CLASSIFICATION OF SIMPLE WEIGHT MODULES OVER THE 1-SPATIAL AGEING ALGEBRA

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ABSTRACT. In this paper we use Block's classification of simple modules over the first Weyl algebra to obtain a complete classification of simple weight modules, in particular, of Harish-Chandra modules, over the 1-spatial ageing algebra $\mathfrak{age}(1)$. Most of these modules have infinite dimensional weight spaces and so far the algebra $\mathfrak{age}(1)$ is the only Lie algebra having simple weight modules with infinite dimensional weight spaces for which such a classification exists. As an application we classify all simple weight modules over the $(1+1)$-dimensional space-time Schrödinger algebra $\mathcal{S}$ that have a simple $\mathfrak{age}(1)$-submodule thus constructing many new simple weight $\mathcal{S}$-modules.

Keywords: weight module, Schrödinger algebra, ageing algebra, highest weight modules

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1. Introduction and description of the results

Non-semisimple Lie algebras play an important role in physics where they are frequently used to study various physical systems and explain diverse physical phenomena. For example one could mention Schrödinger algebras and groups, see [DDM], conformal Galilei algebras and groups, see [AI, AIK, CZ, GIl, GI2, HP, NOR, LMZ], and ageing algebras, see [HP, H1, H2, H3, H4, HEP, HU, HS1, HS2, PH, HSSU]. The representation theory of finite dimensional semisimple Lie algebras is fairly well developed, see for example [Hu, Ja, M] and the references therein. In contrast to this, many important methods of the representation theory of semisimple algebra are either not available or not yet developed or, at best, become much more complicated in the non-semisimple case, confer e.g. [DLMZ]. As a consequence, much less in known for non-semisimple case even for Lie algebras of rather small dimension. Some recent results, see e.g. [WZ, D, DLMZ, LMZ], make some progress in studying modules over certain non-semisimple extensions of the Lie algebra $\mathfrak{sl}_2$ motivated by their applications in physics.

The Schrödinger Lie group is the group of symmetries of the free particle Schrödinger equation. The classical *Schrödinger algebra* is the Lie algebra of this group. The main objects of study in this paper is the extended Schrödinger algebra $\mathcal{S}$ in $(1+1)$-dimensional space-time and its ageing subalgebra $\mathfrak{age}(1)$, see precise definitions in Section 2. The name of the latter algebra comes from its use as dynamical symmetry in physical ageing, which can be observed in strongly interacting many-body systems quenched from a disordered initial state to the coexistence regime below the critical temperature $T_c > 0$ where several equivalent equilibrium states exist, see [BR] for details. Various representations of $\mathfrak{age}(1)$ were constructed in the literature, see for example [HS1, HS2, H1, H2, H3, H4, HU, PH, HSSU].

Both $\mathcal{S}$ and $\mathfrak{age}(1)$ have natural Cartan subalgebras which allows to define the notion of weight modules, that is modules on which elements of the Cartan subalgebra act diagonalizably. The main objective of the present paper is to classify all simple weight modules over $\mathfrak{age}(1)$. It turns out that most of these modules have infinite dimensional weight spaces. It seems that the algebra $\mathfrak{age}(1)$ is the first Lie algebra having simple weight modules with infinite dimensional weight spaces for which classification of simple weight modules is completed. As an application we classify all simple weight $\mathcal{S}$-modules that have a simple $\mathfrak{age}(1)$-submodule. This provides many new examples of simple weight $\mathcal{S}$-modules (for other examples, see [D, LMZ]).

The paper is organized as follows. We start with some preliminaries in Section 2. In Section 3, by embedding our algebras into the first Weyl algebra we obtain a very good presentation for the centralizer U_0 of the Cartan subalgebra in the universal enveloping algebra $U(\mathfrak{age}(1))$. This is the key observation needed to classify all simple U_0-modules. As a consequence, we obtain classification of all simple weight modules over $\mathfrak{age}(1)$. In Section 4 we classify all simple weight $\mathcal{S}$-modules that have a simple $\mathfrak{age}(1)$-submodule.

2. Preliminaries

In this paper, we denote by $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}_+$ and $\mathbb{C}$ the sets of integers, positive integers, nonnegative integers and complex numbers, respectively. All vector spaces and Lie algebras are over $\mathbb{C}$. For a Lie algebra $\mathfrak{g}$ we denote by $U(\mathfrak{g})$ its universal enveloping algebra. We write $\otimes$ for $\otimes_{\mathbb{C}}$.

Consider the *extended Schrödinger algebra* $\mathcal{S}$ in $(1+1)$-dimensional space-time, which is a complex Lie algebra with basis $\{f, q, h, e, p, z\}$

and the Lie brackets given as follows:

$$
\begin{array}{llll}
 & [h,e]=2e, & [e,f]=h, & [h,f]=-2f, \\
(2.1) & [e,q]=p, & [e,p]=0, & [h,p]=p, \\
 & [f,p]=q, & [f,q]=0, & [h,q]=-q, \\
 & [p,q]=z. & [z,\mathcal{S}]=0. &
\end{array}
$$

The classical *ageing algebra* $\mathfrak{age}(1)$ is the subalgebra of $\mathcal{S}$ spanned by $\{e,h,p,q,z\}$. The elements h and z span a Cartan subalgebra $\mathfrak{h}$ while the elements p, q and z span a copy of the Heisenberg subalgebra $\mathcal{H}$. We also denote by $\mathfrak{n}$ the subalgebra spanned by h, z, e and p. The subalgebra of $\mathcal{S}$ spanned by e, f and h is isomorphic to the classical algebra $\mathfrak{sl}_2$ and will be identified with the latter.

