Highest weight categories of Lie algebra modules

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Abstract

We study a certain class of categories of Lie algebra modules which includes the well-known categories $\mathcal{O}$ and $\mathcal{O}_S$. We show that all these categories are highest weight categories.

1 Introduction

Let $\mathfrak{L}$ be a complex simple finite-dimensional Lie algebra. It is well-known that the blocks of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ ([1]) of $\mathfrak{L}$-modules are highest weight categories [2]. This means that they are equivalent to the categories of modules for some finite-dimensional algebras which belong to the class of so-called quasi-hereditary algebras. The other known example of such a category whose blocks correspond to some quasi-hereditary algebras is a category $\mathcal{O}_S$ which was introduced in [10]. Both categories $\mathcal{O}$ and $\mathcal{O}_S$ consist of highest weight $\mathfrak{L}$-modules and their extensions.

A category $\mathcal{O}^\alpha$ of $\mathfrak{L}$-modules, which are torsion free for sl(2)–subalgebra corresponding to a simple root α, was studied in [3] and [6]. Clearly, the modules in $\mathcal{O}^\alpha$ have no highest weight. It was shown that there is a block decomposition of $\mathcal{O}^\alpha$ with each block corresponding to a quasi-hereditary algebra.

All categories mentioned above have the BGG duality between the indecomposable projectives, standard modules and the simples.

The main objective of this paper is to provide a general scheme for constructing categories of $\mathfrak{L}$-modules that lead to some quasi-hereditary algebras. We introduce a class of $\mathfrak{L}$-module categories and show that they are highest weight categories under a certain condition. The examples of such categories include the categories $\mathcal{O}$, $\mathcal{O}_S$, a subcategory of the category $\mathcal{O}^\alpha$, a certain category of Harish-Chandra modules ([12]) and a subcategory of Gelfand-Zetlin modules ([9]).

The structure of the paper is the following. In chapter 3 we discuss the admissible categories of modules for semisimple finite-dimensional Lie algebras. For a parabolic subalgebra $\mathcal{P}$ of $\mathfrak{L}$ and an admissible category Λ of modules for a semisimple part of the Levi factor of $\mathcal{P}$ we construct our main category $\mathcal{O}(\mathcal{P},\Lambda)$ of $\mathfrak{L}$-modules. In chapter 4 we prove that $\mathcal{O}(\mathcal{P},\Lambda)$ has enough projective objects. The main result of the paper is Theorem 3

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which shows that if $\mathcal{O}(\mathcal{P},\Lambda)$ has a block decomposition with finitely many simples in each block then those blocks are highest weight categories. The developed technique is applied to different examples in chapter 5.

2 Preliminaries

For a Lie algebra $\mathfrak{A}$ we will denote by $U(\mathfrak{A})$ the universal enveloping algebra of $\mathfrak{A}$ and by $Z(\mathfrak{A})$ the centre of $U(\mathfrak{A})$.

Let B be an abelian subalgebra of $\mathfrak{L}$. An $\mathfrak{L}$-module V is called a *weight* module (with respect to B) if

$$V = \oplus_{\lambda\in B^*} V_\lambda$$

where $V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in B\}$. Denote by $\operatorname{supp}_B V$ the set of all $\lambda \in B^*$ such that $V_\lambda \neq 0$.

Let $\mathfrak{H}$ be a Cartan subalgebra of $\mathfrak{L}$ and let $\mathcal{P}$ be a parabolic subalgebra of $\mathfrak{L}$ with the Levi decomposition $\mathcal{P} = (\mathfrak{A} \oplus \mathfrak{H}_\mathfrak{A}) \oplus \mathfrak{N}$ where $\mathfrak{A}$ is a semisimple Lie algebra, $\mathfrak{H}_\mathfrak{A} \subset \mathfrak{H}$, $[\mathfrak{A}, \mathfrak{H}_\mathfrak{A}] = 0$ and $\mathfrak{N}$ is nilpotent.

Let Δ be the root system of $\mathfrak{L}$, $\mathfrak{L} = \mathfrak{H} \oplus \sum_{\alpha\in\Delta} \mathfrak{L}_\alpha$ be the root decomposition of $\mathfrak{L}$ and $\mathfrak{N} = \sum_{\alpha\in\Delta(\mathfrak{N})} \mathfrak{L}_\alpha$ be the root decomposition of $\mathfrak{N}$. Denote by Q (respectively $Q_+^\mathfrak{A}$) a free abelian group (respectively monoid) generated by $\Delta(\mathfrak{N})$ and let $\tilde{Q}$ (respectively $\tilde{Q_+^\mathfrak{A}}$) be the restriction of Q (respectively $Q_+^\mathfrak{A}$) on $\mathfrak{H}_\mathfrak{A}$. Set $\mathfrak{N}_- = \sum_{\alpha\in-\Delta(\mathfrak{N})} \mathfrak{L}_\alpha$.

