Algebraic categorification and its applications, I

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Categorification in short

"Definition"

Roughly speaking, categorification means an "upgrade" from set theory to category theory, in particular:

sets are upgraded to categories

functions are upgraded to functors

equalities are upgraded to isomorphisms

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Motivation

Question: Why do we need categorification?

Answer: Categories have more structure than sets.

This can be used to get new useful information about objects we study.

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L — diagram of an oriented link

n_+ — number of right crossings

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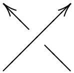

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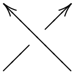

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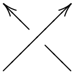

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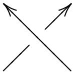

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Example: Khovanov homology — Kauffman bracket

Definition. The Kauffman bracket $\{L\} \in \mathbb{Z}[v, v^{-1}]$ of L is defined via the following rule:

together with $\{\bigcirc L\} = (v + v^{-1})\{L\}$

and normalized by the conditions $\{\emptyset\} = 1$.

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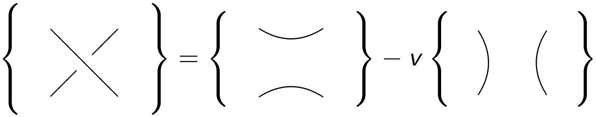

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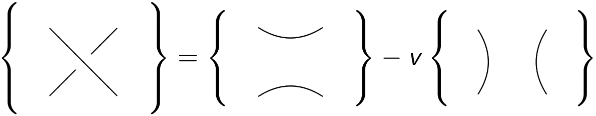

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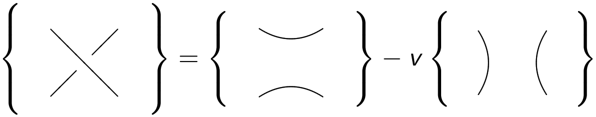

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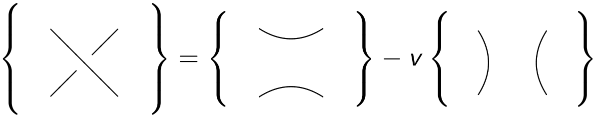

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Example: Khovanov homology — Jones polynomial

Definition. The unnormalized Jones polynomial $\hat{\mathrm{J}}(L)$ of L is defined by

$$\hat{\mathrm{J}}(L) := (-1)^{n_-} v^{n_+ - 2n_-}\{L\} \in \mathbb{Z}[v, v^{-1}]$$

Definition. The (usual) Jones polynomial $\mathrm{J}(L)$ is defined via $(v + v^{-1})\mathrm{J}(L) = \hat{\mathrm{J}}(L)$.

Theorem. [Jones] $\mathrm{J}(L)$ is an invariant of an oriented link.

Example. For the Hopf link

$H :=$

we have $\hat{\mathrm{J}} = (v + v^{-1})(v + v^5)$ and $\mathrm{J}(H) = v + v^5$.

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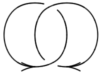
$H :=$ 

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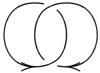
$H :=$ 

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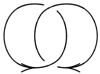
$H :=$ 

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Example: Khovanov homology — characterization of J

Theorem. The Jones polynomial is uniquely determined by the property $\mathrm{J}(\bigcirc) = 1$

and the skein relation

$v^2\mathrm{J}\Bigg($ $\Bigg) - v^{-2}\mathrm{J}\Bigg($ $\Bigg) = (v - v^{-1})\mathrm{J}\Bigg($ $\Bigg)$

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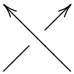
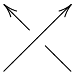
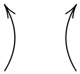
$v^2\mathrm{J}\Bigg($  $\Bigg) - v^{-2}\mathrm{J}\Bigg($  $\Bigg) = (v - v^{-1})\mathrm{J}\Bigg($  $\Bigg)$

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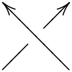
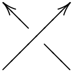
$v^2\mathrm{J}\big($  $\big) - v^{-2}\mathrm{J}\big($  $\big) = (v - v^{-1})\mathrm{J}\big($  $\big)$

