

SERRE FUNCTORS, CATEGORY \mathcal{O} AND SYMMETRIC ALGEBRAS

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1. Serre functors

\mathbb{k} — algebraically closed field.

\mathcal{C} — \mathbb{k} -linear additive category with finite-dimensional morphism spaces.

Definition. (Bondal-Kapranov) An additive functor, $F : \mathcal{C} \rightarrow \mathcal{C}$, is called a *Serre functor* if F is an auto-equivalence of \mathcal{C} and there are isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(X, FY) \cong \mathrm{Hom}_{\mathcal{C}}(Y, X)^*,$$

natural in X and Y .

Example 1. Let X be a smooth projective variety, $n = \dim X$, $\mathcal{A} = \mathcal{D}_{coh}^b(X)$ be the bounded derived category of coherent sheaves on X , $K_X = \Omega_X^n$ be the canonical sheaf. Then $(-) \otimes K_X[n]$ is a Serre functor on \mathcal{A} because of the Serre duality.

Example 2. Let A be a finite-dimensional \mathbb{k} -algebra of finite global dimension, $\mathcal{A} = \mathcal{D}^b(A)$ be the bounded derived category of finite-dimensional A -modules. Then the left derived of the Nakayama functor $A^* \otimes_A -$ is a Serre functor on \mathcal{A} .

Some properties:

- If a Serre functor exists, it is unique up to an isomorphism.
- Let A be a finite-dimensional \mathbb{k} -algebra. Then $\mathcal{D}^b(A)$ has a Serre functor if and only if $\text{gl.dim}(A) < \infty$.
- Let \mathcal{C} be a category, which is equivalent to $A\text{-mod}$ for some finite-dimensional \mathbb{k} -algebra of finite global dimension. If A is not given explicitly, then the Serre functor on $\mathcal{D}^b(\mathcal{C})$ can be very hard to compute.

Main theorem for detection of Serre functors.

Let A be a finite dimensional \mathbb{k} -algebra of finite global dimension. Assume that:

- There are exact sequences

$$0 \rightarrow {}_A A \rightarrow X \rightarrow X', \quad 0 \rightarrow {}_{A^{opp}} A^{opp} \rightarrow Y \rightarrow Y',$$

where X and X' are projective-injective A -modules and Y and Y' are projective-injective A^{opp} -modules.

- The socle and the top of the basic projective-injective modules for A and A^{opp} are isomorphic.

Let $F : A\text{-mod} \rightarrow A\text{-mod}$ be a right exact functor. Then $\mathcal{L}F$ is a Serre functor on $\mathcal{D}^b(A)$ if and only if the following conditions are satisfied:

1. Its left derived functor $\mathcal{L}F : \mathcal{D}^b(A) \rightarrow \mathcal{D}^b(A)$ is an auto-equivalence.
2. F maps projective A -modules to injective A -modules.
3. F preserves the full subcategory \mathcal{PI} of $A\text{-mod}$, consisting of all projective-injective modules, and the restrictions of F and the Nakayama functor to \mathcal{PI} are isomorphic.

2. Category \mathcal{O}

\mathfrak{g} — semi-simple complex finite-dimensional Lie algebra

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — fixed triangular decomposition.

$U(\mathfrak{g})$ — the universal enveloping algebra of \mathfrak{g} .

\mathcal{O} — full subcategory of \mathfrak{g} -mod, consisting of those modules, which are

- finitely generated;
- \mathfrak{h} -diagonalizable;
- $U(\mathfrak{n}_+)$ -finite.

\mathcal{O}_0 — the principal block of \mathcal{O} , that is the indecomposable block, containing the trivial module.

W — the Weyl group of \mathfrak{g} .

$\mathcal{O}_0 \cong A$ -mod for some A , which is not explicitly given.

s — simple reflection in W .

$U(\mathfrak{g})_s$ — the localization of $U(\mathfrak{g})$ with respect to Y_α , where α is the root corresponding to s .

$$F_1^s = U(\mathfrak{g})_s \otimes_{U(\mathfrak{g})} -.$$

$$F_2^s = \text{Coker}(\text{ID} \rightarrow F_1) \text{ induced by } U(\mathfrak{g}) \hookrightarrow U(\mathfrak{g})_s.$$

$T_s : \mathcal{O}_0 \rightarrow \mathcal{O}_0$ is the composition of F_2^s and the inner automorphism of \mathfrak{g} , corresponding to s .

T_s — Arkhipov's twisting functor.

