Homological properties of category O, part I: invariants of structural modules

Volodymyr Mazorchuk

(Uppsala University)

"Enveloping Algebras and Representation Theory"
August 28 – September 1, 2014, St. John's, CANADA

Setup and notation

$\mathfrak{g}$ — a semi-simple finite dimensional Lie algebra over $\mathbb{C}$

For example: $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — fixed triangular decomposition

For example, if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, then

$\mathfrak{n}_-$	—	lower triangular matrices
$\mathfrak{h}$	—	diagonal matrices
$\mathfrak{n}_+$	—	upper triangular matrices

for a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ its universal enveloping algebra

Setup and notation

$\mathfrak{g}$ — a semi-simple finite dimensional Lie algebra over $\mathbb{C}$

For example: $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — fixed triangular decomposition

For example, if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, then

$\mathfrak{n}_-$	—	lower triangular matrices
$\mathfrak{h}$	—	diagonal matrices
$\mathfrak{n}_+$	—	upper triangular matrices

for a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ its universal enveloping algebra

Setup and notation

$\mathfrak{g}$ — a semi-simple finite dimensional Lie algebra over $\mathbb{C}$

For example: $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — fixed triangular decomposition

For example, if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, then

$\mathfrak{n}_-$	—	lower triangular matrices
$\mathfrak{h}$	—	diagonal matrices
$\mathfrak{n}_+$	—	upper triangular matrices

for a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ its universal enveloping algebra

Setup and notation

$\mathfrak{g}$ — a semi-simple finite dimensional Lie algebra over $\mathbb{C}$

For example: $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — fixed triangular decomposition

For example, if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, then

$\mathfrak{n}_-$	—	lower triangular matrices
$\mathfrak{h}$	—	diagonal matrices
$\mathfrak{n}_+$	—	upper triangular matrices

for a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ its universal enveloping algebra

Setup and notation

$\mathfrak{g}$ — a semi-simple finite dimensional Lie algebra over $\mathbb{C}$

For example: $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — fixed triangular decomposition

For example, if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, then

$\mathfrak{n}_-$	—	lower triangular matrices
$\mathfrak{h}$	—	diagonal matrices
$\mathfrak{n}_+$	—	upper triangular matrices

for a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ its universal enveloping algebra

Setup and notation

$\mathfrak{g}$ — a semi-simple finite dimensional Lie algebra over $\mathbb{C}$

For example: $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — fixed triangular decomposition

For example, if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, then

$\mathfrak{n}_-$	—	lower triangular matrices
$\mathfrak{h}$	—	diagonal matrices
$\mathfrak{n}_+$	—	upper triangular matrices

for a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ its universal enveloping algebra

Setup and notation

$\mathfrak{g}$ — a semi-simple finite dimensional Lie algebra over $\mathbb{C}$

For example: $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — fixed triangular decomposition

For example, if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, then

$\mathfrak{n}_-$	—	lower triangular matrices
$\mathfrak{h}$	—	diagonal matrices
$\mathfrak{n}_+$	—	upper triangular matrices

for a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ its universal enveloping algebra

Setup and notation

$\mathfrak{g}$ — a semi-simple finite dimensional Lie algebra over $\mathbb{C}$

For example: $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — fixed triangular decomposition

For example, if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, then

$\mathfrak{n}_-$	—	lower triangular matrices
$\mathfrak{h}$	—	diagonal matrices
$\mathfrak{n}_+$	—	upper triangular matrices

for a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ its universal enveloping algebra

Setup and notation

$\mathfrak{g}$ — a semi-simple finite dimensional Lie algebra over $\mathbb{C}$

For example: $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — fixed triangular decomposition

For example, if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, then

$\mathfrak{n}_-$	—	lower triangular matrices
$\mathfrak{h}$	—	diagonal matrices
$\mathfrak{n}_+$	—	upper triangular matrices

for a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ its universal enveloping algebra

Setup and notation

$\mathfrak{g}$ — a semi-simple finite dimensional Lie algebra over $\mathbb{C}$

For example: $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — fixed triangular decomposition

For example, if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, then

$\mathfrak{n}_-$	—	lower triangular matrices
$\mathfrak{h}$	—	diagonal matrices
$\mathfrak{n}_+$	—	upper triangular matrices

for a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ its universal enveloping algebra

Category $\mathcal{O}$

Definition. BGG category $\mathcal{O}$ is defined as the full subcategory of the category of all $\mathfrak{g}$-modules containing all modules M satisfying the following conditions:

- M is finitely generated;
- the action of $\mathfrak{h}$ on M is diagonalizable (i.e M is a weight module);
- the action of $U(\mathfrak{n}_+)$ on M is locally finite.

$M(\lambda)$ — the Verma module with highest weight $\lambda \in \mathfrak{h}^*$

Fact. $M(\lambda) \in \mathcal{O}$

$L(\lambda)$ — the simple top of $M(\lambda)$

Fact. $L(\lambda) \in \mathcal{O}$

Fact. Every simple object in $\mathcal{O}$ is isomorphic to some $L(\lambda)$

Category $\mathcal{O}$

Definition. BGG category $\mathcal{O}$ is defined as the full subcategory of the category of all $\mathfrak{g}$-modules containing all modules M satisfying the following conditions:

- M is finitely generated;
- the action of $\mathfrak{h}$ on M is diagonalizable (i.e M is a weight module);
- the action of $U(\mathfrak{n}_+)$ on M is locally finite.

$M(\lambda)$ — the Verma module with highest weight $\lambda \in \mathfrak{h}^*$

Fact. $M(\lambda) \in \mathcal{O}$

$L(\lambda)$ — the simple top of $M(\lambda)$

Fact. $L(\lambda) \in \mathcal{O}$

Fact. Every simple object in $\mathcal{O}$ is isomorphic to some $L(\lambda)$

Category $\mathcal{O}$

Definition. BGG category $\mathcal{O}$ is defined as the full subcategory of the category of all $\mathfrak{g}$-modules containing all modules M satisfying the following conditions:

- M is finitely generated;
- the action of $\mathfrak{h}$ on M is diagonalizable (i.e M is a weight module);
- the action of $U(\mathfrak{n}_+)$ on M is locally finite.

$M(\lambda)$ — the Verma module with highest weight $\lambda \in \mathfrak{h}^*$

Fact. $M(\lambda) \in \mathcal{O}$

$L(\lambda)$ — the simple top of $M(\lambda)$

Fact. $L(\lambda) \in \mathcal{O}$

Fact. Every simple object in $\mathcal{O}$ is isomorphic to some $L(\lambda)$

Category $\mathcal{O}$

Definition. BGG category $\mathcal{O}$ is defined as the full subcategory of the category of all $\mathfrak{g}$-modules containing all modules M satisfying the following conditions:

- M is finitely generated;
- the action of $\mathfrak{h}$ on M is diagonalizable (i.e M is a weight module);
- the action of $U(\mathfrak{n}_+)$ on M is locally finite.

$M(\lambda)$ — the Verma module with highest weight $\lambda \in \mathfrak{h}^*$

Fact. $M(\lambda) \in \mathcal{O}$

$L(\lambda)$ — the simple top of $M(\lambda)$

Fact. $L(\lambda) \in \mathcal{O}$

Fact. Every simple object in $\mathcal{O}$ is isomorphic to some $L(\lambda)$

Category $\mathcal{O}$

Definition. BGG category $\mathcal{O}$ is defined as the full subcategory of the category of all $\mathfrak{g}$-modules containing all modules M satisfying the following conditions:

- M is finitely generated;
- the action of $\mathfrak{h}$ on M is diagonalizable (i.e M is a weight module);
- the action of $U(\mathfrak{n}_+)$ on M is locally finite.

$M(\lambda)$ — the Verma module with highest weight $\lambda \in \mathfrak{h}^*$

Fact. $M(\lambda) \in \mathcal{O}$

$L(\lambda)$ — the simple top of $M(\lambda)$

Fact. $L(\lambda) \in \mathcal{O}$

Fact. Every simple object in $\mathcal{O}$ is isomorphic to some $L(\lambda)$

Category $\mathcal{O}$

Definition. BGG category $\mathcal{O}$ is defined as the full subcategory of the category of all $\mathfrak{g}$-modules containing all modules M satisfying the following conditions:

- M is finitely generated;
- the action of $\mathfrak{h}$ on M is diagonalizable (i.e M is a weight module);
- the action of $U(\mathfrak{n}_+)$ on M is locally finite.

$M(\lambda)$ — the Verma module with highest weight $\lambda \in \mathfrak{h}^*$

Fact. $M(\lambda) \in \mathcal{O}$

$L(\lambda)$ — the simple top of $M(\lambda)$

Fact. $L(\lambda) \in \mathcal{O}$

Fact. Every simple object in $\mathcal{O}$ is isomorphic to some $L(\lambda)$

Category $\mathcal{O}$

Definition. BGG category $\mathcal{O}$ is defined as the full subcategory of the category of all $\mathfrak{g}$-modules containing all modules M satisfying the following conditions:

- M is finitely generated;
- the action of $\mathfrak{h}$ on M is diagonalizable (i.e M is a weight module);
- the action of $U(\mathfrak{n}_+)$ on M is locally finite.

$M(\lambda)$ — the Verma module with highest weight $\lambda \in \mathfrak{h}^*$

Fact. $M(\lambda) \in \mathcal{O}$

$L(\lambda)$ — the simple top of $M(\lambda)$

Fact. $L(\lambda) \in \mathcal{O}$

Fact. Every simple object in $\mathcal{O}$ is isomorphic to some $L(\lambda)$

Definition. BGG category $\mathcal{O}$ is defined as the full subcategory of the category of all $\mathfrak{g}$-modules containing all modules M satisfying the following conditions:

- M is finitely generated;
- the action of $\mathfrak{h}$ on M is diagonalizable (i.e M is a weight module);
- the action of $U(\mathfrak{n}_+)$ on M is locally finite.

$M(\lambda)$ — the Verma module with highest weight $\lambda \in \mathfrak{h}^*$

Fact. $M(\lambda) \in \mathcal{O}$

$L(\lambda)$ — the simple top of $M(\lambda)$

Fact. $L(\lambda) \in \mathcal{O}$

Fact. Every simple object in $\mathcal{O}$ is isomorphic to some $L(\lambda)$

Category $\mathcal{O}$

Definition. BGG category $\mathcal{O}$ is defined as the full subcategory of the category of all $\mathfrak{g}$-modules containing all modules M satisfying the following conditions:

- M is finitely generated;
- the action of $\mathfrak{h}$ on M is diagonalizable (i.e M is a weight module);
- the action of $U(\mathfrak{n}_+)$ on M is locally finite.

$M(\lambda)$ — the Verma module with highest weight $\lambda \in \mathfrak{h}^*$

Fact. $M(\lambda) \in \mathcal{O}$

$L(\lambda)$ — the simple top of $M(\lambda)$

Fact. $L(\lambda) \in \mathcal{O}$

Fact. Every simple object in $\mathcal{O}$ is isomorphic to some $L(\lambda)$

Category $\mathcal{O}$

Definition. BGG category $\mathcal{O}$ is defined as the full subcategory of the category of all $\mathfrak{g}$-modules containing all modules M satisfying the following conditions:

- M is finitely generated;
- the action of $\mathfrak{h}$ on M is diagonalizable (i.e M is a weight module);
- the action of $U(\mathfrak{n}_+)$ on M is locally finite.

$M(\lambda)$ — the Verma module with highest weight $\lambda \in \mathfrak{h}^*$

Fact. $M(\lambda) \in \mathcal{O}$

$L(\lambda)$ — the simple top of $M(\lambda)$

Fact. $L(\lambda) \in \mathcal{O}$

Fact. Every simple object in $\mathcal{O}$ is isomorphic to some $L(\lambda)$

Category $\mathcal{O}$

Definition. BGG category $\mathcal{O}$ is defined as the full subcategory of the category of all $\mathfrak{g}$-modules containing all modules M satisfying the following conditions:

- M is finitely generated;
- the action of $\mathfrak{h}$ on M is diagonalizable (i.e M is a weight module);
- the action of $U(\mathfrak{n}_+)$ on M is locally finite.

$M(\lambda)$ — the Verma module with highest weight $\lambda \in \mathfrak{h}^*$

Fact. $M(\lambda) \in \mathcal{O}$

$L(\lambda)$ — the simple top of $M(\lambda)$

Fact. $L(\lambda) \in \mathcal{O}$

Fact. Every simple object in $\mathcal{O}$ is isomorphic to some $L(\lambda)$

$\mathcal{O}$ and finite dimensional associative algebras

$Z(\mathfrak{g})$ — the center of $U(\mathfrak{g})$

Central character decomposition: $\mathcal{O} = \bigoplus_{\mathtt{m} \in \mathrm{Max}(Z(\mathfrak{g}))} \mathcal{O}_{\mathtt{m}}$

$\mathcal{O}_{\mathtt{m}}$ is the full subcategory of $\mathcal{O}$ consisting of all modules on which $\mathtt{m}$ acts locally nilpotently

Fact. $\mathcal{O}_{\mathtt{m}} \cong A_{\mathtt{m}}\text{-mod}$ for some finite dimensional basic associative algebra $A_{\mathtt{m}}$

Fact. $\mathcal{O}$ is a highest weight category in the sense of Cline-Parshall-Scott with Verma modules being standard modules

Reformulation. $A_{\mathtt{m}}$ is quasi-hereditary in the sense of Dlab-Ringel

Note. $A_{\mathtt{m}}$ is usually decomposable (for $\mathfrak{sl}_n$ all indecomposable summands of $A_{\mathtt{m}}$ are isomorphic)

$\mathcal{O}$ and finite dimensional associative algebras

$Z(\mathfrak{g})$ — the center of $U(\mathfrak{g})$

Central character decomposition: $\mathcal{O} = \bigoplus_{\mathtt{m} \in \mathrm{Max}(Z(\mathfrak{g}))} \mathcal{O}_{\mathtt{m}}$

$\mathcal{O}_{\mathtt{m}}$ is the full subcategory of $\mathcal{O}$ consisting of all modules on which $\mathtt{m}$ acts locally nilpotently

Fact. $\mathcal{O}_{\mathtt{m}} \cong A_{\mathtt{m}}\text{-mod}$ for some finite dimensional basic associative algebra $A_{\mathtt{m}}$

Fact. $\mathcal{O}$ is a highest weight category in the sense of Cline-Parshall-Scott with Verma modules being standard modules

Reformulation. $A_{\mathtt{m}}$ is quasi-hereditary in the sense of Dlab-Ringel

Note. $A_{\mathtt{m}}$ is usually decomposable (for $\mathfrak{sl}_n$ all indecomposable summands of $A_{\mathtt{m}}$ are isomorphic)

$\mathcal{O}$ and finite dimensional associative algebras

$Z(\mathfrak{g})$ — the center of $U(\mathfrak{g})$

Central character decomposition: $\mathcal{O} = \bigoplus_{\mathtt{m}\in\mathrm{Max}(Z(\mathfrak{g}))} \mathcal{O}_{\mathtt{m}}$

$\mathcal{O}_{\mathtt{m}}$ is the full subcategory of $\mathcal{O}$ consisting of all modules on which $\mathtt{m}$ acts locally nilpotently

