Involutions on generalized Weyl algebras preserving the principal grading

Volodymyr Mazorchuk and Lyudmyla Turowska

Abstract

We classify all $*$-representations of generalized Weyl algebras by bounded operators with respect to $*$ preserving the principal grading.

Keywords: generalized Weyl algebra, involution, bounded operator, unitary equivalence, grading, tensor product.

1 Introduction and preliminaries

Fix a positive integer, n, and set $N_n = \{1, 2, \ldots, n\}$. Let R be a commutative unital complex algebra, σ_i, $i \in N_n$, a set of pairwise commuting automorphisms of R and $0 \neq t_i \in R$, $i \in N_n$, such that $\sigma_j(t_i) = t_i$, $i \neq j \in N_n$. The *generalized Weyl algebra* $A = A(R, \{\sigma_i\}, \{t_i\})$ is defined as an R-algebra, generated over R by elements X_i, Y_i, $i \in N_n$, subject to the following relations ([1]):

- $X_i r = \sigma_i(r) X_i$, $i \in N_n$, $r \in R$;

- $r Y_i = Y_i \sigma_i(r)$, $i \in N_n$, $r \in R$;

- $Y_i X_i = t_i$, $X_i Y_i = \sigma_i(t_i)$, $i \in N_n$;

- $X_i Y_j = Y_j X_i$, $X_i X_j = X_j X_i$, $Y_i Y_j = Y_j Y_i$, $i \neq j \in N_n$.

We will call these relations the *defining relations*.

Let $\mathbb{Z}^n$ be a free abelian group of rank n with canonical generators e_i, $i \in N_n$. Then A has a natural $\mathbb{Z}^n$-grading, which we will call *principal*, by assigning $\deg(R) = 0$, $\deg(X_i) = e_i$, $\deg(Y_i) = -e_i$, $i \in N_n$. It is well known that many classical algebras can be realized as generalized Weyl algebras (e.g. see examples in [1]). Representations of generalized Weyl algebras have been studied intensively (see e.g. [1, 2] and references therein), in particular, it has been shown that the corresponding classification problem is usually wild.

A natural problem for generalized Weyl algebras is to study involutions over them and $*$-representations of the corresponding $*$-algebras. We remark that by an involution on A we mean a $\mathbb{C}$-skewlinear map, $*$, from A to A satisfying $(ab)^* = b^* a^*$, $a, b \in A$, and

$(a^*)^* = a$, $a \in A$. For the involutions satisfying $(X_i)^* = \pm Y_i$ the corresponding problem was solved in [3] for a much wider class of algebras. In the present paper we will be interested in involutions preserving the principal grading on A, i.e. $(A_g)^* = A_g$ for any $g \in \mathbb{Z}^n$. We give a complete answer to the classification problem, listing all irreducible $*$-representations up to unitary equivalence.

In Section 2 we introduce the involutions on A, which we will be interested in and show that these involutions exhaust all involutions preserving the principal grading provided R is a domain. In Section 3 we deal with the case $n = 1$ and in Section 4 we consider the general situation. The main result is a classification of all $*$-representations of A up to unitary equivalence. Our answer for the general case uses the classification obtained for $n = 1$.

2 Involutions

If $*$ is an involution on A preserving the principal grading then $(A_0 = R)^* = A_0 = R$ and hence it induces an involution on the algebra R. It is well known ([1]) that the elements $Z_1^{k_1} Z_2^{k_2} \ldots Z_n^{k_n}$, $Z_i = X_i$ or $Z_i = Y_i$, form a basis of A as a free left (or right) R-module. Clearly, this basis is graded, hence all graded components of A are free R-modules of rank 1. In particular, if $*$ preserves the principal grading, then $(X_i)^* = r_i X_i$ and $(Y_i)^* = s_i Y_i$, $i \in N_n$, for some $r_i, s_i \in R$.

Theorem 2.1. *Let $*$ be an involution on R and $r_i, s_i \in R$, $i \in N_n$. If the following conditions are satisfied:*

1. $\sigma_i^{-1}(r^*) = \sigma_i(r)^*$ *for all $i \in N_n$, $r \in R$;*

2. $r_i \sigma_i(r_i^*) = 1$ *and* $s_i \sigma^{-1}(s_i^*) = 1$, $i \in N_n$.