Let $\Omega_\mathcal{P}$ denote the set of representatives of the isomorphism classes of simple $\mathfrak{A} \oplus \mathfrak{H}_\mathfrak{A}$-modules. Since $\mathfrak{H}_\mathfrak{A}$ is abelian it act on any simple $V \in \Omega_\mathcal{P}$ via some $\lambda \in \mathfrak{H}_\mathfrak{A}^*$, i.e. $hv = \lambda(h)v$ for all $v \in V$ and $h \in \mathfrak{H}_\mathfrak{A}$. We define a partial order on $\Omega_\mathcal{P}$ as follows. Let $V_i \in \Omega_\mathcal{P}$ and $\mathfrak{H}_\mathfrak{A}$ acts on V_i via $\lambda_i \in \mathfrak{H}_\mathfrak{A}^*$, $i = 1, 2$. We say that $V_1 < V_2$ if $\lambda_1 = \lambda_2 - \varphi$ for some $\varphi \in \tilde{Q_+^\mathfrak{A}} \setminus \{0\}$.

Let $V \in \Omega_\mathcal{P}$. Then we can consider V as a $\mathcal{P}$-module with a trivial action of $\mathfrak{N}$ and construct an $\mathfrak{L}$-module

$$M_\mathcal{P}(V) = U(\mathfrak{L}) \otimes_{U(\mathcal{P})} V$$

which is called a *generalized Verma module.*

The main properties of generalized Verma modules are collected in the following proposition.

Proposition 1. *Let $\mathcal{P} = (\mathfrak{A} \oplus \mathfrak{H}_\mathfrak{A}) \oplus \mathfrak{N}$ be a parabolic subalgebra of $\mathfrak{L}$ and $V \in \Omega_\mathcal{P}$.*

1. *$M_\mathcal{P}$ is a free $U(\mathfrak{N}_-)$-module isomorphic to $U(\mathfrak{N}_-) \otimes V$ as a vector space;*
2. *$M_\mathcal{P}(V)$ is a weight module with respect to $\mathfrak{H}_\mathfrak{A}$ and $M_\mathcal{P}(V)_\lambda \simeq V$ where $\operatorname{supp}_{\mathfrak{H}_\mathfrak{A}} V = \{\lambda\}$;*
3. *$M_\mathcal{P}(V)$ has a unique maximal submodule;*

4. *Let W be an $\mathfrak{L}$-module generated by a simple $\mathcal{P}$-submodule V on which $\mathfrak{N}$ acts trivially. Then W is a homomorphic image of $M_{\mathcal{P}}(V)$.*

Proof. Follows from the construction of the module $M_{\mathcal{P}}(V)$ and the universal properties of the tensor product. □

We will denote by $L_{\mathcal{P}}(V)$ a unique irreducible quotient of $M_{\mathcal{P}}(V)$.

For a fixed basis S of the root system of $\mathfrak{A}$ one can consider the S-homomorphism φ_S, which Harish-Chandra defined in [5]. Since, in fact, φ_S does not depend on the choice of S (it depends on the triple $\mathfrak{L}, \mathfrak{H}, \mathfrak{A}$), we will call it a *generalized Harish-Chandra homomorphism.*

Let $S(\mathfrak{H}_{\mathfrak{A}})$ denote the symmetric algebra of $\mathfrak{H}_{\mathfrak{A}}$ and $K = Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}})$. Let $i : Z(\mathfrak{L}) \to K$ be the restriction of the generalized Harish-Chandra homomorphism on to $Z(\mathfrak{L})$. It induces a natural map $i^* : K^* \to Z(\mathfrak{L})^*$ and the cardinal $|(i^*)^{-1}(\theta)|$ is finite for any $\theta \in Z^*(\mathfrak{L})$.

A category Λ of Lie algebra modules is said to have a *block decomposition* if

$$\Lambda = \oplus_i \Lambda_i$$

is a direct sum of full subcategories Λ_i, each of which has only finitely many simple modules.

For a category Λ we denote by $\mathrm{Irr}(\Lambda)$ the set of isomorphism classes of simple objects in Λ.

Definition 1. *A $\mathfrak{L}$-module V is said to have a standard filtration if there exists a sequence*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_s = V$$

with $V_i/V_{i-1} \simeq M_{\mathcal{P}}(W_i)$ for some simple $\mathcal{P}$-module W_i.

Let D be a finite-dimensional algebra, $\mathrm{Mod}(D)$ the category of all finite-dimensional right D-modules and let S be a finite poset in bijective correspondence with the elements of $\mathrm{Irr}(\mathrm{Mod}(D))$. For each $s \in S$, denote by $L(s)$ a simple module from the isomorphism class, corresponding to s. For $V \in \mathrm{Mod}(D)$, $(V : L(s))$ denotes the multiplicity of $L(s)$ in a composition series of V.

Definition 2 ([7]). 1. *A choice of Verma modules for* $\mathrm{Mod}(D)$ *is a collection of modules $M(s)$, $s \in S$ such that $M(s)/\operatorname{rad} M(s) \simeq L(s)$, $(M(s) : L(s)) = 1$ and $(M(s) : L(t)) = 0$ unless $t \leq s$.*

2. *A Verma flag of a module V with respect to a given choice of Verma modules $\{M(s) : s \in S\}$ is a filtration*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_\tau = V$$

such that $V_i/V_{i-1} \simeq M(s_i)$ for some $s_i \in S$, $i = 1, 2, \ldots, \tau$.