Fact. $\mathcal{O}_{\mathtt{m}} \cong A_{\mathtt{m}}\text{-mod}$ for some finite dimensional basic associative algebra $A_{\mathtt{m}}$

Fact. $\mathcal{O}$ is a highest weight category in the sense of Cline-Parshall-Scott with Verma modules being standard modules

Reformulation. $A_{\mathtt{m}}$ is quasi-hereditary in the sense of Dlab-Ringel

Note. $A_{\mathtt{m}}$ is usually decomposable (for $\mathfrak{sl}_n$ all indecomposable summands of $A_{\mathtt{m}}$ are isomorphic)

$\mathcal{O}$ and finite dimensional associative algebras

$Z(\mathfrak{g})$ — the center of $U(\mathfrak{g})$

Central character decomposition: $\mathcal{O} = \bigoplus_{\mathtt{m}\in \mathrm{Max}(Z(\mathfrak{g}))} \mathcal{O}_{\mathtt{m}}$

$\mathcal{O}_{\mathtt{m}}$ is the full subcategory of $\mathcal{O}$ consisting of all modules on which $\mathtt{m}$ acts locally nilpotently

Fact. $\mathcal{O}_{\mathtt{m}} \cong A_{\mathtt{m}}\text{-mod}$ for some finite dimensional basic associative algebra $A_{\mathtt{m}}$

Fact. $\mathcal{O}$ is a highest weight category in the sense of Cline-Parshall-Scott with Verma modules being standard modules

Reformulation. $A_{\mathtt{m}}$ is quasi-hereditary in the sense of Dlab-Ringel

Note. $A_{\mathtt{m}}$ is usually decomposable (for $\mathfrak{sl}_n$ all indecomposable summands of $A_{\mathtt{m}}$ are isomorphic)

$\mathcal{O}$ and finite dimensional associative algebras

$Z(\mathfrak{g})$ — the center of $U(\mathfrak{g})$

Central character decomposition: $\mathcal{O} = \bigoplus_{\mathtt{m} \in \mathrm{Max}(Z(\mathfrak{g}))} \mathcal{O}_{\mathtt{m}}$

$\mathcal{O}_{\mathtt{m}}$ is the full subcategory of $\mathcal{O}$ consisting of all modules on which $\mathtt{m}$ acts locally nilpotently

Fact. $\mathcal{O}_{\mathtt{m}} \cong A_{\mathtt{m}}\text{-mod}$ for some finite dimensional basic associative algebra $A_{\mathtt{m}}$

Fact. $\mathcal{O}$ is a highest weight category in the sense of Cline-Parshall-Scott with Verma modules being standard modules

Reformulation. $A_{\mathtt{m}}$ is quasi-hereditary in the sense of Dlab-Ringel

Note. $A_{\mathtt{m}}$ is usually decomposable (for $\mathfrak{sl}_n$ all indecomposable summands of $A_{\mathtt{m}}$ are isomorphic)

$\mathcal{O}$ and finite dimensional associative algebras

$Z(\mathfrak{g})$ — the center of $U(\mathfrak{g})$

Central character decomposition: $\mathcal{O} = \bigoplus_{\mathtt{m}\in \mathrm{Max}(Z(\mathfrak{g}))} \mathcal{O}_{\mathtt{m}}$

$\mathcal{O}_{\mathtt{m}}$ is the full subcategory of $\mathcal{O}$ consisting of all modules on which $\mathtt{m}$ acts locally nilpotently

Fact. $\mathcal{O}_{\mathtt{m}} \cong A_{\mathtt{m}}\text{-mod}$ for some finite dimensional basic associative algebra $A_{\mathtt{m}}$

Fact. $\mathcal{O}$ is a highest weight category in the sense of Cline-Parshall-Scott with Verma modules being standard modules

Reformulation. $A_{\mathtt{m}}$ is quasi-hereditary in the sense of Dlab-Ringel

Note. $A_{\mathtt{m}}$ is usually decomposable (for $\mathfrak{sl}_n$ all indecomposable summands of $A_{\mathtt{m}}$ are isomorphic)

$\mathcal{O}$ and finite dimensional associative algebras

$Z(\mathfrak{g})$ — the center of $U(\mathfrak{g})$

Central character decomposition: $\mathcal{O} = \bigoplus_{\mathtt{m} \in \mathrm{Max}(Z(\mathfrak{g}))} \mathcal{O}_{\mathtt{m}}$

$\mathcal{O}_{\mathtt{m}}$ is the full subcategory of $\mathcal{O}$ consisting of all modules on which $\mathtt{m}$ acts locally nilpotently

Fact. $\mathcal{O}_{\mathtt{m}} \cong A_{\mathtt{m}}\text{-mod}$ for some finite dimensional basic associative algebra $A_{\mathtt{m}}$

Fact. $\mathcal{O}$ is a highest weight category in the sense of Cline-Parshall-Scott with Verma modules being standard modules

Reformulation. $A_{\mathtt{m}}$ is quasi-hereditary in the sense of Dlab-Ringel

Note. $A_{\mathtt{m}}$ is usually decomposable (for $\mathfrak{sl}_n$ all indecomposable summands of $A_{\mathtt{m}}$ are isomorphic)

$\mathcal{O}$ and finite dimensional associative algebras

$Z(\mathfrak{g})$ — the center of $U(\mathfrak{g})$

Central character decomposition: $\mathcal{O} = \bigoplus_{\mathtt{m} \in \mathrm{Max}(Z(\mathfrak{g}))} \mathcal{O}_{\mathtt{m}}$

$\mathcal{O}_{\mathtt{m}}$ is the full subcategory of $\mathcal{O}$ consisting of all modules on which $\mathtt{m}$ acts locally nilpotently

Fact. $\mathcal{O}_{\mathtt{m}} \cong A_{\mathtt{m}}\text{-mod}$ for some finite dimensional basic associative algebra $A_{\mathtt{m}}$

Fact. $\mathcal{O}$ is a highest weight category in the sense of Cline-Parshall-Scott with Verma modules being standard modules

Reformulation. $A_{\mathtt{m}}$ is quasi-hereditary in the sense of Dlab-Ringel

Note. $A_{\mathtt{m}}$ is usually decomposable (for $\mathfrak{sl}_n$ all indecomposable summands of $A_{\mathtt{m}}$ are isomorphic)

Structural modules in $\mathcal{O}$

$\lambda \in \mathfrak{h}^*$

$L(\lambda)$ — simple module with highest weight λ

$P(\lambda)$ — indecomposable projective cover of $L(\lambda)$

$I(\lambda)$ — indecomposable injective envelope of $L(\lambda)$

$\Delta(\lambda) = M(\lambda)$ — standard quotient of $P(\lambda)$

$\nabla(\lambda)$ — costandard submodule of $I(\lambda)$

$T(\lambda)$ — tilting envelope of $\Delta(\lambda)$

Fact. $T(\lambda)$ — tilting cover of $\nabla(\lambda)$

Structural modules in $\mathcal{O}$

$\lambda \in \mathfrak{h}^*$

$L(\lambda)$ — simple module with highest weight λ

$P(\lambda)$ — indecomposable projective cover of $L(\lambda)$

$I(\lambda)$ — indecomposable injective envelope of $L(\lambda)$

$\Delta(\lambda) = M(\lambda)$ — standard quotient of $P(\lambda)$

$\nabla(\lambda)$ — costandard submodule of $I(\lambda)$

$T(\lambda)$ — tilting envelope of $\Delta(\lambda)$

Fact. $T(\lambda)$ — tilting cover of $\nabla(\lambda)$

Structural modules in $\mathcal{O}$

$\lambda \in \mathfrak{h}^*$

$L(\lambda)$ — simple module with highest weight λ

$P(\lambda)$ — indecomposable projective cover of $L(\lambda)$

$I(\lambda)$ — indecomposable injective envelope of $L(\lambda)$

$\Delta(\lambda) = M(\lambda)$ — standard quotient of $P(\lambda)$

$\nabla(\lambda)$ — costandard submodule of $I(\lambda)$

$T(\lambda)$ — tilting envelope of $\Delta(\lambda)$

Fact. $T(\lambda)$ — tilting cover of $\nabla(\lambda)$

Structural modules in $\mathcal{O}$

$\lambda \in \mathfrak{h}^*$

$L(\lambda)$ — simple module with highest weight λ

$P(\lambda)$ — indecomposable projective cover of $L(\lambda)$

$I(\lambda)$ — indecomposable injective envelope of $L(\lambda)$

$\Delta(\lambda) = M(\lambda)$ — standard quotient of $P(\lambda)$

$\nabla(\lambda)$ — costandard submodule of $I(\lambda)$

$T(\lambda)$ — tilting envelope of $\Delta(\lambda)$

Fact. $T(\lambda)$ — tilting cover of $\nabla(\lambda)$

Structural modules in $\mathcal{O}$

$\lambda \in \mathfrak{h}^*$

$L(\lambda)$ — simple module with highest weight λ

$P(\lambda)$ — indecomposable projective cover of $L(\lambda)$

$I(\lambda)$ — indecomposable injective envelope of $L(\lambda)$

$\Delta(\lambda) = M(\lambda)$ — standard quotient of $P(\lambda)$

$\nabla(\lambda)$ — costandard submodule of $I(\lambda)$

$T(\lambda)$ — tilting envelope of $\Delta(\lambda)$

Fact. $T(\lambda)$ — tilting cover of $\nabla(\lambda)$

Structural modules in $\mathcal{O}$

$\lambda \in \mathfrak{h}^*$

$L(\lambda)$ — simple module with highest weight λ

$P(\lambda)$ — indecomposable projective cover of $L(\lambda)$

$I(\lambda)$ — indecomposable injective envelope of $L(\lambda)$

$\Delta(\lambda) = M(\lambda)$ — standard quotient of $P(\lambda)$

$\nabla(\lambda)$ — costandard submodule of $I(\lambda)$

$T(\lambda)$ — tilting envelope of $\Delta(\lambda)$

Fact. $T(\lambda)$ — tilting cover of $\nabla(\lambda)$

Structural modules in $\mathcal{O}$

$\lambda \in \mathfrak{h}^*$

$L(\lambda)$ — simple module with highest weight λ

$P(\lambda)$ — indecomposable projective cover of $L(\lambda)$

$I(\lambda)$ — indecomposable injective envelope of $L(\lambda)$

$\Delta(\lambda) = M(\lambda)$ — standard quotient of $P(\lambda)$

$\nabla(\lambda)$ — costandard submodule of $I(\lambda)$

$T(\lambda)$ — tilting envelope of $\Delta(\lambda)$

Fact. $T(\lambda)$ — tilting cover of $\nabla(\lambda)$

Structural modules in $\mathcal{O}$

$\lambda \in \mathfrak{h}^*$

$L(\lambda)$ — simple module with highest weight λ

$P(\lambda)$ — indecomposable projective cover of $L(\lambda)$

$I(\lambda)$ — indecomposable injective envelope of $L(\lambda)$

$\Delta(\lambda) = M(\lambda)$ — standard quotient of $P(\lambda)$

$\nabla(\lambda)$ — costandard submodule of $I(\lambda)$

$T(\lambda)$ — tilting envelope of $\Delta(\lambda)$

Fact. $T(\lambda)$ — tilting cover of $\nabla(\lambda)$

Structural modules in $\mathcal{O}$

$\lambda \in \mathfrak{h}^*$

$L(\lambda)$ — simple module with highest weight λ

$P(\lambda)$ — indecomposable projective cover of $L(\lambda)$

$I(\lambda)$ — indecomposable injective envelope of $L(\lambda)$

$\Delta(\lambda) = M(\lambda)$ — standard quotient of $P(\lambda)$

$\nabla(\lambda)$ — costandard submodule of $I(\lambda)$

$T(\lambda)$ — tilting envelope of $\Delta(\lambda)$

Fact. $T(\lambda)$ — tilting cover of $\nabla(\lambda)$

Structural modules in $\mathcal{O}$

$\lambda \in \mathfrak{h}^*$

$L(\lambda)$ — simple module with highest weight λ

$P(\lambda)$ — indecomposable projective cover of $L(\lambda)$

$I(\lambda)$ — indecomposable injective envelope of $L(\lambda)$

$\Delta(\lambda) = M(\lambda)$ — standard quotient of $P(\lambda)$

$\nabla(\lambda)$ — costandard submodule of $I(\lambda)$

$T(\lambda)$ — tilting envelope of $\Delta(\lambda)$

Fact. $T(\lambda)$ — tilting cover of $\nabla(\lambda)$

The problem

Theorem (BGG). $\mathrm{gl.dim}\mathcal{O} \leq$ number of roots of $\mathfrak{g}$

(the last remark of the BGG paper is that the equality holds, no proof)

Question. Determine projective dimension of all indecomposable structural modules

Note. For almost all λ we have

$$L(\lambda) = P(\lambda) = I(\lambda) = \Delta(\lambda) = \nabla(\lambda) = T(\lambda)$$

and the answer to the question for these λ is: 0

Still open. $\mathrm{proj.dim}(L(\lambda)) =?$, $\mathrm{proj.dim}(\Delta(\lambda)) =?$ (for singular λ)

The problem

Theorem (BGG). $\mathrm{gl.dim}\mathcal{O} \leq$ number of roots of $\mathfrak{g}$

(the last remark of the BGG paper is that the equality holds, no proof)

Question. Determine projective dimension of all indecomposable structural modules

Note. For almost all λ we have

$$L(\lambda) = P(\lambda) = I(\lambda) = \Delta(\lambda) = \nabla(\lambda) = T(\lambda)$$

and the answer to the question for these λ is: 0

Still open. $\mathrm{proj.dim}(L(\lambda)) =?$, $\mathrm{proj.dim}(\Delta(\lambda)) =?$ (for singular λ)

The problem

Theorem (BGG). $\mathrm{gl.dim}\mathcal{O} \leq$ number of roots of $\mathfrak{g}$

(the last remark of the BGG paper is that the equality holds, no proof)

Question. Determine projective dimension of all indecomposable structural modules

Note. For almost all λ we have

$$L(\lambda) = P(\lambda) = I(\lambda) = \Delta(\lambda) = \nabla(\lambda) = T(\lambda)$$

and the answer to the question for these λ is: 0

Still open. $\mathrm{proj.dim}(L(\lambda)) =?$, $\mathrm{proj.dim}(\Delta(\lambda)) =?$ (for singular λ)

The problem

Theorem (BGG). $\mathrm{gl.dim}\mathcal{O} \leq$ number of roots of $\mathfrak{g}$

(the last remark of the BGG paper is that the equality holds, no proof)

Question. Determine projective dimension of all indecomposable structural modules

Note. For almost all λ we have

$$L(\lambda) = P(\lambda) = I(\lambda) = \Delta(\lambda) = \nabla(\lambda) = T(\lambda)$$

and the answer to the question for these λ is: 0

Still open. $\mathrm{proj.dim}(L(\lambda)) =?$, $\mathrm{proj.dim}(\Delta(\lambda)) =?$ (for singular λ)

The problem

Theorem (BGG). $\mathrm{gl.dim}\mathcal{O} \leq$ number of roots of $\mathfrak{g}$