3. $r_i \sigma_i(s_i) \sigma_i(t_i) = t_i^*$, $i \in N_n$;

4. $r_j \sigma_j(r_i) = r_i \sigma_i(r_j)$, $i, j \in N_n$;

5. $s_j \sigma_j^{-1}(s_i) = s_i \sigma_i^{-1}(s_j)$, $i, j \in N_n$;

6. $s_j \sigma_j^{-1}(r_i) = r_i \sigma_i^{-1}(s_j)$, $i \neq j \in N_n$.

then setting $(X_i)^* = r_i X_i$, $(Y_i)^* = s_i Y_i$, $i \in N_n$, defines an involution on A. Moreover, if R is a domain and $*$ is an involution on A then the restriction of $*$ to R satisfies all conditions which are listed above.

Proof. First we note that $r_i \sigma_i(r_i^*) = 1$ and $s_i \sigma^{-1}(s_i^*) = 1$, $i \in N_n$, is equivalent to $(a^*)^* = a$ for all $a \in A$.

Assume that $*$ is an involution on R satisfying the conditions of the theorem. We have to show that the defining relations are stable under the $*$ defined by $(X_i)^* = r_i X_i$, $(Y_i)^* = s_i Y_i$, $i \in N_n$. We will check the relation $X_i r = \sigma_i(r) X_i$. Applying $*$ we get $r^* r_i X_i = r_i X_i \sigma_i(r)^*$

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which can be rewritten as $r^* r_i X_i = r_i \sigma_i (\sigma_i(r)^*) X_i$. The last is true because of the first condition of the theorem. All other relations can be checked by analogous arguments.

Conversely, we assume that R is a domain. Then, if $*$ is an involution on A such that $(X_i)^* = r_i X_i$, $(Y_i)^* = s_i Y_i$, $i \in N_n$, the arguments above imply $r^* r_i X_i = r_i \sigma_i(\sigma_i(r)^*) X_i$. As A is a free R module and R does not have zero divisors, this equality is equivalent to $r^* = \sigma_i(\sigma_i(r)^*)$, which is the first condition of the theorem. Analogous arguments applied to other relations will give us the rest. This completes the proof. $\square$

We have to remark that such involutions really exist. For example, taking $s_i = r_i = 1$, $i \in N_n$, we reduce the list of conditions of our theorem to the following: $\sigma_i^{-1}(r^*) = \sigma_i(r)^*$, $r \in R$, and $\sigma_i(t_i) = t_i^*$, $i \in N_n$. One can easily construct a plenty of examples of such involutions. Below we present some classical examples as well as some new ones.

Example 1. Let $n = 1$, $R = \mathbb{C}[H, T]$, $t_1 = T$, $\sigma_1(H) = H - 1$, $\sigma_1(T) = T + H$, $H^* = H$, $T^* = T + H$. Then all the conditions of Theorem 2.1 are satisfied and $X^* = X$, $Y^* = Y$ defines an involution on A. In this case $A \simeq U(\mathfrak{sl}(2, \mathbb{C}))$ and $(A, *)$ corresponds to the real form $\mathfrak{sl}(2, \mathbb{R})$.

Example 2. Let $n = 1$, $q \in \mathbb{R}$ (resp. $q \in \mathbb{C}$, $|q| = 1$), $R = \mathbb{C}[T, k, k^{-1}]$, $t_1 = T$, $\sigma_1(k) = q^{-1}k$, $\sigma_1(T) = T + \dfrac{k^2 - k^{-2}}{q - q^{-1}}$, $k^* = k^{-1}$ (resp. $k^* = k$), $T^* = T + \dfrac{k^2 - k^{-2}}{q - q^{-1}}$. Then setting $X^* = X$, $Y^* = Y$ defines an involution on A. In this case $A \simeq U_q(\mathfrak{sl}(2, \mathbb{C}))$ and $(A, *)$ corresponds to the real form $\mathfrak{sl}_q(2, \mathbb{R})$.

Example 3. Let $n = 1$, $q \in \mathbb{R}$ (resp. $q \in \mathbb{C}$, $|q| = 1$), $R = \mathbb{C}[T, k, k^{-1}]$, $t_1 = T$, $\sigma_1(k) = q^{-2}k$, $\sigma_1(T) = T + \dfrac{k - k^{-1}}{q - q^{-1}}$, $k^* = k^{-1}$ (resp. $k^* = k$), $T^* = T + \dfrac{k - k^{-1}}{q - q^{-1}}$. If we put $X^* = q^{-1}kX$, $Y^* = qYk^{-1}$, we get an involution on A. In this case $A \simeq U_q'(\mathfrak{sl}(2, \mathbb{C}))$ and $(A, *)$ corresponds to the real form $\mathfrak{sl}_q'(2, \mathbb{R})$. We remark that the corresponding quantum algebra is slightly different from the one, considered in the previous example.