By Schur's lemma, the element z acts as a scalar on any simple module over $\mathfrak{age}(1)$ or $\mathcal{S}$. For a module V on which z acts as a scalar $\dot{z}$ we denote by $\mathrm{supp}(V)$ the set of h-eigenvalues on V and will call these eigenvalues *weights*. All h-eigenvectors will be called *weight vectors*. For a weight $\dot{h}$ we denote by $V_{\dot{h}}$ the corresponding *weight space*, that is the space of all h-eigenvectors with eigenvalue $\dot{h}$. If V is a simple weight $\mathfrak{age}(1)$- or $\mathcal{S}$-module, then, as usual, we have $\mathrm{supp}(V)\subset \dot{h}+\mathbb{Z}$ for any $\dot{h}\in\mathrm{supp}(V)$.

Let $U_0=\{x\in U(\mathfrak{age}(1))|[h,x]=0\}$ be the centralizer of $\mathfrak{h}$ in $U(\mathfrak{age}(1))$. Then, as usual, for every simple weight $\mathfrak{age}(1)$-module V and any $\lambda\in\mathrm{supp}(V)$ the U_0-module V_λ is simple.

Each $(\dot{z},\dot{h})\in\mathbb{C}^2$ defines the one-dimensional $\mathfrak{n}$-module $\mathbb{C}w$ with the action $hw=\dot{h}w$, $zw=\dot{z}w$ and $pw=ew=0$. Using it we define, as usual, the *Verma* $\mathfrak{age}(1)$-module

$$M_{\mathfrak{age}(1)}(\dot{z},\dot{h})=\mathrm{Ind}_{\mathfrak{n}}^{\mathfrak{age}(1)}\,\mathbb{C}w.$$

Denote by $\bar{M}_{\mathfrak{age}(1)}(\dot{z},\dot{h})$ the unique simple quotient of $M_{\mathfrak{age}(1)}(\dot{z},\dot{h})$ (which exists by standard arguments, see e.g. [Di, Chapter 7]). Similarly we may define the Verma modules $M_{\mathcal{S}}(\dot{z},\dot{h})$, $M_{\mathcal{H}}(\dot{z})$, $M_{\mathfrak{sl}_2}(\dot{h})$ and the corresponding simple quotients $\bar{M}_{\mathcal{S}}(\dot{z},\dot{h})$, $\bar{M}_{\mathcal{H}}(\dot{z})$, $\bar{M}_{\mathfrak{sl}_2}(\dot{h})$ over $\mathcal{S}$, $\mathcal{H}$, and $\mathfrak{sl}_2$, respectively. Analogously one defines the *lowest Verma modules* $M^-_{\mathcal{S}}(\dot{z},\dot{h})$, $M^-_{\mathfrak{age}(1)}(\dot{z},\dot{h})$, $M^-_{\mathcal{H}}(\dot{z})$ and $M^-_{\mathfrak{sl}_2}(\dot{h})$ and their corresponding simple quotients $\bar{M}^-_{\mathcal{S}}(\dot{z},\dot{h})$, $\bar{M}^-_{\mathfrak{age}(1)}(\dot{z},\dot{h})$, $\bar{M}^-_{\mathcal{H}}(\dot{z})$ and $\bar{M}^-_{\mathfrak{sl}_2}(\dot{h})$.

3. Simple weight modules over $\mathfrak{age}(1)$

We start by recalling some results from [B] adjusted to our setup. Let $\mathcal{K}$ be the associative algebra $\mathbb{C}(t)[s]$ where $st-ts=1$. The algebra $\mathcal{K}$ is a noncommutative principal left and right ideal domain, in fact, it is an Euclidean domain (both left and right). The space $\mathbb{C}(t)$ becomes a faithful $\mathbb{C}(t)[s]$-module by defining the action of $\mathbb{C}(t)$ via multiplication and the action of s via $\frac{d}{dt}$. Consider the subalgebra R_0 of $\mathcal{K}$ generated by t and t^2s. For every $\dot{z} \in \mathbb{C}^*$ we also consider the subalgebra $R_{\dot{z}}$ of $\mathcal{K}$ generated by t and ts (note that this subalgebra does not really depend on $\dot{z}$).

Lemma 1.

(*a*) *Let* $\alpha = \sum_{j=0}^{n} \alpha_j s^j$, *where* $\alpha_j \in \mathbb{C}(t)$ *with* $\alpha_0 = 1$, *be an irreducible element in* $\mathcal{K}$. *Then we have the following:*

(*i*) *If the rational function* α_j *has a zero at* 0 *of order at least* $j+1$, *then* $R_1/(R_1 \cap \mathcal{K}\alpha)$ *is a simple* R_1*-module. Up to isomorphism, every* $\mathbb{C}[t]$*-torsion-free simple* R_1*-module arises in this way.*

(*ii*) *If the rational function* α_j *has a zero at* 0 *of order at least* $2j+1$, *then* $R_0/(R_0 \cap \mathcal{K}\alpha)$ *is a simple* R_0*-module. Up to isomorphism, every* $\mathbb{C}[t]$*-torsion-free simple* R_0*-module arises in this way.*

(*b*) *Any simple quotient of the* $R_{\dot{z}}$*-module* $R_{\dot{z}}/R_{\dot{z}}t$ *is 1-dimensional.*

(*c*) *For any* $\lambda \in \mathbb{C}^*$, *the* $R_{\dot{z}}$*-module* $R_{\dot{z}}/R_{\dot{z}}(t-\lambda)$ *is simple.*

(*d*) *Let* V *be a simple* $\mathbb{C}[t]$*-torsion-free* $R_{\dot{z}}$*-module. Then* t *acts bijectively on* V.