Let $[V : M(s)]$ be the number of subquotients in a Verma flag of V which are isomorphic to $M(s)$.

Definition 3. *A category* $\mathrm{Mod}(D)$ *is called a highest weight category if*

1. *There is a choice of Verma modules* $\{M(s) \,|\, s \in S\}$ *for* $\mathrm{Mod}(D)$ *such that each projective indecomposable module has a Verma flag with respect to this choice.*

2. *Any module* V *with* $V/\operatorname{rad} V \simeq L(s)$ *and with other composition subquotients of the form* $L(t)$, $t \leq s$, *is a homomorphic image of* $M(s)$.

Definition 4. *Let* $\mathfrak{A}$ *be a Lie algebra and* $C(\mathfrak{A})$ *be a category of* $\mathfrak{A}$*-modules. A module* $M \in C(\mathfrak{A})$ *is called generic if* $M \otimes F$ *is completely reducible in* $C(\mathfrak{A})$ *for any finite-dimensional* $\mathfrak{A}$*-module* F.

Note that, for example, any module in the category of finite-dimensional modules for a semisimple complex Lie algebra is generic (by the Weyl theorem).

Lemma 1. *If* M *is a generic* $\mathfrak{A}$*-module and* F *is a finite-dimensional* $\mathfrak{A}$*-module then any submodule of* $M \otimes F$ *is generic.*

Proof. Follows from the associativity of the tensor product. □

Theorem 1 ([8]). *Let* V *be a simple* $\mathfrak{A}$*-module with infinitesimal character* $\chi = \chi_\lambda$, $\lambda \in \mathfrak{H}^*$ *and let* F *be a finite-dimensional* $\mathfrak{A}$*-module. Then for any* $z \in Z(\mathfrak{A})$ *and* $v \in V$

$$\prod_{\mu \in \mathrm{supp}_{\mathfrak{H}} V} (z - \chi_{\lambda+\mu}(z))v = 0.$$

Remark 1. *Suppose that for any simple* $X, Y \in C(\mathfrak{A})$, *a non-trivial extension of* X *by* Y *has no infinitesimal character. Then any module from* $C(\mathfrak{A})$ *in general position is generic. Indeed, it follows from Theorem 1 that for any* $V \in C(\mathfrak{A})$ *in general position and any finite-dimensional* $\mathfrak{A}$*-module* F, $Z(\mathfrak{A})$ *is diagonalizable on* $V \otimes F$. *Hence* $V \otimes F$ *is completely reducible by our assumption.*

3 Admissible categories of Lie algebra modules

Let $\mathfrak{A}$ be a semisimple complex finite-dimensional Lie algebra and let Λ be a category of $\mathfrak{A}$-modules.

Definition 5. *A category* Λ *is called admissible if the following conditions are satisfied:*

1. $\mathrm{Ext}^1_{\mathfrak{A}}(X, Y) = 0$ *for all non-isomorphic simple modules* X *and* Y *in* Λ.

2. *Any simple module* $X \in \Lambda$ *is generic in* Λ.

Example 1.

If Λ consists of all finite-dimensional $\mathfrak{A}$-modules then Λ is admissible (by the Weyl theorem).

Example 2.

Let $\mathfrak{A} = sl(2,\mathbb{C})$ with a standard basis $\{e,f,h\}$ and let $c = (h+1)^2 + 4fe$ be a Casimir element. Let Λ be a category of all the weight (with respect to $\mathbb{C}h$), torsion-free (i.e. e and f act injectively), $\mathfrak{A}$-modules. It is well-known that such simple modules are parametrized by pairs $(\tilde{\lambda},\gamma)$ where $\tilde{\lambda} \in \mathbb{C}/2\mathbb{Z}$ is the set of all eigenvalues of h, γ is the unique eigenvalue of c and $\gamma \neq (\lambda+1)^2$ for all $\lambda \in \tilde{\lambda}$.

Let X_1 and X_2 be simple modules in Λ parametrized by $(\tilde{\lambda}_1,\gamma_1)$ and $(\tilde{\lambda}_2,\gamma_2)$ respectively. Suppose that $\mathrm{Ext}^1(X_1,X_2) \neq 0$. Since c belongs to the centre of $U(\mathfrak{A})$ we immediately obtain that $\gamma_1 = \gamma_2$. Also note that if V is an indecomposable weight $\mathfrak{A}$-module then $\operatorname{supp} V \subset \tilde{\mu}$ for some $\tilde{\mu} \in \mathbb{C}/2\mathbb{Z}$ implying that $\tilde{\lambda}_1 = \tilde{\lambda}_2$ and $X_1 \simeq X_2$. Also, there is no non-trivial self-extensions of a simple module in Λ having an infinitesimal character.

