σ-unitarizability of simple weight modules

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Abstract

We prove that any simple weight modules with finite-dimensional weight spaces over a semi-simple complex finite-dimensional Lie algebra is unitarizable with respect to a non-degenerate symmetric inner product in the case of trivial involution. In the case of non-trivial involution σ we classify all σ-unitarizable simple weight modules with finite-dimensional weight spaces.

1 Origins

A principal step in the theory of $*$-representations of $*$-algebras is to come from the representation theory in spaces with positively defined inner product (or metric), see [OS], to the representation theory in spaces with indefinite metric, see [KS]. In particular, this usually drastically increases the complexity of the problem and also involves the study of non-simple modules, the last problem being considered as very difficult (wild) in a lot of cases. In [MT] the authors proposed an easy algebraic criterion to distinguish those representations of $*$-algebras which can be endowed with some indefinite inner product and proved the uniqueness of this inner product up to a natural equivalence. The method we used there works for finite-dimensional representations and for weight representations with finite-dimensional weight spaces with respect to a $*$-stable subalgebra.

One of the most natural and extensively studied examples of the situation, which is described above, is the case of weight (with respect to a Cartan subalgebra) modules with finite-dimensional weight spaces over a semi-simple complex finite-dimensional Lie algebra. In this paper, basing on the recent classification of simple weight modules with finite-dimensional weight spaces, [F, M], we classify all such modules admitting an indefinite inner product.

2 Main results

Let $\mathfrak{g}$ denote a semi-simple complex finite-dimensional Lie algebra; $\mathfrak{h}$ its fixed Cartan subalgebra; Δ the corresponding root system; π some fixed basis of Δ; X_α, $\alpha \in \Delta$, H_α,

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$\alpha \in \pi$, some fixed Weyl-Chevalley basis of $\mathfrak{g}$ and $*$ the Chevalley involution on $\mathfrak{g}$ which can be naturally extended to the involution on the universal enveloping algebra $U(\mathfrak{g})$. Let M be a simple weight (with respect to $\mathfrak{h}$) $U(\mathfrak{g})$-module with finite-dimensional weight spaces. Our first result in this paper is.

Theorem 1. *There exists a non-degenerate, symmetric inner product, $(\cdot,\cdot)$, on M such that $(uv, w) = (v, u^*w)$ for all $v, w \in M$ and all $u \in U(\mathfrak{g})$. It is unique up to a scalar multiple.*

Denote by σ an automorphism of $\mathbb{C}$ of order 2 (e.g. the complex conjugation). A form, $(\cdot,\cdot)$, on a complex vector space, V, is called a *σ-form* if it is linear with respect to the first argument and σ-linear with respect to the second one. A σ-form, $(\cdot,\cdot)$, is called *σ-symmetric* provided $(v, w) = \sigma((w, v))$ for any $v, w \in V$. Our second result is the following (here we assume $*$ to be σ-linear).

Theorem 2. *There exists a non-degenerate, σ-symmetric σ-form, $(\cdot,\cdot)$, on M such that $(uv, w) = (v, u^*w)$ for all $v, w \in M$ and all $u \in U(\mathfrak{g})$ if and only if the support and the central character of M are stable under σ. This σ-form is unique up to a scalar multiple.*

3 Proof of main results

First we remark that, as any simple $U(\mathfrak{g})$-module has only scalar endomorphisms ([D, Proposition 2.6.8]), the uniqueness will follow from [MT, Section 4] provided that the existence is proved.

Denote by $M^{\#}$ the *finitistic dual* to M, which is constructed as follows: we fix a weight basis, $\{v_i, i \in I\}$, of M and define $M^{\#}$ to be generated by the dual basis $\{v_i^{\#}, i \in I\}$ (here $v_i^{\#}$ is a σ-linear map from M to $\mathbb{C}$ such that $v_i^{\#}(v_j) = \delta_{i,j}$) with the action of $U(\mathfrak{g})$ given by $(u \cdot f)(m) = f(u^*m)$, $u \in U(\mathfrak{g})$, $m \in M$, $f \in M^{\#}$. According to [MT, Theorem 1] to prove our results it is sufficient to show that $M \simeq M^{\#}$ (this will guarantee the existence of a bilinear from, which, however can be antisymmetric for trivial σ), and, in the case of trivial σ, that there exists an element of non-zero length in M. The necessity of the condition of Theorem 2 is obvious, so in this case we will prove only the "if" part.

