Homological properties of category O, part II: Alexandru conjecture

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Some basic homological algebra

$\mathcal{A}$ — an abelian category

$\mathrm{Ext}^n_{\mathcal{A}}(N, M)$: equivalence classes of exact sequences

$$0 \to M \to X_1 \to X_2 \to \cdots \to X_n \to N \to 0$$

$\mathcal{B}$ — another abelian category

$\mathcal{A} \subset \mathcal{B}$ with exact inclusion $\mathbf{i}$

Fact. $\mathbf{i}$ induces a homomorphism $\mathbf{i}_n : \mathrm{Ext}^n_{\mathcal{A}}(N, M) \to \mathrm{Ext}^n_{\mathcal{B}}(N, M)$

Fact. $\mathbf{i}_n$ is usually neither injective nor surjective

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Extension full subcategories

Definition. $\mathcal{A}$ is extension full in $\mathcal{B}$ provided that $\mathbf{i}_n$ is iso for all n.

Note. Ext^0-full = full

Note. Ext^1-full $\sim$ Serre subcategory

Motivating? example.

- A — quasi-hereditary algebra w.r.t. $e_1 < e_2 < \cdots < e_n$
- Ae_nA — heredity ideal
- $B = A/Ae_nA$ (also quasi-hereditary w.r.t. $e_1 < e_2 < \cdots < e_{n-1}$)
- Theorem. (CPS) B-mod is extension full in A-mod

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Some categories of $\mathfrak{g}$-modules

$\mathfrak{g}$ — semi-simple complex finite dimensional Lie algebra

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — triangular decomposition

$\mathcal{O}$ — corresponding BGG category $\mathcal{O}$

$\mathfrak{g}$-Mod — the category of all $\mathfrak{g}$-modules

$\mathcal{W}$ — the category of all weight (i.e. $\mathfrak{h}$-diagonalizable) $\mathfrak{g}$-modules

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Thick category $\mathcal{O}$

Definition. Thick category $\widetilde{\mathcal{O}}$ is the full subcategory of $\mathfrak{g}$-Mod containing all M such that

- M is finitely generated;
- M is locally $U(\mathfrak{h})$-finite
- M is locally $U(\mathfrak{n}_+)$-finite

Alternative to the last two: M is locally $U(\mathfrak{b})$-finite for $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$

Difference to $\mathcal{O}$: category $\widetilde{\mathcal{O}}$ has no projectives

Note. $\mathcal{O}$ is not extension full in $\widetilde{\mathcal{O}}$ (not even Ext^1-full)

Note. $\widetilde{\mathcal{O}}$ is the Serre subcategory of $\mathfrak{g}$-Mod generated by $\mathcal{O}$

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Main results (Coulembier-M.)

Theorem 1. $\mathcal{O}$ is extension full in $\mathcal{W}$.

Theorem 2. $\widetilde{\mathcal{O}}$ is extension full in $\mathcal{GW}$.

Theorem 3. $\mathcal{GW}$ is extension full in $\mathfrak{g}$-Mod.

Theorem 4. Theorems 1, 2 and 3 are true for basic classical Lie superalgebras.

Corollary. $\mathrm{gl.dim}\,\widetilde{\mathcal{O}} = \mathrm{gl.dim}\,\mathcal{GW} = \dim \mathfrak{g}$ $(= \mathrm{gl.dim}\,\mathfrak{g}\text{-Mod})$

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Very rough idea of the proof of Theorem 1.

Theorem 1. $\mathcal{O}$ is extension full in $\mathcal{W}$.

"Easy" case: Both categories have projectives.

Use:

- Frobenius reciprocity ($=$ adjunction of Ind and Res)
- BGG's construction of projectives in $\mathcal{O}$
- Comparison of these projectives to projectives in $\mathcal{W}$.
- the next lemma

Lemma. Assume $\mathcal{A} \subset \mathcal{B}$ with exact inclusion, both have enough projectives, and any projective $P \in \mathcal{A}$ is acyclic for the functor $\mathrm{Hom}_{\mathcal{B}}(\_, K)$ for any $K \in \mathcal{A}$. Then $\mathcal{A}$ is extension full in $\mathcal{B}$.

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Very rough idea of the proof of Theorem 1.

Theorem 1. $\mathcal{O}$ is extension full in $\mathcal{W}$.

"Easy" case: Both categories have projectives.

Use:

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Theorem 2. $\widetilde{\mathcal{O}}$ is extension full in $\mathcal{GW}$.

Note: None of the categories have projectives.

Steps:

- Restrict the size of Jordan cells allowed for the action of $\mathfrak{h}$ to get $\widetilde{\mathcal{O}}^{(n)}$ and $\mathcal{GW}^{(n)}$
- Both $\widetilde{\mathcal{O}}^{(n)}$ and $\mathcal{GW}^{(n)}$ have projectives
- Use proof of Theorem 1 to show that $\widetilde{\mathcal{O}}^{(n)}$ is extension full in $\mathcal{GW}^{(n)}$
- Take limit $n \to \infty$
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Very rough idea of the proof of Theorem 3.

Theorem 3. $\mathcal{GW}$ is extension full in $\mathfrak{g}$-Mod.

Note: $\mathfrak{g}$-Mod has projectives while $\mathcal{GW}$ does not.