(the last remark of the BGG paper is that the equality holds, no proof)

Question. Determine projective dimension of all indecomposable structural modules

Note. For almost all λ we have

$$L(\lambda) = P(\lambda) = I(\lambda) = \Delta(\lambda) = \nabla(\lambda) = T(\lambda)$$

and the answer to the question for these λ is: 0

Still open. $\mathrm{proj.dim}(L(\lambda)) = ?$, $\mathrm{proj.dim}(\Delta(\lambda)) = ?$ (for singular λ)

The problem

Theorem (BGG). $\mathrm{gl.dim}\mathcal{O} \leq$ number of roots of $\mathfrak{g}$

(the last remark of the BGG paper is that the equality holds, no proof)

Question. Determine projective dimension of all indecomposable structural modules

Note. For almost all λ we have

$$L(\lambda) = P(\lambda) = I(\lambda) = \Delta(\lambda) = \nabla(\lambda) = T(\lambda)$$

and the answer to the question for these λ is: 0

Still open. $\mathrm{proj.dim}(L(\lambda)) =?$, $\mathrm{proj.dim}(\Delta(\lambda)) =?$ (for singular λ)

The problem

Theorem (BGG). $\mathrm{gl.dim}\mathcal{O} \leq$ number of roots of $\mathfrak{g}$

(the last remark of the BGG paper is that the equality holds, no proof)

Question. Determine projective dimension of all indecomposable structural modules

Note. For almost all λ we have

$$L(\lambda) = P(\lambda) = I(\lambda) = \Delta(\lambda) = \nabla(\lambda) = T(\lambda)$$

and the answer to the question for these λ is: 0

Still open. $\mathrm{proj.dim}(L(\lambda)) =?$, $\mathrm{proj.dim}(\Delta(\lambda)) =?$ (for singular λ)

The principal block

$\mathfrak{m}_0$ — the maximal ideal in $Z(\mathfrak{g})$ annihilating the trivial $\mathfrak{g}$-module

$\mathcal{O}_0 := \mathcal{O}_{\mathfrak{m}_0}$ — the principal block of $\mathcal{O}$

Fact. $\mathcal{O}_0$ is indecomposable

W — the Weyl group of $\mathfrak{g}$

W acts on $\mathfrak{h}^*$ in the usual way

ρ — the half of the sum of all positive roots

W acts on $\mathfrak{h}^*$ via the ρ-shifted dot action

Fact. Simples in $\mathcal{O}_0$ are $L(w \cdot 0)$, $w \in W$ (all different).

Notation. $L(w) := L(w \cdot 0)$, $P(w) := P(w \cdot 0)$ and so on

The principal block

$\mathtt{m}_0$ — the maximal ideal in $Z(\mathfrak{g})$ annihilating the trivial $\mathfrak{g}$-module

$\mathcal{O}_0 := \mathcal{O}_{\mathtt{m}_0}$ — the principal block of $\mathcal{O}$

Fact. $\mathcal{O}_0$ is indecomposable

W — the Weyl group of $\mathfrak{g}$

W acts on $\mathfrak{h}^*$ in the usual way

ρ — the half of the sum of all positive roots

W acts on $\mathfrak{h}^*$ via the ρ-shifted dot action

Fact. Simples in $\mathcal{O}_0$ are $L(w \cdot 0)$, $w \in W$ (all different).

Notation. $L(w) := L(w \cdot 0)$, $P(w) := P(w \cdot 0)$ and so on

The principal block

$\mathfrak{m}_0$ — the maximal ideal in $Z(\mathfrak{g})$ annihilating the trivial $\mathfrak{g}$-module

$\mathcal{O}_0 := \mathcal{O}_{\mathfrak{m}_0}$ — the principal block of $\mathcal{O}$

Fact. $\mathcal{O}_0$ is indecomposable

W — the Weyl group of $\mathfrak{g}$

W acts on $\mathfrak{h}^*$ in the usual way

ρ — the half of the sum of all positive roots

W acts on $\mathfrak{h}^*$ via the ρ-shifted dot action

Fact. Simples in $\mathcal{O}_0$ are $L(w \cdot 0)$, $w \in W$ (all different).

Notation. $L(w) := L(w \cdot 0)$, $P(w) := P(w \cdot 0)$ and so on

The principal block

$\mathfrak{m}_0$ — the maximal ideal in $Z(\mathfrak{g})$ annihilating the trivial $\mathfrak{g}$-module

$\mathcal{O}_0 := \mathcal{O}_{\mathfrak{m}_0}$ — the principal block of $\mathcal{O}$

Fact. $\mathcal{O}_0$ is indecomposable

W — the Weyl group of $\mathfrak{g}$

W acts on $\mathfrak{h}^*$ in the usual way

ρ — the half of the sum of all positive roots

W acts on $\mathfrak{h}^*$ via the ρ-shifted dot action

Fact. Simples in $\mathcal{O}_0$ are $L(w \cdot 0)$, $w \in W$ (all different).

Notation. $L(w) := L(w \cdot 0)$, $P(w) := P(w \cdot 0)$ and so on

The principal block

$\mathfrak{m}_0$ — the maximal ideal in $Z(\mathfrak{g})$ annihilating the trivial $\mathfrak{g}$-module

$\mathcal{O}_0 := \mathcal{O}_{\mathfrak{m}_0}$ — the principal block of $\mathcal{O}$

Fact. $\mathcal{O}_0$ is indecomposable

W — the Weyl group of $\mathfrak{g}$

W acts on $\mathfrak{h}^*$ in the usual way

ρ — the half of the sum of all positive roots

W acts on $\mathfrak{h}^*$ via the ρ-shifted dot action

Fact. Simples in $\mathcal{O}_0$ are $L(w \cdot 0)$, $w \in W$ (all different).

Notation. $L(w) := L(w \cdot 0)$, $P(w) := P(w \cdot 0)$ and so on

The principal block

$\mathfrak{m}_0$ — the maximal ideal in $Z(\mathfrak{g})$ annihilating the trivial $\mathfrak{g}$-module

$\mathcal{O}_0 := \mathcal{O}_{\mathfrak{m}_0}$ — the principal block of $\mathcal{O}$

Fact. $\mathcal{O}_0$ is indecomposable

W — the Weyl group of $\mathfrak{g}$

W acts on $\mathfrak{h}^*$ in the usual way

ρ — the half of the sum of all positive roots

W acts on $\mathfrak{h}^*$ via the ρ-shifted dot action

Fact. Simples in $\mathcal{O}_0$ are $L(w \cdot 0)$, $w \in W$ (all different).

Notation. $L(w) := L(w \cdot 0)$, $P(w) := P(w \cdot 0)$ and so on

The principal block

$\mathfrak{m}_0$ — the maximal ideal in $Z(\mathfrak{g})$ annihilating the trivial $\mathfrak{g}$-module

$\mathcal{O}_0 := \mathcal{O}_{\mathfrak{m}_0}$ — the principal block of $\mathcal{O}$

Fact. $\mathcal{O}_0$ is indecomposable

W — the Weyl group of $\mathfrak{g}$

W acts on $\mathfrak{h}^*$ in the usual way

ρ — the half of the sum of all positive roots

W acts on $\mathfrak{h}^*$ via the ρ-shifted dot action

Fact. Simples in $\mathcal{O}_0$ are $L(w \cdot 0)$, $w \in W$ (all different).

Notation. $L(w) := L(w \cdot 0)$, $P(w) := P(w \cdot 0)$ and so on

The principal block

$\mathfrak{m}_0$ — the maximal ideal in $Z(\mathfrak{g})$ annihilating the trivial $\mathfrak{g}$-module

$\mathcal{O}_0 := \mathcal{O}_{\mathfrak{m}_0}$ — the principal block of $\mathcal{O}$

Fact. $\mathcal{O}_0$ is indecomposable

W — the Weyl group of $\mathfrak{g}$

W acts on $\mathfrak{h}^*$ in the usual way

ρ — the half of the sum of all positive roots

W acts on $\mathfrak{h}^*$ via the ρ-shifted dot action

Fact. Simples in $\mathcal{O}_0$ are $L(w \cdot 0)$, $w \in W$ (all different).

Notation. $L(w) := L(w \cdot 0)$, $P(w) := P(w \cdot 0)$ and so on

The principal block

$\mathfrak{m}_0$ — the maximal ideal in $Z(\mathfrak{g})$ annihilating the trivial $\mathfrak{g}$-module

$\mathcal{O}_0 := \mathcal{O}_{\mathfrak{m}_0}$ — the principal block of $\mathcal{O}$

Fact. $\mathcal{O}_0$ is indecomposable

W — the Weyl group of $\mathfrak{g}$

W acts on $\mathfrak{h}^*$ in the usual way

ρ — the half of the sum of all positive roots

W acts on $\mathfrak{h}^*$ via the ρ-shifted dot action

Fact. Simples in $\mathcal{O}_0$ are $L(w \cdot 0)$, $w \in W$ (all different).

Notation. $L(w) := L(w \cdot 0)$, $P(w) := P(w \cdot 0)$ and so on

The principal block

$\mathfrak{m}_0$ — the maximal ideal in $Z(\mathfrak{g})$ annihilating the trivial $\mathfrak{g}$-module

$\mathcal{O}_0 := \mathcal{O}_{\mathfrak{m}_0}$ — the principal block of $\mathcal{O}$

Fact. $\mathcal{O}_0$ is indecomposable

W — the Weyl group of $\mathfrak{g}$

W acts on $\mathfrak{h}^*$ in the usual way

ρ — the half of the sum of all positive roots

W acts on $\mathfrak{h}^*$ via the ρ-shifted dot action

Fact. Simples in $\mathcal{O}_0$ are $L(w \cdot 0)$, $w \in W$ (all different).

Notation. $L(w) := L(w \cdot 0)$, $P(w) := P(w \cdot 0)$ and so on

The principal block

$\mathfrak{m}_0$ — the maximal ideal in $Z(\mathfrak{g})$ annihilating the trivial $\mathfrak{g}$-module

$\mathcal{O}_0 := \mathcal{O}_{\mathfrak{m}_0}$ — the principal block of $\mathcal{O}$

Fact. $\mathcal{O}_0$ is indecomposable

W — the Weyl group of $\mathfrak{g}$

W acts on $\mathfrak{h}^*$ in the usual way

ρ — the half of the sum of all positive roots

W acts on $\mathfrak{h}^*$ via the ρ-shifted dot action

Fact. Simples in $\mathcal{O}_0$ are $L(w \cdot 0)$, $w \in W$ (all different).

Notation. $L(w) := L(w \cdot 0)$, $P(w) := P(w \cdot 0)$ and so on

Preliminary observations and reduction

Obvious: $\mathrm{proj.dim}(P(\lambda)) = 0$ for all λ

Soergel's combinatorial description of blocks in terms of root systems: Each block of $\mathcal{O}$ can be reduced, up to equivalence, to an integral block (maybe for some other $\mathfrak{g}$)

Note. $\mathcal{O}_0$ is a regular integral block

Note. Regular and singular integral blocks are connected via projective functors (exact, with adjunction morphisms) whose combinatorics is known

Preliminary observations and reduction

Obvious: $\mathrm{proj.dim}(P(\lambda)) = 0$ for all λ

Soergel's combinatorial description of blocks in terms of root systems: Each block of $\mathcal{O}$ can be reduced, up to equivalence, to an integral block (maybe for some other $\mathfrak{g}$)

Note. $\mathcal{O}_0$ is a regular integral block

Note. Regular and singular integral blocks are connected via projective functors (exact, with adjunction morphisms) whose combinatorics is known

Preliminary observations and reduction

Obvious: $\mathrm{proj.dim}(P(\lambda)) = 0$ for all λ

Soergel's combinatorial description of blocks in terms of root systems: Each block of $\mathcal{O}$ can be reduced, up to equivalence, to an integral block (maybe for some other $\mathfrak{g}$)

Note. $\mathcal{O}_0$ is a regular integral block

Note. Regular and singular integral blocks are connected via projective functors (exact, with adjunction morphisms) whose combinatorics is known

Preliminary observations and reduction

Obvious: $\mathrm{proj.dim}(P(\lambda)) = 0$ for all λ

Soergel's combinatorial description of blocks in terms of root systems: Each block of $\mathcal{O}$ can be reduced, up to equivalence, to an integral block (maybe for some other $\mathfrak{g}$)

Note. $\mathcal{O}_0$ is a regular integral block

Note. Regular and singular integral blocks are connected via projective functors (exact, with adjunction morphisms) whose combinatorics is known

Preliminary observations and reduction

Obvious: $\mathrm{proj.dim}(P(\lambda)) = 0$ for all λ

Soergel's combinatorial description of blocks in terms of root systems: Each block of $\mathcal{O}$ can be reduced, up to equivalence, to an integral block (maybe for some other $\mathfrak{g}$)

Note. $\mathcal{O}_0$ is a regular integral block

Note. Regular and singular integral blocks are connected via projective functors (exact, with adjunction morphisms) whose combinatorics is known

Preliminary observations and reduction

Obvious: $\mathrm{proj.dim}(P(\lambda)) = 0$ for all λ

Soergel's combinatorial description of blocks in terms of root systems: Each block of $\mathcal{O}$ can be reduced, up to equivalence, to an integral block (maybe for some other $\mathfrak{g}$)

Note. $\mathcal{O}_0$ is a regular integral block

Note. Regular and singular integral blocks are connected via projective functors (exact, with adjunction morphisms) whose combinatorics is known

Projective dimension of Verma modules in $\mathcal{O}_0$

$\mathbf{l} : W \to \{0, 1, 2, \ldots\}$ — the usual length function

w_0 — the longest element in W

e — the identity element in W

Proposition. $\mathrm{proj.dim}(\Delta(w)) = \mathbf{l}(w)$ for all $w \in W$

The inequality $\leq$ (BGG): We have $\Delta(e) = P(e)$ and hence $\mathrm{proj.dim}(\Delta(e)) = 0$. We have $\mathrm{Ker} \hookrightarrow P(w) \twoheadrightarrow \Delta(w)$ where Ker has a filtration with subquotients $\Delta(x)$, $\mathbf{l}(x) < \mathbf{l}(w)$. Now use the long exact sequence and the standard dimension shift argument.

The inequality $\geq$: Enough to prove: $\mathrm{proj.dim}(\Delta(w_0)) = \mathbf{l}(w_0)$ (postponed).

Note. $\Delta(w_0) = \nabla(w_0) = L(w_0) = T(w_0)$

Projective dimension of Verma modules in $\mathcal{O}_0$

$\mathbf{l} : W \to \{0, 1, 2, \dots\}$ — the usual length function

w_0 — the longest element in W

e — the identity element in W

Proposition. $\mathrm{proj.dim}(\Delta(w)) = \mathbf{l}(w)$ for all $w \in W$

The inequality $\leq$ (BGG): We have $\Delta(e) = P(e)$ and hence $\mathrm{proj.dim}(\Delta(e)) = 0$. We have $\mathrm{Ker} \hookrightarrow P(w) \twoheadrightarrow \Delta(w)$ where Ker has a filtration with subquotients $\Delta(x)$, $\mathbf{l}(x) < \mathbf{l}(w)$. Now use the long exact sequence and the standard dimension shift argument.