It is easy to see now that a simple torsion free $\mathfrak{A}$-module V parametrized by $(\tilde{\lambda},\gamma)$ is generic if and only if its infinitesimal character does not appear among the infinitesimal characters of finite-dimensional $\mathfrak{A}$-modules, i.e. $\gamma \neq k^2$ for $k \in \mathbb{Z}$.

A finitely generated module in Λ will be called generic if all its simple subquotients are generic.

Let $\tilde{\Lambda}$ be a full subcategory of Λ consisting of all the generic modules. The discussion above immediately implies that $\tilde{\Lambda}$ is an admissible category.

Example 3.

Let $\mathfrak{A} = sl(2,\mathbb{C})$, $K = SO(2) \subset SL(2,\mathbb{R})$. Let Λ be a category of Harish-Chandra $(\mathfrak{A},K)$-modules. Choose the following basis in $\mathfrak{A}$:

$$\tilde{h} = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tilde{e} = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right],$$

$$\tilde{f} = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

and consider a Cartan subalgebra $\tilde{\mathfrak{H}} = \mathbb{C}\tilde{h}$. Let $\tilde{\Lambda}$ be the full subcategory of Λ consisting of torsion-free (with respect to the action of $\tilde{e}$ and $\tilde{f}$) modules. It is known ([12]) that $\tilde{\Lambda}$ coincides with the category of finitely-generated weight (with respect to $\tilde{\mathfrak{H}}$) torsion-free modules with integer weights. If V is a simple module in $\tilde{\Lambda}$ and γ is an eigenvalue of c on V then $\gamma \neq k^2$ for $k \in \mathbb{Z}$ implying that V is generic. Hence $\tilde{\Lambda}$ is an admissible category.

Let $\mathcal{P} = (\mathfrak{A} \oplus \mathfrak{H}_{\mathfrak{A}}) \oplus \mathfrak{N}$ be a parabolic subalgebra of $\mathfrak{L}$ with the Levi factor $\mathfrak{A} \oplus \mathfrak{H}_{\mathfrak{A}}$ where $\mathfrak{A}$ is semisimple.

Let Λ be an admissible category of $\mathfrak{A}$-modules. Denote by $\mathcal{O}(\mathcal{P},\Lambda)$ the full subcategory of the category of $\mathfrak{L}$-modules consisting of modules that are

1. finitely generated;

2. weight with respect to $\mathfrak{H}_{\mathfrak{A}}$;

3. completely reducible $\mathfrak{A}$-modules with simple submodules in Λ;

4. $\mathfrak{N}$-finite.

Proposition 2. *1. $\mathcal{O}(\mathcal{P},\Lambda)$ is closed under the operations of taking submodules, quotients and finite direct sums.*

2. Modules $M_{\mathcal{P}}(W)$ and $L_{\mathcal{P}}(W)$ are the objects of $\mathcal{O}(\mathcal{P},\Lambda)$ for any simple $W \in \Lambda$.

3. If V is a simple module in $\mathcal{O}(\mathcal{P},\Lambda)$ then $V \simeq L_{\mathcal{P}}(W)$ for some simple $W \in \Lambda$.

Proof. Statement (1) is obvious. To prove (2) it is enough to show that $M_{\mathcal{P}}(W)$ is a completely reducible $\mathfrak{A}$-module with simple $\mathfrak{A}$-submodules in Λ. This follows from the fact that $M_{\mathcal{P}}(W) \simeq U(\mathfrak{N}_-) \otimes W$ as a vector space by Proposition 1 and $U(\mathfrak{N}_-)$ is a direct sum of finite-dimensional $\mathfrak{A}$-modules with respect to the adjoint action. We conclude that $M_{\mathcal{P}}(W) \in \mathcal{O}(\mathcal{P},\Lambda)$ and also $L_{\mathcal{P}}(W) \in \mathcal{O}(\mathcal{P},\Lambda)$.

Let V be a simple module in $\mathcal{O}(\mathcal{P},\Lambda)$. Since V is $\mathfrak{N}$-finite and $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable there exists a non-zero element $v \in V$ such that $\mathfrak{N}v = 0$ and $hv = \lambda(h)v$ for all $h \in \mathfrak{H}_{\mathfrak{A}}$ and some $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$. Then $V_\lambda = U(\mathfrak{A})v$ and $\mathfrak{N}w = 0$ for any $w \in V_\lambda$ implying that V_λ is irreducible $\mathfrak{A}$-module and $V \simeq L_{\mathcal{P}}(V_\lambda)$ by Proposition 1. This completes the proof. □

Proposition 3. *If $\mathcal{O}(\mathcal{P},\Lambda)$ has a block decomposition*

$$\mathcal{O}(\mathcal{P},\Lambda) = \bigoplus_i \mathcal{O}_i$$

then every module in $\mathcal{O}(\mathcal{P},\Lambda)$ has finite length.