We start with Theorem 1. So, we have to prove that $M \simeq M^{\#}$ for any simple weight M with finite-dimensional weight spaces. As a first step we claim that it is enough to consider the case of *dense* (or cuspidal) module M (i.e. all X_α, $\alpha \in \Delta$ act bijectively on M). Indeed, according to the Fernando Theorem ([F]) any simple weight module with finite-dimensional weight spaces is either dense or a simple quotients of a generalized Verma module induced from a simple dense module over a parabolic subalgebra. As in the case of trivial σ the weight lattice is obviously preserved, everything reduces to the dense module, from which the generalized Verma module was induced.

Now we recall that the classification of simple dense modules with finite-dimensional weight spaces was recently completed in [M]. According to this classification such modules are parametrized by certain simple highest weight modules and certain cosets in $\mathfrak{h}^*$ (weight lattices). In the case of trivial σ both parameters are naturally preserved which gives us necessary isomorphism $M \simeq M^{\#}$.

The existence of an element of non-zero length can be checked on any $\mathfrak{sl}(2,\mathbb{C})$-submodule of M, where it follows from an easy $\mathfrak{sl}(2,\mathbb{C})$-computation.

Now we prove Theorem 2 or, more precisely, we show that $M \simeq M^{\#}$ provided that its support and central character are stable under σ. As above the question naturally reduces to the case of dense modules. Denote by $L(\lambda)$ a simple weight module, associated with M, see [M]. According to [M, Proposition 4.8], $L(\lambda)$ defines a coherent family, $\mathcal{EXT}(L(\lambda))$, whose simple submodules have the same central character as M, the last being stable under σ. Now the dual family $\mathcal{EXT}(L(\lambda))^{\#}$ is also a coherent and equals $\mathcal{EXT}(L(\lambda)^{\#})$. If λ is integral or half-integral, we obviously have $L(\lambda) \simeq L(\lambda)^{\#}$, that is $\mathcal{EXT}(L(\lambda)) \cong \mathcal{EXT}(L(\lambda))^{\#}$. The only case left is the one considered in [M, Lemma 8.3(iii)]. However, in this case $\mathcal{EXT}(L(\lambda)) \cong \mathcal{EXT}(L(\lambda))^{\#}$ follows from [M, Lemma 8.3(iii)] and [M, Theorem 8.6]. Finally, since any module of a coherent family is uniquely determined by its support, which is preserved by σ by assumptions, we deduce that $M \simeq M^{\#}$.

4 Remarks about Gelfand-Zetlin modules

Let $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$ and Γ be the Gelfand-Zetlin subalgebra of $U(\mathfrak{g})$ (see [DFO, O]). Γ is commutative and is generated by elements, stable with respect to $*$. Hence, if χ is a character of Γ such that there exists precisely one simple $U(\mathfrak{g})$-module $M(\chi)$, extending χ (some sufficient conditions for this can be found in [O, Corollary 5]), the same arguments as in Section 3 show that the module $M(\chi)$ will be isomorphic to $M(\chi)^{\#}$ if and only if its Γ-support is stable with respect to σ. This proves

Theorem 3. *Assume that χ satisfies the condition above. Then $M(\chi)$ admits a non-degenerate σ-symmetric inner product, which is unique up to a scalar multiple.*

In particular, all generic Gelfand-Zetlin modules, [MO], and simple quotients of generalized Verma modules induced from generic Gelfand-Zetlin modules do admit a non-degenerate σ-symmetric inner product for trivial σ.

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