The inequality $\geq$: Enough to prove: $\mathrm{proj.dim}(\Delta(w_0)) = \mathbf{l}(w_0)$ (postponed).

Note. $\Delta(w_0) = \nabla(w_0) = L(w_0) = T(w_0)$

Projective dimension of Verma modules in $\mathcal{O}_0$

$\mathbf{l} : W \to \{0, 1, 2, \ldots\}$ — the usual length function

w_0 — the longest element in W

e — the identity element in W

Proposition. $\mathrm{proj.dim}(\Delta(w)) = \mathbf{l}(w)$ for all $w \in W$

The inequality $\leq$ (BGG): We have $\Delta(e) = P(e)$ and hence $\mathrm{proj.dim}(\Delta(e)) = 0$. We have $\mathrm{Ker} \hookrightarrow P(w) \twoheadrightarrow \Delta(w)$ where Ker has a filtration with subquotients $\Delta(x)$, $\mathbf{l}(x) < \mathbf{l}(w)$. Now use the long exact sequence and the standard dimension shift argument.

The inequality $\geq$: Enough to prove: $\mathrm{proj.dim}(\Delta(w_0)) = \mathbf{l}(w_0)$ (postponed).

Note. $\Delta(w_0) = \nabla(w_0) = L(w_0) = T(w_0)$

Projective dimension of Verma modules in $\mathcal{O}_0$

$\mathbf{l} : W \to \{0, 1, 2, \dots\}$ — the usual length function

w_0 — the longest element in W

e — the identity element in W

Proposition. $\mathrm{proj.dim}(\Delta(w)) = \mathbf{l}(w)$ for all $w \in W$

The inequality $\leq$ (BGG): We have $\Delta(e) = P(e)$ and hence $\mathrm{proj.dim}(\Delta(e)) = 0$. We have $\mathrm{Ker} \hookrightarrow P(w) \twoheadrightarrow \Delta(w)$ where Ker has a filtration with subquotients $\Delta(x)$, $\mathbf{l}(x) < \mathbf{l}(w)$. Now use the long exact sequence and the standard dimension shift argument.

The inequality $\geq$: Enough to prove: $\mathrm{proj.dim}(\Delta(w_0)) = \mathbf{l}(w_0)$ (postponed).

Note. $\Delta(w_0) = \nabla(w_0) = L(w_0) = T(w_0)$

Projective dimension of Verma modules in $\mathcal{O}_0$

$\mathbf{l}: W \to \{0, 1, 2, \dots\}$ — the usual length function

w_0 — the longest element in W

e — the identity element in W

Proposition. $\mathrm{proj.dim}(\Delta(w)) = \mathbf{l}(w)$ for all $w \in W$

The inequality $\leq$ (BGG): We have $\Delta(e) = P(e)$ and hence $\mathrm{proj.dim}(\Delta(e)) = 0$. We have $\mathrm{Ker} \hookrightarrow P(w) \twoheadrightarrow \Delta(w)$ where Ker has a filtration with subquotients $\Delta(x)$, $\mathbf{l}(x) < \mathbf{l}(w)$. Now use the long exact sequence and the standard dimension shift argument.

The inequality $\geq$: Enough to prove: $\mathrm{proj.dim}(\Delta(w_0)) = \mathbf{l}(w_0)$ (postponed).

Note. $\Delta(w_0) = \nabla(w_0) = L(w_0) = T(w_0)$

Projective dimension of Verma modules in $\mathcal{O}_0$

$\mathbf{l} : W \to \{0, 1, 2, \dots\}$ — the usual length function

w_0 — the longest element in W

e — the identity element in W

Proposition. $\mathrm{proj.dim}(\Delta(w)) = \mathbf{l}(w)$ for all $w \in W$

The inequality $\leq$ (BGG): We have $\Delta(e) = P(e)$ and hence $\mathrm{proj.dim}(\Delta(e)) = 0$. We have $\mathrm{Ker} \hookrightarrow P(w) \twoheadrightarrow \Delta(w)$ where Ker has a filtration with subquotients $\Delta(x)$, $\mathbf{l}(x) < \mathbf{l}(w)$. Now use the long exact sequence and the standard dimension shift argument.

The inequality $\geq$: Enough to prove: $\mathrm{proj.dim}(\Delta(w_0)) = \mathbf{l}(w_0)$ (postponed).

Note. $\Delta(w_0) = \nabla(w_0) = L(w_0) = T(w_0)$

Projective dimension of Verma modules in $\mathcal{O}_0$

$\mathbf{l} : W \to \{0, 1, 2, \dots\}$ — the usual length function

w_0 — the longest element in W

e — the identity element in W

Proposition. $\mathrm{proj.dim}(\Delta(w)) = \mathbf{l}(w)$ for all $w \in W$

The inequality $\leq$ (BGG): We have $\Delta(e) = P(e)$ and hence $\mathrm{proj.dim}(\Delta(e)) = 0$. We have $\mathrm{Ker} \hookrightarrow P(w) \twoheadrightarrow \Delta(w)$ where Ker has a filtration with subquotients $\Delta(x)$, $\mathbf{l}(x) < \mathbf{l}(w)$. Now use the long exact sequence and the standard dimension shift argument.

The inequality $\geq$: Enough to prove: $\mathrm{proj.dim}(\Delta(w_0)) = \mathbf{l}(w_0)$ (postponed).

Note. $\Delta(w_0) = \nabla(w_0) = L(w_0) = T(w_0)$

Projective dimension of Verma modules in $\mathcal{O}_0$

$\mathbf{l} : W \to \{0, 1, 2, \dots\}$ — the usual length function

w_0 — the longest element in W

e — the identity element in W

Proposition. $\mathrm{proj.dim}(\Delta(w)) = \mathbf{l}(w)$ for all $w \in W$

The inequality $\leq$ (BGG): We have $\Delta(e) = P(e)$ and hence $\mathrm{proj.dim}(\Delta(e)) = 0$. We have $\mathrm{Ker} \hookrightarrow P(w) \twoheadrightarrow \Delta(w)$ where Ker has a filtration with subquotients $\Delta(x)$, $\mathbf{l}(x) < \mathbf{l}(w)$. Now use the long exact sequence and the standard dimension shift argument.

The inequality $\geq$: Enough to prove: $\mathrm{proj.dim}(\Delta(w_0)) = \mathbf{l}(w_0)$ (postponed).

Note. $\Delta(w_0) = \nabla(w_0) = L(w_0) = T(w_0)$

Projective dimension of Verma modules in $\mathcal{O}_0$

$\mathbf{l} : W \to \{0, 1, 2, \dots\}$ — the usual length function

w_0 — the longest element in W

e — the identity element in W

Proposition. $\mathrm{proj.dim}(\Delta(w)) = \mathbf{l}(w)$ for all $w \in W$

The inequality $\leq$ (BGG): We have $\Delta(e) = P(e)$ and hence $\mathrm{proj.dim}(\Delta(e)) = 0$. We have $\mathrm{Ker} \hookrightarrow P(w) \twoheadrightarrow \Delta(w)$ where Ker has a filtration with subquotients $\Delta(x)$, $\mathbf{l}(x) < \mathbf{l}(w)$. Now use the long exact sequence and the standard dimension shift argument.

The inequality $\geq$: Enough to prove: $\mathrm{proj.dim}(\Delta(w_0)) = \mathbf{l}(w_0)$ (postponed).

Note. $\Delta(w_0) = \nabla(w_0) = L(w_0) = T(w_0)$

Projective dimension of simple modules in $\mathcal{O}_0$

Proposition. $\mathrm{proj.dim}(L(w)) = 2\mathbf{l}(w_0) - \mathbf{l}(w)$ for all $w \in W$

The inequality $\leq$ (BGG): For w_0 this is already done before. We have $\mathrm{Ker} \hookrightarrow \Delta(w) \twoheadrightarrow L(w)$ where Ker has a filtration with subquotients $L(x)$, $\mathbf{l}(x) > \mathbf{l}(w)$. Now use the long exact sequence and the standard dimension shift argument.

The inequality $\geq$: Enough to prove: $\mathrm{proj.dim}(L(e)) = 2\mathbf{l}(w_0)$

Proof.

- Let $\mathcal{X}^\bullet$ be the BGG resolution of $L(e)$ by Verma modules. It has length $\mathbf{l}(w_0)$ and $\mathcal{X}^{-\mathbf{l}(w_0)} = \Delta(w_0)$, the simple Verma module.
- The dual $\mathcal{Y}^\bullet$ of $\mathcal{X}^\bullet$ is a coresolution of $L(e)$ by costandard modules, it has length $\mathbf{l}(w_0)$ and $\mathcal{Y}^{\mathbf{l}(w_0)} = \nabla(w_0) = \Delta(w_0)$.
- Standard and costandard modules are homologically orthogonal and hence all derived homs are realized already in the homotopy category.
- The identity on $\Delta(w_0)$ gives a homomorphism from $\mathcal{X}^\bullet$ to $\mathcal{Y}^\bullet[2\mathbf{l}(w_0)]$ which is clearly not homotopic to zero.

Projective dimension of simple modules in $\mathcal{O}_0$

Proposition. $\mathrm{proj.dim}(L(w)) = 2\mathbf{l}(w_0) - \mathbf{l}(w)$ for all $w \in W$

The inequality $\leq$ (BGG): For w_0 this is already done before. We have $\mathrm{Ker} \hookrightarrow \Delta(w) \twoheadrightarrow L(w)$ where Ker has a filtration with subquotients $L(x)$, $\mathbf{l}(x) > \mathbf{l}(w)$. Now use the long exact sequence and the standard dimension shift argument.

The inequality $\geq$: Enough to prove: $\mathrm{proj.dim}(L(e)) = 2\mathbf{l}(w_0)$

Proof.

- Let $\mathcal{X}^\bullet$ be the BGG resolution of $L(e)$ by Verma modules. It has length $\mathbf{l}(w_0)$ and $\mathcal{X}^{-\mathbf{l}(w_0)} = \Delta(w_0)$, the simple Verma module.
- The dual $\mathcal{Y}^\bullet$ of $\mathcal{X}^\bullet$ is a coresolution of $L(e)$ by costandard modules, it has length $\mathbf{l}(w_0)$ and $\mathcal{Y}^{\mathbf{l}(w_0)} = \nabla(w_0) = \Delta(w_0)$.
- Standard and costandard modules are homologically orthogonal and hence all derived homs are realized already in the homotopy category.
- The identity on $\Delta(w_0)$ gives a homomorphism from $\mathcal{X}^\bullet$ to $\mathcal{Y}^\bullet[2\mathbf{l}(w_0)]$ which is clearly not homotopic to zero.

Projective dimension of simple modules in $\mathcal{O}_0$

Proposition. $\mathrm{proj.dim}(L(w)) = 2\mathbf{l}(w_0) - \mathbf{l}(w)$ for all $w \in W$

The inequality $\leq$ (BGG): For w_0 this is already done before. We have $\mathrm{Ker} \hookrightarrow \Delta(w) \twoheadrightarrow L(w)$ where Ker has a filtration with subquotients $L(x)$, $\mathbf{l}(x) > \mathbf{l}(w)$. Now use the long exact sequence and the standard dimension shift argument.

The inequality $\geq$: Enough to prove: $\mathrm{proj.dim}(L(e)) = 2\mathbf{l}(w_0)$

Proof.

- Let $\mathcal{X}^\bullet$ be the BGG resolution of $L(e)$ by Verma modules. It has length $\mathbf{l}(w_0)$ and $\mathcal{X}^{-\mathbf{l}(w_0)} = \Delta(w_0)$, the simple Verma module.
- The dual $\mathcal{Y}^\bullet$ of $\mathcal{X}^\bullet$ is a coresolution of $L(e)$ by costandard modules, it has length $\mathbf{l}(w_0)$ and $\mathcal{Y}^{\mathbf{l}(w_0)} = \nabla(w_0) = \Delta(w_0)$.
- Standard and costandard modules are homologically orthogonal and hence all derived homs are realized already in the homotopy category.
- The identity on $\Delta(w_0)$ gives a homomorphism from $\mathcal{X}^\bullet$ to $\mathcal{Y}^\bullet[2\mathbf{l}(w_0)]$ which is clearly not homotopic to zero.

Projective dimension of simple modules in $\mathcal{O}_0$

Proposition. $\mathrm{proj.dim}(L(w)) = 2\mathbf{l}(w_0) - \mathbf{l}(w)$ for all $w \in W$

The inequality $\leq$ (BGG): For w_0 this is already done before. We have $\mathrm{Ker} \hookrightarrow \Delta(w) \twoheadrightarrow L(w)$ where Ker has a filtration with subquotients $L(x)$, $\mathbf{l}(x) > \mathbf{l}(w)$. Now use the long exact sequence and the standard dimension shift argument.

The inequality $\geq$: Enough to prove: $\mathrm{proj.dim}(L(e)) = 2\mathbf{l}(w_0)$

Proof.

- Let $\mathcal{X}^\bullet$ be the BGG resolution of $L(e)$ by Verma modules. It has length $\mathbf{l}(w_0)$ and $\mathcal{X}^{-\mathbf{l}(w_0)} = \Delta(w_0)$, the simple Verma module.
- The dual $\mathcal{Y}^\bullet$ of $\mathcal{X}^\bullet$ is a coresolution of $L(e)$ by costandard modules, it has length $\mathbf{l}(w_0)$ and $\mathcal{Y}^{\mathbf{l}(w_0)} = \nabla(w_0) = \Delta(w_0)$.
- Standard and costandard modules are homologically orthogonal and hence all derived homs are realized already in the homotopy category.
- The identity on $\Delta(w_0)$ gives a homomorphism from $\mathcal{X}^\bullet$ to $\mathcal{Y}^\bullet[2\mathbf{l}(w_0)]$ which is clearly not homotopic to zero.

Projective dimension of simple modules in $\mathcal{O}_0$

Proposition. $\mathrm{proj.dim}(L(w)) = 2\mathbf{l}(w_0) - \mathbf{l}(w)$ for all $w \in W$

The inequality $\leq$ (BGG): For w_0 this is already done before. We have $\mathrm{Ker} \hookrightarrow \Delta(w) \twoheadrightarrow L(w)$ where Ker has a filtration with subquotients $L(x)$, $\mathbf{l}(x) > \mathbf{l}(w)$. Now use the long exact sequence and the standard dimension shift argument.

The inequality $\geq$: Enough to prove: $\mathrm{proj.dim}(L(e)) = 2\mathbf{l}(w_0)$

Proof.

- Let $\mathcal{X}^\bullet$ be the BGG resolution of $L(e)$ by Verma modules. It has length $\mathbf{l}(w_0)$ and $\mathcal{X}^{-\mathbf{l}(w_0)} = \Delta(w_0)$, the simple Verma module.
- The dual $\mathcal{Y}^\bullet$ of $\mathcal{X}^\bullet$ is a coresolution of $L(e)$ by costandard modules, it has length $\mathbf{l}(w_0)$ and $\mathcal{Y}^{\mathbf{l}(w_0)} = \nabla(w_0) = \Delta(w_0)$.
- Standard and costandard modules are homologically orthogonal and hence all derived homs are realized already in the homotopy category.
- The identity on $\Delta(w_0)$ gives a homomorphism from $\mathcal{X}^\bullet$ to $\mathcal{Y}^\bullet[2\mathbf{l}(w_0)]$ which is clearly not homotopic to zero.

