Algebraic categorification and its applications, II

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Complex representations of symmetric groups

S_n — the symmetric group on $\{1, 2, \dots, n\}$

$\mathbf{P}_n := \{\lambda = (\lambda_1, \dots, \lambda_k) : \lambda_1 \geq \dots \geq \lambda_k, \lambda_1 + \dots + \lambda_k = n\}$

$\lambda \in \mathbf{P}_n$ is called a partition of n, denoted $\lambda \vdash n$

$\mathcal{S}^\lambda$ — the Specht module associated to λ

Theorem. $\{\mathcal{S}^\lambda : \lambda \vdash n\}$ is a cross-section of isomorphism classes of simple S_n-modules.

Examples:

- $\mathcal{S}^{(n)}$ is the trivial module
- $\mathcal{S}^{(1,1,\dots,1)}$ is the sign module
- $\mathcal{S}^{(n)} \oplus \mathcal{S}^{(n-1,1)}$ is the natural module

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$\mathfrak{g} := \mathfrak{sl}_n(\mathbb{C})$

$\mathcal{O}$ — BGG category $\mathcal{O}$

$\mathcal{O}_0$ — principal block of $\mathcal{O}$

S_n — Weyl group of $\mathfrak{g}$

$M(\mu)$ — Verma module with highest weight μ

$L(\mu)$ — unique simple quotient of $M(\mu)$

Theorem. $\{L(w) := L(w \cdot 0) : w \in S_n\}$ is a cross-section of isomorphism classes of simple objects in $\mathcal{O}_0$

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Grothendieck group and different bases

Corollary. $\mathrm{Gr}(\mathcal{O}_0) \cong \mathbb{Z}[S_n]$.

Note. $\{[L(w)] : w \in S_n\}$ is the natural basis in $\mathrm{Gr}(\mathcal{O}_0)$.

$\Delta(w) := M(w \cdot 0)$

Fact. $\{[\Delta(w)] : w \in S_n\}$ is the standard basis in $\mathrm{Gr}(\mathcal{O}_0)$.

Reason: $[\Delta(x) : L(y)] \neq 0$ implies $x \leq y$ and $[\Delta(x) : L(x)] = 1$.

Fact. $\mathcal{O}_0$ has finite global dimension.

$P(w)$ — the indecomposable projective cover of $L(w)$

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Tilting modules

Theorem.[Collingwood-Irving, Ringel] For $w \in S_n$ there is a unique indecomposable module $T(w)$ such that

- $\Delta(w) \subset T(w)$ and the cokernel has a Verma flag;
- $T(w)$ is self-dual.

$T(w)$ — tilting module

Fact. $\{[T(w)] : w \in S_n\}$ is a basis in $\mathrm{Gr}(\mathcal{O}_0)$.

Reason: Extensions between Vermas are directed.

Question. Which bases in $\mathbb{Z}[S_n]$ correspond to:

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- $\Delta(w) \subset T(w)$ and the cokernel has a Verma flag;
- $T(w)$ is self-dual.

$T(w)$ — tilting module

Fact. $\{[T(w)] : w \in S_n\}$ is a basis in $\mathrm{Gr}(\mathcal{O}_0)$.

Reason: Extensions between Vermas are directed.

Question. Which bases in $\mathbb{Z}[S_n]$ correspond to:

- $\{[L(w)] : w \in S_n\}$?
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Projective functors

Recall: A projective functor $\theta : \mathcal{O}_0 \to \mathcal{O}_0$ is a direct summand of $V \otimes_{\mathbb{C}} \_$, where V is finite dimensional

Theorem. [Bernstein-S. Gelfand]

(a) There is a unique (up to isomorphism) indecomposable projective functor θ_w such that $\theta_w P(e) \cong P(w)$.

(b) $\{\theta_w : w \in S_n\}$ is a cross-section of isomorphism classes of indecomposable projective endofunctors of $\mathcal{O}_0$.

Definition. $\mathcal{P}$ — the category of projective functors.

Note: $\mathcal{P}$ is additive, idempotent split with finitely many indecomposables up to isomorphism, it has the structure of a tensor category.

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Observation. For s simple reflection and $w \in S_n$ there are s.e.s.

$\Delta(ws) \hookrightarrow \theta_s \Delta(w) \twoheadrightarrow \Delta(w)$ if $ws > w$,

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Fact. $\mathcal{P}$ is generated by θ_s, s simple reflection, as a tensor category.

Corollary. $\mathrm{Gr}_{\oplus}(\mathcal{P}) \cong \mathbb{Z}[S_n]$

Question. Which basis of $\mathbb{Z}[S_n]$ is $\{[\theta_w], w \in S_n\}$?

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Categorification of the right regular $\mathbb{Z}[S_n]$-module

Note. Projective functors are exact.

Consequence. Each $[\theta_w]$ is an endomorphism of $\mathrm{Gr}(\mathcal{O}_0)$

Identify: $\mathrm{Gr}(\mathcal{O}_0)$ with $\mathbb{Z}[S_n]$ via $[\Delta(w)] \mapsto w$.

Identify: $\mathrm{Gr}_{\oplus}(\mathcal{P})$ with $\mathbb{Z}[S_n]$ via $[\theta_s] \mapsto (e + s)$.

Theorem The action of $\mathcal{P}$ on $\mathcal{O}_0$ is a categorification of the right regular $\mathbb{Z}[S_n]$-module.