T_s 's satisfy braid relations.

$$w \in W, w = s_1 s_2 \cdots s_k.$$

$$T_w = T_{s_1} T_{s_2} \cdots T_{s_k}.$$

w_0 — the longest element of W .

Theorem. $\mathcal{L}T_{w_0}T_{w_0}$ is the Serre functor on $\mathcal{D}^b(\mathcal{O}_0)$.

Idea of the proof: Use our main theorem and known properties of Arkhipov's functors (mostly due to H. Andersen and C. Stroppel).

Remark: It is well known that \mathcal{O}_0 is equivalent to the category of perverse sheaves on certain flag variety. For the corresponding bounded derived category the structure of the Serre functor was conjectured by Bondal and Kapranov. This conjecture was proved by Beilinson, Bezrukavnikov and Mirkovic in 2003 (published 2004).

Remark: The Braid group acts on $\mathcal{D}^b(\mathcal{O}_0)$ via auto-equivalences as follows: $\sigma_s \rightarrow \mathcal{L}T_s$. The Serre functor $\mathcal{L}T_{w_0}T_{w_0}$ is a part of this action, namely it corresponds to the element w_0^2 , which generates the center of the braid group (at least for type A). Note that the Serre functor commutes with all auto-equivalences, and hence if it appears in some group action, it must correspond to a central element (in the image).

3. Parabolic category \mathcal{O}

\mathfrak{p} — a parabolic subalgebra of \mathfrak{g} , containing $\mathfrak{h} \oplus \mathfrak{n}_+$.

$\mathcal{O}^{\mathfrak{p}}$ the subcategory of \mathcal{O} , consisting of $U(\mathfrak{p})$ -locally finite modules (the parabolic category of Rocha-Caridi).

$W^{\mathfrak{p}}$ — the Weyl group of \mathfrak{p} .

$w_0^{\mathfrak{p}}$ — the longest element in $W^{\mathfrak{p}}$.

Theorem. $\mathcal{L}(T_{w_0}^2)[-2l(w_0^{\mathfrak{p}})]$ is the Serre functor on $\mathcal{D}^b(\mathcal{O}_0^{\mathfrak{p}})$.

Idea of the proof: The algebra of $\mathcal{O}_0^{\mathfrak{p}}$ is a quotient of that of \mathcal{O}_0 , given by what is known as *Zuckerman's functor*. We already know the Serre functor on \mathcal{O}_0 , so using rather careful calculation one shows that it “induces” a Serre functor on $\mathcal{O}_0^{\mathfrak{p}}$. The length $2l(w_0^{\mathfrak{p}})$ of the shift in the formulation is nothing more than the derived length of Zuckerman's functor.

4. Application to symmetric algebras

A finite-dimensional algebra, A , is called *symmetric* if there exists a *symmetrizing form* on A , that is a linear map, $\lambda : A \rightarrow \mathbb{k}$, such that $\lambda(xy) = \lambda(yx)$ for all $x, y \in A$ and whose kernel does not contain any non-zero left or right ideal of A (equivalently if $A \cong A^*$ as A -bimodules).

Theorem. (The positive answer to a conjecture of M. Khovanov)
The endomorphism algebra of a basic projective-injective module in $\mathcal{O}_0^{\mathfrak{p}}$ is symmetric.

Explanation:

Let B be the endomorphism algebra of a basic projective-injective module in $\mathcal{O}_0^{\mathfrak{p}}$.

We know the Serre functor for $\mathcal{D}^b(\mathcal{O}_0^{\mathfrak{p}})$.

Thanks to Andersen and Stroppel, we know that our Serre functor naturally commutes with translation functors.

The functor F of partial coapproximation (which has appeared in an earlier work of the author and O. Khomenko) with respect to the basic projective-injective module also naturally commutes with translation functors.

A direct computation shows that the Serre functor and F^2 coincide on the dominant Verma module.

Using recent results of O. Khomenko on functors, naturally commuting with translation functors, it follows that $\mathcal{L}F^2$ and the Serre functor coincide.

By the definition of the partial coapproximation, this functor induces the identity functor on the additive closure of the original projective-injective module.

Hence the Serre functor induces the identity functor on the additive closure of the original projective-injective module.

Hence for $\mathcal{K}^{perf}(B)$ the Serre functor is the identity functor.

This means that $B \cong B^*$, namely that B is symmetric.

Using our main theorem one can compute Serre functors for various categories of Harish-Chandra bimodules, for certain rational Cherednik algebras and for certain Schur algebras.