Projective dimension of simple modules in $\mathcal{O}_0$

Proposition. $\mathrm{proj.dim}(L(w)) = 2\mathbf{l}(w_0) - \mathbf{l}(w)$ for all $w \in W$

The inequality $\leq$ (BGG): For w_0 this is already done before. We have $\mathrm{Ker} \hookrightarrow \Delta(w) \twoheadrightarrow L(w)$ where Ker has a filtration with subquotients $L(x)$, $\mathbf{l}(x) > \mathbf{l}(w)$. Now use the long exact sequence and the standard dimension shift argument.

The inequality $\geq$: Enough to prove: $\mathrm{proj.dim}(L(e)) = 2\mathbf{l}(w_0)$

Proof.

- Let $\mathcal{X}^\bullet$ be the BGG resolution of $L(e)$ by Verma modules. It has length $\mathbf{l}(w_0)$ and $\mathcal{X}^{-\mathbf{l}(w_0)} = \Delta(w_0)$, the simple Verma module.
- The dual $\mathcal{Y}^\bullet$ of $\mathcal{X}^\bullet$ is a coresolution of $L(e)$ by costandard modules, it has length $\mathbf{l}(w_0)$ and $\mathcal{Y}^{\mathbf{l}(w_0)} = \nabla(w_0) = \Delta(w_0)$.
- Standard and costandard modules are homologically orthogonal and hence all derived homs are realized already in the homotopy category.
- The identity on $\Delta(w_0)$ gives a homomorphism from $\mathcal{X}^\bullet$ to $\mathcal{Y}^\bullet[2\mathbf{l}(w_0)]$ which is clearly not homotopic to zero.

Projective dimension of simple modules in $\mathcal{O}_0$

Proposition. $\mathrm{proj.dim}(L(w)) = 2\mathbf{l}(w_0) - \mathbf{l}(w)$ for all $w \in W$

The inequality $\leq$ (BGG): For w_0 this is already done before. We have $\mathrm{Ker} \hookrightarrow \Delta(w) \twoheadrightarrow L(w)$ where Ker has a filtration with subquotients $L(x)$, $\mathbf{l}(x) > \mathbf{l}(w)$. Now use the long exact sequence and the standard dimension shift argument.

The inequality $\geq$: Enough to prove: $\mathrm{proj.dim}(L(e)) = 2\mathbf{l}(w_0)$

Proof.

- Let $\mathcal{X}^\bullet$ be the BGG resolution of $L(e)$ by Verma modules. It has length $\mathbf{l}(w_0)$ and $\mathcal{X}^{-\mathbf{l}(w_0)} = \Delta(w_0)$, the simple Verma module.
- The dual $\mathcal{Y}^\bullet$ of $\mathcal{X}^\bullet$ is a coresolution of $L(e)$ by costandard modules, it has length $\mathbf{l}(w_0)$ and $\mathcal{Y}^{\mathbf{l}(w_0)} = \nabla(w_0) = \Delta(w_0)$.
- Standard and costandard modules are homologically orthogonal and hence all derived homs are realized already in the homotopy category.
- The identity on $\Delta(w_0)$ gives a homomorphism from $\mathcal{X}^\bullet$ to $\mathcal{Y}^\bullet[2\mathbf{l}(w_0)]$ which is clearly not homotopic to zero.

Projective dimension of simple modules in $\mathcal{O}_0$

Proposition. $\mathrm{proj.dim}(L(w)) = 2\mathbf{l}(w_0) - \mathbf{l}(w)$ for all $w \in W$

The inequality $\leq$ (BGG): For w_0 this is already done before. We have $\mathrm{Ker} \hookrightarrow \Delta(w) \twoheadrightarrow L(w)$ where Ker has a filtration with subquotients $L(x)$, $\mathbf{l}(x) > \mathbf{l}(w)$. Now use the long exact sequence and the standard dimension shift argument.

The inequality $\geq$: Enough to prove: $\mathrm{proj.dim}(L(e)) = 2\mathbf{l}(w_0)$

Proof.

- Let $\mathcal{X}^\bullet$ be the BGG resolution of $L(e)$ by Verma modules. It has length $\mathbf{l}(w_0)$ and $\mathcal{X}^{-\mathbf{l}(w_0)} = \Delta(w_0)$, the simple Verma module.
- The dual $\mathcal{Y}^\bullet$ of $\mathcal{X}^\bullet$ is a coresolution of $L(e)$ by costandard modules, it has length $\mathbf{l}(w_0)$ and $\mathcal{Y}^{\mathbf{l}(w_0)} = \nabla(w_0) = \Delta(w_0)$.
- Standard and costandard modules are homologically orthogonal and hence all derived homs are realized already in the homotopy category.
- The identity on $\Delta(w_0)$ gives a homomorphism from $\mathcal{X}^\bullet$ to $\mathcal{Y}^\bullet[2\mathbf{l}(w_0)]$ which is clearly not homotopic to zero.

Projective dimension of simple modules in $\mathcal{O}_0$

Proposition. $\mathrm{proj.dim}(L(w)) = 2\mathbf{l}(w_0) - \mathbf{l}(w)$ for all $w \in W$

The inequality $\leq$ (BGG): For w_0 this is already done before. We have $\mathrm{Ker} \hookrightarrow \Delta(w) \twoheadrightarrow L(w)$ where Ker has a filtration with subquotients $L(x)$, $\mathbf{l}(x) > \mathbf{l}(w)$. Now use the long exact sequence and the standard dimension shift argument.

The inequality $\geq$: Enough to prove: $\mathrm{proj.dim}(L(e)) = 2\mathbf{l}(w_0)$

Proof.

- Let $\mathcal{X}^\bullet$ be the BGG resolution of $L(e)$ by Verma modules. It has length $\mathbf{l}(w_0)$ and $\mathcal{X}^{-\mathbf{l}(w_0)} = \Delta(w_0)$, the simple Verma module.
- The dual $\mathcal{Y}^\bullet$ of $\mathcal{X}^\bullet$ is a coresolution of $L(e)$ by costandard modules, it has length $\mathbf{l}(w_0)$ and $\mathcal{Y}^{\mathbf{l}(w_0)} = \nabla(w_0) = \Delta(w_0)$.
- Standard and costandard modules are homologically orthogonal and hence all derived homs are realized already in the homotopy category.
- The identity on $\Delta(w_0)$ gives a homomorphism from $\mathcal{X}^\bullet$ to $\mathcal{Y}^\bullet[2\mathbf{l}(w_0)]$ which is clearly not homotopic to zero.

Projective dimension of costandard modules in $\mathcal{O}_0$

Proposition. $\mathrm{proj.dim}(\nabla(w)) = 2\mathbf{l}(w_0) - \mathbf{l}(w)$ for all $w \in W$

Proof.

- We have $\nabla(w_0) = L(w_0)$ so in this case the claim is already established.
- Now do induction using standard dimension shift argument and the short exact sequence $L(w) \hookrightarrow \nabla(w) \twoheadrightarrow \mathrm{Coker}$ where Coker is filtered by $L(x)$ with $\mathbf{l}(x) > \mathbf{l}(w)$.

Projective dimension of costandard modules in $\mathcal{O}_0$

Proposition. $\mathrm{proj.dim}(\nabla(w)) = 2\mathbf{l}(w_0) - \mathbf{l}(w)$ for all $w \in W$

Proof.

- We have $\nabla(w_0) = L(w_0)$ so in this case the claim is already established.
- Now do induction using standard dimension shift argument and the short exact sequence $L(w) \hookrightarrow \nabla(w) \twoheadrightarrow \mathrm{Coker}$ where Coker is filtered by $L(x)$ with $\mathbf{l}(x) > \mathbf{l}(w)$.

Projective dimension of costandard modules in $\mathcal{O}_0$

Proposition. $\mathrm{proj.dim}(\nabla(w)) = 2\mathbf{l}(w_0) - \mathbf{l}(w)$ for all $w \in W$

Proof.

- We have $\nabla(w_0) = L(w_0)$ so in this case the claim is already established.
- Now do induction using standard dimension shift argument and the short exact sequence $L(w) \hookrightarrow \nabla(w) \twoheadrightarrow \mathrm{Coker}$ where Coker is filtered by $L(x)$ with $\mathbf{l}(x) > \mathbf{l}(w)$.

Projective dimension of costandard modules in $\mathcal{O}_0$

Proposition. $\mathrm{proj.dim}(\nabla(w)) = 2\mathbf{l}(w_0) - \mathbf{l}(w)$ for all $w \in W$

Proof.

- We have $\nabla(w_0) = L(w_0)$ so in this case the claim is already established.
- Now do induction using standard dimension shift argument and the short exact sequence $L(w) \hookrightarrow \nabla(w) \twoheadrightarrow \mathrm{Coker}$ where Coker is filtered by $L(x)$ with $\mathbf{l}(x) > \mathbf{l}(w)$.

Projective dimension of costandard modules in $\mathcal{O}_0$

Proposition. $\mathrm{proj.dim}(\nabla(w)) = 2\mathbf{l}(w_0) - \mathbf{l}(w)$ for all $w \in W$

Proof.

- We have $\nabla(w_0) = L(w_0)$ so in this case the claim is already established.
- Now do induction using standard dimension shift argument and the short exact sequence $L(w) \hookrightarrow \nabla(w) \twoheadrightarrow \mathrm{Coker}$ where Coker is filtered by $L(x)$ with $\mathbf{l}(x) > \mathbf{l}(w)$.

Projective dimension of costandard modules in $\mathcal{O}_0$

Proposition. $\mathrm{proj.dim}(\nabla(w)) = 2\mathbf{l}(w_0) - \mathbf{l}(w)$ for all $w \in W$

Proof.

- We have $\nabla(w_0) = L(w_0)$ so in this case the claim is already established.
- Now do induction using standard dimension shift argument and the short exact sequence $L(w) \hookrightarrow \nabla(w) \twoheadrightarrow \mathrm{Coker}$ where Coker is filtered by $L(x)$ with $\mathbf{l}(x) > \mathbf{l}(w)$.

Kazhdan-Lusztig combinatorics

$\mathbb{Z}[W]$ — the integral group algebra of W

$\{H_w := w\}$ — the standard basis of $\mathbb{Z}[W]$

$\{\underline{H}_w\}$ — the Kazhdan-Lusztig basis of $\mathbb{Z}[W]$

Fact. $\underline{H}_x \underline{H}_y = \sum_z c_{x,y}^z \underline{H}_z$ with all $c_{x,y}^z \in \{0, 1, 2, \dots\}$.

Left KL-order: $z \geq_L y$ if there is x such that $c_{x,y}^z > 0$.

Left KL-cell: $y \sim_L z$ if $y \geq_L z$ and $z \geq_L y$.

Note: Similarly right order $\geq_R$ and right cells $\sim_R$

Note: Similarly two-sided order $\geq_J$ and two-sided cells $\sim_J$

Kazhdan-Lusztig combinatorics

$\mathbb{Z}[W]$ — the integral group algebra of W

$\{H_w := w\}$ — the standard basis of $\mathbb{Z}[W]$

$\{\underline{H}_w\}$ — the Kazhdan-Lusztig basis of $\mathbb{Z}[W]$

Fact. $\underline{H}_x \underline{H}_y = \sum_z c_{x,y}^z \underline{H}_z$ with all $c_{x,y}^z \in \{0, 1, 2, \dots\}$.

Left KL-order: $z \geq_L y$ if there is x such that $c_{x,y}^z > 0$.

Left KL-cell: $y \sim_L z$ if $y \geq_L z$ and $z \geq_L y$.

Note: Similarly right order $\geq_R$ and right cells $\sim_R$

Note: Similarly two-sided order $\geq_J$ and two-sided cells $\sim_J$

Kazhdan-Lusztig combinatorics

$\mathbb{Z}[W]$ — the integral group algebra of W

$\{H_w := w\}$ — the standard basis of $\mathbb{Z}[W]$

$\{\underline{H}_w\}$ — the Kazhdan-Lusztig basis of $\mathbb{Z}[W]$

Fact. $\underline{H}_x \underline{H}_y = \sum_z c_{x,y}^z \underline{H}_z$ with all $c_{x,y}^z \in \{0, 1, 2, \dots\}$.

Left KL-order: $z \geq_L y$ if there is x such that $c_{x,y}^z > 0$.

Left KL-cell: $y \sim_L z$ if $y \geq_L z$ and $z \geq_L y$.

Note: Similarly right order $\geq_R$ and right cells $\sim_R$

Note: Similarly two-sided order $\geq_J$ and two-sided cells $\sim_J$

Kazhdan-Lusztig combinatorics

$\mathbb{Z}[W]$ — the integral group algebra of W

$\{H_w := w\}$ — the standard basis of $\mathbb{Z}[W]$

$\{\underline{H}_w\}$ — the Kazhdan-Lusztig basis of $\mathbb{Z}[W]$

Fact. $\underline{H}_x \underline{H}_y = \sum_z c^z_{x,y} \underline{H}_z$ with all $c^z_{x,y} \in \{0, 1, 2, \dots\}$.

Left KL-order: $z \geq_L y$ if there is x such that $c^z_{x,y} > 0$.

Left KL-cell: $y \sim_L z$ if $y \geq_L z$ and $z \geq_L y$.

Note: Similarly right order $\geq_R$ and right cells $\sim_R$

Note: Similarly two-sided order $\geq_J$ and two-sided cells $\sim_J$

Kazhdan-Lusztig combinatorics

$\mathbb{Z}[W]$ — the integral group algebra of W

$\{H_w := w\}$ — the standard basis of $\mathbb{Z}[W]$

$\{\underline{H}_w\}$ — the Kazhdan-Lusztig basis of $\mathbb{Z}[W]$

Fact. $\underline{H}_x \underline{H}_y = \sum_z c^z_{x,y} \underline{H}_z$ with all $c^z_{x,y} \in \{0, 1, 2, \dots\}$.

Left KL-order: $z \geq_L y$ if there is x such that $c^z_{x,y} > 0$.

Left KL-cell: $y \sim_L z$ if $y \geq_L z$ and $z \geq_L y$.

Note: Similarly right order $\geq_R$ and right cells $\sim_R$

Note: Similarly two-sided order $\geq_J$ and two-sided cells $\sim_J$

Kazhdan-Lusztig combinatorics

$\mathbb{Z}[W]$ — the integral group algebra of W

$\{H_w := w\}$ — the standard basis of $\mathbb{Z}[W]$

$\{\underline{H}_w\}$ — the Kazhdan-Lusztig basis of $\mathbb{Z}[W]$

Fact. $\underline{H}_x \underline{H}_y = \sum_z c^z_{x,y} \underline{H}_z$ with all $c^z_{x,y} \in \{0, 1, 2, \dots\}$.

Left KL-order: $z \geq_L y$ if there is x such that $c^z_{x,y} > 0$.

Left KL-cell: $y \sim_L z$ if $y \geq_L z$ and $z \geq_L y$.

