Green's relations on $\mathcal{FP}^+(S_n)$

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1 Introduction

Description of Green's relations on the semigroup $\mathcal{B}(X)$ of all binary relations on a finite set X forms a classical part of the semigroup theory. This description was obtained and studied by several authors (see for example [PW, Z] and references therein). Recently, an approximative subsemigroup $\mathcal{FP}^+(S_n)$ in $\mathcal{B}(X)$ for $X = \{1, 2, \dots, n\}$ was introduced in [GM1] in a natural way. It was shown that this subsemigroup has a lot of nice properties ([GM2, GM3, M]), for example it inherits the property of $\mathcal{B}(X)$ to have only inner automorphisms. In this paper we study Green's relations on $\mathcal{FP}^+(S_n)$.

The paper is organized as follows: In Section 2 we collect all necessary preliminaries. In Section 3 we prove several technical lemmas necessary for the proof of our main result presented in Section 4. Finally in Section 5 we describe a lattice of a natural family of ideals in $\mathcal{FP}^+(S_n)$.

2 Preliminaries

For a fixed positive integer n let X denote an n-element set $\{1, 2, \dots, n\}$. Let S_n be the symmetric group on X. Consider the Boolean $\mathfrak{B}_n$ of S_n as a semigroup under natural operation induced from S_n and define an equivalence relation $\sim$ on $\mathfrak{B}_n$ as follows: for A_1 and A_2 from $\mathfrak{B}_n$ we set $A_1 \sim A_2$ if and only if for any $x \in X$ the sets $\{\sigma(x) : \sigma \in A_1\}$ and $\{\sigma(x) : \sigma \in A_2\}$ coincide. It is straightforward that $\sim$ is a well-defined congruence on $\mathfrak{B}_n$. The corresponding quotient semigroup $\mathfrak{B}_n/_\sim$ is called the factorpower of S_n and denoted by $\mathcal{FP}(S_n)$. The last semigroup has an empty set class as an outher zero element. Throwing this element away one obtains a semigroup $\mathcal{FP}^+(S_n)$ which we will also call the factorpower of S_n. In what follows we will consider the semigroup $\mathcal{FP}^+(S_n)$ only.

Let $\mathcal{B}(X)$ denote the semigroup of all binary relations on X. $\mathcal{FP}^+(S_n)$ can be identified with a subsemigroup of $\mathcal{B}(X)$ in a natural way. To each $A \in \mathcal{FP}^+(S_n)$ we associate a binary relation on X which consists of all pairs $(x, \sigma(x))$ where x runs throug X and σ runs throug A. One can show that this is in fact a monomorphism of semigroups. Thus, the elements of $\mathcal{FP}^+(S_n)$ can be written down as usual permutations:

$$A = \begin{pmatrix} 1 & 2 & \dots & n \\ A_1 & A_2 & \dots & A_n \end{pmatrix}$$

where $A \in \mathcal{FP}^+(S_n)$ and $A_x = \{\sigma(x) \,|\, \sigma \in A\}$ for all $x \in X$. Using these notations the elements of $\mathcal{FP}^+(S_n)$ can be multiplied also as permutations. Namely, for A and B in $\mathcal{FP}^+(S_n)$ one has

$$\begin{pmatrix} 1 & 2 & \dots & n \\ A_1 & A_2 & \dots & A_n \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & \dots & n \\ B_1 & B_2 & \dots & B_n \end{pmatrix} = \begin{pmatrix} 1 & 2 & \dots & n \\ \bigcup\limits_{x \in B_1} A_x & \bigcup\limits_{x \in B_2} A_x & \dots & \bigcup\limits_{x \in B_n} A_x \end{pmatrix}.$$

Consider a set $\mathfrak{D}$ consisting of all vectors $(l_1, l_2, \dots, l_n)$ with positive integer coefficients. For an element $l \in \mathfrak{D}$ and for $1 \leq i \leq n$ by l_i (or $(l)_i$) we will denote the i-th coefficient of l. There are two natural partial preorders on $\mathfrak{D}$. For l and m from $\mathfrak{D}$ we will write $l < m$ if $l_i \leq m_i$ for all i and we will write $l \prec m$ if there exist a permutation $\sigma \in S_n$ such that $l_i \leq m_{\sigma(i)}$ for all i. Clearly, $\prec$ is a partial preorder on $\mathfrak{D}$ and $<$ is a (reflexive) partial order on $\mathfrak{D}$. One also has that $l < m$ implies $l \prec m$.