Example: Khovanov homology — idea and ingredients

Main idea: [Khovanov] Upgrade Kauffman bracket to a new bracket $[\![\cdot]\!]$

$\mathbb{C}$-mod — category of finite dimensional $\mathbb{C}$-vector spaces

$\mathbb{C}$-gmod — category of finite dimensional graded $\mathbb{C}$-vector spaces

$\mathrm{Com}^b(\mathbb{C}\text{-gmod})$ — category of finite complexes over $\mathbb{C}$-gmod

$[\![\cdot]\!]$ takes values in $\mathrm{Com}^b(\mathbb{C}\text{-mod})$

V — $\mathbb{C}$ in degree 1 $\quad\oplus\quad$ $\mathbb{C}$ in degree -1

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Example: Khovanov homology — categorification

Categorification of normalization conditions:

$[\![\varnothing]\!] = 0 \to \mathbb{C} \to 0$

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Categorification of the Kauffman bracket:

Main difficulty: Definition of d.

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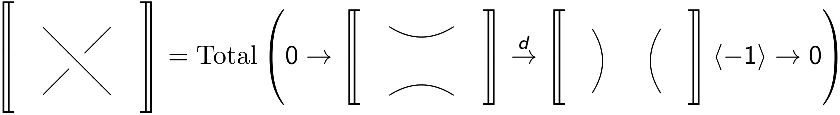

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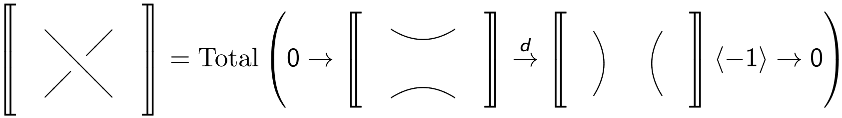

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Example: Khovanov homology — the result

$[\cdot]$ — shift in homological position

$\langle\cdot\rangle$ — shift in grading

Theorem. [Khovanov]
Homology of $[\![\cdot]\!][n_-]\langle n_+ - 2n_-\rangle$ is an invariant of an oriented link.

Note: $[\![\cdot]\!][n_-]\langle n_+ - 2n_-\rangle$ is not an invariant of an oriented link.

Decategorification theorem. [Khovanov]
Graded Euler characteristic of $[\![L]\!][n_-]\langle n_+ - 2n_-\rangle$ equals $\hat{\mathrm{J}}(L)$.

Benefit: Khovanov homology is a strictly stronger invariant than Jones polynomial. For example:

Theorem. [Kronheimer-Mrowka] Khovanov homology detects the unknot.

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Alternative approach — Alexander theorem

Theorem. [Alexander] Every link is a closure of a braid

Elementary diagrams:

the cup diagram the cap diagram right crossing left crossing

Corollary. Every oriented link is a composition of elementary diagrams.

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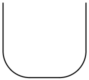

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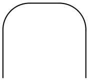
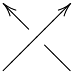


the cap diagram



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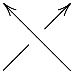

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Corollary. Every oriented link is a composition of elementary diagrams.

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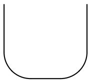
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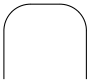
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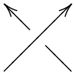
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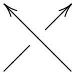
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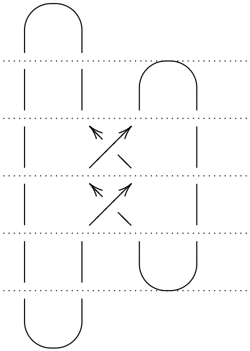

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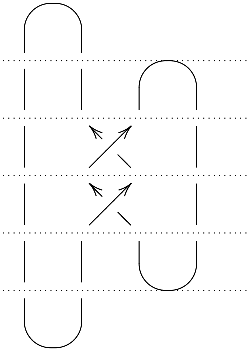

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$\mathfrak{g}$ — simple finite dimensional Lie algebra

$U(\mathfrak{g})$ — the universal enveloping algebra of $\mathfrak{g}$

Fact. $U(\mathfrak{g})$ is a cocommutative Hopf algebra.

Consequence. The isomorphism $V \otimes W \cong W \otimes V$ is involutive.

$U_v(\mathfrak{g})$ — the quantum enveloping algebra of $\mathfrak{g}$

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Alternative approach — tangles

Tang — the category of oriented tangles

Objects: Non-negative integers

Informally: $n \in \{0, 1, 2, \dots\}$ should be thought of as a collection of n points.