Note: Similarly right order $\geq_R$ and right cells $\sim_R$

Note: Similarly two-sided order $\geq_J$ and two-sided cells $\sim_J$

Kazhdan-Lusztig combinatorics

$\mathbb{Z}[W]$ — the integral group algebra of W

$\{H_w := w\}$ — the standard basis of $\mathbb{Z}[W]$

$\{\underline{H}_w\}$ — the Kazhdan-Lusztig basis of $\mathbb{Z}[W]$

Fact. $\underline{H}_x \underline{H}_y = \sum_z c^z_{x,y} \underline{H}_z$ with all $c^z_{x,y} \in \{0, 1, 2, \dots\}$.

Left KL-order: $z \geq_L y$ if there is x such that $c^z_{x,y} > 0$.

Left KL-cell: $y \sim_L z$ if $y \geq_L z$ and $z \geq_L y$.

Note: Similarly right order $\geq_R$ and right cells $\sim_R$

Note: Similarly two-sided order $\geq_J$ and two-sided cells $\sim_J$

Kazhdan-Lusztig combinatorics

$\mathbb{Z}[W]$ — the integral group algebra of W

$\{H_w := w\}$ — the standard basis of $\mathbb{Z}[W]$

$\{\underline{H}_w\}$ — the Kazhdan-Lusztig basis of $\mathbb{Z}[W]$

Fact. $\underline{H}_x \underline{H}_y = \sum_z c_{x,y}^z \underline{H}_z$ with all $c_{x,y}^z \in \{0, 1, 2, \dots\}$.

Left KL-order: $z \geq_L y$ if there is x such that $c_{x,y}^z > 0$.

Left KL-cell: $y \sim_L z$ if $y \geq_L z$ and $z \geq_L y$.

Note: Similarly right order $\geq_R$ and right cells $\sim_R$

Note: Similarly two-sided order $\geq_J$ and two-sided cells $\sim_J$

Kazhdan-Lusztig combinatorics

$\mathbb{Z}[W]$ — the integral group algebra of W

$\{H_w := w\}$ — the standard basis of $\mathbb{Z}[W]$

$\{\underline{H}_w\}$ — the Kazhdan-Lusztig basis of $\mathbb{Z}[W]$

Fact. $\underline{H}_x\underline{H}_y = \sum_z c^z_{x,y}\underline{H}_z$ with all $c^z_{x,y} \in \{0, 1, 2, \dots\}$.

Left KL-order: $z \geq_L y$ if there is x such that $c^z_{x,y} > 0$.

Left KL-cell: $y \sim_L z$ if $y \geq_L z$ and $z \geq_L y$.

Note: Similarly right order $\geq_R$ and right cells $\sim_R$

Note: Similarly two-sided order $\geq_J$ and two-sided cells $\sim_J$

Kazhdan-Lusztig combinatorics

$\mathbb{Z}[W]$ — the integral group algebra of W

$\{H_w := w\}$ — the standard basis of $\mathbb{Z}[W]$

$\{\underline{H}_w\}$ — the Kazhdan-Lusztig basis of $\mathbb{Z}[W]$

Fact. $\underline{H}_x \underline{H}_y = \sum_z c_{x,y}^z \underline{H}_z$ with all $c_{x,y}^z \in \{0, 1, 2, \dots\}$.

Left KL-order: $z \geq_L y$ if there is x such that $c_{x,y}^z > 0$.

Left KL-cell: $y \sim_L z$ if $y \geq_L z$ and $z \geq_L y$.

Note: Similarly right order $\geq_R$ and right cells $\sim_R$

Note: Similarly two-sided order $\geq_J$ and two-sided cells $\sim_J$

Type A example ($\mathfrak{g} = \mathfrak{sl}_n$)

$W = S_n$

$\lambda \vdash n$

SYT_λ — the set of standard Young tableaux of shape λ

Robinson-Schensted correspondence. $S_n \xrightarrow{\mathbf{RS}} \bigcup_{\lambda \vdash n} \mathrm{SYT}_\lambda \times \mathrm{SYT}_\lambda$

Fact: Two-sided cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same shape

Fact: Left cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same first component

Fact: Right cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same second component

Type A example ($\mathfrak{g} = \mathfrak{sl}_n$)

$W = S_n$

$\lambda \vdash n$

SYT_λ — the set of standard Young tableaux of shape λ

Robinson-Schensted correspondence. $S_n \xrightarrow{\mathbf{RS}} \bigcup_{\lambda \vdash n} \mathrm{SYT}_\lambda \times \mathrm{SYT}_\lambda$

Fact: Two-sided cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same shape

Fact: Left cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same first component

Fact: Right cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same second component

Type A example ($\mathfrak{g} = \mathfrak{sl}_n$)

$W = S_n$

$\lambda \vdash n$

SYT_λ — the set of standard Young tableaux of shape λ

Robinson-Schensted correspondence. $S_n \xrightarrow{\mathbf{RS}} \bigcup_{\lambda \vdash n} \mathrm{SYT}_\lambda \times \mathrm{SYT}_\lambda$

Fact: Two-sided cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same shape

Fact: Left cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same first component

Fact: Right cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same second component

Type A example ($\mathfrak{g} = \mathfrak{sl}_n$)

$W = S_n$

$\lambda \vdash n$

SYT_λ — the set of standard Young tableaux of shape λ

Robinson-Schensted correspondence. $S_n \xrightarrow{\mathbf{RS}} \bigcup_{\lambda \vdash n} \mathrm{SYT}_\lambda \times \mathrm{SYT}_\lambda$

Fact: Two-sided cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same shape

Fact: Left cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same first component

Fact: Right cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same second component

Type A example ($\mathfrak{g} = \mathfrak{sl}_n$)

$W = S_n$

$\lambda \vdash n$

SYT_λ — the set of standard Young tableaux of shape λ

Robinson-Schensted correspondence. $S_n \xrightarrow{\mathbf{RS}} \bigcup_{\lambda \vdash n} \mathrm{SYT}_\lambda \times \mathrm{SYT}_\lambda$

Fact: Two-sided cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same shape

Fact: Left cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same first component

Fact: Right cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same second component

Type A example ($\mathfrak{g} = \mathfrak{sl}_n$)

$W = S_n$

$\lambda \vdash n$

SYT_λ — the set of standard Young tableaux of shape λ

Robinson-Schensted correspondence. $S_n \xrightarrow{\mathbf{RS}} \bigcup_{\lambda \vdash n} \mathrm{SYT}_\lambda \times \mathrm{SYT}_\lambda$

Fact: Two-sided cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same shape

Fact: Left cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same first component

Fact: Right cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same second component

Type A example ($\mathfrak{g} = \mathfrak{sl}_n$)

$W = S_n$

$\lambda \vdash n$

SYT_λ — the set of standard Young tableaux of shape λ

Robinson-Schensted correspondence. $S_n \xrightarrow{\mathbf{RS}} \bigcup_{\lambda \vdash n} \mathrm{SYT}_\lambda \times \mathrm{SYT}_\lambda$

Fact: Two-sided cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same shape

Fact: Left cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same first component

Fact: Right cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same second component

Type A example ($\mathfrak{g} = \mathfrak{sl}_n$)

$W = S_n$

$\lambda \vdash n$

SYT_λ — the set of standard Young tableaux of shape λ

Robinson-Schensted correspondence. $S_n \xrightarrow{\mathbf{RS}} \bigcup_{\lambda \vdash n} \mathrm{SYT}_\lambda \times \mathrm{SYT}_\lambda$

Fact: Two-sided cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same shape

Fact: Left cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same first component

Fact: Right cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same second component

Type A example ($\mathfrak{g} = \mathfrak{sl}_n$)

$W = S_n$

$\lambda \vdash n$

SYT_λ — the set of standard Young tableaux of shape λ

Robinson-Schensted correspondence. $S_n \xrightarrow{\mathbf{RS}} \bigcup_{\lambda \vdash n} \mathrm{SYT}_\lambda \times \mathrm{SYT}_\lambda$

Fact: Two-sided cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same shape

Fact: Left cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same first component

Fact: Right cell of w: all x such that $\mathbf{RS}(x)$ and $\mathbf{RS}(w)$ have the same second component

Lusztig's **a**-function in type A

$W = S_n$

Fact: Each two-sided cell of w contains the longest element for some parabolic (Young) subgroup

Fact: If some 2-sided cell contains longest elements for different parabolic subgroups, all these elements have the same length

Definition of a-function: $\mathbf{a} : W \to \{0, 1, 2, \dots\}$ is the unique function such that

- $\mathbf{a}$ is constant on 2-sided cells;
- $\mathbf{a}(w) = \mathbf{l}(w)$ if w is the longest element in a parabolic subgroup

Example. For $S_3 = \{e, s, t, st, ts, sts = tst = w_0\}$ we have

w	e	s	t	st	ts	sts
$\mathbf{a}(w)$	0	1	1	1	1	3

Lusztig's **a**-function in type A

$W = S_n$

Fact: Each two-sided cell of w contains the longest element for some parabolic (Young) subgroup

Fact: If some 2-sided cell contains longest elements for different parabolic subgroups, all these elements have the same length

Definition of a-function: $\mathbf{a} : W \to \{0, 1, 2, \dots\}$ is the unique function such that

- $\mathbf{a}$ is constant on 2-sided cells;
- $\mathbf{a}(w) = \mathbf{l}(w)$ if w is the longest element in a parabolic subgroup

Example. For $S_3 = \{e, s, t, st, ts, sts = tst = w_0\}$ we have

w	e	s	t	st	ts	sts
$\mathbf{a}(w)$	0	1	1	1	1	3

Lusztig's **a**-function in type A

$W = S_n$

Fact: Each two-sided cell of w contains the longest element for some parabolic (Young) subgroup

Fact: If some 2-sided cell contains longest elements for different parabolic subgroups, all these elements have the same length

Definition of a-function: $\mathbf{a} : W \to \{0, 1, 2, \dots\}$ is the unique function such that

- $\mathbf{a}$ is constant on 2-sided cells;
- $\mathbf{a}(w) = \mathbf{l}(w)$ if w is the longest element in a parabolic subgroup

Example. For $S_3 = \{e, s, t, st, ts, sts = tst = w_0\}$ we have

w	e	s	t	st	ts	sts
$\mathbf{a}(w)$	0	1	1	1	1	3

Lusztig's **a**-function in type A

$W = S_n$

Fact: Each two-sided cell of w contains the longest element for some parabolic (Young) subgroup

Fact: If some 2-sided cell contains longest elements for different parabolic subgroups, all these elements have the same length

Definition of a-function: $\mathbf{a} : W \to \{0, 1, 2, \dots\}$ is the unique function such that

- $\mathbf{a}$ is constant on 2-sided cells;
- $\mathbf{a}(w) = \mathbf{l}(w)$ if w is the longest element in a parabolic subgroup

Example. For $S_3 = \{e, s, t, st, ts, sts = tst = w_0\}$ we have

w	e	s	t	st	ts	sts
$\mathbf{a}(w)$	0	1	1	1	1	3

Lusztig's **a**-function in type A

$W = S_n$

Fact: Each two-sided cell of w contains the longest element for some parabolic (Young) subgroup

Fact: If some 2-sided cell contains longest elements for different parabolic subgroups, all these elements have the same length

Definition of a-function: $\mathbf{a} : W \to \{0, 1, 2, \dots\}$ is the unique function such that

- $\mathbf{a}$ is constant on 2-sided cells;
- $\mathbf{a}(w) = \mathbf{l}(w)$ if w is the longest element in a parabolic subgroup

Example. For $S_3 = \{e, s, t, st, ts, sts = tst = w_0\}$ we have

w	e	s	t	st	ts	sts
$\mathbf{a}(w)$	0	1	1	1	1	3

Lusztig's **a**-function in type A

$W = S_n$

Fact: Each two-sided cell of w contains the longest element for some parabolic (Young) subgroup

Fact: If some 2-sided cell contains longest elements for different parabolic subgroups, all these elements have the same length

Definition of a-function: $\mathbf{a} : W \to \{0, 1, 2, \dots\}$ is the unique function such that

- $\mathbf{a}$ is constant on 2-sided cells;
- $\mathbf{a}(w) = \mathbf{l}(w)$ if w is the longest element in a parabolic subgroup

Example. For $S_3 = \{e, s, t, st, ts, sts = tst = w_0\}$ we have

w	e	s	t	st	ts	sts
$\mathbf{a}(w)$	0	1	1	1	1	3

Lusztig's **a**-function in type A

$W = S_n$

Fact: Each two-sided cell of w contains the longest element for some parabolic (Young) subgroup

Fact: If some 2-sided cell contains longest elements for different parabolic subgroups, all these elements have the same length

Definition of a-function: $\mathbf{a} : W \to \{0, 1, 2, \dots\}$ is the unique function such that

- $\mathbf{a}$ is constant on 2-sided cells;
- $\mathbf{a}(w) = \mathbf{l}(w)$ if w is the longest element in a parabolic subgroup

Example. For $S_3 = \{e, s, t, st, ts, sts = tst = w_0\}$ we have

w	e	s	t	st	ts	sts
$\mathbf{a}(w)$	0	1	1	1	1	3

Lusztig's **a**-function in type A

$W = S_n$

Fact: Each two-sided cell of w contains the longest element for some parabolic (Young) subgroup

Fact: If some 2-sided cell contains longest elements for different parabolic subgroups, all these elements have the same length

Definition of a-function: $\mathbf{a} : W \to \{0, 1, 2, \dots\}$ is the unique function such that

- $\mathbf{a}$ is constant on 2-sided cells;
- $\mathbf{a}(w) = \mathbf{l}(w)$ if w is the longest element in a parabolic subgroup

Example. For $S_3 = \{e, s, t, st, ts, sts = tst = w_0\}$ we have

w	e	s	t	st	ts	sts
$\mathbf{a}(w)$	0	1	1	1	1	3

Lusztig's **a**-function in type A

$W = S_n$

Fact: Each two-sided cell of w contains the longest element for some parabolic (Young) subgroup

Fact: If some 2-sided cell contains longest elements for different parabolic subgroups, all these elements have the same length

Definition of a-function: $\mathbf{a} : W \to \{0, 1, 2, \dots\}$ is the unique function such that

- $\mathbf{a}$ is constant on 2-sided cells;
- $\mathbf{a}(w) = \mathbf{l}(w)$ if w is the longest element in a parabolic subgroup

Example. For $S_3 = \{e, s, t, st, ts, sts = tst = w_0\}$ we have

w	e	s	t	st	ts	sts
$\mathbf{a}(w)$	0	1	1	1	1	3

Lusztig's **a**-function in type A

$W = S_n$

Fact: Each two-sided cell of w contains the longest element for some parabolic (Young) subgroup

Fact: If some 2-sided cell contains longest elements for different parabolic subgroups, all these elements have the same length

Definition of a-function: $\mathbf{a} : W \to \{0, 1, 2, \dots\}$ is the unique function such that

- $\mathbf{a}$ is constant on 2-sided cells;
- $\mathbf{a}(w) = \mathbf{l}(w)$ if w is the longest element in a parabolic subgroup

Example. For $S_3 = \{e, s, t, st, ts, sts = tst = w_0\}$ we have

w	e	s	t	st	ts	sts
$\mathbf{a}(w)$	0	1	1	1	1	3

KL-combinatorics and $\mathcal{O}$

Fact (Bernstein-S. Gelfand):

$$\mathcal{O}_0 \circlearrowleft_{\mathcal{P}} \quad \overset{\text{Grothendieck group}}{\longmapsto} \quad \mathbb{Z}[W] \circlearrowleft_{\mathbb{Z}[W]}$$

Obvious fact: $\mathrm{Gr}(\mathcal{O}_0) \cong \mathbb{Z}[W]$ via $[\Delta(w)] \mapsto H_w$.

