Zomological propertief of category O, part I: invariants of structural modules

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For example: $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$

 $\mathfrak{g} = \mathfrak{n}_{-} \oplus \mathfrak{h} \oplus \mathfrak{n}_{+}$ — fixed triangular decomposition

For example, if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, then

- h diagonal matrices
- \mathfrak{n}_+ upper triangular matrices

for a Lie algebra \mathfrak{a} we denote by $U(\mathfrak{a})$ its universal enveloping algebra

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Definition. BGG category \mathcal{O} is defined as the full subcategory of the category of all g-modules containing all modules M satisfying the following conditions:

- ► *M* is finitely generated;
- ▶ the action of h on *M* is diagonalizable (i.e *M* is a weight module);
- the action of $U(n_+)$ on M is locally finite

 $M(\lambda)$ — the Verma module with highest weight $\lambda \in \mathfrak{h}^*$

Fact. $M(\lambda) \in \mathcal{O}$

 $L(\lambda)$ — the simple top of $M(\lambda)$

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$Z(\mathfrak{g})$ — the center of $U(\mathfrak{g})$



 \mathcal{O}_m is the full subcategory of $\mathcal O$ consisting of all modules on which m acts locally nilpotently

Fact. $\mathcal{O}_m \cong A_m$ -mod for some finite dimensional basic associative algebra A_m

Fact. \mathcal{O} is a highest weight category in the sense of Cline-Parshall-Scott with Verma modules being standard modules

Reformulation. A_m is quasi-hereditary in the sense of Dlab-Ringel

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Structural modules in $\ensuremath{\mathcal{O}}$

 $\lambda \in \mathfrak{h}^*$

- $L(\lambda)$ simple module with highest weight λ
- $P(\lambda)$ indecomposable projective cover of $L(\lambda)$
- $I(\lambda)$ indecomposable injective envelope of $L(\lambda)$
- $\Delta(\lambda) = M(\lambda)$ standard quotient of $P(\lambda)$
- $\nabla(\lambda)$ costandard submodule of $I(\lambda)$
- $T(\lambda)$ tilting envelope of $\Delta(\lambda)$
- **Fact.** $T(\lambda)$ tilting cover of $\nabla(\lambda)$

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(the last remark of the BGG paper is that the equality holds, no proof)

Question. Determine projective dimension of all indecomposable structural modules

Note. For almost all λ we have

$$L(\lambda) = P(\lambda) = I(\lambda) = \Delta(\lambda) = \nabla(\lambda) = T(\lambda)$$

and the answer to the question for these λ is: 0

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Theorem (BGG). $gl.dim\mathcal{O} \leq number of roots of g$ (the last remark of the BGG paper is that the equality holds, no proof)

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 m_0 — the maximal ideal in Z(g) annihilating the trivial g-module

 $\mathcal{O}_0 := \mathcal{O}_{m_0}$ — the principal block of \mathcal{O}

Fact. \mathcal{O}_0 is indecomposable

W — the Weyl group of $\mathfrak g$

W acts on \mathfrak{h}^* in the usual way

 ρ — the half of the sum of all positive roots

W acts on \mathfrak{h}^* via the $\rho\text{-shifted}$ dot action

Fact. Simples in \mathcal{O}_0 are $L(w \cdot 0)$, $w \in W$ (all different).

Notation. $L(w) := L(w \cdot 0)$, $P(w) := P(w \cdot 0)$ and so on

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 ρ — the half of the sum of all positive roots

W acts on \mathfrak{h}^* via the $\rho\text{-shifted}$ dot action

Fact. Simples in \mathcal{O}_0 are $L(w \cdot 0)$, $w \in W$ (all different).

Notation. $L(w) := L(w \cdot 0)$, $P(w) := P(w \cdot 0)$ and so on

 m_0 — the maximal ideal in $Z(\mathfrak{g})$ annihilating the trivial \mathfrak{g} -module

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Soergel's combinatorial description of blocks in terms of root systems: Each block of \mathcal{O} can be reduced, up to equivalence, to an integral block (maybe for some other g)

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Proposition. proj.dim $(\Delta(w)) = I(w)$ for all $w \in W$

The inequality \leq **(BGG):** We have $\Delta(e) = P(e)$ and hence proj.dim $(\Delta(e)) = 0$. We have Ker $\leftrightarrow P(w) \Rightarrow \Delta(w)$ where Ker has a filtration with subquotients $\Delta(x)$, I(x) < I(w). Now use the long exact sequence and the standard dimension shift argument.

The inequality \geq : Enough to prove: $\operatorname{proj.dim}(\Delta(w_0)) = I(w_0)$ (postponed).

