0. An Example

Consider $\mathfrak{sl}_2(\mathbb{C})$ with the standard generators

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The (unique) irreducible 3-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module has a basis with the following action of generators:

$$\begin{tikzcd} v_{-2} \arrow[rr, bend left, "X=2"] & & v_0 \arrow[ll, bend left, "Y=1"] \arrow[rr, bend left, "X=1"] & & v_2 \arrow[ll, bend left, "Y=2"] \end{tikzcd} \tag{1}$$

The generators X and Y satisfy the following defining relations for $\mathfrak{sl}_2(\mathbb{C})$:

$$[[X, Y], X] = 2X, \qquad [[X, Y], Y] = -2Y. \tag{2}$$

Consider another diagram:

$$\begin{tikzcd}[column sep=huge]
\mathbb{C}\text{−mod} \arrow[r, bend left, "F=\mathrm{Ind}"] & \mathbb{C}[x]/(x^2)\text{−mod} \arrow[l, bend left, "G=\mathrm{Res}"] \arrow[r, bend left, "F=\mathrm{Res}"] & \mathbb{C}\text{−mod} \arrow[l, bend left, "G=\mathrm{Ind}"]
\end{tikzcd} \qquad (3)$$

The functors F and G satisfy the following relations:

$$\begin{aligned} FGF \oplus FGF &\cong F \oplus F \oplus GFF \oplus FFG, \\ GFG \oplus GFG &\cong G \oplus G \oplus FGG \oplus GGF. \end{aligned} \qquad (4)$$

Question:

Is there any connection between (1), (2) *and* (3), (4)*?*

Answer:

To get (1) *just take the Grothendieck group of* (3)
(and tensor with $\mathbb{C}$*).*

Then the isomorphisms (4) *induce the relations* (2).

This is what is called ***DECATEGORIFICATION***

The inverse procedure is called ***CATEGORIFICATION***

CATEGORIFICATION OF THE REPRESENTATION THEORY OF THE SYMMETRIC GROUP

Volodymyr Mazorchuk

(Uppsala University)

joint with Catharina Stroppel (University of Glasgow, UK)
and (partially) Mikhail Khovanov (Columbia University, USA)

1. Decategorification and Categorification

$\mathscr{C}$ — an abelian, additive or triangulated category.

$\mathbb{F}$ — a field

$[\ ]_{Gr}$ — (split) Grothendieck group

Definition. An $\mathbb{F}$-*decategorification* of $\mathscr{C}$ is the vector space

$$[\mathscr{C}]_{\mathbb{F}} := \mathbb{F} \otimes_{\mathbb{Z}} [\mathscr{C}]_{Gr}$$

$F : \mathscr{C}_1 \to \mathscr{C}_2$ – an exact, additive or triangulated functor.

Definition. An $\mathbb{F}$-*decategorification* of F is the linear operator

$$[F]_{\mathbb{F}} : [\mathscr{C}_1]_{\mathbb{F}} \to [\mathscr{C}_2]_{\mathbb{F}}$$

V — an $\mathbb{F}$-vector space.

Definition. A *categorification* of V is an abelian (or additive or triangulated) category $\mathscr{C}$ together with a fixed isomorphism

$$\varphi : V \xrightarrow{\sim} [\mathscr{C}]_{\mathbb{F}}.$$

$f : V_1 \to V_2$ — a linear map between $\mathbb{F}$-vector spaces.

$(\mathscr{C}_i, \varphi_i)$ — categorifications of V_i, $i = 1, 2$.

Definition. A *categorification* of f is an exact (or additive or triangulated) functor $F : \mathscr{C}_1 \to \mathscr{C}_2$ such that the following diagram commutes:

$$\begin{tikzcd}
V_1 \arrow[r, "f"] \arrow[d, "\varphi_1"'] & V_2 \arrow[d, "\varphi_2"] \\
{[\mathscr{C}_1]_{\mathbb{F}}} \arrow[r, "{[F]_{\mathbb{F}}}"] & {[\mathscr{C}_2]_{\mathbb{F}}}
\end{tikzcd}$$

$A = \langle a_1, \ldots \mid R_i(a_1, \ldots) = 0 \rangle$ — an $\mathbb{F}$-algebra with a fixed presentation.

M — an A-module.

Mathematical definition:

Definition. A *weak categorification* of M is a categorification of the vector space M together with categorification of each linear operator $a_i : M \to M$.

Philosophical definition:

Definition. A *categorification* of M is a weak categorification of M together with some interpretation of each relation $R_i(a_1, \ldots) = 0$ in terms of isomorphism of functors.

Idea:

"Definition." A *strong categorification* of M is a categorification of M plus some extra data, which guarantees some nice properties (for example some kind of uniqueness).

2. Historical remarks

The word *categorification* was introduced by Louis Crane about 15 years ago and refers to the process of *replacing set-theoretic theorems/notions by category-theoretic analogues.*

Set-theoretic	$\leftrightarrow$	Category-theoretic
set	$\leftrightarrow$	category
element	$\leftrightarrow$	object
number	$\leftrightarrow$	dimension
function	$\leftrightarrow$	functor
equation	$\leftrightarrow$	natural isomorphism

The original idea seems to come from Igor Frenkel.

3. Why categorify?

Disadvantage: Categorification makes everything more complicated.

Advantage: Categorification **might** give more structure.

Example 1: Khovanov's categorification of Jones polynomial resulted in construction of new stronger invariants of knots and links.