Example 4. Let $n = 1$, $R = \mathbb{C}[H]$, $\sigma_1(H) = H + 1$, $H^* = -H$, $t_1 = f(H^2 - H)$, where f is a fixed polynomial with real coefficients. Then it follows from Theorem 2.1 that $X^* = X$, $Y^* = Y$ defines an involution on A.

As an example of higher rank one can take, say, a tensor product of some rank one examples. In the rest of the paper we will assume that the conditions of Theorem 2.1 are satisfied.

3   Rank one case

In this section we will deal with the case $n = 1$ and set $\sigma = \sigma_1$, $t = t_1$, $X = X_1$ and $Y = Y_1$, $r = r_1$, $s = s_1$. Let $\pi : A \to \mathcal{B}(H)$ be a $*$-representation of A by bounded linear operators on a separable Hilbert space, H. To simplify the notation we will denote the operators $\pi(a)$, $a \in A$, simply by a and hence all equalities below will be operator equalities in $\mathcal{B}(H)$.

Lemma 3.1. *The elements X^2 and Y^2 commute with all $a \in A$.*

Proof. We will prove the statement for X^2, the case of Y^2 is analogous. We start with elements from R. As R is commutative any $h \in R$ is normal. Write $h = h_1 + ih_2$, where $h_1 = (h + h^*)/2$, $h_2 = (h - h^*)/(2i)$. Then the operators h_1, h_2 are self-adjoint. For $i = 1, 2$ we have $Xh_i = \sigma(h_i)X$. Applying the Fuglede-Putnam-Rosenblum Theorem ([4, Theorem 12.16]) we get $Xh_i^* = Xh_i = \sigma(h_i)^*X$. Hence $\sigma(h_i)X = \sigma(h_i)^*X$. Using $\sigma(h_i)^* = \sigma^{-1}(h_i^*) = \sigma^{-1}(h_i)$ we get $(\sigma(h_i) - \sigma^{-1}(h_i))X = 0$, which implies $(\sigma(h) - \sigma^{-1}(h))X = 0$ for any $h \in R$. Substituting h with $\sigma(h)$ we get $(\sigma^2(h) - h)X = 0$ and $(\sigma^2(h) - h)X^2 = 0$. The last is equivalent to $hX^2 = X^2h$.

For $a = X$ the statement is clear. The only case left is $a = Y$. Using the equality $(\sigma^2(h) - h)X = 0$ obtained above, we have $X^2Y = X\sigma(t) = \sigma^2(t)X = tX = YX^2$. $\square$

Lemma 3.2. *For any $h \in R$ the elements $h\sigma(h)$ and $h + \sigma(h)$ commute with all $a \in A$.*

Proof. Using $(\sigma^2(h) - h)X = 0$ we have that $X\sigma(h)h = \sigma^2(h)\sigma(h)X = h\sigma(h)X$ and analogously $h\sigma(h)Y = Yh\sigma(h)$, $(h+\sigma(h))X = X(h+\sigma(h))$ and $(h+\sigma(h))Y = Y(h+\sigma(h))$, completing the proof. $\square$

Lemma 3.3. *Assume that π is irreducible. Then the spectrum of all elements $h \in R$ is discrete, in particular, there exists a non-zero common (for R) eigenvector $v \in H$.*

Proof. Assume that π is irreducible. Fix $h \in R$. As $h\sigma(h)$ and $h + \sigma(h)$ commute with the action of A and π is irreducible these operators act as scalars, c_1 and c_2 correspondingly, on H. Hence $h^2 - c_2 h + c_1 = 0$, which implies our statement. $\square$

Lemma 3.4. *If $X^2 \neq 0$ then H coincides with the linear span of v and Xv. Otherwise, H coincides with the linear span of v and Yv. In particular, H is at most two-dimensional.*

Proof. As X^2 is central, $X^2 = cI$, where I is the identity operator. First assume $c \neq 0$. Then the linear span L of v and Xv is invariant with respect to R and X. We have $Y(av + bXv) = c^{-1}YX^2(av + bXv) = c^{-1}tX(av + bXv) \in L$. This proves the first statement.