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Proposition. proj.dim $(L(w)) = 2I(w_0) - I(w)$ for all $w \in W$

The inequality \leq **(BGG)**: For w_0 this is already done before. We have $\text{Ker} \hookrightarrow \Delta(w) \twoheadrightarrow L(w)$ where Ker has a filtration with subquotients L(x), I(x) > I(w). Now use the long exact sequence and the standard dimension shift argument.

The inequality \geq : Enough to prove: proj.dim(L(e)) = 2I(w_0)

Proof.

- ► Let \mathcal{X}^{\bullet} be the BGG resolution of L(e) by Verma modules. It has length $I(w_0)$ and $\mathcal{X}^{-I(w_0)} = \Delta(w_0)$, the simple Verma module.
- ▶ The dual \mathcal{Y}^{\bullet} of \mathcal{X}^{\bullet} is a coresolution of L(e) by costandard modules, it has length $I(w_0)$ and $\mathcal{Y}^{I(w_0)} = \nabla(w_0) = \Delta(w_0)$.
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San

Projective dimension of simple modules in $\mathcal{O}_{\mathbf{0}}$

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 $\{H_w := w\}$ — the standard basis of $\mathbb{Z}[W]$

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Fact. $\underline{H}_{x}\underline{H}_{y} = \sum_{z} c_{x,y}^{z}\underline{H}_{z}$ with all $c_{x,y}^{z} \in \{0, 1, 2, \dots\}$.

Left KL-order: $z \ge_L y$ if there is x such that $c_{x,y}^z > 0$.

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Note: Similarly right order \geq_R and right cells \sim_R

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Note: Similarly right order \geq_R and right cells \sim_R

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Fact: Two-sided cell of w: all x such that RS(x) and RS(w) have the same shape

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Lusztig's **a**-function in type A

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Fact: Each two-sided cell of w contains the longest element for some parabolic (Young) subgroup

Fact: If some 2-sided cell contains longest elements for different parabolic subgroups, all these elements have the same length

Definition of a-function: $\mathbf{a}: W \to \{0, 1, 2, ...\}$ is the unique function such that

- ► a is constant on 2-sided cells;
- a(w) = I(w) if w is the longest element in a parabolic subgroup

Example. For $S_3 = \{e, s, t, st, ts, sts = tst = w_0\}$ we have

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Obvious fact: Gr(\mathcal{O}_0) $\cong \mathbb{Z}[W]$ via $[\Delta(w)] \mapsto H_w$.

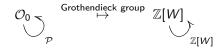
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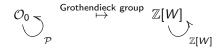
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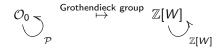
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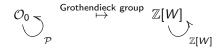
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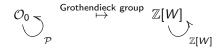
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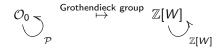
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Lusztig's **a**-function in general

Left cell in W gives rise to a cell representation of W (in type A: Specht module)

This can be modeled (i.e. categorified) via action of ${\cal P}$ on the additive category of certain self-dual modules in ${\cal O}_0$

Such indecomposable self-dual modules are naturally indexed (for all cells at the same time) by $w \in W$, say $w \mapsto Q(w)$

Definition. $\mathbf{a}(w) = \frac{1}{2}(\text{Loewy.Length}(Q(w)) - 1)$

- ▶ a is constant on 2-sided cells;
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Theorem. Let $w \in W$.

- ▶ proj.dim $(T(w)) = \mathbf{a}(w)$
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Step 1. Both $\operatorname{proj.dim}(T(w))$ and $\operatorname{proj.dim}(I(w))$ are constant on two-sided cells.

Why:

- projective functors preserve both the additive category of tilting modules, the additive category of projective modules and the additive category of injective modules,
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- use projective functors to related projective resolutions of indecomposable (tilting or injective modules) inside the same two-sided cell

Theorem. Let $w \in W$.

- ▶ proj.dim $(T(w)) = \mathbf{a}(w)$
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for w the longest element in a parabolic subcategory, the value $\operatorname{proj.dim}(\mathcal{T}(w))$ can be computed using Ringel self-duality of \mathcal{O} , the Koszul duality of Beilinson-Ginzburg-Soergel and computations of Loewy lengths of certain structural modules in \mathcal{O} by Irving

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This implies the result in type A, that is for $\mathfrak{g} = \mathfrak{sl}_n$

General case:

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Open problem: Determine proj.dim $(\theta_x L(y))$ for all $x, y \in W$

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injectives: computable using the regular case and projective functors

simples: open in general (known for the antidominant and the left cell of the dominant)

Koszul dual problem: graded length of an indecomposable projective in a parabolic $\ensuremath{\mathcal{O}}$

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THANK YOU!!!

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