For an element $A \in \mathcal{FP}^+(S_n)$ by its signature $\mathrm{Sgn}(A)$ we will mean an element $(|A_1|, |A_2|, \dots, |A_n|) \in \mathfrak{D}$.

3 Some technical lemmas

Lemma 1. *Let $A \in \mathcal{FP}^+(S_n)$ and $Y \subset X$ be a non-empty subset. Then*

$$\left| \bigcup_{y \in Y} A_y \right| \geq |Y|.$$

Proof. Let Z denote the union of all A_y, where y runs through Y. By the definition of $\mathcal{FP}^+(S_n)$, A is an equivalence class in $\mathfrak{B}$, which differs from empty set. Take any representative in A and choose a permutation $\sigma \in S_n$ in it. Clearly, Z contains as a subset the set $Z(\sigma) = \{\sigma(y) \,:\, y \in Y\}$. Since σ is a permutation, one has $|Z(\sigma)| = |Y|$. Hence $|Z| \geq |Y|$ as required. □

Lemma 2. *Suppose that $A, B, C \in \mathcal{FP}^+(S_n)$ such that $AB = C$. Then* $\mathrm{Sgn}(B) < \mathrm{Sgn}(C)$.

Proof. Fix $x \in X$. By definition of the multiplication in $\mathcal{FP}^+(S_n)$ we have

$$C_x = \bigcup_{y \in B_x} A_y.$$

From Lemma 1 it follows that $|C_x| \geq |B_x|$ and thus $\mathrm{Sgn}(B) < \mathrm{Sgn}(C)$ as required. □

Lemma 3. *Suppose that $A, B, C \in \mathcal{FP}^+(S_n)$ such that $AB = C$. Then* $\mathrm{Sgn}(A) \prec \mathrm{Sgn}(C)$.

Proof. Fix a representative of B in $\mathfrak{B}_n$ and choose a permutation σ in it. Fix $x \in X$. Since

$$C_x = \bigcup_{y \in B_x} A_y$$

and $\sigma(x) \in B_x$ it follows immediately that $|C_x| \geq |A_{\sigma(x)}|$. Hence $\mathrm{Sgn}(A) \prec \mathrm{Sgn}(C)$ as required. □

Corollary 1. *Suppose that* $A, B, C \in \mathcal{FP}^+(S_n)$ *such that* $AB = C$. *Then* $\mathrm{Sgn}(A) \prec \mathrm{Sgn}(C)$ *and* $\mathrm{Sgn}(B) \prec \mathrm{Sgn}(C)$.

4 Green's relations on $\mathcal{FP}^+(S_n)$

Theorem 1. *Let* $A, B \in \mathcal{FP}^+(S_n)$. *Then*

1. $A\mathcal{L}B$ *if and only if* $A = \sigma B$ *for some* $\sigma \in S_n$;
2. $A\mathcal{R}B$ *if and only if* $A = B\sigma$ *for some* $\sigma \in S_n$;
3. $A\mathcal{H}B$ *if and only if* $A = \sigma B = B\tau$ *for some* $\sigma, \tau \in S_n$;
4. $A\mathcal{D}B$ *if and only if* $A = \sigma B\tau$ *for some* $\sigma, \tau \in S_n$;
5. $\mathcal{J} = \mathcal{D}$.

Proof. Clearly, it is sufficient to prove only two first statements.

Suppose that $A = \sigma B$ ($A = B\sigma$) for some $\sigma \in S_n$. Clearly, this implies $A\mathcal{L}B$ ($A\mathcal{R}B$). So, now suppose that $A\mathcal{L}B$. This means that $A = XB$ and $B = YA$ for some $X, Y \in \mathcal{FP}^+(S_n)$. In particular, this implies $B \prec A$ and $A \prec B$, or, in other words, for any $1 \leq i \leq n$ we have

$$|\{x : |A_x| = i\}| = |\{x : |B_x| = i\}|.$$

Our goual is to show that one can choose $X \in S_n$.