Using tricks and Frobenius reciprocity, Theorem 3 can be reduced to:

Lemma. Assume $\mathcal{A} \subset \mathcal{B}$ with exact inclusion.
Assume $\mathcal{A}$ has a full subcategory $\mathcal{A}_0$ such that

- $\mathcal{A}$ is the Serre subcategory of $\mathcal{B}$ generated by $\mathcal{A}_0$;
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Then $\mathcal{A}$ is extension full in $\mathcal{B}$ if and only if the natural map

$$\mathrm{Ext}^n_{\mathcal{A}}(P, K) \to \mathrm{Ext}^n_{\mathcal{B}}(P, K)$$

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Definition. $\mathrm{p.dim}_{\widetilde{\mathcal{O}}} M := \sup\{k : \mathrm{Ext}^k_{\widetilde{\mathcal{O}}}(M, N) \neq 0$ for some $N \in \widetilde{\mathcal{O}}\}$

Theorem 5. (Coulembier-M.) $\mathrm{p.dim}_{\widetilde{\mathcal{O}}} M \geq \dim \mathfrak{h}$ for $M \in \widetilde{\mathcal{O}}$

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Motivation: Alexandru conjecture, part I

Source: Pierre-Yves Gaillard. Statement of the Alexandru Conjecture. arXiv:math/0003070

$\mathcal{A}$ — abelian length category

$\mathrm{Irr}(\mathcal{A})$ — set of isoclasses of simple objects in $\mathcal{A}$

$<$ — smallest partial order on $\mathrm{Irr}(\mathcal{A})$ such that

$$\mathrm{p.dim}\, L = \mathrm{p.dim}\, L' + 1 \quad \text{and} \quad \mathrm{Ext}^1_{\mathcal{A}}(L, L') \neq 0 \quad \text{imply} \quad L_i < L_j$$

Definition. A Serre subcategory $\mathcal{B} \subset \mathcal{A}$ is an initial segment if $L_j \in \mathcal{B}$ and $L_i < L_j$ implies $L_i \in \mathcal{B}$

Definition. $\mathcal{A}$ is Guichardet if any initial segment is extension full in $\mathcal{A}$

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Motivation: Alexandru conjecture, part II

Weak Alexandru conjecture: The principal block of the category of Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-modules is Guichardet.

Motivation: $\mathcal{O}_0$ is Guichardet.

Explanation:

- we know explicitly $\mathrm{p.dim}$ of all simples in $\mathcal{O}_0$;
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Motivation: Alexandru conjecture, part II

Weak Alexandru conjecture: The principal block of the category of Harish-Chandra $(\mathfrak{g},\mathfrak{k})$-modules is Guichardet.

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Alexandru conjecture for $\widetilde{\mathcal{O}}$ and for $\mathcal{H}$

Theorem 7. (Coulembier-M.) $\widetilde{\mathcal{O}}_0$ is Guichardet.

$\mathcal{H}$ — the category of Harish-Chandra bimodules for $\mathfrak{g}$

Note: $\mathcal{H} \in \mathfrak{g} \oplus \mathfrak{g}$-mod

BG-equivalences. $\mathcal{O}_0 \cong {}^{\infty}_{0}\mathcal{H}^{1}_{0}$ and $\widetilde{\mathcal{O}}_0 \cong {}^{\infty}_{0}\mathcal{H}^{\infty}_{0}$

Corollary. ${}^{\infty}_{\chi}\mathcal{H}^{1}_{\chi}$, ${}^{1}_{\chi}\mathcal{H}^{\infty}_{\chi}$ and ${}^{\infty}_{0}\mathcal{H}^{\infty}_{0}$ are Guichardet.

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Alexandru conjecture for singular blocks of $\mathcal{O}$

Observation. Singular blocks of $\mathcal{O}$ for $\mathfrak{sl}_3$ are not always Guichardet

Given by:

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\delta"] \arrow[r, shift left, "\beta"] & 3 \arrow[l, shift left, "\gamma"] \end{tikzcd}$$

with relations $\beta\gamma = 0$ and $\gamma\beta = \alpha\delta$.

Easy: $\mathrm{p.dim}\, L_1 = 1$ and $\mathrm{p.dim}\, L_2 = \mathrm{p.dim}\, L_3 = 2$

Note: $\mathrm{Serre}\langle L_3\rangle$ is an initial segment (and is semi-simple).

Note: $\mathrm{Ext}^2_{\mathcal{O}}(L_3, L_3) \neq 0$

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Saturated Alexandru conjectures

Source: Alain Fuser. The Alexandru conjectures. Prepublication de l'Institut Elie Cartan, Nancy, 1997

$\mathcal{A}$ — abelian length category

$\preceq$ — smallest partial pre-order on $\mathrm{Irr}(\mathcal{A})$ such that

- $L_i < L_j$ implies $L_i \preceq L_j$;
- $\mathrm{p.dim}\, L = \mathrm{p.dim}\, L'$ and $\mathrm{Ext}^1(L, L') \neq 0$ or $\mathrm{Ext}^1(L', L) \neq 0$ implies $L \preceq L'$.

Definition. A Serre subcategory $\mathcal{B} \subset \mathcal{A}$ is a saturated initial segment if $L_j \in \mathcal{B}$ and $L_i \preceq L_j$ implies $L_i \in \mathcal{B}$

Definition. $\mathcal{A}$ is saturated Guichardet if any saturated initial segment is extension full in $\mathcal{A}$

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Reason: Saturated initial segments and initial segments coincide.

Observation: All blocks of $\mathcal{O}_0$ for $\mathfrak{sl}_3$ are saturated Guichardet.

Reason: Saturated initial segments are coideals in the Bruhat order.

Unknown: Is any block of $\mathcal{O}$ saturated Guichardet?

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Saturated Alexandru conjectures for $\mathcal{O}$

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Why difficult: We do not know projective dimensions of simples in $\mathcal{O}$!

Our guess: Some blocks of $\mathcal{O}$ are not saturated Guichardet.

Another difficulty: Difficult to estimate extensions in the saturated Serre subcategories.

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THANK YOU!!!