θ_w — indecomposable projective functor s.t. $\theta_w P(e) = P(w)$, $w \in W$

Kazhdan-Lusztig conjecture=theorem: $\mathrm{Gr}_{\oplus}(\mathcal{P}) \cong \mathbb{Z}[W]$ via $[\theta_w] \mapsto \underline{H}_w$.

Note: Recent algebraic proof by Elias and Williamson.

KL-combinatorics and $\mathcal{O}$

Fact (Bernstein-S. Gelfand):

$$\begin{tikzcd} \mathcal{O}_0 \arrow[loop below, "\mathcal{P}"] \arrow[r, mapsto, "\text{Grothendieck group}"] & \mathbb{Z}[W] \arrow[loop below, "\mathbb{Z}[W]"] \end{tikzcd}$$

Obvious fact: $\mathrm{Gr}(\mathcal{O}_0) \cong \mathbb{Z}[W]$ via $[\Delta(w)] \mapsto H_w$.

θ_w — indecomposable projective functor s.t. $\theta_w P(e) = P(w)$, $w \in W$

Kazhdan-Lusztig conjecture=theorem: $\mathrm{Gr}_{\oplus}(\mathcal{P}) \cong \mathbb{Z}[W]$ via $[\theta_w] \mapsto \underline{H}_w$.

Note: Recent algebraic proof by Elias and Williamson.

KL-combinatorics and $\mathcal{O}$

Fact (Bernstein-S. Gelfand):

$$\begin{tikzcd} \mathcal{O}_0 \arrow[loop below, "\mathcal{P}"] \arrow[rr, mapsto, "\text{Grothendieck group}"] & & \mathbb{Z}[W] \arrow[loop below, "\mathbb{Z}[W]"] \end{tikzcd}$$

Obvious fact: $\mathrm{Gr}(\mathcal{O}_0) \cong \mathbb{Z}[W]$ via $[\Delta(w)] \mapsto H_w$.

θ_w — indecomposable projective functor s.t. $\theta_w P(e) = P(w)$, $w \in W$

Kazhdan-Lusztig conjecture=theorem: $\mathrm{Gr}_{\oplus}(\mathcal{P}) \cong \mathbb{Z}[W]$ via $[\theta_w] \mapsto \underline{H}_w$.

Note: Recent algebraic proof by Elias and Williamson.

KL-combinatorics and $\mathcal{O}$

Fact (Bernstein-S. Gelfand):

$$\begin{tikzcd} \mathcal{O}_0 \arrow[loop below, "\mathcal{P}"] \arrow[rr, mapsto, "\text{Grothendieck group}"] & & \mathbb{Z}[W] \arrow[loop below, "\mathbb{Z}[W]"] \end{tikzcd}$$

Obvious fact: $\mathrm{Gr}(\mathcal{O}_0) \cong \mathbb{Z}[W]$ via $[\Delta(w)] \mapsto H_w$.

θ_w — indecomposable projective functor s.t. $\theta_w P(e) = P(w)$, $w \in W$

Kazhdan-Lusztig conjecture=theorem: $\mathrm{Gr}_{\oplus}(\mathcal{P}) \cong \mathbb{Z}[W]$ via $[\theta_w] \mapsto \underline{H}_w$.

Note: Recent algebraic proof by Elias and Williamson.

KL-combinatorics and $\mathcal{O}$

Fact (Bernstein-S. Gelfand):

$$\mathcal{P} \circlearrowright \mathcal{O}_0 \quad \xmapsto{\text{Grothendieck group}} \quad \mathbb{Z}[W] \circlearrowleft \mathbb{Z}[W]$$

Obvious fact: $\mathrm{Gr}(\mathcal{O}_0) \cong \mathbb{Z}[W]$ via $[\Delta(w)] \mapsto H_w$.

θ_w — indecomposable projective functor s.t. $\theta_w P(e) = P(w)$, $w \in W$

Kazhdan-Lusztig conjecture=theorem: $\mathrm{Gr}_{\oplus}(\mathcal{P}) \cong \mathbb{Z}[W]$ via $[\theta_w] \mapsto \underline{H}_w$.

Note: Recent algebraic proof by Elias and Williamson.

KL-combinatorics and $\mathcal{O}$

Fact (Bernstein-S. Gelfand):

$$\mathcal{O}_0 \circlearrowleft_{\mathcal{P}} \quad \overset{\text{Grothendieck group}}{\mapsto} \quad \mathbb{Z}[W] \circlearrowright_{\mathbb{Z}[W]}$$

Obvious fact: $\mathrm{Gr}(\mathcal{O}_0) \cong \mathbb{Z}[W]$ via $[\Delta(w)] \mapsto H_w$.

θ_w — indecomposable projective functor s.t. $\theta_w P(e) = P(w)$, $w \in W$

Kazhdan-Lusztig conjecture=theorem: $\mathrm{Gr}_{\oplus}(\mathcal{P}) \cong \mathbb{Z}[W]$ via $[\theta_w] \mapsto \underline{H}_w$.

Note: Recent algebraic proof by Elias and Williamson.

KL-combinatorics and $\mathcal{O}$

Fact (Bernstein-S. Gelfand):

$$\begin{tikzcd} \mathcal{O}_0 \arrow[loop below, "\mathcal{P}"] \arrow[r, mapsto, "\text{Grothendieck group}"] & \mathbb{Z}[W] \arrow[loop below, "\mathbb{Z}[W]"] \end{tikzcd}$$

Obvious fact: $\mathrm{Gr}(\mathcal{O}_0) \cong \mathbb{Z}[W]$ via $[\Delta(w)] \mapsto H_w$.

θ_w — indecomposable projective functor s.t. $\theta_w P(e) = P(w)$, $w \in W$

Kazhdan-Lusztig conjecture=theorem: $\mathrm{Gr}_\oplus(\mathcal{P}) \cong \mathbb{Z}[W]$ via $[\theta_w] \mapsto \underline{H}_w$.

Note: Recent algebraic proof by Elias and Williamson.

Lusztig's **a**-function in general

Left cell in W gives rise to a cell representation of W (in type A: Specht module)

This can be modeled (i.e. categorified) via action of $\mathcal{P}$ on the additive category of certain self-dual modules in $\mathcal{O}_0$

Such indecomposable self-dual modules are naturally indexed (for all cells at the same time) by $w \in W$, say $w \mapsto Q(w)$

Definition. $\mathbf{a}(w) = \frac{1}{2}(\mathrm{Loewy.Length}(Q(w)) - 1)$

Easy facts:

- $\mathbf{a}$ is constant on 2-sided cells;
- $\mathbf{a}(w) = \mathbf{l}(w)$ if w is the longest element in a parabolic subgroup

Lusztig's **a**-function in general

Left cell in W gives rise to a cell representation of W (in type A: Specht module)

This can be modeled (i.e. categorified) via action of $\mathcal{P}$ on the additive category of certain self-dual modules in $\mathcal{O}_0$

Such indecomposable self-dual modules are naturally indexed (for all cells at the same time) by $w \in W$, say $w \mapsto Q(w)$

Definition. $\mathbf{a}(w) = \frac{1}{2}(\mathrm{Loewy.Length}(Q(w)) - 1)$

Easy facts:

- $\mathbf{a}$ is constant on 2-sided cells;
- $\mathbf{a}(w) = \mathbf{l}(w)$ if w is the longest element in a parabolic subgroup

Lusztig's **a**-function in general

Left cell in W gives rise to a cell representation of W (in type A: Specht module)

This can be modeled (i.e. categorified) via action of $\mathcal{P}$ on the additive category of certain self-dual modules in $\mathcal{O}_0$

Such indecomposable self-dual modules are naturally indexed (for all cells at the same time) by $w \in W$, say $w \mapsto Q(w)$

Definition. $\mathbf{a}(w) = \frac{1}{2}(\mathrm{Loewy.Length}(Q(w)) - 1)$

Easy facts:

- $\mathbf{a}$ is constant on 2-sided cells;
- $\mathbf{a}(w) = \mathbf{l}(w)$ if w is the longest element in a parabolic subgroup

Lusztig's **a**-function in general

Left cell in W gives rise to a cell representation of W (in type A: Specht module)

This can be modeled (i.e. categorified) via action of $\mathcal{P}$ on the additive category of certain self-dual modules in $\mathcal{O}_0$

Such indecomposable self-dual modules are naturally indexed (for all cells at the same time) by $w \in W$, say $w \mapsto Q(w)$

Definition. $\mathbf{a}(w) = \frac{1}{2}(\mathrm{Loewy.Length}(Q(w)) - 1)$

Easy facts:

- $\mathbf{a}$ is constant on 2-sided cells;
- $\mathbf{a}(w) = \mathbf{l}(w)$ if w is the longest element in a parabolic subgroup

Lusztig's **a**-function in general

Left cell in W gives rise to a cell representation of W (in type A: Specht module)

This can be modeled (i.e. categorified) via action of $\mathcal{P}$ on the additive category of certain self-dual modules in $\mathcal{O}_0$

Such indecomposable self-dual modules are naturally indexed (for all cells at the same time) by $w \in W$, say $w \mapsto Q(w)$

Definition. $\mathbf{a}(w) = \frac{1}{2}(\mathrm{Loewy.Length}(Q(w)) - 1)$

Easy facts:

- $\mathbf{a}$ is constant on 2-sided cells;
- $\mathbf{a}(w) = \mathbf{l}(w)$ if w is the longest element in a parabolic subgroup

Lusztig's **a**-function in general

Left cell in W gives rise to a cell representation of W (in type A: Specht module)

This can be modeled (i.e. categorified) via action of $\mathcal{P}$ on the additive category of certain self-dual modules in $\mathcal{O}_0$

Such indecomposable self-dual modules are naturally indexed (for all cells at the same time) by $w \in W$, say $w \mapsto Q(w)$

Definition. $\mathbf{a}(w) = \frac{1}{2}(\mathrm{Loewy.Length}(Q(w)) - 1)$

Easy facts:

- **a** is constant on 2-sided cells;
- $\mathbf{a}(w) = \mathbf{l}(w)$ if w is the longest element in a parabolic subgroup

Lusztig's **a**-function in general

Left cell in W gives rise to a cell representation of W (in type A: Specht module)

This can be modeled (i.e. categorified) via action of $\mathcal{P}$ on the additive category of certain self-dual modules in $\mathcal{O}_0$

Such indecomposable self-dual modules are naturally indexed (for all cells at the same time) by $w \in W$, say $w \mapsto Q(w)$

Definition. $\mathbf{a}(w) = \frac{1}{2}(\mathrm{Loewy.Length}(Q(w)) - 1)$

Easy facts:

- $\mathbf{a}$ is constant on 2-sided cells;
- $\mathbf{a}(w) = \mathbf{l}(w)$ if w is the longest element in a parabolic subgroup

Lusztig's **a**-function in general

Left cell in W gives rise to a cell representation of W (in type A: Specht module)

This can be modeled (i.e. categorified) via action of $\mathcal{P}$ on the additive category of certain self-dual modules in $\mathcal{O}_0$

Such indecomposable self-dual modules are naturally indexed (for all cells at the same time) by $w \in W$, say $w \mapsto Q(w)$

Definition. $\mathbf{a}(w) = \frac{1}{2}(\mathrm{Loewy.Length}(Q(w)) - 1)$

Easy facts:

- $\mathbf{a}$ is constant on 2-sided cells;
- $\mathbf{a}(w) = \mathbf{l}(w)$ if w is the longest element in a parabolic subgroup

Lusztig's **a**-function in general

Left cell in W gives rise to a cell representation of W (in type A: Specht module)

This can be modeled (i.e. categorified) via action of $\mathcal{P}$ on the additive category of certain self-dual modules in $\mathcal{O}_0$

Such indecomposable self-dual modules are naturally indexed (for all cells at the same time) by $w \in W$, say $w \mapsto Q(w)$

Definition. $\mathbf{a}(w) = \frac{1}{2}(\mathrm{Loewy.Length}(Q(w)) - 1)$

Easy facts:

- $\mathbf{a}$ is constant on 2-sided cells;
- $\mathbf{a}(w) = \mathbf{l}(w)$ if w is the longest element in a parabolic subgroup

The main result

Theorem. Let $w \in W$.

- $\mathrm{proj.dim}(T(w)) = \mathbf{a}(w)$
- $\mathrm{proj.dim}(I(w)) = 2\mathbf{a}(w_0 w)$

Step 1. Both $\mathrm{proj.dim}(T(w))$ and $\mathrm{proj.dim}(I(w))$ are constant on two-sided cells.

Why:

- projective functors preserve both the additive category of tilting modules, the additive category of projective modules and the additive category of injective modules,
- use projective functors to related projective resolutions of indecomposable (tilting or injective modules) inside the same two-sided cell

The main result

Theorem. Let $w \in W$.

- $\mathrm{proj.dim}(T(w)) = \mathbf{a}(w)$
- $\mathrm{proj.dim}(I(w)) = 2\mathbf{a}(w_0 w)$

Step 1. Both $\mathrm{proj.dim}(T(w))$ and $\mathrm{proj.dim}(I(w))$ are constant on two-sided cells.

Why:

- projective functors preserve both the additive category of tilting modules, the additive category of projective modules and the additive category of injective modules,
- use projective functors to related projective resolutions of indecomposable (tilting or injective modules) inside the same two-sided cell

The main result

Theorem. Let $w \in W$.

- $\mathrm{proj.dim}(T(w)) = \mathbf{a}(w)$
- $\mathrm{proj.dim}(I(w)) = 2\mathbf{a}(w_0 w)$

Step 1. Both $\mathrm{proj.dim}(T(w))$ and $\mathrm{proj.dim}(I(w))$ are constant on two-sided cells.

Why:

- projective functors preserve both the additive category of tilting modules, the additive category of projective modules and the additive category of injective modules,
- use projective functors to related projective resolutions of indecomposable (tilting or injective modules) inside the same two-sided cell

The main result

Theorem. Let $w \in W$.

- $\mathrm{proj.dim}(T(w)) = \mathbf{a}(w)$
- $\mathrm{proj.dim}(I(w)) = 2\mathbf{a}(w_0 w)$

Step 1. Both $\mathrm{proj.dim}(T(w))$ and $\mathrm{proj.dim}(I(w))$ are constant on two-sided cells.

Why:

- projective functors preserve both the additive category of tilting modules, the additive category of projective modules and the additive category of injective modules,
- use projective functors to related projective resolutions of indecomposable (tilting or injective modules) inside the same two-sided cell

The main result

Theorem. Let $w \in W$.

- $\mathrm{proj.dim}(T(w)) = \mathbf{a}(w)$
- $\mathrm{proj.dim}(I(w)) = 2\mathbf{a}(w_0 w)$

Step 1. Both $\mathrm{proj.dim}(T(w))$ and $\mathrm{proj.dim}(I(w))$ are constant on two-sided cells.