Morphisms: Oriented diagrams generated by (oriented) elementary diagrams (up to isotopy), connecting the corresponding points, read from bottom to top.

Composition: Concatenation

Example 1: An oriented cup diagram is a morphism from 0 to 2.

Example 2: An oriented cap diagram is a morphism from 2 to 0.

Example 3: An oriented crossing is a morphism from 2 to 2.

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Alternative approach — idea of quantum knot invariants

Idea of quantum knot invariants. [Reshetikhin-Turaev]

Consider some $U_v(\mathfrak{g})$.

V — the "natural" $U_v(\mathfrak{g})$-module

Define a functor $\mathrm{F} : \mathbf{Tang} \to U_v(\mathfrak{g})$-mod

$\mathrm{F}(n) := V^{\otimes n}$, where $\mathrm{F}(0) := \mathbb{C}(v)$

F(elementary diagram) := certain explicit homomorphisms of $U_v(\mathfrak{g})$-modules

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Consequence: $\mathrm{F}(T_L)(1)$ is an invariant of L.

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Definition: $U_v(\mathfrak{sl}_2)$ has generators E, F, K, K^{-1} and relations

$$KE = v^2 EK, \qquad KF = v^{-2} FK, \qquad KK^{-1} = K^{-1}K = 1,$$

$$EF - FE = \frac{K - K^{-1}}{v - v^{-1}}.$$

Hopf structure:

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}.$$

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$$KE = v^2EK, \qquad KF = v^{-2}FK, \qquad KK^{-1} = K^{-1}K = 1,$$

$$EF - FE = \frac{K - K^{-1}}{v - v^{-1}}.$$

Hopf structure:

$$\Delta(E) = 1\otimes E + E\otimes K, \quad \Delta(F) = K^{-1}\otimes F + F\otimes 1, \quad \Delta(K^{\pm 1}) = K^{\pm 1}\otimes K^{\pm 1}.$$

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Quantum invariants — natural $U_v(\mathfrak{sl}_2)$-module

Quantum numbers: $[a] := \frac{v^a - v^{-a}}{v - v^{-1}}$, $a \in \mathbb{Z}$

V — the "natural" $U_v(\mathfrak{sl}_2)$-module

Basis: w_0 and w_1

Action:

$$Ew_k = [k+1]w_{k+1}, \qquad Fw_k = -[n-k+1]w_{k-1},$$
$$K^{\pm 1}w_k = -v^{\pm(2k-n)}w_k$$

Notation: $w_{i_1} \otimes w_{i_2} \otimes \cdots \otimes w_{i_k}$ denoted by $i_1 i_2 \dots i_k$

Consequence: Basis in $V^{\otimes n}$ consists of 0-1–sequences of length n.

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Quantum invariants — action of tangles

Definition. The functor $\mathrm{F} : \mathbf{Tang} \to U_v(\mathfrak{sl}_2)$-mod is given by:

$\cup : \mathbb{C}(v) \to \hat{\mathcal{V}}_1^{\otimes 2}$ is given by:

$$1 \mapsto 01 + v10.$$

$\cap : \hat{\mathcal{V}}_1^{\otimes 2} \to \mathbb{C}(v)$ is given by:

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right crossing: $\hat{\mathcal{V}}_1^{\otimes 2} \to \hat{\mathcal{V}}_1^{\otimes 2}$ is given by:

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Quantum $U_v(\mathfrak{sl}_2)$-invariants and Jones polynomial

Theorem. [Reshetikhin-Turaev]

Let L be an oriented link. Then

the polynomials $\mathrm{F}(T_L)(1)$ and $\hat{\mathrm{J}}(L)$ coincide.

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Cat — category of categories

Idea: Construct a functor from **Tang** to **Cat**?