If $c = 0$ we have $X^2 = 0$, in particular, $X^*X = rX^2 = 0$ which implies $X = 0$. Now if $Y^2 = 0$ then $Y = 0$ by the same arguments and the statement is clear. Otherwise, the second statement follows by the similar arguments as the first one. $\square$

To state our main classification result in the rank one case we introduce two sets of parameters, Γ_1 and Γ_2. Let $\hat{R}$ denote the set of all characters of R. Put

$$\Gamma_1 = \{\chi \in \hat{R} \,|\, \chi(h) = \chi(\sigma^2(h)) \text{ for all } h \in R, \chi(h) \neq \chi(\sigma(h)) \text{ for some } h\};$$

$$\Gamma_2 = \{\chi \in \hat{R} \,|\, \chi(h) = \chi(\sigma(h)) \text{ for all } h \in R\}.$$

Theorem 3.1. *Any irreducible representation of A is one or two-dimensional and unitarily equivalent to one of the following representations:*

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1. $\pi_0(\chi)$: $X = Y = 0$, $h = \chi(h)$, where $\chi \in \hat{R}$ such that $\chi(t) = \chi(\sigma(t)) = 0$;

2. $\pi_1(\chi, c, d)$: $X = c$, $Y = d$, $h = \chi(h)$, where $\chi \in \Gamma_2$, $c, d \in \mathbb{C}$ such that $cd = \chi(t)$, $|c| + |d| \neq 0$, $\chi(r)c = \bar{c}$, $\chi(s)d = \bar{d}$;

3. $\pi_2(\chi, d)$:

$$X = 0; \quad Y = \begin{pmatrix} 0 & \sqrt{d/\chi(s)} \\ \sqrt{d\chi(s)} & 0 \end{pmatrix}; \quad h = \begin{pmatrix} \chi(h) & 0 \\ 0 & \chi(\sigma(h)) \end{pmatrix},$$

where $\chi \in \Gamma_1$, $d \in \mathbb{C}$ such that $\chi(t) = \chi(\sigma(t)) = 0$, $\chi(s)d > 0$.

4. $\pi_3(\chi, c)$:

$$X = \begin{pmatrix} 0 & \sqrt{c/\chi(r)} \\ \sqrt{c\chi(r)} & 0 \end{pmatrix}; \quad Y = 0; \quad h = \begin{pmatrix} \chi(h) & 0 \\ 0 & \chi(\sigma(h)) \end{pmatrix},$$

where $\chi \in \Gamma_1$, $c \in \mathbb{C}$ such that $\chi(t) = \chi(\sigma(t)) = 0$, $\chi(r)c > 0$.

5. $\pi_4(\chi, c, d)$:

$$X = \begin{pmatrix} 0 & \sqrt{c/\chi(r)} \\ \sqrt{c\chi(r)} & 0 \end{pmatrix}; \quad Y = \begin{pmatrix} 0 & \chi(t)/\sqrt{c/\chi(r)} \\ \chi(\sigma(t))\sqrt{\chi(r)/c} & 0 \end{pmatrix};$$

$$h = \begin{pmatrix} \chi(h) & 0 \\ 0 & \chi(\sigma(h)) \end{pmatrix},$$

where $\chi \in \Gamma_1$, $c, d \in \mathbb{C}$, such that $\chi(r)c > 0$, $\chi(s)d > 0$, $\chi(t\sigma(t)) = cd$.

The representations $\pi_i(\chi_1, c_1, d_1)$ and $\pi_j(\chi_2, c_2, d_2)$ (the presence of c_i and d_i depends on i, j) are unitarily equivalent if and only one of the following holds:

1. $i = j = 1$ and $\chi_1 = \chi_2$;

2. $i = j = 2$, $c_1 = c_2$, $d_1 = d_2$ and $\chi_1 = \chi_2$;

3. $i = j = 3$, $c_1 = c_2$ and $\chi_1 = \chi_2$ or $\chi_1 = \chi_2 \circ \sigma$;

4. $i = j = 4$, $d_1 = d_2$ and $\chi_1 = \chi_2$ or $\chi_1 = \chi_2 \circ \sigma$;

5. $i = j = 5$, $c_1 = c_2$, $d_1 = d_2$ and $\chi_1 = \chi_2$ or $\chi_1 = \chi_2 \circ \sigma$.

Proof. Let π be an irreducible $*$-representation of A. From Lemmas 3.1-3.4 we know that π is at most two-dimensional and X^2, Y^2 act as scalars, say c and d respectively, on H.

First we assume $c = d = 0$. From the proof of Lemma 3.4 we have $X = Y = 0$ and hence $\pi(A)$ is commutative. Therefore π is one-dimensional and is completely determined by $\chi \in \hat{R}$. From $X = Y = 0$ we also have $t = \sigma(t) = 0$ and π is equivalent to π_0.

