A GENERALIZATION OF THE IDENTITY FUNCTOR

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1. What I am going to generalize

Classical situation: A — f.dim. associative algebra,

$$\mathrm{ID} : A\text{-mod} \to A\text{-mod}.$$

Restrictive situation: gl.dim.$(A) < \infty$, then the above functor $\mathrm{ID} : A\text{-mod} \to A\text{-mod}$ induces an equivalence

$$\mathrm{ID} : \mathcal{P}^{<\infty}(A) \to \mathcal{I}^{<\infty}(A),$$

where

$$\mathcal{P}^{<\infty}(A) = \{M \in A\text{-mod} : \text{p.d.}(M) < \infty\},$$
$$\mathcal{I}^{<\infty}(A) = \{M \in A\text{-mod} : \text{i.d.}(M) < \infty\}.$$

Our situation: A — quasi-hereditary (hence gl.dim.$(A) < \infty$).

2. Stratified algebras

P — projectives, I — injectives, L — simples.

If $\mathcal{M} = \{M_1, \dots, M_k\} \subset A$−mod then

$\mathcal{F}(\mathcal{M}) = \{M \in A$−mod : M admits a filtration with subquotients from $\mathcal{M}\}$.

$e_1, \dots, e_n$ — a complete set of primitive idempotents for A.

For $M, N \in A$−mod set $\mathrm{Tr}_M(N) = \sum_{f:M\to N} \mathrm{Im}(f) \subset N$.

$$P^{>i} = \oplus_{j>i} P(j)$$

$$\Delta(i) = P(i)/\mathrm{Tr}_{P^{>i}}(P(i)), \qquad \nabla(i) = D(\Delta^{(A^{opp})}(i)).$$

$$\overline{\Delta}(i) = \Delta(i)/\mathrm{Tr}_{\Delta(i)}(\mathrm{rad}\Delta(i)), \qquad \overline{\nabla}(i) = D(\overline{\Delta}^{(A^{opp})}(i)).$$

$\Delta = \{\Delta_1, \dots, \Delta_n\}$ and so on.

Definition. A is *strongly standardly stratified* (SSS) provided that ${}_AA \in \mathcal{F}(\Delta)$; and A is quasi-hereditary provided that it is SSS and $\Delta(i) = \overline{\Delta}(i)$ for all i.

3. The Ringel dual

A — SSS algebra.

Theorem. (Ágoston-Happel-Lucács-Unger) $\mathcal{F}(\Delta) \cap \mathcal{F}(\overline{\nabla})$ is closed under taking direct summands. Indecomposable modules $T(i)$ in this category (called *tilting modules*) correspond bijectively to $\Delta(i)$, from which their Δ-flag starts. Set $T = \oplus_i T(i)$ and $R = R(A) = \mathrm{End}_A(T)$ (the *Ringel dual*). Then the algebra R^{opp} is SSS and $R(R^{opp})^{opp}$ is Morita equivalent to A. The *Ringel duality functor* $F(\_) = \mathrm{Hom}_A(T, \_)$ induces an equivalence between $\mathcal{F}(\overline{\nabla}^{(A)})$ and $\mathcal{F}(\overline{\Delta}^{(R)})$.

Remark. The module T above is a *generalized tilting module* in the sense that $\mathrm{Ext}_A^{>0}(T,T) = 0$, p.d.$(T) < \infty$ and ${}_AA$ admits a finite coresolution $0 \to {}_AA \to T_0 \to T_1 \to \cdots \to T_l \to 0$, where all $T_i \in \mathrm{Add}(T)$.

4. Properly stratified algebras

Definition. (Dlab) A is said to be **properly stratified** provided that both A and A^{opp} are SSS-algebras.

5. Two-step Ringel duality

A — SSS algebra. Assume that $R(A)$ is properly stratified.

Theorem. (Frisk-M.) Let A be SSS with properly stratified R and $H = F^{-1}(T^{(R)})$. Set $B = B(A) = \mathrm{End}_A(H)$ (the *two-step Ringel dual* of A). Then B^{opp} is SSS, the Ringel dual of B^{opp} is $R(A)^{opp}$ and thus is properly stratified, and $B(B^{opp})^{opp}$ is Morita equivalent to A. Moreover, the *two-step Ringel duality* functor $G(\_) = D \circ \mathrm{Hom}_A(\_, H)$ induces an equivalence between

$$\mathcal{P}^{<\infty}(A) \quad \text{and} \quad \mathcal{I}^{<\infty}(B).$$

Remark. If A is quasi-hereditary, then $H = \oplus_{i=1}^{n} I(i)$, B is Morita equivalent to A and G is isomorphic to the identity functor. Same if A is properly stratified with tilting=cotilting.

Remark. The fact that A is properly stratified does not guarantee that B is properly stratified.

6. Applications

Corollary. Let A be SSS with a properly stratified Ringel dual. Then $\mathcal{P}^{<\infty}(A)$ is contravariantly finite in A-mod.

Corollary. Let A be SSS with a properly stratified Ringel dual. Then $\text{fin.dim.}(A) = \text{p.d.}(H)$.

Corollary. Let A be properly stratified having a simple preserving duality, and assume that R is properly stratified as well. Then $\text{fin.dim.}(A) = 2 \cdot \text{p.d.}(T^{(R)}) \leq 2 \cdot \text{p.d.}(T)$.

Corollary. Let A be properly stratified having a simple preserving duality, and assume that R is properly stratified and has a simple preserving duality as well. Then $\text{fin.dim.}(A) = 2 \cdot \text{p.d.}(T)$.

7. An example

$\mathfrak{g}$ — s.s. f.d. Lie algebra over $\mathbb{C}$.

$U(\mathfrak{g})$ — the universal enveloping algebra.

$Z(\mathfrak{g})$ — the center of $U(\mathfrak{g})$.

${}_0^\infty\mathcal{H}_0^n$, $0 < n < \infty$, — a category of Harish-Chandra $U(\mathfrak{g})$-bimodules (finitely generated, direct sum of f.d. $\mathfrak{g}$-modules under the diagonal action, annihilated by the n-the power of $Z(\mathfrak{g})^+$ from the right and by some power of $Z(\mathfrak{g})^+$ from the left).

Theorem. ${}_0^\infty\mathcal{H}_0^n$ is equivalent to A-mod for some properly stratified A having a simple preserving duality. Moreover, $R(A)$ is Morita equivalent to A. Hence $B(A)$ is Morita equivalent to A as well and the two-step Ringel duality functor G induces a covariant exact equivalence between $\mathcal{P}^{<\infty}(A)$ and $\mathcal{I}^{<\infty}(A)$ ($\neq \mathcal{P}^{<\infty}(A)$ if $n > 1$). This equivalence sends H to I and ${}_AA$ to $D(H^{(A^{opp})})$.

Corollary. The category $\mathcal{P}^{<\infty}(A)$ from the previous theorem is equivalent to $\mathcal{P}^{<\infty}(A)^{opp}$.