Finitary 2-categories and their 2-representations

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Finite dimensional algebras and $\Bbbk$-linear categories

$\Bbbk$ — algebraically closed field

A — finite dimensional algebra over $\Bbbk$

$\mathrm{mod}\text{-}A$ — the category of right finitely generated A-modules

$P_1, P_2, \dots, P_k$ — indecomposable projectives in $\mathrm{mod}\text{-}A$ up to iso.

$\mathcal{P}$ — the full subcategory of $\mathrm{mod}\text{-}A$ with objects $P_1, P_2, \dots, P_k$

$\mathcal{P}$ — $\Bbbk$-linear category (i.e. enriched over $\Bbbk\text{-}\mathrm{mod}$)

$\mathcal{P}\text{-}\mathrm{mod}$ — the category of $\Bbbk$-linear functors from $\mathcal{P}$ to $\Bbbk\text{-}\mathrm{mod}$

Theorem. $\mathcal{P}\text{-}\mathrm{mod} \cong \mathrm{mod}\text{-}A$.

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Definition. A 2-category is a category enriched over the monoidal category **Cat** of small categories (in the latter the monoidal structure is induced by the cartesian product).

This means that a 2-category $\mathscr{C}$ is given by the following data:

- objects of $\mathscr{C}$;
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Principal example. The category **Cat** is a 2-category.

- Objects of **Cat** are small categories.
- 1-morphisms in **Cat** are functors.
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Definition: A 2-category $\mathscr{C}$ is called finitary (over $\Bbbk$) provided that

- $\mathscr{C}$ has finitely many objects;
- each $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is additive, $\Bbbk$-linear, idempotent split, with finitely many indecomposables (up to isomorphism);
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Example 1: projective functors on A-mod

A — finite dimensional connected associative $\Bbbk$-algebra

$A\text{-mod}$ — the category of finitely generated left A-modules

Definition: $F : A\text{-mod} \to A\text{-mod}$ is projective is it is isomorphic to tensoring with a projective bimodule.

Definition: [M.-Miemietz] The 2-category $\mathscr{P}_A$ is defined as follows:

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Remark: $\mathscr{P}_A$ is a "simple" finitary 2-category ($\sim$ Artin-Wedderburn)

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Γ — finite acyclic quiver

$\Bbbk\Gamma$ — the path algebra of Γ

For $\mathtt{i} \in \Gamma$ let $\mathrm{F_i} : \Bbbk\Gamma\text{-mod} \to \Bbbk\Gamma\text{-mod}$ be the $\mathtt{i}$-th projection functor "factor out the trace of the $\mathtt{i}$-th simple module"

Fact: $\mathrm{F_i} : \Bbbk\Gamma\text{-inj} \to \Bbbk\Gamma\text{-inj}$.

$\mathrm{G_i} : \Bbbk\Gamma\text{-mod} \to \Bbbk\Gamma\text{-mod}$ — the unique (up to iso) left exact functor such that $\mathrm{G_i}|_{\Bbbk\Gamma\text{-inj}} \cong \mathrm{F_i}|_{\Bbbk\Gamma\text{-inj}}$

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Problem: Classify indecomposable 1-morphisms in $\mathscr{G}_\Gamma$.

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Example 3: Soergel bimodules

(W, S) — finite Coxeter system

C — the coinvariant algebra of a fixed geometric realization of (W, S)

B_w — the Soergel **C**-**C**-bimodule corresponding to $w \in W$

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Inspired by: W. Soergel's combinatorial description of projective functors acting on the regular block of the BGG category $\mathcal{O}$.

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Example 3: Soergel bimodules

(W, S) — finite Coxeter system

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2-functors and 2-representations

$\mathscr{A}$ and $\mathscr{C}$ — two 2-categories

Definition. A 2-functor $\mathrm{F} : \mathscr{A} \to \mathscr{C}$ is a functor which sends 1-morphisms to 1-morphisms and 2-morphisms to 2-morphisms in a way that is coordinated with all the categorical structures (domains, codomains, identities and compositions).

Definition. A 2-representation of a 2-category $\mathscr{C}$ is a 2-functor from $\mathscr{C}$ to some "classical" 2-category.

Example. Categories $\mathscr{P}_A$, $\mathscr{G}_\Gamma$ and $\mathscr{S}_{(W,S)}$ were defined using the corresponding defining 2-representation

"Classical" 2-representations:

- in **Cat**;
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$\mathcal{A}$ — additive category

Definition. The split Gorthendieck group $[\mathcal{A}]_\oplus$ of $\mathcal{A}$ is the quotient of the free abelian group generated by $[M]$, $M \in \mathcal{A}$, modulo the relations $[X] - [Y] - [Z]$ whenever $X \cong Y \oplus Z$ in $\mathcal{A}$.

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- $[\mathscr{S}_{(W,S)}](\clubsuit,\clubsuit) \cong \mathbb{Z}W$.
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A 2-representation of $\mathscr{A}$ can often be understood as a functorial action of $\mathscr{A}$ on a collection of certain categories.

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2-Morita Theory

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Definition. A progenerator is a projective 2-representation **P** of $\mathscr{C}$ such that any other projective 2-representation is a retract of some 2-representation from $\mathrm{add}(\mathbf{P})$.

Example. $\bigoplus_{\mathtt{i}\in\mathscr{C}} \mathbb{P}_{\mathtt{i}}$ is a progenerator.