Proof. Claim (ai) is [B, Theorem 2]. Claim (aii) follows from [B, Theorem 4.3].

To prove claim (b), note that $R_{\dot{z}}t$ is a two-sided ideal of $R_{\dot{z}}$ for any value of $\dot{z}$. The algebra $R_{\dot{z}}/R_{\dot{z}}t$ is easily checked to be the commutative algebra $\mathbb{C}[ts]$ if $\dot{z} \neq 0$. Similarly, if $\dot{z} = 0$, we have that $R_{\dot{z}}/R_{\dot{z}}t \cong \mathbb{C}[t^2s]$. All simple modules over the latter commutative algebras have dimension one.

To prove claim (c), observe that $\{(ts)^i | i \in \mathbb{Z}_+\}$ is a basis in $R_{\dot{z}}/R_{\dot{z}}(t-\lambda)$ if $\dot{z} \neq 0$. Moreover, if $\dot{z} = 0$, then $\{(t^2s)^i | i \in \mathbb{Z}_+\}$ is a basis in the quotient $R_{\dot{z}}/R_{\dot{z}}(t-\lambda)$. Using this it is easy to verify that the $R_{\dot{z}}$-module $R_{\dot{z}}/R_{\dot{z}}(t-\lambda)$ is simple.

Claim (d) follows from the fact that tV is an $R_{\dot{z}}$-submodule of V and hence is equal to V. □

Next we characterize the associative algebra U_0. Denote by $\mathbf{A}$ the first Weyl algebra, which we realize as the unital subalgebra of the algebra of all linear operators on $\mathbb{C}[t]$ generated by the linear operator $\frac{d}{dt}$ and the linear operator of multiplication by t (which we will denote simply by t). Alternatively, one can also think of the algebra $\mathbf{A}$ as the unital subalgebra of the algebra $\mathcal{K}$ generated by t and s. In particular, we can view R_0 as a subalgebra of $\mathbf{A}$ generated by t and $t^2\frac{d}{dt}$. Similarly, we can view $R_{\dot z}$ as a subalgebra of $\mathbf{A}$ generated by t and $t\frac{d}{dt}$.

Lemma 2.

(a) *The associative algebra U_0 is generated by eq^2, pq, h, and z and*

$$\{(eq^2)^{i_1}(pq)^{i_2}h^{i_3}z^{i_4}|i_1,i_2,i_3,i_4\in\mathbb{Z}_+\}$$

is a basis of U_0. In particular, for any $\dot z,\dot h\in\mathbb{C}$, the image of $\{(eq^2)^{i_1}(pq)^{i_2}|i_1,i_2\in\mathbb{Z}_+\}$ in $U_0/\langle h-\dot h, z-\dot z\rangle$ forms a basis there.

(b) *For any $\dot z,\dot h\in\mathbb{C}$, there is a unique homomorphism of associative algebras $\varphi_{\dot z,\dot h}:U_0\to\mathbf{A}$ such that $\varphi_{\dot z,\dot h}(z)=\dot z$, $\varphi_{\dot z,\dot h}(h)=\dot h$ and*

(i) $\varphi_{0,\dot h}(pq)=t,\ \varphi_{0,\dot h}(eq^2)=t^2\frac{d}{dt}$ *if* $\dot z=0$*;*

(ii) $\varphi_{\dot z,\dot h}(pq)=2\dot z t\frac{d}{dt},\ \varphi_{\dot z,\dot h}(eq^2)=t+\dot z t\frac{d}{dt}+2\dot z(t\frac{d}{dt})^2$ *if* $\dot z\neq 0$.

(c) *We have $\varphi_{\dot z,\dot h}(U_0)=R_{\dot z}$ and $U_0/\langle h-\dot h,z-\dot z\rangle\cong R_{\dot z}$.*

(d) *The isomorphism in* (c) *induces a natural bijection between isomorphism classes of simple U_0-modules on which h acts as $\dot h$ and z acts as $\dot z$ and isomorphism classes of simple $R_{\dot z}$-modules.*

Proof. To prove claim (a) we argue that, by the PBW Theorem, the algebra U_0 has a basis

$$\{e^{i_1}q^{2i_1+i_2}p^{i_2}z^{i_3}h^{i_4}|i_1,i_2,i_3,i_4\in\mathbb{Z}_+\}=$$
$$\{(e^{i_1}q^{2i_1})(q^{i_2}p^{i_2})z^{i_3}h^{i_4}|i_1,i_2,i_3,i_4\in\mathbb{Z}_+\}.$$

Step 1: The element q^ip^i can be written as a linear combination of elements of the form $(pq)^jz^{i-j}$ for $j\in\mathbb{Z}_+$.

This follows by induction on i from the following computation:

$$q^{i+1}p^{i+1}=q^{i+1}pp^i=pqq^ip^i-(i+1)q^ip^iz.$$

Step 2: The element e^iq^{2i} can be written as a linear combination of elements of the form $(eq^2)^k(pq)^jz^{i-k-j}$ for $k,j\in\mathbb{Z}_+$ with $i\geq k+j$.