Proof. Let $V \in \mathcal{O}(\mathcal{P},\Lambda)$, $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$. Since V is finitely generated any $\mathfrak{A}$-module V_λ has finite length. We can assume that $V \in \mathcal{O}_i$. Since $\mathcal{O}_i$ has only finitely many simple objects we immediately conclude that V has finite length. □

Proposition 4. $\mathcal{O}(\mathcal{P},\Lambda) = \oplus_{\theta \in Z^*(\mathfrak{L})} \mathcal{O}(\mathcal{P},\Lambda)_\theta$*, where a full subcategory $\mathcal{O}(\mathcal{P},\Lambda)_\theta$ consists of modules with a generalized infinitesimal character θ.*

Proof. Follows from the fact that any module in $\mathcal{O}(\mathcal{P},\Lambda)$ is finitely generated and $\mathfrak{N}$-finite. □

4 Projective objects in $\mathcal{O}(\mathcal{P},\Lambda)$

¿From now on we will assume that the category $\mathcal{O}(\mathcal{P},\Lambda)$ has a block decomposition

$$\mathcal{O}(\mathcal{P},\Lambda) = \bigoplus_{j \in J} \mathcal{O}_j.$$

Consider a category Λ^0 of $\mathfrak{A} \oplus \mathfrak{H}_{\mathfrak{A}}$-modules which are $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable and belong to Λ as $\mathfrak{A}$-modules.

Proposition 5. *For any simple $V \in \Lambda^0$ there exists a $\mathfrak{L}$-module $P = P_V \in \mathcal{O}(\mathcal{P}, \Lambda)$ such that for every $M \in \mathcal{O}(\mathcal{P}, \Lambda)$ there is a canonical isomorphism between* $\mathrm{Hom}_{\mathfrak{L}}(P_V, M)$ *and* $\mathrm{Hom}_{\mathfrak{A} \oplus \mathfrak{H}_{\mathfrak{A}}}(V, M)$.

Proof. Since V is simple, $\mathfrak{H}_{\mathfrak{A}}$ acts on V by means of some $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$. Also note that $V \in \Lambda^0$ defines uniquely a simple module $L \in \mathcal{O}(\mathcal{P}, \Lambda)$ and hence a corresponding block $\mathcal{O}_j$ in $\mathcal{O}(\mathcal{P}, \Lambda)$ with finitely many simples. For a positive integer m denote by $U(\mathfrak{N})^{(m)}$ a subspace in $U(\mathfrak{N})$ spanned by all the monomials of length $\geq m$. Thus there is a non-negative integer k, depending only on V, such that for any $W \in \mathcal{O}_j$ and any $w \in W_\lambda$, $U(\mathfrak{N})^{(k)} w = 0$. Set

$$P_V = U(\mathfrak{L}) \bigotimes_{U(\mathcal{P})} \left((U(\mathfrak{N})/U(\mathfrak{N})^{(k)}) \otimes V \right).$$

Using the same arguments as in Proposition 2,(2) one can easily show that $P \in \mathcal{O}(\mathcal{P}, \Lambda)$.
The rest of the proof follows the general lines of the proof of Proposition 1 in [11]. □

Proposition 5 immediately implies:

Corollary 1. *For any simple $V \in \Lambda^0$, P_V is a projective object of $\mathcal{O}(\mathcal{P}, \Lambda)$.*

Theorem 2.

1. *Every object in $\mathcal{O}(\mathcal{P}, \Lambda)$ is a quotient of a projective in $\mathcal{O}(\mathcal{P}, \Lambda)$.*
2. *Every projective in $\mathcal{O}(\mathcal{P}, \Lambda)$ has a standard filtration.*
3. *There is a one-to-one correspondence between the simple objects in $\mathcal{O}(\mathcal{P}, \Lambda)$ and the indecomposable projectives in $\mathcal{O}(\mathcal{P}, \Lambda)$.*

Proof. The proof is analogous to the proofs of Corollary 3, Corollary 10 and Corollary 13 in [11]. □

We will denote by $I(V)$ the projective cover of a simple object $V \in \mathcal{O}(\mathcal{P}, \Lambda)$. Set

$$P_j = \sum_{V \in \mathrm{Irr}(\mathcal{O}_j)} I(V) \text{ and } R_j = \mathrm{Hom}_{\mathcal{O}(\mathcal{P},\Lambda)}(P_j, P_j).$$

It is well-known that there is a canonical equivalence between $\mathcal{O}_j$ and $\mathrm{Mod}(R_j)$.
For $j \in J$ denote by $\Omega_{\mathcal{P}}^j$ a subset in $\Omega_{\mathcal{P}}$ consisting of all those V that parametrize the simples in $\mathcal{O}_j$. The order on $\Omega_{\mathcal{P}}$ induces the structure of a poset on $\Omega_{\mathcal{P}}^j$.

Proposition 6. *$\{M_{\mathcal{P}}(V) : V \in \Omega_{\mathcal{P}}^j\}$ is a choice of Verma modules for* $\mathrm{Mod}(R_j)$.

Proof. Follows from the construction of $M(V)$. □

Remark 2.