Now let $|c|+|d| \neq 0$ and v be a normed common eigenvector for R, given by Lemma 3.3. This means $hv = \chi(h)v$, $h \in R$, for some $\chi \in \hat{R}$. As $|c| + |d| \neq 0$ either X^2 or Y^2 is non-zero. Assume $X^2 \neq 0$. Then from $X^2 h = \sigma^2(h)X^2$ we get $h = \sigma^2(h)$ for any $h \in R$ (for Y^2 the arguments are similar). Now we consider two cases: $\chi \in \Gamma_1$ or $\chi \in \Gamma_2$. If $\chi \in \Gamma_2$ then $\pi(A)$ is commutative and therefore π is one-dimensional. We have $X = c'$, $Y = d'$, $XY = t$ implies $c'd' = \chi(t)$, $X^* = rX$ (resp. $Y^* = sY$) implies $\bar{c}' = \chi(r)c'$ (resp. $\bar{d}' = \chi(s)d'$) and π is equivalent to π_1.

If $\chi \in \Gamma_1$ then we consider three cases: $c = 0$, $d = 0$, $c \neq 0$ and $d \neq 0$. If $c = 0$ then v and Yv generate H and they are orthogonal as eigenvectors of a normal operator ($h \in R$ such that $\chi(h) \neq \chi(\sigma(h))$) with different eigenvalues. Writing our representation in the orthonormal basis v, $Yv/\|Yv\|$ ($\|Yv\| = \sqrt{d\chi(s)}$) we get the representation π_2. Since $X^*X = rX^2$ and $Y^*Y = sY^2$ we have $\chi(r)c \geq 0$ and $\chi(s)d \geq 0$. Analogously, if $d = 0$ one gets that π is equivalent to π_3 by the similar arguments (for the basis v, Xv). If $c \neq 0$ and $d \neq 0$ then we get $\pi \simeq \pi_4$, using the basis v, Xv.

The statement about unitary equivalence is obvious. $\qquad\square$

4   Case of arbitrary rank

We fix an irreducible representation, π, of A by bounded operators on a separable Hilbert space, H, and denote by a the image of a under π for all $a \in A$.

Lemma 4.1. *The elements X_i^2 and Y_i^2, $i \in N_n$, commute with all $a \in A$.*

Proof. Follows from Lemma 3.1 and the defining relations. $\qquad\square$

Set $G = \prod_{i \in N_n} \mathbb{Z}_2^{(i)}$ (n copies of $\mathbb{Z}_2$). For $h \in R$ and $g = (\varepsilon_1, \ldots, \varepsilon_n) \in G$ set $g(h) = \sigma_1^{\varepsilon_1} \circ \cdots \circ \sigma_n^{\varepsilon_n}(h)$. Write $G = \{g_1, g_2, \ldots, g_{2^n}\}$. For $h \in R$ and a symmetric polynomial, f, in 2^n variables put $u(f, h) = f(g_1(h), g_2(h), \ldots, g_{2^n}(h))$.

Lemma 4.2. *The elements $u(f, h)$ commute with all $a \in A$. In particular, all $h \in R$ have a discrete spectrum and there exists a non-zero common (for R) eigenvector, $v \in H$.*

Proof. Clearly, $u(f, h)$ commute with R. From the proof of Lemma 3.1 it follows that $(\sigma_i^2(h) - h)X_i = 0$. This implies $X_i u(f, h) = u(f, \sigma_i(h))X_i = u(f, h)X_i$. The arguments for Y_i are similar completing the proof of the first statement.

From the first statement it follows that all $u(f, h)$ are scalar operators on H. Let $f_1, \ldots, f_{2^n}$ be elementary symmetric polynomials. Then each $u(f_i, h)$ is a scalar, say $\gamma_i = \gamma_i(h)$, and we have $h^{2^n} - \gamma_1 h^{2^n - 1} + \cdots + \gamma_{2^n} = 0$. This implies the second statement. $\qquad\square$

Let X_i^2 (resp. Y_i^2) act with the scalar c_i (resp. d_i) on H. Set $N(\pi) = \{i : |c_i|+|d_i| \neq 0\}$ and for each $i \in N(\pi)$ fix $Z_i = X_i$ if $c_i \neq 0$ and $Z_i = Y_i$ otherwise.