This follows by induction on i from the following computation:

$$e^{i+1}q^{2(i+1)}=ee^iqq^{2i+1}=eqe^iq^{2i+1}+ie^ipq^{2i+1}=$$
$$eq^2e^iq^{2i}+2ie^ipq^{2i+1}=(eq^2)(e^iq^{2i})+2i((e^iq^{2i})(pq)+2ie^iq^{2i}z).$$

Claim (a) follows easily from Steps 1 and 2.

To prove claim (b), we only need to check

$$\varphi_{\dot z,\dot h}([eq^2,pq])=[\varphi_{\dot z,\dot h}(eq^2),\varphi_{\dot z,\dot h}(pq)]$$

for any $\dot{z}$ and $\dot{h}$. Note that

$$\begin{aligned}
[eq^2, pq] &= [eq^2, p]q + p[eq^2, q] \\
&= -2zeq^2 + p(pq)q \\
&= -2zeq^2 + zpq + (pq)^2.
\end{aligned}$$

Therefore we have $[\varphi_{0,\dot{h}}(eq^2), \varphi_{0,\dot{h}}(pq)] = [t^2\frac{d}{dt}, t] = t^2$ while

$$\varphi_{0,\dot{h}}([eq^2, pq]) = \varphi_{0,\dot{h}}(-2zeq^2 + zpq + (pq)^2) = t^2,$$

which implies (bi).

Similarly, for $\dot{z} \neq 0$ we have

$$\begin{aligned}
\varphi_{\dot{z},\dot{h}}([eq^2, pq]) &= \varphi_{\dot{z},\dot{h}}((-2zeq^2 + zpq + (pq)^2)) \\
&= -2\dot{z}(t + \dot{z}t\tfrac{d}{dt} + 2\dot{z}(t\tfrac{d}{dt})^2) + \dot{z}2\dot{z}t\tfrac{d}{dt} + (2\dot{z}t\tfrac{d}{dt})^2 \\
&= -2\dot{z}t,
\end{aligned}$$

while

$$[\varphi_{\dot{z},\dot{h}}(eq^2), \varphi_{\dot{z},\dot{h}}(pq)] = [t + \dot{z}t\frac{d}{dt} + 2\dot{z}(t\frac{d}{dt})^2, 2\dot{z}t\frac{d}{dt}] = -2\dot{z}t,$$

which implies (bii).

Claim (c) follows from claims (a) and (b). Claim (d) follows from claim (c). □

Now we address the structure of Verma modules over $\mathfrak{age}(1)$.

Lemma 3.

(*a*) *Let* $\dot{z}, \dot{h} \in \mathbb{C}$. *Then the* $\mathfrak{age}(1)$*-module* $M_{\mathfrak{age}(1)}(\dot{z}, \dot{h})$ *is simple if and only if* $\dot{z} \neq 0$.

(*b*) *Let* $\dot{h} \in \mathbb{C}$. *Then we have* $\dim \bar{M}_{\mathfrak{age}(1)}(0, \dot{h}) = 1$.

Proof. If $\dot{z} \neq 0$, then the module $M_{\mathfrak{age}(1)}(\dot{z}, \dot{h})$ is obviously simple already as an $\mathcal{H}$-module, which implies (a). If $\dot{z} = 0$, we have a simple module $\mathbb{C}v$ with action $ev = pv = qv = zv = 0$ and $hv = \dot{h}v$. By the universal property of Verma modules, $M_{\mathfrak{age}(1)}(0, \dot{h})$ surjects onto $\mathbb{C}v$ and hence $\bar{M}_{\mathfrak{age}(1)}(0, \dot{h}) = \mathbb{C}v$, which implies (b). □

Remark 4. Lemma 3 describes all simple highest weight $\mathfrak{age}(1)$-modules. Let V be a simple highest weight $\mathfrak{age}(1)$-module. Consider the decomposition $V = \oplus_{\dot{h}\in\mathbb{C}} V_{\dot{h}}$ and note that all $V_{\dot{h}}$ are finite dimensional. As usual, the space $V^\star = \oplus_{\dot{h}\in\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(V_{\dot{h}}, \mathbb{C})$ has the natural structure of an $\mathfrak{age}(1)$-module defied using the canonical involution $x \mapsto -x$, $x \in \mathfrak{age}(1)$. The module $V^\star$ is a simple lowest weight $\mathfrak{age}(1)$-module and the correspondence $V \mapsto V^\star$ is a bijection between the sets of isomorphism classes of simple highest weight and simple lowest weight modules.

Before constructing some other weight $\mathfrak{age}(1)$-modules, we have to define some automorphisms of the associative algebras U_0 and $R_{\dot z}$.

Lemma 5. *For any $i \in \mathbb{Z}$ there is a unique $\tau_i \in \mathrm{Aut}(U_0)$ such that*

$$\tau_i(pq) = pq + iz, \quad \tau_i(h) = h - i, \quad \tau_i(z) = z,$$
$$\tau_i(eq^2) = eq^2 + ipq + \frac{i(i+1)}{2} z.$$

Proof. We have

$$\begin{aligned} [\tau_i(eq^2), \tau_i(pq)] &= [eq^2 + ipq + \tfrac{i(i+1)}{2} z, pq + iz] \\ &= [eq^2, pq] \\ &= -2zeq^2 + zpq + (pq)^2 \end{aligned}$$

while

$$\begin{aligned} \tau_i([eq^2, pq]) &= \tau_i(-2zeq^2 + zpq + (pq)^2) \\ &= -2z(eq^2 + ipq + \tfrac{i(i+1)}{2} z) + z(pq + iz) + (pq + iz)^2 \\ &= -2zeq^2 + zpq + (pq)^2. \end{aligned}$$

Thus $[\tau_i(eq^2), \tau_i(pq)] = \tau_i([eq^2, pq])$. All other relations are checked similarly. □

The proof of the next lemma is a straightforward computation which is left to the reader.