Diagrammatically:

$$\mathcal{O}_0 \circlearrowleft_{\mathcal{P}} \quad \xrightarrow{\mathrm{Gr}} \quad \mathbb{Z}[S_n] \circlearrowleft_{\mathbb{Z}[S_n]}$$

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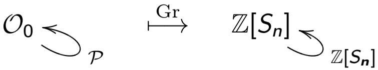

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Kazhdan-Lusztig basis

Note. The action of $\mathcal{P}$ categorifies $\mathbb{Z}[S_n]$ and not S_n.

Question. What is $\{[\theta_w], w \in S_n\}$?

Answer. This is the Kazhdan-Lusztig basis.

Remark. This is equivalent to Kazhdan-Lusztig conjecture (=theorem).

Remark. Recent algebraic proof by Elias-Williamson.

Remark. To define Kazhdan-Lusztig basis one needs to deform $\mathbb{Z}[S_n]$ to the Hecke algebra.

Categorically this means to introduce a grading on $\mathcal{O}_0$.

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Alternative approach: coinvariants

$\mathbb{C}[x_1, x_2, \dots, x_n]$ — polynomial algebra

grading: $\deg(x_i) = 2$

S_n acts on $\mathbb{C}[x_1, x_2, \dots, x_n]$ by permuting indices

$\mathbb{C}[x_1, x_2, \dots, x_n]_i^{S_n}$ — invariant homogeneous polynomials of degree i

$\mathbb{C}[x_1, x_2, \dots, x_n]_+^{S_n} = \bigoplus_{i>0} \mathbb{C}[x_1, x_2, \dots, x_n]_i^{S_n}$

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Alternative approach: combinatorial description of $\mathcal{P}$

B_w, $w \in S_n$ — Soergel bimodules

Note: For s simple reflection, $B_s \otimes_{\mathbf{C}} B_s \cong B_s \oplus B_s$.

Fact. For $x, y \in S_n$, each direct summand of the $\mathbf{C}$-$\mathbf{C}$-bimodule $B_x \otimes_{\mathbf{C}} B_y$ is isomorphic to B_z for some $z \in S_n$.

Definition. $\mathcal{S}$ is the (additive) tensor category of Soergel bimodules.

Theorem. [Soergel's combinatorial description]
The categories $\mathcal{P}$ and $\mathcal{S}$ are equivalent as tensor categories.

Corollary. $\mathrm{Gr}_{\oplus}[\mathcal{S}] \cong \mathbb{Z}[S_n]$.

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Categorification of permutation modules

$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ — composition of n

$\mathfrak{g}_\lambda \subset \mathfrak{g}$ — corresponding parabolic subalgebra.

W_λ — corresponding Young subgroup of S_n

${}_\lambda\mathrm{Long}$ — longest representatives in $W_\lambda \backslash W$

$\mathcal{X}_\lambda$ — Serre subcategory of $\mathcal{O}_0$ generated by $L(w)$, $w \notin {}_\lambda\mathrm{Long}$

Fact: $\mathcal{P}$ preserves $\mathcal{X}_\lambda$

Theorem. [M.-Stroppel] The induced action of $\mathcal{P}$ on $\mathcal{O}_0/\mathcal{X}_\lambda$ categorifies the permutation module $\mathrm{Ind}_{W_\lambda}^{W}\mathrm{triv}$.

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Theorem. [M.-Stroppel] The induced action of $\mathcal{P}$ on $\mathcal{O}_0/\mathcal{X}_\lambda$ categorifies the permutation module $\mathrm{Ind}_{W_\lambda}^{W}\mathrm{triv}$.

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$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ — composition of n

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Question. Are categorifications of $\mathcal{S}^\lambda$ via $\mathcal{X}'_\lambda$ and $\mathcal{Y}_{\lambda^t}/\mathcal{Y}'_{\lambda^t}$ equivalent?

Need: An equivalence between these two categories which naturally commutes with the action of $\mathcal{P}$

Theorem. [M.-Stroppel]
These two categorifications are indeed equivalent (using derived completion functors).

Theorem. [M.-Miemietz]
Simple transitive categorification of a Specht module (using $\mathcal{P}$) is unique up to equivalence.

Note. The latter works only in type A.

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Kazhdan-Lusztig's cell modules for S_n

Note. This requires a generalization of parabolic category $\mathcal{O}$

Induced cell modules $\mathrm{Ind}_{W_\lambda}^{W}\mathrm{cell}$

Note. Uses combinatorially defined subquotients of $\mathcal{O}_0$

Note. Permutation and induced sign modules are special cases

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Summary of bonuses provided by categorification

Various bases in $\mathbb{Z}[S_n]$ and in other categorified modules

Different submodules in $\mathbb{Z}[S_n]$

Filtration of $\mathbb{Z}[S_n]$ using Gelfand-Kirillov dimension of simples in $\mathcal{O}$

Uniqueness of categorification allows to compare different categories of $\mathfrak{g}$-modules

Explicit knowledge of categorification provides explicit (partial) information on the underlying category
(e.g. quiver and relations for blocks of $\mathcal{O}$)

Summary of bonuses provided by categorification

Various bases in $\mathbb{Z}[S_n]$ and in other categorified modules

Different submodules in $\mathbb{Z}[S_n]$

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THANK YOU!!!