Algebraic categorification and its applications, III

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2-categories

Definition. A 2-category is a category enriched over the monoidal category **Cat** of small categories (in the latter the monoidal structure is induced by the cartesian product).

This means that a 2-category $\mathscr{C}$ is given by the following data:

- objects of $\mathscr{C}$;
- small categories $\mathscr{C}(\mathtt{i},\mathtt{j})$ of morphisms;
- bifunctorial composition $\mathscr{C}(\mathtt{j},\mathtt{k}) \times \mathscr{C}(\mathtt{i},\mathtt{j}) \to \mathscr{C}(\mathtt{i},\mathtt{k})$;
- identity objects $\mathbb{1}_{\mathtt{j}}$;

which are subject to the obvious set of (strict) axioms.

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Terminology and the principal example

Terminology.

- An object in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 1-morphism of $\mathscr{C}$.
- A morphism in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 2-morphism of $\mathscr{C}$.
- Composition in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called vertical and denoted $\circ_1$.
- Composition in $\mathscr{C}$ is called horizontal and denoted $\circ_0$.

Principal example. The category **Cat** is a 2-category.

- Objects of **Cat** are small categories.
- 1-morphisms in **Cat** are functors.
- 2-morphisms in **Cat** are natural transformations.
- Composition is the usual composition.
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Decategorification of additive 2-categories

Definition. A 2-category $\mathscr{C}$ is additive if:

- Each $\mathscr{C}(\mathtt{i},\mathtt{j})$ is additive and idempotent split.
- Horizontal composition is biadditive.

Definition. The split Gorthendieck group $[\mathcal{A}]_{\oplus}$ of an additive category $\mathcal{A}$ is the quotient of the free abelian group generated by $[X]$, where X is an object of $\mathcal{A}$, modulo relations $[X]=[Y]+[Z]$ whenever $X\cong Y\oplus Z$.

Definition. The decategorification $[\mathscr{C}]$ of an additive 2-category $\mathscr{C}$ is the (usual 1-) category defined as follows:

- $[\mathscr{C}]$ has the same objects as $\mathscr{C}$;
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Categorification

$\mathscr{C}$ — additive 2-category

$[\mathscr{C}]$ — decategorification of $\mathscr{C}$

Definition. $\mathscr{C}$ is called a categorification of $[\mathscr{C}]$.

Put differently: Categorification is just the formal "inverse" of decategorification.

Warning: Categorification is "multi-valued" in general.

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Example: projective functors on $\mathcal{O}_0$

$\mathfrak{g} = \mathfrak{sl}_n$

$\mathcal{O}_0$ — principal block of category $\mathcal{O}$ for $\mathfrak{g}$

$\mathscr{S}$ — the 2-category of projective functors on $\mathcal{O}_0$, that is:

- $\mathscr{S}$ has one object ♣ (identified with some small category $\mathcal{C} \cong \mathcal{O}_0$);
- 1-morphisms in $\mathscr{S}$ are functors isomorphic to projective functors;
- 2-morphisms in $\mathscr{S}$ are natural transformations of functors;
- horizontal composition in $\mathscr{S}$ is composition of functors.

Fact. $\mathscr{S}$ is an additive 2-category.

Theorem. $[\mathscr{S}](\clubsuit, \clubsuit) \cong \mathbb{Z}[S_n]$

Consequence: In this sense $\mathscr{S}$ is a categorification of $\mathbb{Z}[S_n]$.

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Fact. $\mathscr{S}$ is an additive 2-category.

Theorem. $[\mathscr{S}](\clubsuit, \clubsuit) \cong \mathbb{Z}[S_n]$

Consequence: In this sense $\mathscr{S}$ is a categorification of $\mathbb{Z}[S_n]$.

Example: projective functors on $\mathcal{O}_0$

$\mathfrak{g} = \mathfrak{sl}_n$

$\mathcal{O}_0$ — principal block of category $\mathcal{O}$ for $\mathfrak{g}$

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$\Bbbk$ — field

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- $\mathcal{A}$ is idempotent split;
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Example: A-proj for a finite dimensional $\Bbbk$-algebra A.

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Examples:

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Example: projective functors for finite dimensional algebras.