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Approach via category $\mathcal{O}$

$\mathfrak{gl}_n$ — reductive Lie algebra over $\mathbb{C}$

$\mathfrak{gl}_n = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — standard triangular decomposition

$\mathcal{O}$ — BGG category $\mathcal{O}$

S_n — the Weyl group of $\mathfrak{gl}_n$

Fact: S_n acts on $\mathfrak{h}^*$ in the natural way

$M(\lambda)$ — Verma module with highest weight $\lambda - \rho$

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Blocks in $\mathcal{O}$

$\mathcal{O}_0$ — the principal block of $\mathcal{O}$

$k \in \{0, 1, 2, \dots, n\}$

$S_k \times S_{n-k} \subset S_N$ — maximal Young subgroup

$\mathfrak{p}_k$ — corresponding parabolic subalgebra

$\mathcal{O}_0^{(k,n-k)}$ — parabolic subcategory of locally $\mathfrak{p}_k$-finite modules

Definition: $\mathcal{C}_n := \bigoplus_{k=0}^{n} \mathcal{O}_0^{(k,n-k)}$

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Definition: $\mathrm{Gr}(\mathcal{C}_n)$ is the Grothendieck group of $\mathcal{C}_n$

Blocks in $\mathcal{O}$

$\mathcal{O}_0$ — the principal block of $\mathcal{O}$

$k \in \{0, 1, 2, \dots, n\}$

$S_k \times S_{n-k} \subset S_N$ — maximal Young subgroup

$\mathfrak{p}_k$ — corresponding parabolic subalgebra

$\mathcal{O}_0^{(k,n-k)}$ — parabolic subcategory of locally $\mathfrak{p}_k$-finite modules

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Categorification of $V^{\otimes n}$ for $v = 1$

Observation: $\dim V^{\otimes n} = \mathrm{rank}(\mathrm{Gr}(\mathcal{C}_n))$

$\mathfrak{p} \subset \mathfrak{q}$ — parabolic subalgebras

$\mathrm{I}_{(\mathfrak{p},\mathfrak{q})} : \mathcal{O}^{\mathfrak{q}} \subset \mathcal{O}^{\mathfrak{p}}$ — natural inclusion

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Note: $\mathrm{Z}_{(\mathfrak{p},\mathfrak{q})}$ is only right exact

Action: $\mathrm{E} : \mathcal{D}^b(\mathcal{O}^{(k,n-k)}) \xrightarrow{\mathrm{I}} \mathcal{D}^b(\mathcal{O}^{(k,1,n-k-1)}) \xrightarrow{\mathcal{L}\mathrm{Z}} \mathcal{D}^b(\mathcal{O}^{(k+1,n-k-1)})$

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Theorem.[Bernstein-Frenkel-Khovanov] This categorifies $V^{\otimes n}$ for $v = 1$.

Meaning: Taking the Grothendieck group results in $V^{\otimes n}$

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Question: Where can we find v?

Answer: Introduce grading.

Theorem. [Soergel] Each block of (parabolic) $\mathcal{O}$ is equivalent to the category of (ungraded) modules over a finite dimensional positively graded and even Koszul algebra.

$\tilde{\mathcal{C}}_n$ — graded version of $\mathcal{C}_n$

Theorem. [Stroppel] The action of graded Zuckerman functors on $\mathcal{D}^b(\tilde{\mathcal{C}}_n)$ categorifies $V^{\otimes n}$

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Definition. [Bernstein-S. Gelfand] A projective functor on $\mathcal{O}$ is a functor isomorphic to a direct summand of tensoring with a finite dimensional module.

Fact. Projective functors commute with Zuckerman functors.

Need: Categorification of $V^{\otimes m}$ for $m < n$

Use: Singular and singular-parabolic blocks of $\mathcal{O}$

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Categorification of quantum $U_v(\mathfrak{sl}_2)$-invariants — shuffling functors

$s \in S_n$ — simple reflection

θ_s — wall-crossing functor

Fact There are adjunctions $\theta_s \to \mathrm{Id}$ and $\mathrm{Id} \to \theta_s$

Definition.[Carlin] Shuffling functor $C_s := \mathrm{Coker}(\mathrm{Id} \to \theta_s)$ (adjoint: coshuffling)

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Theorem.[Stroppel] For L oriented link, the functor $\mathcal{F}(T_L)[n_-]\langle n_+ - 2n_-\rangle$ is an invariant of L.

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