Why:

- projective functors preserve both the additive category of tilting modules, the additive category of projective modules and the additive category of injective modules,
- use projective functors to related projective resolutions of indecomposable (tilting or injective modules) inside the same two-sided cell

The main result

Theorem. Let $w \in W$.

- $\mathrm{proj.dim}(T(w)) = \mathbf{a}(w)$
- $\mathrm{proj.dim}(I(w)) = 2\mathbf{a}(w_0 w)$

Step 1. Both $\mathrm{proj.dim}(T(w))$ and $\mathrm{proj.dim}(I(w))$ are constant on two-sided cells.

Why:

- projective functors preserve both the additive category of tilting modules, the additive category of projective modules and the additive category of injective modules,
- use projective functors to related projective resolutions of indecomposable (tilting or injective modules) inside the same two-sided cell

The main result

Theorem. Let $w \in W$.

- $\mathrm{proj.dim}(T(w)) = \mathbf{a}(w)$
- $\mathrm{proj.dim}(I(w)) = 2\mathbf{a}(w_0 w)$

Step 1. Both $\mathrm{proj.dim}(T(w))$ and $\mathrm{proj.dim}(I(w))$ are constant on two-sided cells.

Why:

- projective functors preserve both the additive category of tilting modules, the additive category of projective modules and the additive category of injective modules,
- use projective functors to related projective resolutions of indecomposable (tilting or injective modules) inside the same two-sided cell

The main result

Theorem. Let $w \in W$.

- $\mathrm{proj.dim}(T(w)) = \mathbf{a}(w)$
- $\mathrm{proj.dim}(I(w)) = 2\mathbf{a}(w_0 w)$

Step 1. Both $\mathrm{proj.dim}(T(w))$ and $\mathrm{proj.dim}(I(w))$ are constant on two-sided cells.

Why:

- projective functors preserve both the additive category of tilting modules, the additive category of projective modules and the additive category of injective modules,
- use projective functors to related projective resolutions of indecomposable (tilting or injective modules) inside the same two-sided cell

The main result

Theorem. Let $w \in W$.

- $\mathrm{proj.dim}(T(w)) = \mathbf{a}(w)$
- $\mathrm{proj.dim}(I(w)) = 2\mathbf{a}(w_0 w)$

Step 1. Both $\mathrm{proj.dim}(T(w))$ and $\mathrm{proj.dim}(I(w))$ are constant on two-sided cells.

Why:

- projective functors preserve both the additive category of tilting modules, the additive category of projective modules and the additive category of injective modules,
- use projective functors to related projective resolutions of indecomposable (tilting or injective modules) inside the same two-sided cell

Step 2

for w the longest element in a parabolic subcategory, the value $\mathrm{proj.dim}(T(w))$ can be computed using Ringel self-duality of $\mathcal{O}$, the Koszul duality of Beilinson-Ginzburg-Soergel and computations of Loewy lengths of certain structural modules in $\mathcal{O}$ by Irving

computation of $\mathrm{proj.dim}(I(w))$ is similar

This implies the result in type A, that is for $\mathfrak{g} = \mathfrak{sl}_n$

General case:

- uses crucially the Koszul duality for $\mathcal{O}$
- uses generalized parabolic categories associated with right KL-cells
- uses delicate computation of Loewy (=graded) lengths of certain modules of the form $\theta_x L(y)$ and the fact that the set of these modules is Koszul self-dual

Open problem: Determine $\mathrm{proj.dim}(\theta_x L(y))$ for all $x, y \in W$

Step 2

for w the longest element in a parabolic subcategory, the value $\mathrm{proj.dim}(T(w))$ can be computed using Ringel self-duality of $\mathcal{O}$, the Koszul duality of Beilinson-Ginzburg-Soergel and computations of Loewy lengths of certain structural modules in $\mathcal{O}$ by Irving

computation of $\mathrm{proj.dim}(I(w))$ is similar

This implies the result in type A, that is for $\mathfrak{g} = \mathfrak{sl}_n$

General case:

- uses crucially the Koszul duality for $\mathcal{O}$
- uses generalized parabolic categories associated with right KL-cells
- uses delicate computation of Loewy (=graded) lengths of certain modules of the form $\theta_x L(y)$ and the fact that the set of these modules is Koszul self-dual

Open problem: Determine $\mathrm{proj.dim}(\theta_x L(y))$ for all $x, y \in W$

Step 2

for w the longest element in a parabolic subcategory, the value $\mathrm{proj.dim}(T(w))$ can be computed using Ringel self-duality of $\mathcal{O}$, the Koszul duality of Beilinson-Ginzburg-Soergel and computations of Loewy lengths of certain structural modules in $\mathcal{O}$ by Irving

computation of $\mathrm{proj.dim}(I(w))$ is similar

This implies the result in type A, that is for $\mathfrak{g} = \mathfrak{sl}_n$

General case:

- uses crucially the Koszul duality for $\mathcal{O}$
- uses generalized parabolic categories associated with right KL-cells
- uses delicate computation of Loewy (=graded) lengths of certain modules of the form $\theta_x L(y)$ and the fact that the set of these modules is Koszul self-dual

Open problem: Determine $\mathrm{proj.dim}(\theta_x L(y))$ for all $x, y \in W$

Step 2

for w the longest element in a parabolic subcategory, the value $\mathrm{proj.dim}(T(w))$ can be computed using Ringel self-duality of $\mathcal{O}$, the Koszul duality of Beilinson-Ginzburg-Soergel and computations of Loewy lengths of certain structural modules in $\mathcal{O}$ by Irving

computation of $\mathrm{proj.dim}(I(w))$ is similar

This implies the result in type A, that is for $\mathfrak{g} = \mathfrak{sl}_n$

General case:

- uses crucially the Koszul duality for $\mathcal{O}$
- uses generalized parabolic categories associated with right KL-cells
- uses delicate computation of Loewy (=graded) lengths of certain modules of the form $\theta_x L(y)$ and the fact that the set of these modules is Koszul self-dual

Open problem: Determine $\mathrm{proj.dim}(\theta_x L(y))$ for all $x, y \in W$

Step 2

for w the longest element in a parabolic subcategory, the value $\mathrm{proj.dim}(T(w))$ can be computed using Ringel self-duality of $\mathcal{O}$, the Koszul duality of Beilinson-Ginzburg-Soergel and computations of Loewy lengths of certain structural modules in $\mathcal{O}$ by Irving

computation of $\mathrm{proj.dim}(I(w))$ is similar

This implies the result in type A, that is for $\mathfrak{g} = \mathfrak{sl}_n$

General case:

- uses crucially the Koszul duality for $\mathcal{O}$
- uses generalized parabolic categories associated with right KL-cells
- uses delicate computation of Loewy (=graded) lengths of certain modules of the form $\theta_x L(y)$ and the fact that the set of these modules is Koszul self-dual

Open problem: Determine $\mathrm{proj.dim}(\theta_x L(y))$ for all $x, y \in W$

Step 2

for w the longest element in a parabolic subcategory, the value $\mathrm{proj.dim}(T(w))$ can be computed using Ringel self-duality of $\mathcal{O}$, the Koszul duality of Beilinson-Ginzburg-Soergel and computations of Loewy lengths of certain structural modules in $\mathcal{O}$ by Irving

computation of $\mathrm{proj.dim}(I(w))$ is similar

This implies the result in type A, that is for $\mathfrak{g} = \mathfrak{sl}_n$

General case:

- uses crucially the Koszul duality for $\mathcal{O}$
- uses generalized parabolic categories associated with right KL-cells
- uses delicate computation of Loewy (=graded) lengths of certain modules of the form $\theta_x L(y)$ and the fact that the set of these modules is Koszul self-dual

Open problem: Determine $\mathrm{proj.dim}(\theta_x L(y))$ for all $x, y \in W$

Step 2

for w the longest element in a parabolic subcategory, the value $\mathrm{proj.dim}(T(w))$ can be computed using Ringel self-duality of $\mathcal{O}$, the Koszul duality of Beilinson-Ginzburg-Soergel and computations of Loewy lengths of certain structural modules in $\mathcal{O}$ by Irving

computation of $\mathrm{proj.dim}(I(w))$ is similar

This implies the result in type A, that is for $\mathfrak{g} = \mathfrak{sl}_n$

General case:

- uses crucially the Koszul duality for $\mathcal{O}$
- uses generalized parabolic categories associated with right KL-cells
- uses delicate computation of Loewy (=graded) lengths of certain modules of the form $\theta_x L(y)$ and the fact that the set of these modules is Koszul self-dual

Open problem: Determine $\mathrm{proj.dim}(\theta_x L(y))$ for all $x, y \in W$

Step 2

for w the longest element in a parabolic subcategory, the value $\mathrm{proj.dim}(T(w))$ can be computed using Ringel self-duality of $\mathcal{O}$, the Koszul duality of Beilinson-Ginzburg-Soergel and computations of Loewy lengths of certain structural modules in $\mathcal{O}$ by Irving

computation of $\mathrm{proj.dim}(I(w))$ is similar

This implies the result in type A, that is for $\mathfrak{g} = \mathfrak{sl}_n$

General case:

- uses crucially the Koszul duality for $\mathcal{O}$
- uses generalized parabolic categories associated with right KL-cells
- uses delicate computation of Loewy (=graded) lengths of certain modules of the form $\theta_x L(y)$ and the fact that the set of these modules is Koszul self-dual

Open problem: Determine $\mathrm{proj.dim}(\theta_x L(y))$ for all $x, y \in W$

Step 2

for w the longest element in a parabolic subcategory, the value $\mathrm{proj.dim}(T(w))$ can be computed using Ringel self-duality of $\mathcal{O}$, the Koszul duality of Beilinson-Ginzburg-Soergel and computations of Loewy lengths of certain structural modules in $\mathcal{O}$ by Irving

computation of $\mathrm{proj.dim}(I(w))$ is similar

This implies the result in type A, that is for $\mathfrak{g} = \mathfrak{sl}_n$

General case:

- uses crucially the Koszul duality for $\mathcal{O}$
- uses generalized parabolic categories associated with right KL-cells
- uses delicate computation of Loewy (=graded) lengths of certain modules of the form $\theta_x L(y)$ and the fact that the set of these modules is Koszul self-dual

Open problem: Determine $\mathrm{proj.dim}(\theta_x L(y))$ for all $x, y \in W$

Step 2

for w the longest element in a parabolic subcategory, the value $\mathrm{proj.dim}(T(w))$ can be computed using Ringel self-duality of $\mathcal{O}$, the Koszul duality of Beilinson-Ginzburg-Soergel and computations of Loewy lengths of certain structural modules in $\mathcal{O}$ by Irving

computation of $\mathrm{proj.dim}(I(w))$ is similar

This implies the result in type A, that is for $\mathfrak{g} = \mathfrak{sl}_n$

General case:

- uses crucially the Koszul duality for $\mathcal{O}$
- uses generalized parabolic categories associated with right KL-cells
- uses delicate computation of Loewy (=graded) lengths of certain modules of the form $\theta_x L(y)$ and the fact that the set of these modules is Koszul self-dual

Open problem: Determine $\mathrm{proj.dim}(\theta_x L(y))$ for all $x, y \in W$

Singular case

projectives: obvious

tilting: computable using the regular case and projective functors

injectives: computable using the regular case and projective functors

simples: open in general (known for the antidominant and the left cell of the dominant)

Koszul dual problem: graded length of an indecomposable projective in a parabolic $\mathcal{O}$

standard: open in general

Koszul dual problem: graded length of a standard in a parabolic $\mathcal{O}$

costandard: open in general

Singular case

projectives: obvious

tilting: computable using the regular case and projective functors

injectives: computable using the regular case and projective functors

simples: open in general (known for the antidominant and the left cell of the dominant)

Koszul dual problem: graded length of an indecomposable projective in a parabolic $\mathcal{O}$

standard: open in general

Koszul dual problem: graded length of a standard in a parabolic $\mathcal{O}$

costandard: open in general

Singular case

projectives: obvious

tilting: computable using the regular case and projective functors

injectives: computable using the regular case and projective functors

simples: open in general (known for the antidominant and the left cell of the dominant)

Koszul dual problem: graded length of an indecomposable projective in a parabolic $\mathcal{O}$

standard: open in general

Koszul dual problem: graded length of a standard in a parabolic $\mathcal{O}$

costandard: open in general

Singular case

projectives: obvious

tilting: computable using the regular case and projective functors

injectives: computable using the regular case and projective functors

simples: open in general (known for the antidominant and the left cell of the dominant)

Koszul dual problem: graded length of an indecomposable projective in a parabolic $\mathcal{O}$

standard: open in general

Koszul dual problem: graded length of a standard in a parabolic $\mathcal{O}$

costandard: open in general

Singular case

projectives: obvious

tilting: computable using the regular case and projective functors

injectives: computable using the regular case and projective functors

simples: open in general (known for the antidominant and the left cell of the dominant)

Koszul dual problem: graded length of an indecomposable projective in a parabolic $\mathcal{O}$

standard: open in general

Koszul dual problem: graded length of a standard in a parabolic $\mathcal{O}$

costandard: open in general

Singular case

projectives: obvious

tilting: computable using the regular case and projective functors

injectives: computable using the regular case and projective functors

simples: open in general (known for the antidominant and the left cell of the dominant)

Koszul dual problem: graded length of an indecomposable projective in a parabolic $\mathcal{O}$

standard: open in general

Koszul dual problem: graded length of a standard in a parabolic $\mathcal{O}$

costandard: open in general

Singular case

projectives: obvious

tilting: computable using the regular case and projective functors

injectives: computable using the regular case and projective functors

simples: open in general (known for the antidominant and the left cell of the dominant)

Koszul dual problem: graded length of an indecomposable projective in a parabolic $\mathcal{O}$

standard: open in general

Koszul dual problem: graded length of a standard in a parabolic $\mathcal{O}$

costandard: open in general

Singular case

projectives: obvious

tilting: computable using the regular case and projective functors

injectives: computable using the regular case and projective functors

simples: open in general (known for the antidominant and the left cell of the dominant)

Koszul dual problem: graded length of an indecomposable projective in a parabolic $\mathcal{O}$

standard: open in general

Koszul dual problem: graded length of a standard in a parabolic $\mathcal{O}$

costandard: open in general

Singular case

projectives: obvious

tilting: computable using the regular case and projective functors

injectives: computable using the regular case and projective functors

simples: open in general (known for the antidominant and the left cell of the dominant)

Koszul dual problem: graded length of an indecomposable projective in a parabolic $\mathcal{O}$

standard: open in general

Koszul dual problem: graded length of a standard in a parabolic $\mathcal{O}$

costandard: open in general

Singular case

projectives: obvious

tilting: computable using the regular case and projective functors

injectives: computable using the regular case and projective functors

simples: open in general (known for the antidominant and the left cell of the dominant)

Koszul dual problem: graded length of an indecomposable projective in a parabolic $\mathcal{O}$

standard: open in general

Koszul dual problem: graded length of a standard in a parabolic $\mathcal{O}$

costandard: open in general

THANK YOU!!!