1. *One can see that $\{L_{\mathcal{P}}(V) : V \in \Omega_{\mathcal{P}}^j\}$ is another choice of Verma modules for* $\mathrm{Mod}(R_j)$.
2. *A standard filtration for an indecomposable projective module P_W, $W \in \Omega_{\mathcal{P}}^j$ is a Verma flag with respect to $\{M_{\mathcal{P}}(V) : V \in \Omega_{\mathcal{P}}^j\}$.*

Theorem 3. *Let $\mathcal{P} \subset \mathfrak{L}$ be a parabolic subalgebra, $\mathfrak{A}$ a semisimple subalgebra of $\mathcal{P}$ and Λ an admissible category of $\mathfrak{A}$-modules. Suppose that*

$$\mathcal{O}(\mathcal{P}, \Lambda) = \oplus_{j \in J} \mathcal{O}_j$$

is a block decomposition. Then $\mathcal{O}_j$ is a highest weight category for any $j \in J$.

Proof. It follows from Proposition 6, Proposition 2,2 and Remark 2 that $\{M_{\mathcal{P}}(V) : V \in \Omega^j_{\mathcal{P}}\}$ is a choice of Verma modules for $\mathcal{O}_j$ and each projective indecomposable module has a Verma flag with respect to this choice.

Let $W \in \mathcal{O}_j$ be such that $W / \operatorname{rad} W \simeq L_{\mathcal{P}}(V)$ and if $L_{\mathcal{P}}(V')$ is a composition subquotient of W then $V' \leq V$. Since W is completely reducible as an $\mathfrak{A}$-module, it has an $\mathfrak{A} \oplus \mathfrak{H}_{\mathfrak{A}}$-submodule $\tilde{V}$ isomorphic to V. Suppose that $\mathfrak{H}_{\mathfrak{A}}$ acts on $\tilde{V}$ via $\lambda \in \mathfrak{H}^*_{\mathfrak{A}}$. Since W has a unique maximal submodule, it follows that $W = U(\mathfrak{L}) W_\lambda$ and $W_\lambda = \tilde{V}$. Hence W is a homomorphic image of $M_{\mathcal{P}}(V)$ by proposition 1.

We conclude that $\mathcal{O}_j$ is a highest weight category. □

Corollary 2. *If $\mathcal{O}(\mathcal{P}, \Lambda)$ is self-dual, i.e. if there is a contravariant, exact involutive functor on $\mathcal{O}(\mathcal{P}, \Lambda)$ which preserves the simple objects, then the BGG-duality holds in $\mathcal{O}(\mathcal{P}, \Lambda)$:*

$$[P_V : M(W)] = (M(W) : L(V))$$

for any simple modules V and W in Λ^0.

5 Examples

5.1 Category $\mathcal{O}$

If $\mathfrak{A} = 0$ then $\mathcal{P} = \mathfrak{H} \oplus \mathfrak{N}$ and $V \simeq \mathbb{C}$ for any $V \in \Omega_{\mathcal{P}}$. In this case $V = V_\lambda$ for some $\lambda \in \mathfrak{H}^*$ and $M_{\mathcal{P}}(V)$ is a Verma module of a highest weight λ. The category $\mathcal{O}(\mathcal{P}, \emptyset)$ coincides with a well-known category $\mathcal{O}$ which has a block decomposition

$$\mathcal{O} = \oplus_{\theta \in Z^*(\mathfrak{L})} \mathcal{O}_\theta$$

where each block $\mathcal{O}_\theta$ has no more non-isomorphic simple modules than the order of the Weyl group of $\mathfrak{L}$. We conclude, by Theorem 3, that $\mathcal{O}_\theta$ is a highest weight category for all $\theta \in Z^*(\mathfrak{L})$.

5.2 Category $\mathcal{O}_S$

Let $\mathcal{P} = (\mathfrak{A} \oplus \mathfrak{H}_{\mathfrak{A}}) \oplus \mathfrak{N}$ be a parabolic subalgebra of $\mathfrak{L}$. Suppose that $\mathfrak{A} \neq 0$ and consider a category Λ of finite-dimensional $\mathfrak{A}$-modules. It follows from the Weyl theorem that Λ is admissible.

The category $\mathcal{O}(\mathcal{P}, \Lambda)$, in this case, coincides with the category $\mathcal{O}_S$ ([10]) and is a subcategory of the category $\mathcal{O}$. Hence it has a block decomposition with blocks being highest weight categories by Theorem 3.

5.3 Category $\tilde{\mathcal{O}}^\alpha$

Let $\alpha \in \Delta$, and let $\mathfrak{A} \simeq sl(2)$ be a subalgebra of $\mathfrak{L}$ generated by $\mathfrak{L}_{\pm\alpha}$ and let $\mathcal{P} = (\mathfrak{A} \oplus \mathfrak{H}_{\mathfrak{A}}) \oplus \mathfrak{N}$ be a parabolic subalgebra of $\mathfrak{L}$.