Theorem: [M.-Miemietz] $\mathscr{A}$, $\mathscr{C}$ — finitary 2-categories. Then TFAE:

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Multisemigroup of a finitary category

F, G are composable indecomposable 1-morphisms in $\mathscr{C}$, then

$$F \circ G \cong \sum_{H \text{ indec.}} H^{\oplus m_{F,G}^{H}}.$$

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Definition. The multisemigroup $(S(\mathscr{C}), \diamond)$ of $\mathscr{C}$ is defined as follows: $S(\mathscr{C})$ is the set of isomorphism classes of 1-morphisms in $\mathscr{C}$ (including 0),

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Multisemigroup of a finitary category

F, G are composable indecomposable 1-morphisms in $\mathscr{C}$, then

$$F \circ G \cong \sum_{H \text{ indec.}} H^{\oplus m_{F,G}^{H}}.$$

$\mathscr{C}$ — finitary 2-category

Definition. The multisemigroup $(S(\mathscr{C}), \diamond)$ of $\mathscr{C}$ is defined as follows: $S(\mathscr{C})$ is the set of isomorphism classes of 1-morphisms in $\mathscr{C}$ (including 0),

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Cells

$\mathscr{C}$ — finitary 2-category

$(S(\mathscr{C}), \diamond)$ — the multisemigroup of $\mathscr{C}$

Definition. $[F] \sim_L [G]$ if $S(\mathscr{C}) \diamond [F] = S(\mathscr{C}) \diamond [G]$

Definition. Equivalence classes of $\sim_L$ are called left cells.

Similarly: $\sim_R$ (right cells) and $\sim_J$ (two-sided cells)

Examples:

- if $(S(\mathscr{C}), \diamond)$ is a semigroup, we get Green's relations
- for $\mathscr{S}_{(W,S)}$ we get Kazhdan-Lusztig cells

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Definition. The abelianization 2-functor $\bar{\cdot}$ is defined as follows:

given $\mathbf{M} \in \mathscr{C}\text{-afmod}$ and $\mathtt{i} \in \mathscr{C}$ the category $\overline{\mathbf{M}}(\mathtt{i})$ has objects

$$X \xrightarrow{\alpha} Y, \qquad X, Y \in \mathbf{M}(\mathtt{i}), \quad \alpha : X \to Y;$$

and morphisms

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\beta"'] & Y \arrow[d, "\gamma"] \\ X' \arrow[r, "\alpha'"] & Y' \end{tikzcd} \qquad \text{modulo} \qquad \begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\beta"'] & Y \arrow[d, "\alpha'\xi"] \arrow[dl, "\xi"'] \\ X' \arrow[r, "\alpha'"] & Y' \end{tikzcd}$$

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Fiat 2-categories

$\mathscr{C}$ — finitary 2-category

Definition. $\mathscr{C}$ is fiat (finitary - involution - adjunction - two category) provided that it has

- a weak object preserving involution $\star$;
- adjunction morphisms $F \circ F^\star \to \mathbb{1}_{\mathtt{i}}$ and $\mathbb{1}_{\mathtt{j}} \to F^\star \circ F$.

Examples.

- $\mathscr{S}_{(W,S)}$ is fiat;
- $\mathscr{P}_A$ is fiat iff A is self-injective and weakly symmetric;
- $\mathscr{G}_\Gamma$ is not fiat.

Fact. If $\mathscr{C}$ is fiat, then each left cell of $\mathscr{C}$ has a unique Duflo involution

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Cell 2-representation

$\mathscr{C}$ — fiat category;

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$\overline{\mathbb{P}}_{\mathtt{i}}$ — its abelianization

$G_{\mathcal{L}}$ – Duflo involution

$L_{G_{\mathcal{L}}}$ – the corresponding simple module in $\overline{\mathbb{P}}_{\mathtt{i}}(\mathtt{i})$

Theorem. $\mathcal{X} := \mathrm{add}\{F\, L_{G_{\mathcal{L}}} : F \in \mathcal{L}\}$ is closed under the action of $\mathscr{C}$

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$L_{G_{\mathcal{L}}}$ – the corresponding simple module in $\overline{\mathbb{P}}_{\mathtt{i}}(\mathtt{i})$

Theorem. $\mathcal{X} := \mathrm{add}\{F\, L_{G_{\mathcal{L}}} : F \in \mathcal{L}\}$ is closed under the action of $\mathscr{C}$

Definition. The cell 2-representation $\mathrm{C}_{\mathcal{L}}$ of $\mathscr{C}$ corresponding to $\mathcal{L}$ is the finitary 2-representation obtained by restricting the action of $\mathscr{C}$ to $\mathcal{X}$.

Properties of cell 2-representation

Assume:

- $\mathcal{J}$ be a 2-sided cell of $\mathscr{C}$;
- different left cells inside $\mathcal{J}$ are not comparable w.r.t. the left order;
- for any $\mathcal{L}, \mathcal{R} \subset \mathcal{J}$ we have $|\mathcal{L} \cap \mathcal{R}| = 1$;
- the function $F \mapsto m_F$, where $F^* \circ F = m_F H$ is constant on right cells of $\mathcal{J}$.

Theorem. [M.-Miemietz]

- For any two left cells $\mathcal{L}$ and $\mathcal{L}'$ of $\mathcal{J}$ the corresponding cell 2-representations $\mathbf{C}_{\mathcal{L}}$ and $\mathbf{C}_{\mathcal{L}'}$ are equivalent.
- $\mathrm{End}_{\mathscr{C}} \mathbf{C}_{\mathcal{L}} \cong \Bbbk\text{-mod}$.
- If $\mathscr{C}$ admits a positive grading, then the last technical assumption is redundant.

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- find most suitable setup;
- what are simple 2-representations?
- any Jordan-Hölder theory?
- Morita theory for abelian representations;
- general categorification algorithms;
- any homological methods for 2-representations?
- understand combinatorics of available examples;
- many more...

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THANK YOU!!!