We have $A = XB$ for some $X \in \mathcal{FP}^+(S_n)$. Choose a representative for X, say $\{\sigma_1, \dots, \sigma_k\} \subset S_n$ and consider the elements $A(j) = \{\sigma_1, \dots, \sigma_j\} \cdot B$ for $1 \leq j \leq k$. One has $A(k) = A$. From Corollary 1 we have $\mathrm{Sgn}(B) \prec \mathrm{Sgn}(A(j))$ for all j. Since

$$A(j+1)_t = A(j)_t \cup \bigcup_{f \in B_t} \{\sigma_{j+1}(f)\},$$

it follows that $\mathrm{Sgn}(A(j)) < \mathrm{Sgn}(A(j+1))$ for all j. But we recall that $\mathrm{Sgn}(A) \prec \mathrm{Sgn}(B)$, which implies that, in fact, $\mathrm{Sgn}(A(j)) = \mathrm{Sgn}(A(j+1))$ for all j, hence $A(j) = A(j+1)$ for all j and thus $A = A(1) = \sigma_1 \cdot B$. This completes the proof for the $\mathcal{L}$ relation.

Now suppose that $A\mathcal{R}B$. This means that $A = BX$ and $B = AY$ for some $X, Y \in \mathcal{FP}^+(S_n)$. In particular, this implies $B \prec A$ and $A \prec B$. Again we will show that one can choose $X \in S_n$.

We have $A = BX$ for some $X \in \mathcal{FP}^+(S_n)$. Choose a representative for X, say $\{\sigma_1, \dots, \sigma_k\} \subset S_n$ and consider the elements $A(j) = B \cdot \{\sigma_1, \dots, \sigma_j\}$ for $1 \leq j \leq k$. One has $A(k) = A$. From Corollary 1 we have $\mathrm{Sgn}(B) \prec \mathrm{Sgn}(A(j))$ for all j. Since

$$A(j+1)_t = A(j)_t \cup B_{\sigma_{j+1}(t)},$$

it follows that $\mathrm{Sgn}(A(j)) < \mathrm{Sgn}(A(j+1))$ for all j. At the same way as above we conclude that $\mathrm{Sgn}(A(j)) = \mathrm{Sgn}(A(j+1))$ for all j, hence $A(j) = A(j+1)$ for all j and thus $A = A(1) = B \cdot \sigma_1$. This completes the proof of the theorem. $\square$

5 Signature ideals in $\mathcal{FP}^+(S_n)$

The problem to describe the ideal structure of $\mathcal{FP}^+(S_n)$ is still open. Nevertheless, thechnical lemmas presented in Section 3 enables one to describe a natural family of ideals defined by using the notion of signature.

Set $\mathfrak{a} = (1, 1, \dots, 1) \in \mathfrak{D}$ and $\mathfrak{b} = (n, n, \dots, n) \in \mathfrak{D}$. Consider an interval $\mathfrak{D}\{\mathfrak{a}, \mathfrak{b}\} = [\mathfrak{a}, \mathfrak{b}]$ with respect to the preoder $\prec$. Let $\mathfrak{D}(\mathfrak{a}, \mathfrak{b})$ denote a subset of $\mathfrak{D}\{\mathfrak{a}, \mathfrak{b}\}$ which consists of all those $(l_1, l_2, \dots, l_n)$ such that

$$\max_{1 \le i \le n} l_i \le n - |\{x : l_i = 1\}|.$$

Let $\tilde{\mathfrak{D}}(\mathfrak{a}, \mathfrak{b})$ denote the poset associated with $\mathfrak{D}(\mathfrak{a}, \mathfrak{b})$ in which the induced relation $\prec$ becames a partial order. Let $\tilde{\mathfrak{D}}$ be a subset of $\mathfrak{D}$ consisting of all those vectors, whose coefficients do not decrease. Then $\tilde{\mathfrak{D}}$ is a poset with respect to $<$. One can easily show that the interval $[\mathfrak{a}, \mathfrak{B}]$ in $\tilde{\mathfrak{D}}$ is isomorphic to $\tilde{\mathfrak{D}}(\mathfrak{a}, \mathfrak{b})$.