Let $\tilde{\Lambda}$ be an admissible category of $\mathfrak{A}$-modules from Example 2. Consider a category $\tilde{\mathcal{O}}^\alpha = \mathcal{O}(\mathcal{P}, \Lambda)$. It follows from Proposition 4 that

$$\tilde{\mathcal{O}}^\alpha = \oplus_{\theta \in Z^*(\mathfrak{L})} \tilde{\mathcal{O}}^\alpha_\theta.$$

Note that all modules in $\tilde{\mathcal{O}}^\alpha$ are weight (with respect to $\mathfrak{H}$) modules. Fix $\tilde{\lambda} \in \mathfrak{H}^*/Q$ and consider a full subcategory $\tilde{\mathcal{O}}^\alpha_{\theta,\tilde{\lambda}}$ in $\tilde{\mathcal{O}}^\alpha_\theta$ consisting of modules V such that $\operatorname{supp}_{\mathfrak{H}} V \subset \tilde{\lambda}$. Applying the generalized Harish-Chandra homomorphism and using the description of simple modules in $\tilde{\Lambda}$ in Example 3, we obtain that

$$\tilde{\mathcal{O}}^\alpha = \oplus \tilde{\mathcal{O}}^\alpha_{\theta,\tilde{\lambda}}, \theta \in Z^*(\mathfrak{L}), \tilde{\lambda} \in \mathfrak{H}^*/Q$$

is a block decomposition. Hence each block $\tilde{\mathcal{O}}^\alpha_{\theta,\tilde{\lambda}}$ is a highest weight category.

Remark 3. *Note that the category $\tilde{\mathcal{O}}^\alpha$ is a full subcategory of $\mathcal{O}^\alpha$ ([6]).*

5.4 Category of Harish-Chandra modules

Let G be a linear reductive real Lie group and let K be a maximal compact subgroup in G. Denote by $\mathfrak{A}_0$ the corresponding Lie algebra of G and by $\mathfrak{A}$ its complexification.

Definition 6. 1. *An infinitesimal character $\chi = \chi(\lambda) \in Z^*$, where λ is a highest weight of Verma module, is called singular if $\lambda + \rho_{\mathfrak{A}}$ lies on the wall of a Weyl chamber (here $\rho_{\mathfrak{A}}$ denotes a half-sum of positive roots of $\mathfrak{A}$).*

2. *An infinitesimal character $\chi = \chi(\lambda) \in Z^*$, where λ is a highest weight of Verma module $M(\lambda)$, is called strongly non-singular if infinitesimal characters of all simple subquotients of $M(\lambda) \otimes F$ are non-singular for any finite-dimensional $\mathfrak{A}$-module F.*

Clearly, for any Harish-Chandra $(\mathfrak{A}, K)$-module in general position with infinitesimal character χ, χ is strongly non-singular. Moreover, for any two non-isomorphic such simples X and Y we have $\operatorname{Ext}^1_{\mathfrak{A}}(X, Y) = 0$. A Harish-Chandra module will be called strongly non-singular if all its simple subquotients have strongly non-singular infinitesimal characters. Since by [12, Lemma 9.5.2] there is no non-trivial extensions of a simple module X by itself having a non-singular infinitesimal character, we conclude, by Remark 1, that any Harish-Chandra strongly non-singular $(\mathfrak{A}, K)$-module in general position is generic.

Let Λ be an admissible category of all generic Harish-Chandra $(\mathfrak{A}, K)$-modules such that $\operatorname{Ext}^1_{\mathfrak{A}}(X, Y) = 0$ for all non-isomorphic simple modules X and Y in Λ.

Remark 4. *In the case when $\mathfrak{A} = sl(2)$, an admissible category constructed in Example 3 is a subcategory of Λ.*

Let $\mathcal{P} = (\mathfrak{A} \oplus \mathfrak{H}_{\mathfrak{A}}) \oplus \mathfrak{N}$ be a parabolic subalgebra of $\mathfrak{L}$ with Levi factor $\mathfrak{A} \oplus \mathfrak{H}_{\mathfrak{A}}$.

The category $\mathcal{O}(\mathcal{P}, \Lambda)$ has a decomposition $\mathcal{O}(\mathcal{P}, \Lambda) = \oplus_{\theta \in Z^*(\mathfrak{L})} \mathcal{O}(\mathcal{P}, \Lambda)_\theta$ by Proposition 4. Fix $\tilde{\lambda} \in \mathfrak{H}_{\mathfrak{A}}^*/\tilde{Q}$ and consider a full subcategory $\mathcal{O}(\mathcal{P}, \Lambda)_{\theta, \tilde{\lambda}}$ of $\mathcal{O}(\mathcal{P}, \Lambda)_\theta$ consisting of modules V such that $\operatorname{supp}_{\mathfrak{H}_{\mathfrak{A}}} V \subset \tilde{\lambda}$.

Applying the generalized Harish-Chandra homomorphism and using the fact that there exist only finitely many non-isomorphic simple modules in Λ with a given infinitesimal character ([12]), we conclude that $|\operatorname{Irr}(\mathcal{O}(\mathcal{P}, \Lambda)_{\theta, \tilde{\lambda}})| < \infty$ for all $\theta \in Z^*(\mathfrak{L})$ and $\tilde{\lambda} \in \mathfrak{H}_{\mathfrak{A}}^*/\tilde{Q}$. Hence

$$\mathcal{O}(\mathcal{P}, \Lambda) = \oplus \mathcal{O}(\mathcal{P}, \Lambda)_{\theta, \tilde{\lambda}}, \theta \in Z^*(\mathfrak{L}), \tilde{\lambda} \in \mathfrak{H}_{\mathfrak{A}}^*/\tilde{Q}$$

is a block decomposition and each block is a highest weight category by Theorem 3.