Lemma 6. *For any $i \in \mathbb{Z}$ and $\dot z \in \mathbb{C}$ there is a unique $\tau_{i,\dot z} \in \mathrm{Aut}(R_{\dot z})$ such that*

$$\tau_{i,0}(t) = t, \quad \tau_{i,0}(t^2 \frac{d}{dt}) = t^2 \frac{d}{dt} + it \quad \text{if} \quad \dot z = 0;$$
$$\tau_{i,\dot z}(t) = t, \quad \tau_{i,\dot z}(t \frac{d}{dt}) = t \frac{d}{dt} + \frac{i}{2} \quad \text{if}\ \dot z \neq 0.$$

The next step is to construct some weight $\mathfrak{age}(1)$-modules using $R_{\dot z}$-modules. From now on in this section N denotes an $R_{\dot z}$-module. We define the $\mathfrak{age}(1)$-module $N_{\dot z, \dot h}$ as follows: as a vector space we set $N_{\dot z, \dot h} = N \otimes \mathbb{C}[x, x^{-1}]$ and then for $v \in N$ define

$$q(v \otimes x^i) = v \otimes x^{i+1}, \quad p(v \otimes x^i) = (\varphi_{\dot z, \dot h}(pq + (i-1)z)v) \otimes x^{i-1}, \tag{3.1}$$

$$e(v \otimes x^i) = (\varphi_{\dot z, \dot h}(eq^2 + (i-2)pq + \frac{(i-1)(i-2)}{2} z)v) \otimes x^{i-2}, \tag{3.2}$$

$$h(v \otimes x^i) = ((\dot h - i)v) \otimes x^i, \quad z(v \otimes x^i) = (\dot z v) \otimes x^i. \tag{3.3}$$

By a straightforward computation we get the following:

Lemma 7.

(*a*) *Formulae* (3.1)*,* (3.2) *and* (3.3) *define on $N_{\dot z, \dot h}$ the structure of an $\mathfrak{age}(1)$-module.*

(*b*) *For* $\dot{z}=0$ *the action in* (3.1), (3.2) *and* (3.3) *reads as follows*

$$\begin{cases} h(v\otimes x^i)=(\dot{h}-i)v\otimes x^i, \quad z(v\otimes x^i)=0,\\ q(v\otimes x^i)=v\otimes x^{i+1}, \quad p(v\otimes x^i)=(tv)\otimes x^{i-1},\\ e(v\otimes x^i)=((t^2\frac{d}{dt}+(i-2)t)v)\otimes x^{i-2}. \end{cases} \tag{3.4}$$

(*c*) *For* $\dot{z}\neq 0$ *the action in* (3.1), (3.2) *and* (3.3) *reads as follows*

$$\begin{cases} h(v\otimes x^i)=(\dot{h}-i)v\otimes x^i, \quad z(v\otimes x^i)=\dot{z}v\otimes x^i,\\ q(v\otimes x^i)=v\otimes x^{i+1}, \quad p(v\otimes x^i)=((2\dot{z}t\frac{d}{dt}+(i-1)\dot{z})v)\otimes x^{i-1},\\ e(v\otimes x^i)=((t+(2i-3)\dot{z}t\frac{d}{dt}+2\dot{z}(t\frac{d}{dt})^2+\frac{(i-1)(i-2)}{2}\dot{z})v)\otimes x^{i-2}. \end{cases} \tag{3.5}$$

For an associative algebra A, an A-module V and $\sigma\in \mathrm{Aut}(A)$, we denote by V^σ the module obtained from V via twisting the action of A by σ, that is $a\cdot v=\sigma(a)v$ for $a\in A$ and $v\in V$.

For each $i\in\mathbb{Z}$ the space $N\otimes x^i$ is naturally a U_0-module. We may even make $N\otimes x^i$ into an $R_{\dot{z}}$-module as follows:

If $\dot{z}=0$, then for any $v\in N$, we set

$$\begin{aligned} t(v\otimes x^i)&=(pq)(v\otimes x^i)=(tv)\otimes x^i,\\ (t^2\frac{d}{dt})(v\otimes x^i)&=(eq^2)(v\otimes x^i)=((t^2\frac{d}{dt}+it)v)\otimes x^i. \end{aligned} \tag{3.6}$$

If $\dot{z}\neq 0$, then for any $v\in N$, we set

$$\begin{aligned} t(v\otimes x^i)&=(eq^2-\frac{pq}{2}-2\dot{z}(\frac{pq}{2\dot{z}})^2)(v\otimes x^i)=(tv)\otimes x^i,\\ (t\frac{d}{dt})(v\otimes x^i)&=(\frac{pq}{2\dot{z}})(v\otimes x^i)=((t\frac{d}{dt}+\frac{i}{2})v)\otimes x^i. \end{aligned} \tag{3.7}$$

The following standard result asserts that any simple weight module is uniquely determined by any of its nonzero weight space.