$\mathbb{k}$ — algebraically closed field

A — finite dimensional $\mathbb{k}$-algebra

Definition. A projective endofunctor of A-mod is tensoring with a projective A–A-bimodule, up to isomorphism

$\mathcal{C}$ — a small category equivalent to A-mod

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$\Sigma(\mathscr{C})$ — isoclasses of indecomposable 1-morphisms in $\mathscr{C}$

Fact: $\Sigma(\mathscr{C})$ is a multisemigroup under

$$\mathrm{F} \star \mathrm{G} = \{\mathrm{H} : \mathrm{H} \text{ is isomorphic to a direct summand of } \mathrm{FG}\}$$

Left preorder: $\mathrm{F} \geq_L \mathrm{G}$ if $\mathrm{F} \in \Sigma(\mathscr{C}) \star \mathrm{G}$

Left cells: equivalence classes w.r.t. $\geq_L$ (a.k.a. Green's $\mathcal{L}$-classes)

Similarly: right and two-sided preorders $\geq_R$ and $\geq_J$ and right and two-sided cells

Example: For Soergel bimodules (projective functors on $\mathcal{O}_0$) these are Kazhdan-Lusztig orders and cells

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Left preorder: $\mathrm{F} \geq_L \mathrm{G}$ if $\mathrm{F} \in \Sigma(\mathscr{C}) \star \mathrm{G}$

Left cells: equivalence classes w.r.t. $\geq_L$ (a.k.a. Green's $\mathcal{L}$-classes)

Similarly: right and two-sided preorders $\geq_R$ and $\geq_J$ and right and two-sided cells

Example: For Soergel bimodules (projective functors on $\mathcal{O}_0$) these are Kazhdan-Lusztig orders and cells

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More detailed example: $\mathcal{C}_A$

A — basic, connected finite dimensional $\Bbbk$-algebra

$1 = e_1 + e_2 + \dots + e_n$ — primitive decomposition of $1 \in A$

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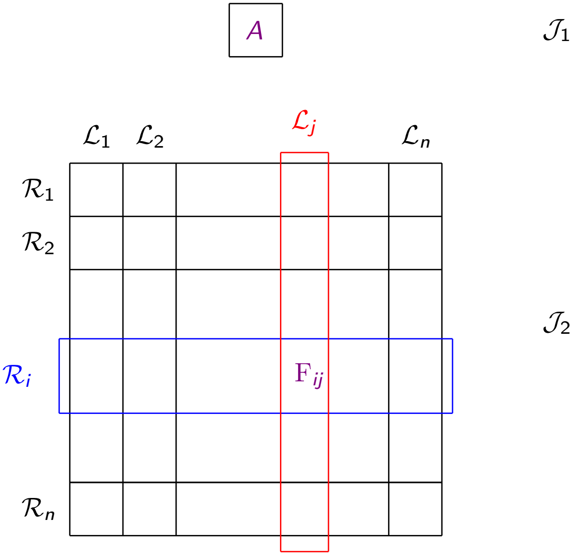

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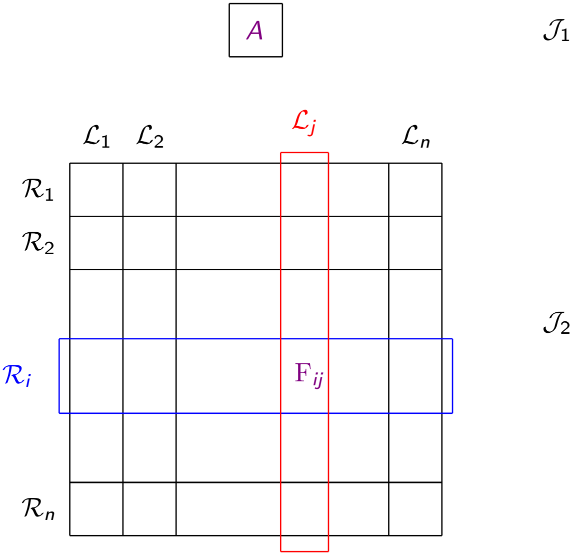

Fiat 2-categories

$\mathscr{C}$ — finitary 2-category

Definition: $\mathscr{C}$ is fiat if

- there is a weak involution $* : \mathscr{C} \to \mathscr{C}$;
- there are adjunction 2-morphisms $\alpha : \mathbb{1}_{\mathtt{i}} \to \mathrm{F}\mathrm{F}^*$ and $\beta : \mathrm{F}^*\mathrm{F} \to \mathbb{1}_{\mathtt{j}}$ such that

$$\mathrm{F}(\beta) \circ_1 \alpha_{\mathrm{F}} = \mathrm{id}_{\mathrm{F}} \quad \text{and} \quad \beta_{\mathrm{F}^*} \circ_1 \mathrm{F}^*(\alpha) = \mathrm{id}_{\mathrm{F}^*}$$

Note: This makes F and F^* always biadjoint

Examples:

- Soergel bimodules (projective functors on $\mathcal{O}_0$)
- $\mathscr{C}_A$ for A self-injective and weakly symmetric

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Example: 2-Kac-Moody algebras

$\mathfrak{g}$ — Kac-Moody algebra

$U_q(\mathfrak{g})$ — the corresponding quantum group

$\dot{\mathrm{U}}$ — the idempotent completion of $U_q(\mathfrak{g})$

$\dot{\mathrm{U}}_{\mathbb{Z}}$ — the integral form for $\dot{\mathrm{U}}$

There is a number of 2-categories associated to $\mathfrak{g}$.