Lemma 4.3. *The set $B = \{\prod_{i \in N(\pi)} Z_i^{\varepsilon_i} v\}$, $\varepsilon_i \in \{0, 1\}$, generates H. In particular, we have $\dim(H) \leq 2^{|N(\pi)|} \leq 2^n$.*

Proof. The invariance of the linear span of B under A follows from Lemma 3.4 and the relations $X_iX_j = X_jX_i$, $Y_iY_j = Y_jY_i$ and $X_iY_j = Y_jX_i$, $i \neq j$, in A. $\qquad\square$

Now we can start the classification. Our representations will be indexed by vectors $\mathfrak{v} = (\tau_i)_{i \in N_n}$, where each τ_i is an irreducible representation of the generalized Weyl algebra $A_i = A_i(R, \sigma_i, t_i)$, $i \in N_n$, associated with a common character, $\chi \in \hat{R}$. Let H_i denote the Hilbert space of the representation τ_i. Let N be the set of all i such that $\dim(H_i) = 2$. For $i \notin N$ fix a normed element, $v_i^0 \in H_i$. For $i \in N$ fix an orthonormal basis v_i^0, v_i^1 in H_i provided by Theorem 3.1. Set $H(\mathfrak{v}) = H_1 \otimes H_2 \otimes \cdots \otimes H_n$. Then $\dim(H(\mathfrak{v})) = 2^{|N|}$ and $\{v_{(\varepsilon_1,\ldots,\varepsilon_n)} = v_1^{\varepsilon_1} \otimes \cdots \otimes v_n^{\varepsilon_n} \mid \varepsilon_i = 0 \text{ for } i \notin N \text{ and } \varepsilon_i = 0,1 \text{ for } i \in N\}$ is an orthonormal basis of $H(\mathfrak{v})$. Set

$$hv_{(\varepsilon_1,\ldots,\varepsilon_n)} = \chi(\sigma_1^{\varepsilon_1} \circ \cdots \circ \sigma_n^{\varepsilon_n}(h))v_{(\varepsilon_1,\ldots,\varepsilon_n)}, \quad h \in H(\mathfrak{v});$$
$$X_iv_{(\varepsilon_1,\ldots,\varepsilon_n)} = v_1^{\varepsilon_1} \otimes \cdots \otimes v_{i-1}^{\varepsilon_{i-1}} \otimes \tau_i(X_i)v_i^{\varepsilon_i} \otimes v_{i+1}^{\varepsilon_{i+1}} \otimes \cdots \otimes v_n^{\varepsilon_n}; \quad i \in N_n;$$
$$Y_iv_{(\varepsilon_1,\ldots,\varepsilon_n)} = v_1^{\varepsilon_1} \otimes \cdots \otimes v_{i-1}^{\varepsilon_{i-1}} \otimes \tau_i(Y_i)v_i^{\varepsilon_i} \otimes v_{i+1}^{\varepsilon_{i+1}} \otimes \cdots \otimes v_n^{\varepsilon_n}; \quad i \in N_n.$$

We will denote this representation by $\pi(\mathfrak{v})$.

Lemma 4.4. $\pi(\mathfrak{v})$ *is a $*$-representation of A.*

Proof. Recall that τ_i are representations of A_i which correspond to the same $\chi \in \hat{R}$. The relations $X_iY_j = Y_jX_i$, $X_iX_j = X_jX_i$ and $Y_iY_j = Y_jY_i$ are obvious. Now consider $X_ih = \sigma_i(h)X_i$ and apply this equality to $v = v_{(\varepsilon_1,\ldots,\varepsilon_n)}$. If $X_i = Y_i = 0$, the relation is obvious. If $\varepsilon_i = 1$ the relation follows directly from the definition of the action of R. Otherwise we have $\chi(\sigma_i^2(h)) = \chi(h)$ by Theorem 3.1 and also $X_ihv = \chi(\sigma_1^{\varepsilon_1} \circ \cdots \circ \sigma_n^{\varepsilon_n}(h))X_iv$ and $\sigma_i(h)X_iv = \chi(\sigma_1^{\varepsilon_1} \circ \cdots \circ \sigma_n^{\varepsilon_n}(\sigma_i^2(h)))X_iv$ which is the same. One can use similar arguments to get $hY_i = Y_i\sigma_i(h)$. Using $\sigma_j(t_i) = t_i$, $i \neq j$, the relations $Y_iX_i = t_i$ and $X_iY_i = \sigma_i(t_i)$ can be easily reduced to the same relations between $\tau_i(X_i)$ and $\tau_i(Y_i)$. This completes the proof. $\qquad\square$