5.5 Category of Gelfand-Zetlin modules

Let $\mathfrak{A}' = gl(n, \mathbb{C})$. For $m = 1, \ldots, n$ let $\mathfrak{A}'_m = gl(m, \mathbb{C})$. Let U_m be the universal enveloping algebra of $\mathfrak{A}'_m$ and let Z_m be the centre of U_m. We identify $\mathfrak{A}'_m$, for $m = 1, \ldots, n$, with the Lie subalgebra of $\mathfrak{A}'$ generated by the matrix units $\{e_{ij};\ i, j = 1, \ldots, m\}$. Thus we have the inclusions

$$\mathfrak{A}'_1 \subset \mathfrak{A}'_2 \subset \ldots \mathfrak{A}'_n = \mathfrak{A}'$$

and

$$U_1 \subset U_2 \subset \ldots U_n = U(\mathfrak{A}').$$

Let Γ be a subalgebra of $U(\mathfrak{A}')$ generated by $\{Z_m;\ m = 1, \ldots, n\}$. This subalgebra is called the Gelfand-Zetlin subalgebra of $U(\mathfrak{A}')$ ([9]).

An $\mathfrak{A}'$-module V is called a Gelfand-Zetlin module provided it is a direct sum of finite-dimensional Γ-modules.

Let $\mathfrak{A} = sl(n, \mathbb{C}) \subset \mathfrak{A}'$, $\mathfrak{A}_k = sl(k, \mathbb{C}) \subset \mathfrak{A}'_k$, $k = 1, \ldots, n$. We define Gelfand-Zetlin $\mathfrak{A}$-modules as the restriction on $\mathfrak{A}$ of the Gelfand-Zetlin $\mathfrak{A}'$-modules.

In [9] a category Λ of strongly generic Gelfand-Zetlin modules was introduced. The basis of a strongly generic Gelfand-Zetlin module is given by the tuples $[l_{ij}] \in \mathbb{C}^{n(n+1)/2}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, i$ that satisfy the following conditions:

1. $l_{ij} - l_{ik} \notin \mathbb{Z}$ for all $i \in \{1, 2, \ldots, n\}$ and $j, k \in \{1, 2, \ldots, i\}$;
2. $l_{ij} - l_{i+1\,k} \notin \mathbb{Z}$ for all $i \in \{1, 2, \ldots, n-1\}$, $j \in \{1, 2, \ldots, i\}$ and $k \in \{1, 2, \ldots, i+1\}$.

For any two such non-isomorphic simple modules X and Y it follows that

$$\operatorname{Ext}^1_{\mathfrak{A}}(X, Y) = 0.$$

In fact, Λ has a block decomposition with a single simple module in each block.

Every simple module X is completely reducible as an $\mathfrak{A}_k$-module. Moreover any of its simple submodules is a strongly generic Gelfand-Zetlin $\mathfrak{A}_k$-module. It follows from Theorem 1 that the centre Z_k is diagonalizable on $X \otimes F$ for any finite-dimensional $\mathfrak{A}_k$-module F. This implies that $X \otimes F$ is a Gelfand-Zetlin module and thus is completely reducible. We obtain that any of our X is generic and so Λ is admissible.

Let $\mathcal{P}$ be a parabolic subalgebra of $\mathfrak{L}$ and $\mathcal{P} = (\mathfrak{A}' \oplus \mathfrak{H}_{\mathfrak{A}}) \oplus \mathfrak{N}$ where $\mathfrak{A} \oplus \mathfrak{H}_{\mathfrak{A}}$ is the Levi factor. Applying the generalized Harish-Chandra homomorphism and using similar arguments as in Sections 5.3 and 5.4 one can show that $\mathcal{O}(\mathcal{P}, \Lambda)$ has a block decomposition. It follows from Theorem 3 that any block of such a decomposition is a highest weight category.

Proposition 7. *$\mathcal{O}(\mathcal{P}, \Lambda)$ is self-dual.*

Proof. The proof goes along the lines of the proof of the corresponding result for weight modules in [6]. Let $V \in \mathcal{O}(\mathcal{P}, \Lambda)$ and consider V^* which has the canonical structure of a $\mathfrak{L}$–module. Then V^* contains a unique maximal Gelfand-Zetlin submodule $\tilde{V^*}$ and the correspondence $V \leftrightarrow \tilde{V^*}$ gives an exact contravariant functor from $\mathcal{O}(\mathcal{P}, \Lambda)$ to itself preserving the simples. □

It follows immediately from Proposition 7 and Corollary 2 that the BGG duality holds in the constructed categories of Gelfand-Zetlin modules.

Remark 5. *When $n = 2$ the category $\mathcal{O}(\mathcal{P}, \Lambda)$ coincides with the category from Section 5.3.*

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