Due to: Khovanov-Lauda, Rouquier, Webster, Cautis-Lauda

Some of these categorify $\dot{\mathrm{U}}_{\mathbb{Z}}$.

Remark. They have involution and adjunctions but are not finitary.

Remark. Some of them have finitary quotients.

Example: 2-Kac-Moody algebras

$\mathfrak{g}$ — Kac-Moody algebra

$U_q(\mathfrak{g})$ — the corresponding quantum group

$\dot{\mathrm{U}}$ — the idempotent completion of $U_q(\mathfrak{g})$

$\dot{\mathrm{U}}_{\mathbb{Z}}$ — the integral form for $\dot{\mathrm{U}}$

There is a number of 2-categories associated to $\mathfrak{g}$.

Due to: Khovanov-Lauda, Rouquier, Webster, Cautis-Lauda

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Every simple complex finite-dimensional algebra is isomorphic to $\mathrm{Mat}_{n\times n}(\mathbb{C})$ for some n.

Theorem. [M.-Miemietz]

Every "simple" fiat 2-category with a strongly regular maximal two-sided cell is "essentially" $\mathscr{C}_A$ for A self-injective and weakly symmetric.

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"Definition": A 2-representation of $\mathscr{C}$ is a functorial action of $\mathscr{C}$ on a suitable category(ies).

Example: Principal 2-representation $\mathbf{P}_{\mathtt{i}} := \mathscr{C}(\mathtt{i}, {}_{-})$ for $\mathtt{i} \in \mathscr{C}$

Note: 2-representations of $\mathscr{C}$ form a 2-category where

- 1-morphisms are 2-natural transformations
- 2-morphisms are modifications

Note: There is a natural notion of equivalence for 2-representations

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Cell 2-representations

$\mathscr{C}$ — finitary 2-category

$\mathcal{L}$ — left cell in $\mathscr{C}$

$\mathtt{i}$ — the source for 1-morphisms in $\mathscr{C}$

$\mathbf{P}_\mathtt{i}$ — the $\mathtt{i}$-th principal 2-representation

$\mathbf{Q}_\mathcal{L}$ — 2-subrepresentation of $\mathbf{P}_\mathtt{i}$ generated by $\mathrm{F} \geq_L \mathcal{L}$

$\mathbf{I}$ — the unique maximal $\mathscr{C}$-invariant ideal in $\mathbf{Q}_\mathcal{L}$

Definition: $\mathbf{C}_\mathcal{L} := \mathbf{Q}_\mathcal{L}/\mathbf{I}$ — the cell 2-representation of $\mathscr{C}$ for $\mathcal{L}$

Example: The defining (tautological) 2-representation of $\mathscr{C}_A$ is equivalent to $\mathbf{C}_{\mathcal{L}_j}$ for any $j = 1, 2, \dots, n$.

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Transitive 2-representations

$\mathscr{C}$ — finitary 2-category

M — 2-representation of $\mathscr{C}$

Definition: **M** is finitary if **M**($\mathtt{i}$) is finitary $\Bbbk$-linear for all $\mathtt{i}$

Definition: **M** is transitive if **M** is finitary and for any indecomposable X, Y in **M** there is a 1-morphism F such that X is isomorphic to a direct summand of $\mathrm{F}\,Y$

Intuition: Transitive action of a group (for us: a multisemigroup)

Definition: **M** is simple transitive if **M** is transitive and has no non-trivial $\mathscr{C}$-invariant ideals.

Example: Cell 2-representations are simple transitive.

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Classification of simple transitive 2-representations

Theorem. [M.-Miemietz]

Under some natural assumption, cell 2-representations are the only simple transitive 2-representation.

Applies to:

- Soergel bimodules in type A.
- $\mathscr{C}_A$.
- Finitary quotients of 2-Kac-Moody algebras of finite type.

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Other results and future challenges

Known: Morita theory for finitary 2-categories ([M.-Miemietz]).

Known: Classification of isotypic 2-representations of certain fiat 2-categories ([M.-Miemietz]).

Known: An analogue of Schur's lemma for cell 2-representations of certain fiat 2-categories ([M.-Miemietz]).

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