Lemma 8. *Let* $\mathfrak{g}\in\{\mathfrak{age}(1),\mathcal{S}\}$ *and* V *and* W *be simple weight* $\mathfrak{g}$-*modules such that for some* $\dot{h}\in\mathrm{supp}(V)$ *the* $U(\mathfrak{g})_0$ *modules* $V_{\dot{h}}$ *and* $W_{\dot{h}}$ *are isomorphic. Then* $V\cong W$.

Proof. Set $N=V_{\dot{h}}=W_{\dot{h}}$ and write

$$U(\mathfrak{g})=\bigoplus_{i\in\mathbb{Z}}U_i \quad \text{where} \quad U_i=\{x\in U(\mathfrak{g})\,|\,[h,y]=iy\}.$$

Every U_i is a $U(\mathfrak{g})_0$-$U(\mathfrak{g})_0$-bimodule. Consider the induced module

$$M:=\mathrm{Ind}_{U(\mathfrak{g})_0}^{U(\mathfrak{g})}N=U(\mathfrak{g})\bigotimes_{U(\mathfrak{g})_0}N=\bigoplus_{i\in\mathbb{Z}}(U_i\otimes_{U(\mathfrak{g})_0}N).$$

Then, by adjunction, both V and W are simple quotients of M. The module M is, clearly, a weight $\mathfrak{g}$-module with $M_{\dot{h}} \cong N$, the latter being a simple $U(\mathfrak{g})_0$-module, which implies, using standard arguments, that M has a unique maximal submodule K satisfying $K_{\dot{h}} = 0$ and thus the unique corresponding simple quotient M/K. As both V and W have to be quotients of M/K by adjunction, we get $V \cong W$. □

Now we list some properties of the $\mathfrak{age}(1)$-module $N_{\dot{z},\dot{h}}$.

Lemma 9.

(*a*) *For every $i \in \mathbb{Z}$ the U_0-modules $N \otimes x^i$ and N^{τ_i} are isomorphic.*

(*b*) *The $\mathfrak{age}(1)$-module $N_{\dot{z},\dot{h}}$ is simple if and only if N is a simple $R_{\dot{z}}$-module and one of the following conditions is satisfied:*

(*i*) *$\dot{z} = 0$ and $(\mathbb{C}t + \mathbb{C}t^2\frac{d}{dt})N \neq 0$;*

(*ii*) *$\dot{z} \neq 0$ and $\tau_{i,\dot{z}}(\mathbb{C}t + \mathbb{C}t\frac{d}{dt})N \neq 0$ for all $i \in \mathbb{Z}$.*

(*c*) *Let $\dot{z}, \dot{h}, \dot{z}', \dot{h}' \in \mathbb{C}$. Let further N be an $R_{\dot{z}}$-module and N' be an $R_{\dot{z}'}$-module. Assume that $N_{\dot{z},\dot{h}}$ and $N'_{\dot{z}',\dot{h}'}$ are simple. Then we have $N_{\dot{z},\dot{h}} \cong N'_{\dot{z}',\dot{h}'}$ if and only if $\dot{z} = \dot{z}'$, $i := \dot{h} - \dot{h}' \in \mathbb{Z}$ and $N' \cong N^{\tau_{i,\dot{z}}}$ as $R_{\dot{z}}$-modules.*

Proof. Let $\psi : N \otimes x^i \to N^{\tau_i}$ be the linear map defined as $\psi(v \otimes x^i) = v$ for all $v \in A$. It is straightforward to verify that

$$\psi(pq(v \otimes x^i)) = \psi(((pq + i\dot{z})v) \otimes x^i) = (pq + iz)v = \tau_i(pq)v,$$

$$\psi((eq^2)(v \otimes x^i)) = \psi((eq^2 + ipq + \frac{i(i+1)}{2}\dot{z})v \otimes x^i)$$

$$= (eq^2 + ipq + \frac{i(i+1)}{2}\dot{z})v = \tau_i(eq^2)\psi(v \otimes x^i),$$

$$\psi(h(v \otimes x^i)) = \psi((\dot{h} - i)v \otimes x^i) = (\dot{h} - i)v = \tau_i(h)\psi(v \otimes x^i),$$

$$\psi(z(v \otimes x^i)) = \psi(\dot{z}v \otimes x^i) = \tau_i(z)\psi(v \otimes x^i).$$

This implies claim (a).

To prove claim (b), we observe that simplicity of $N_{\dot{z},\dot{h}}$ clearly requires simplicity of N. Now suppose that N is a simple $R_{\dot{z}}$-module. We study simplicity of $N_{\dot{z},\dot{h}}$ using a case-by-case analysis.

Case 1. Assume $\dot{z} = 0$ and $(\mathbb{C}t + \mathbb{C}t^2\frac{d}{dt})N = 0$.

In this case, from (3.4) we get $eN_{0,\dot{h}} = pN_{0,\dot{h}} = 0$. Hence each nonzero weight element of $N_{0,\dot{h}}$ generates a proper highest weight submodule.

Case 2. Assume $\dot{z} = 0$ and $(\mathbb{C}t + \mathbb{C}t^2\frac{d}{dt})N \neq 0$.

Since N is a simple $R_{\dot{z}}$-module, we have $(\mathbb{C}t + \mathbb{C}t^2\frac{d}{dt})v \neq 0$ for any nonzero $v \in N$. From (3.4) it follows that for each $i \in \mathbb{Z}$ we either have $p(v \otimes x^i) \neq 0$ or $e(v \otimes x^j) \neq 0$. As the action of q on $N_{0,\dot{h}}$ is injective, it follows that any nonzero submodule V of $N_{0,\dot{h}}$ has support $\dot{h} + \mathbb{Z}$. Now from claim (a) we get $V = N_{0,\dot{h}}$, that is any nonzero submodule V coincides with $N_{0,\dot{h}}$ and thus $N_{0,\dot{h}}$ is simple.