Assume that $\pi(\mathfrak{v})$ does exist and χ is the corresponding character of R. Set $N = \{i \in N_n \mid \chi(h) \neq \chi(\sigma_i(h)) \text{ for some } h \in R\} \cap \{i \in N_n \mid X_i \neq 0 \text{ or } Y_i \neq 0\} \subset N_n$. Denote by $\tilde{G} = \prod_{i \in N} \mathbb{Z}_2^{(i)}$ the corresponding subgroup of G and by $W(\mathfrak{v})$ the subgroup of $\tilde{G}$ consisting of all $g \in \tilde{G}$ such that $\chi(g(h)) = \chi(h)$ for all $h \in R$. Clearly, $|W(\mathfrak{v})| = 2^k$ for some k.

Lemma 4.5. $\pi(\mathfrak{v})$ *decomposes into $|W(\mathfrak{v})|$ pairwise unitarily non-equivalent irreducible representations. In particular, $\pi(\mathfrak{v})$ is irreducible if and only if $W(\mathfrak{v})$ is trivial.*

Proof. For each $w \in W(\mathfrak{v})$ we define a monomial, $X(w) \in A$, as follows: let $w = (\varepsilon_i)_{i \in N}$; for each $i \in N$, $\varepsilon_i = 1$, we set $Z_i = X_i$ if X_i is not zero on $H(\mathfrak{v})$ otherwise $Z_i = Y_i$; set $X(w) = \prod_{i \in N} Z_i$. We claim that the elements $X(w)$, $w \in W(\mathfrak{v})$, commute with the action of A on $H(\mathfrak{v})$. Indeed, fix $X(w)$, $w = (\varepsilon_i)_{i \in N} \in W(\mathfrak{v})$. That $X(w)$ commutes with R follows from $\chi(w(h)) = \chi(h)$, $w \in W(\mathfrak{v})$. Fix $i \in N_n$. If $X(w)$ contains X_i (resp. Y_i) then $X(w)X_i = X_iX(w)$ (resp. $X(w)Y_i = Y_iX(w)$). If $X(w)$ contains neither X_i nor Y_i then clearly $X(w)X_i = X_iX(w)$ and $X(w)Y_i = Y_iX(w)$. Now assume that $X(w)$ contains

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X_i (for Y_i the arguments are similar). Decompose $w = \sigma_i \circ w'$, where w' does not contain σ_i. Then $\chi(\sigma_i \circ w'(t_i)) = \chi(t_i)$ and hence $\chi(\sigma_i(t_i)) = \chi(t_i)$. This implies $X_i Y_i = Y_i X_i$ and therefore $X(w)Y_i = Y_i X(w)$, proving our claim. In particular, all $X(w)$ are normal operators.

By the definition of $\pi(\mathfrak{v})$, each $(X(w))^2$ is a scalar operator with some scalar $\lambda(w) \neq 0$. Hence the spectrum of $X(w)$ is a subset of $\{\pm\sqrt{\lambda(w)}\}$. As $v_{(0,\dots,0)}$ is orthogonal to $X(w)v_{(0,\dots,0)}$ we get that spectrum of $X(w)$ coincides with $\{\pm\sqrt{\lambda(w)}\}$. Let $\{w_1,\dots,w_k\}$ be a minimal set of generators of $W(\mathfrak{v})$. Decomposing $\pi(\mathfrak{v})$ with respect to the common spectrum of $X(w_1), \dots, X(w_k)$ we get $|W(\mathfrak{v})|$ non-zero subrepresentations of $\pi(\mathfrak{v})$. From the definition of $W(\mathfrak{v})$ it follows that each of these subrepresentations has a basis, whose elements are separated by the action of R (i.e. correspond to pairwise distinct characters of R). Moreover, as $\pi(\mathfrak{v})$ is cyclic, each subrepresentation of it is also cyclic. Altogether this implies that all our subrepresentations are irreducible. The statement that they are pairwise unitarily non-equivalent is obvious. $\qquad\square$

Denote by W the group, generated by all σ_j, $j \in N_n$. W acts on the set of representations of A_i in a natural way.

Lemma 4.6. *1. $\pi((\tau(1)_i))$ and $\pi((\tau(2)_i))$ are unitarily equivalent if and only if $\tau(1)_i$ and $\tau(2)_i$ belong to the same orbit of the action of W for all $i \in N_n$.*

2. If representations $\pi((\tau(1)_i))$ and $\pi((\tau(2)_i))$ are unitarily non-equivalent then any irreducible component of $\pi((\tau(1)_i))$ is unitarily non-equivalent to any irreducible component of $\pi((\tau(2)_i))$.

