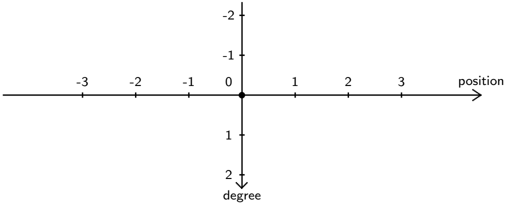

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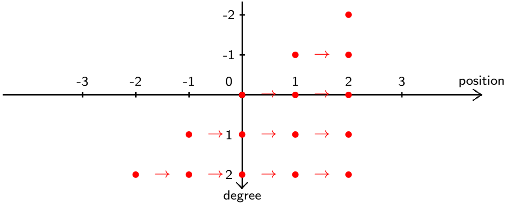

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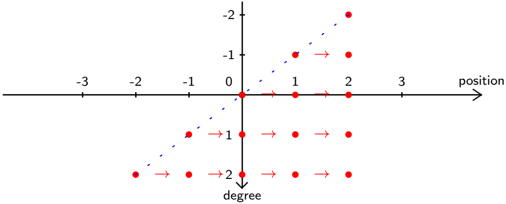

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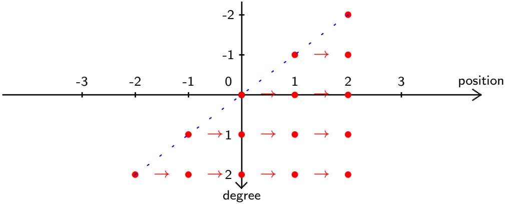

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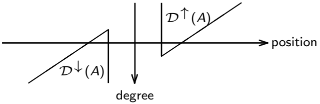

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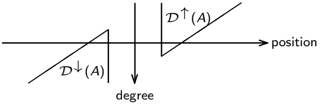

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