Case 3. Assume $\dot{z} \neq 0$ and $\tau_{i,\dot{z}}(\mathbb{C}t + \mathbb{C}t\frac{d}{dt})N = 0$ for some $i \in \mathbb{Z}$.

In this case, we have $N = \mathbb{C}v$ with $tv = 0, (t\frac{d}{dt})v = -\frac{i}{2}v$. From (3.5), we have $p(v \otimes x^{i+1}) = e(v \otimes x^{i+1}) = 0$ and hence $v \otimes x^{i+1}$ generates a proper highest weight submodule of $N_{\dot{z},\dot{h}}$. Therefore $N_{\dot{z},\dot{h}}$ is not simple.

Case 4. Assume $\dot{z} \neq 0$ and $\tau_{i,\dot{z}}(\mathbb{C}t + \mathbb{C}t\frac{d}{dt})N \neq 0$ for all $i \in \mathbb{Z}$.

Since N is a simple $R_{\dot{z}}$-module, have $\tau_{i,\dot{z}}(\mathbb{C}t + \mathbb{C}t\frac{d}{dt})v \neq 0$ for any nonzero $v \in N$. From (3.5) it follows that for each $i \in \mathbb{Z}$ we have either $p(vx^{i+1}) \neq 0$ or $e(vx^{i+2}) \neq 0$ for some $v \in N$. Similarly to Case 2 we deduce that $N_{\dot{z},\dot{h}}$ is simple. This completes the proof of claim (b).

To prove claim (c), first assume $N_{\dot{z},\dot{h}} \cong N'_{\dot{z}',\dot{h}'}$. Then we have $i = h - h' \in \mathbb{Z}$, $\dot{z} = \dot{z}'$ and $N' \cong N \otimes x^i$ as U_0-modules. From (3.6) and (3.7) it follows that $A' \cong N^{\tau_{i,\dot{z}}}$ as $R_{\dot{z}}$-modules.

Now suppose that $\dot{z} = \dot{z}'$, $i = \dot{h} - \dot{h}' \in \mathbb{Z}$ and $N' \cong N^{\tau_{i,\dot{z}}}$ as $R_{\dot{z}}$-modules. From (3.6) and (3.7) it follows that $N' \cong N \otimes x^i$ as U_0-modules. From Lemma 8 we thus get $N_{\dot{z},\dot{h}} \cong N'_{\dot{z}',\dot{h}'}$. This completes the proof. □

Remark 10. Let N be a simple $R_{\dot{z}}$-module. If we have $\dot{z} = 0$ and $(\mathbb{C}t + \mathbb{C}t^2\frac{d}{dt})N = 0$, then $\dim N = 1$. If $\dot{z} \neq 0$ and $\tau_{i,\dot{z}}(\mathbb{C}t + \mathbb{C}t\frac{d}{dt})N = 0$ for some $i \in \mathbb{Z}$, then $\dim N = 1$. Thus, if N is an infinite dimensional simple $R_{\dot{z}}$-module, then $N_{\dot{z},\dot{h}}$ is a simple weight $\mathfrak{age}(1)$-module.

Now we are ready to prove our main result on classification of all simple weight $\mathfrak{age}(1)$-modules.

Theorem 11. *Each simple weight $\mathfrak{age}(1)$-module is isomorphic to one of the following simple modules:*

(*i*) *A simple highest or lowest weight module.*

(*ii*) *The module $Q(\lambda, \dot{h}) = \mathbb{C}[x, x^{-1}]$, where $\dot{h} \in \mathbb{C}$ and $\lambda \in \mathbb{C}^*$, with the action given by:*

$$zx^i = px^i = 0, \quad qx^i = x^{i+1}, \quad ex^i = \lambda x^{i-2}, \quad hx^i = (\dot{h} - i)x^i.$$

(iii) The module $Q'(\dot{z},\dot{h},\lambda)=\mathbb{C}[x,x^{-1}]$, *where* $\dot{h}\in\mathbb{C},\dot{z}\in\mathbb{C}^*$ *and* $\lambda\in\mathbb{C}\backslash\mathbb{Z}$, *with the action given by:*

$$qx^i=x^{i+1},\quad px^i=\dot{z}(\lambda+i)x^{i-1},\quad zx^i=\dot{z}x^i,$$
$$hx^i=(\dot{h}-i)x^i,\quad ex^i=\frac{\dot{z}}{2}(\lambda+i)(\lambda+i-1)x^{i-2}.$$

(iv) The module $N_{\dot{z},\dot{h}}$, *where* N *is an infinite dimensional simple* $R_{\dot{z}}$-*module.*

Proof. It is straightforward to verify that $Q(\lambda,\dot{h})$ in (ii) and $Q'(\dot{z},\dot{h},\lambda)$ in (iii) are simple $\mathfrak{age}(1)$-modules. From Remark 10 we know that the module $N_{\dot{z},\dot{h}}$ in (iv) is a simple $\mathfrak{age}(1)$-module.