Proof. Obvious. $\qquad\square$

Theorem 4.1. *Any irreducible $*$-representation of A is unitarily equivalent to an irreducible subrepresentation of some $\pi(\mathfrak{v})$.*

Proof. Let π be an irreducible representation of A on a Hilbert space H. We claim that it is equivalent to an irreducible subrepresentation of $\pi(\mathfrak{v})$ for a suitable $\mathfrak{v}$. From $X_j X_i = X_i X_j$, $X_j Y_i = Y_i X_j$, $Y_j X_i = Y_i X_j$, $Y_j Y_i = Y_i X_j$ it follows that $\pi|_{A_i}$ decomposes into a direct sum of irreducible representations of A_i which belong to the same orbit of W-action. For each $i \in N_n$ fix one representation from the corresponding orbit, say τ_i. As above, set $N = \{i \in N_n \mid \chi(h) \neq \chi(\sigma_i(h))$ for some $h \in R\} \cap \{i \in N_n \mid X_i \neq 0$ or $Y_i \neq 0\}$. Consider $W(\pi)$ and $X(w)$ associated with π and $w \in W(\pi)$, as defined in the proof of Lemma 4.5. By the same arguments as there we get that $X(w)$ commute with all $a \in A$ and hence are scalar operators on H with scalars $\pm\sqrt{\lambda(w)}$ (the choice of signs is coordinated by the group structure on $W(\pi)$). Consider the quotient group $G(\pi) = (\prod_{i \in N} \mathbb{Z}_2^{(i)})/W(\pi)$ and fix some representatives $\Lambda = \{w_1,\dots,w_{|G(\pi)|}\}$ in the corresponding cosets. For $w \in \Lambda$ define $X(w)$ as above. Let v be a common eigenvector for R in H. Then the linear span of $X(w)v$, $w \in \Lambda$, is an orthogonal basis of H. Indeed, first we show that this linear span is invariant under A. This is clear for R. If $\pi(X_i) = 0$, the statement is obvious for such X_i. Otherwise, using the fact that X_i^2, $X(w)$, $w \in W(\pi)$, are scalars on H, we have $X_i X(w) = cX(w')$

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for some uniquely defined $c \in \mathbb{C}$ and $w' \in \Lambda$. The same argument works for Y_i and we get the necessary invariance. By the construction, all $X(w)v$, $w \in \Lambda$, are common eigenvectors for R. Now the orthogonality follows from the fact, that the corresponding common eigenvalues are distinct (definition of $W(\pi)$) and all operators $h \in R$ are normal.

In particular, we get that the irreducible representation with parameters τ_i, $i \in N_n$, and $\pm\sqrt{\lambda(w)}$, $w \in W(\pi)$, is unique up to a unitary equivalence. Therefore it is unitarily equivalent to an irreducible subrepresentation of $\pi((\tau_i)_{i \in N_n})$. □

Corollary 4.1. *The dimension of any irreducible $*$-representation of A equals 2^k for some $k \in \mathbb{Z}_+$.*

References

[1] *V.Bavula*, Generalized Weyl algebras and their representations. (Russian) Algebra i Analiz 4 (1992), no. 1, 75–97.

[2] *V.Bavula and V.Bekkert*, Indecomposable Representations of Generalized Weyl Algebras, Comm. Alg. 28 (2000), 5067-5100.

[3] *V.Mazorchuk and L.Turowska*, $*$-Representations of twisted generalized Weyl constructions, Preprint TRITA-MAT-1998-MA-04 (1998), Royal Institute of Technology, Department of Mathematics, Stockholm, to appear in Algebr. Represent. Theory.

[4] *W.Rudin*, Functional analysis. Second edition. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991.

Volodymyr Mazorchuk, Algebra, Department of Mechanics and Mathematics, Kyiv Taras Shevchenko University, 64, Volodymyrska st., 01033, Kyiv, Ukraine,
e-mail: `mazor@mail.univ.kiev.ua`, `mazor@mathematik.uni-bielefeld.de`
Present Address: Chalmers Tekniska Högskola, Matematik, SE 412 96, Göteborg, Sweden, e-mail: `volody@math.chalmers.se`

Lyudmyla Turowska, Chalmers Tekniska Högskola, Matematik, SE 412 96, Göteborg, Sweden, e-mail: `turowska@math.chalmers.se`