Example 2: Chuang and Rouquier's strong categorification of finite-dimensional $\mathfrak{sl}_2$-modules resulted in proof of Broué's abelian defect group conjecture for symmetric group. The extra data, which Chuang and Rouquier used, consisted of some conditions on the existence of certain natural transformations between the functors, which produce our categorification.

4. Regular representation of S_n

$\mathcal{O}_0$ — regular block of the BGG category $\mathcal{O}_0$ for $\mathfrak{sl}_n(\mathbb{C})$.

A *projective functor* on $\mathcal{O}_0$ is an appropriate direct summand of the functor $V \otimes_{\mathbb{C}} -$, where V is a finite-dimensional $\mathfrak{sl}_n(\mathbb{C})$-module.

Theorem. ([BG]) Indecomposable projective functors on $\mathcal{O}_0$ are in bijection with the elements of the Weyl group $W = S_n$, $w \leftrightarrow \theta_w$.

$s_i = (i, i+1)$, $i = 1, \dots, n-1$.

$\mathbb{C}[S_n]$ is generated by $\tau_i = 1 + s_i$, subject to the relations:

$$\begin{aligned} \tau_i^2 &= \tau_i + \tau_i, & & \\ \tau_i\tau_j &= \tau_j\tau_i, & i &\neq j \pm 1, \\ \tau_i\tau_j\tau_i + \tau_j &= \tau_j\tau_i\tau_j + \tau_i, & i &= j \pm 1. \end{aligned} \tag{5}$$

$\Delta(w)$ — Verma module $M(w \cdot 0)$ in $\mathcal{O}_0$.

Theorem. (Modern reformulation of [BG]) $\tau_i \mapsto \theta_{s_i}$ and $w \mapsto [\Delta(w)]$ is a categorification of the right regular representation of $\mathbb{C}[S_n]$, where the relations (5) are interpreted as isomorphisms of corresponding functors.

Adavantages: New bases in the regular representation of $\mathbb{C}[S_n]$. They are given by projective, simple, and tilting modules (Kazhdan-Lusztig's combinatorics).

5. Irreducible representation of $\mathbb{C}[S_n]$ (Specht modules)

λ — composition of n (parts do not have to decrease).

$\mathcal{O}_0^\lambda$ — λ-parabolic subcategory of $\mathcal{O}_0$.

$\mathtt{P}_\lambda$ — basic projective-injective module in $\mathcal{O}_0^\lambda$.

Projective functors act on $\mathrm{mod}-A_\lambda$, where $A_\lambda = \mathrm{End}_{\mathcal{O}}(\mathtt{P}_\lambda)$, in the natural way.

Theorem. ([KMS]) This action leads to a categorification of the Specht module associated with λ, in which projective modules correspond to the Kazhdan-Lusztig basis of the Specht module and the relations (5) are interpreted as isomorphisms of corresponding functors.

The unique invariant bilinear form on the Specht module is categorified via the Hom bifunctor and can be used to compute the Cartan matrix of A_λ.

Theorem. ([MS]) The identity functor is the Serre functor on $\mathcal{D}^{perf}(A_\lambda)$. In particular, $\mathcal{D}^{perf}(A_\lambda)$ has Auslander-Reiten triangles and A_λ is a symmetric algebra.

Question: Strong categorification of Specht modules?

6. Cell modules

Let $w \in S_n$.

Let $\mathcal{O}_0^w$ be the full subcategory of $\mathcal{O}_0$, where the only allowed simples are $L(x \cdot 0)$, $x \leq_R w$.

B_w — the endomorphism algebra of the direct sum of all projectives in $\mathcal{O}_0^w$ corresponding to $x \equiv_R w$.

Theorem. ([MS]) Projective functors act naturally on $\mathrm{mod} - B_w$ and this action gives rise to a categorification of the cell module associated to w. $B_w \cong A_\lambda$ for appropriate λ, which also induces an equivalence of the corresponding categorifications of Specht modules.

C_w — the endomorphism algebra of the direct sum of all projectives in $\mathcal{O}_0$ corresponding to $x \equiv_R w$.

Theorem. ([MS]) Projective functors act naturally on $\mathcal{D}^-(C_w)$ and this action gives rise to a categorification of the cell module, associated to w.

Question. Are B_w and C_w derived equivalent?

Question. Which extra conditions are needed to get uniqueness?

7. Induced modules

λ — composition of n.

W_λ — corresponding parabolic subgroup of S_n.

For any W_λ-module M we have the induced module $\mathbb{C}[S_n]\otimes_{\mathbb{C}[W_\lambda]}M$.

Theorem. ([MS]) If M is a cell module, then the induced module $\mathbb{C}[S_n]\otimes_{\mathbb{C}[W_\lambda]} M$ is categorified via the action of projective functors on the principal block of a certain parabolic analogue $\mathcal{O}(\mathfrak{p},\Lambda)$ of $\mathcal{O}$.

Example 1. If M is the sign module, then the induced module $\mathbb{C}[S_n]\otimes_{\mathbb{C}[W_\lambda]} M$ is categorified via $\mathcal{O}_0^\lambda$.

Example 2. If M is the trivial module, then the induced module $\mathbb{C}[S_n]\otimes_{\mathbb{C}[W_\lambda]} M$ (the so-called *permutation module*) is categorified via the principal block of a certain category of Harish-Chandra bimodules ($\mathcal{S}$-subcategory of $\mathcal{O}$).

There is an equivalence of these categorifications analogous to equivalences of cell modules.

Gelfand-Kirillov dimension and finiteness of $X_{-\alpha}$ action for various simple roots α define natural filtrations on induced modules.