For $l \in \mathfrak{D}$ let $I(l)$ denote the set of all elements $A \in \mathcal{FP}^+(S_n)$ such that $l \prec \mathrm{Sgn}(A)$. Clearly, $I(l)$ is not empty if and only if $l < \mathfrak{b}$ ($l \prec \mathfrak{b}$). We will call $I(l)$ the signature ideal corresponding to l. To proceed we need the following lemma:

Lemma 4. *For $l \in \mathfrak{D}$ there exist an element $A \in \mathcal{FP}^+(S_n)$ such that* $\mathrm{Sgn}(A) = l$ *if and only if $l \in \mathfrak{D}(\mathfrak{a}, \mathfrak{b})$.*

Proof. First we prove that $\mathrm{Sgn}(A) \in \mathfrak{D}(\mathfrak{a}, \mathfrak{b})$ for all $A \in \mathcal{FP}^+(S_n)$. Fix $A \in \mathcal{FP}^+(S_n)$. Clearly, $\mathrm{Sgn}(A) \in \mathfrak{D}\{\mathfrak{a}, \mathfrak{b}\}$. Let $1 \le x_1 < x_2 < \dots < x_k \le n$ be all indexes such that $|A_{x_j}| = 1$. It follows immediately, that

$$\bigcup_{y \in X \setminus \{x_1, x_2, \dots, x_k\}} A_y = X \setminus \left(\bigcup_{1 \le j \le k} A_{x_j} \right).$$

Clearly, $A_{x_i} \ne A_{x_j}$ if $i \ne j$. Hence $|A_y| \le n - k$ for any $y \in X \setminus \{x_1, x_2, \dots, x_k\}$. This implies that $\mathrm{Sgn}(A) \in \mathfrak{D}(\mathfrak{a}, \mathfrak{b})$.

Now we prove that for any $l \in \mathfrak{D}(\mathfrak{a}, \mathfrak{b})$ there exists an element $A \in \mathcal{FP}^+(S_n)$ such that $\mathrm{Sgn}(A) = l$. Clearly, we can assume that $\min_{1 \le i \le n} l_i \ge 2$, otherwise one can reduce the statement to the case of smaller n. Set

$$\tilde{A} = \begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ \{1,2\} & \{2,3\} & \{3,4\} & \dots & \{n-1,n\} & \{n,1\} \end{pmatrix}.$$

Clearly, $\tilde{A} \in \mathcal{FP}^+(S_n)$. Now it is enough to prove that for any

$$B = \begin{pmatrix} 1 & 2 & \dots & n \\ B_1 & B_2 & \dots & B_n \end{pmatrix}$$

such that $A_i \subset B_i$ for $1 \le i \le n$ the element B belongs to $\mathcal{FP}^+(S_n)$. Fix $x \in X$ and $y \ne x, x+1$ (here we set $n + 1 = 1$). Clearly, it is enough to show that there exists a permutation $\sigma \in S_n$ such that $\sigma(x) = y$ and $\sigma(i) \in A_i = \{i, i+1\}$ for $i \neq x$. We have $\sigma(x) = y$. Set $\sigma(y) = y+1$, $\sigma(y+1) = y+2$ and so on till $\sigma(x-1) = x$. Also set $\sigma(y-1) = y-1$, $\sigma(y-2) = y-2$ and so on till $\sigma(x+1) = x+1$. Obviously, σ is a permutation. This completes the proof. □

Theorem 2. *1. $I(l)$ is a two-sided ideal of $\mathcal{FP}^+(S_n)$.*

2. Signature ideals form a lattice which is isomorphic to $\tilde{\mathfrak{D}}(\mathfrak{a}, \mathfrak{b})$.

Proof. The first statement follows immediately form Corollary 1.

By virtue of Lemma 4, to prove the rest we first note that $\tilde{\mathfrak{D}}(\mathfrak{a}, \mathfrak{b})$ is in fact a lattice. Indeed, let $l, m \in \tilde{\mathfrak{D}}(\mathfrak{a}, \mathfrak{b})$. Using the isomorphism mentioned above we can assume that the coefficients of l and m do not decrease. Obviously, in this case

$$\min(l, m) = (\min(l_1, m_1), \min(l_2, m_2), \dots, \min(l_n, m_n))$$

and

$$\max(l, m) = (\max(l_1, m_1), \max(l_2, m_2), \dots, \max(l_n, m_n)).$$

This observation and Corollary 1 imply the second statement of our theorem. □

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