ALGEBRAIC CATEGORIFICATION

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1. DECATEGORIFICATION

$\mathscr{C}$ — a category (additive, abelian, or triangulated)

$[\mathscr{C}]$ — Grothendieck group of $\mathscr{C}$

$\mathbb{F}$ — commutative ring with 1

Definition. **The $\mathbb{F}$-module $[\mathscr{C}]^{\mathbb{F}} = \mathbb{F} \otimes_{\mathbb{Z}} [\mathscr{C}]$ is called the $\mathbb{F}$-*decategorification* of $\mathscr{C}$.**

2. PRECATEGORIFICATION AND CATEGORIFICATION

V — $\mathbb{F}$-module

Definition. A ***precategorification*** of V is a pair $(\mathscr{C}, \varphi)$ where $\mathscr{C}$ is a category (additive, abelian, or triangulated) and $\varphi : V \to [\mathscr{C}]^{\mathbb{F}}$ is a monomorphism.

Definition. A precategorification $(\mathscr{C}, \varphi)$ is called a ***categorification*** provided that φ is an isomorphism.

$f \in \mathrm{End}_{\mathbb{F}}(V)$

Definition. **A *categorification* of f is a functor $F : \mathscr{C} \to \mathscr{C}$ (additive, exact, or triangulated) such that $[F] \circ \varphi = \varphi \circ f$, where $[F]$ is the endomorphism of $[\mathscr{C}]^{\mathbb{F}}$, induced by F.**

$A = \langle a_1, a_2, \dots | R_1(a_1, \dots) = 0, \dots \rangle$ — **$\mathbb{F}$-algebra**

V — **A-module**

Definition. **A *weak categorification* of V is a categorification of V and all a_1, $a_2, \dots$**

Definition. **A *categorification* of V is a weak categorification with some functorial interpretation of the relations in A.**

Definition??? **A *strong categorification* of V is a categorification with some other properties which would guarantee some kind of uniqueness.**

Example. **There is a definition of a strong categorification for finite-dimensional $\mathfrak{sl}_2$-modules by Chuang and Rouquier.**

Problem. **Give a "reasonable" definition of a strong categorification in the general case.**

3. EXAMPLE: SPECHT MODULES

n — **positive integer**

λ — **partition of** n

S_n — **symmetric group**

$S(\lambda)^{\mathbb{F}}$ — **Specht module over** $\mathbb{F}[S_n]$

λ' — **dual partition**

$\mathcal{O}_0^{\lambda'}$ — **regular block of the** λ'**-parabolic category** $\mathcal{O}$ **for** $\mathfrak{sl}_n$

Q — **basic projective-injective module in** $\mathcal{O}_0^{\lambda'}$

$\mathscr{C}_1$ — **additive category of projective-injective modules in** $\mathcal{O}_0^{\lambda'}$

$\mathscr{C}_2$ — **abelian category** $\mathrm{End}_{\mathcal{O}_0^{\lambda'}}(Q)$**-mod**

$\varphi : S(\lambda)^{\mathbb{F}} \to [\mathscr{C}]^{\mathbb{F}}$ **given by sending Kazhdan-Lusztig basis elements to corresponding indecomposable projectives**

Theorem (Khovanov-M.-Stroppel) $(\mathscr{C}_2, \varphi)$ **is a precategorification and** $(\mathscr{C}_1, \varphi)$ **is a categorification of** $S(\lambda)^{\mathbb{F}}$**.** $(\mathscr{C}_2, \varphi)$ **is a categorification of** $S(\lambda)^{\mathbb{F}}$ **if** $\mathbb{F}$ **is a field of characteristic zero.**

Difficulty: **Projective modules do not form a basis in the Grothendieck group ($\mathrm{End}_{\mathcal{O}_0^{\lambda'}}(Q)$ is self-injective and hence has infinite global dimension in general).**

Problem: **What is $[\mathscr{C}_2]^{\mathbb{Z}}$ as a $\mathbb{Z}[S_n]$-module?**

Question: **Why do we need $\mathscr{C}_2$?**

Answer: **Because it is abelian and we can derive. In particular, this allows us to lift the S_n-action on the Specht module to the functorial action of the canonical generators of the corresponding braid group (via shuffling functors).**