Let V be any simple weight $\mathfrak{age}(1)$-module. Assume that $\dot{h}\in\operatorname{supp}(V)$ and that z acts on V as the scalar $\dot{z}$. Then $V_{\dot{h}}$ is both, a simple U_0-module and a simple $R_{\dot{z}}$. By Lemma 1 we have that $V_{\dot{h}}$ either is a one-dimensional module with $tV_{\dot{h}}=0$ or is an infinite dimensional module with t acting bijectively on it. If $N=V_{\dot{h}}$ has infinite dimension, then $V\cong N_{\dot{z},\dot{h}}$ by Lemma 8. Therefore it remains to consider the case when all nontrivial weight spaces of V have dimension one.

If V has a nonzero element annihilated by q, then V is clearly a lowest weight module. Therefore we may assume that q acts injectively on V. Let $0\neq w\in V_{\dot{h}}$ and consider first the case $z=0$.

In this case, we have $tw=pqw=0$ and hence $pw=0$ since the action of q is injective. This implies that the ideal $\mathbb{C}p+\mathbb{C}z$ of $\mathfrak{age}(1)$ annihilates V. Let $\mathfrak{a}=\mathfrak{age}(1)/(\mathbb{C}p+\mathbb{C}z)$. Then V is a simple $\mathfrak{a}$-module. Denote by $\bar{x}$ the image of $x\in\mathfrak{age}(1)$ in $\mathfrak{a}$. Note that $\bar{e}\bar{q}^2$ is a central element in $U(\mathfrak{a})$. By Schur's lemma, $\bar{e}\bar{q}^2$ acts on V as a scalar, say $\lambda\in\mathbb{C}$. If $\lambda=0$, then $e(q^2w)=0$ and V is a highest weight module. If $\lambda\neq 0$, then it is easy to check that $V\cong Q(\lambda,\dot{h})$.

Finally, consider the case $z\neq 0$. In this case, w generates an $\mathcal{H}$-submodule of V with nonzero central charge and one-dimensional weight spaces. Consequently, V contains a simple $\mathbb{C}h+\mathcal{H}$ submodule M with nonzero central charge. We make M into an $\mathfrak{age}(1)$-module by setting $ev=\frac{1}{2\dot{z}}p^2v$ and denote the resulting module by $M^{\mathfrak{age}(1)}$. Let $\mathbb{C}v$ be the one-dimensional trivial $\mathbb{C}h+\mathcal{H}$-module. Then [LZ1, Lemma 8 and Theorem 7] yield that V is isomorphic to a simple quotient of

$$\operatorname{Ind}_{\mathbb{C}h+\mathcal{H}}^{\mathfrak{age}(1)}(M\otimes\mathbb{C}v)\cong M^{\mathfrak{age}(1)}\otimes\operatorname{Ind}_{\mathbb{C}h+\mathcal{H}}^{\mathfrak{age}(1)}\mathbb{C}v,$$

which is of the form $M^{\mathfrak{age}(1)}\otimes X$, where X is a simple quotient of $\operatorname{Ind}_{\mathbb{C}h+\mathcal{H}}^{\mathfrak{age}(1)}\mathbb{C}v$. Since $M^{\mathfrak{age}(1)}\otimes X$ is a weight module, we get that X is the trivial module and hence $V\cong M^{\mathfrak{age}(1)}$. This gives that V is either a highest weight module or is isomorphic to $Q'(\dot{z}, \dot{h}, \lambda)$ with $\lambda \in \mathbb{C} \setminus \mathbb{Z}$. This completes the proof. $\square$

The following result is a direct consequence of Theorem 11.

Corollary 12. *Let V be a simple weight $\mathfrak{age}(1)$-module and $\dot{h} \in \mathrm{supp}(V)$. Then either $\dim V_{\dot{h}+i} \leq 1$ for all $i \in \mathbb{Z}$ or $\dim V_{\dot{h}+i} = \infty$ for all $i \in \mathbb{Z}$.*

Next we present a nontrivial example of a simple weight $\mathfrak{age}(1)$-module with infinite dimensional weight spaces.

Example 13. Let $\alpha = 1 - t^3(t-1)\frac{d}{dt} \in \mathcal{K}$. It is easy to see that α is irreducible in $\mathcal{K}$. Then $N = R_0/(R_0 \cap \mathcal{K}\alpha)$ is a simple R_0-module such that t acts bijectively on N. This implies that N is also a simple R_1-module and a simple module over the localization of $\mathbf{A}$ at t. In fact, we have $N = (t-1)^{-1}\mathbb{C}[t, t^{-1}]$ with the following action:

$$\frac{d}{dt} \cdot g(t) = \frac{dg(t)}{dt} + \frac{g(t)}{t^3(t-1)}, \quad t \cdot g(t) = tg(t) \quad \text{for all } g(t) \in N.$$

Thus we have the simple $\mathfrak{age}(1)$-module $N_{\dot{z},\dot{h}} = (t-1)^{-1}\mathbb{C}[t, t^{-1}, x, x^{-1}]$ with the action

$$\begin{aligned} h &:= \dot{h} - \partial_x, \quad z := 0, \quad q := x, \quad p := tx^{-1}, \\ e &:= x^{-2}t\partial_t + \frac{1}{t(t-1)x^2} + tx^{-2}\partial_x - 2tx^{-2}. \end{aligned}$$

if $\dot{z} = 0$, and the action

$$\begin{aligned} h &:= \dot{h} - \partial_x, \quad z := \dot{z}, \quad q := x, \quad p := x^{-1}\dot{z}(2\partial_t + \frac{2}{t^2(t-1)} + \partial_x - 1), \\ e &:= \dot{z}x^{-2}\left(\frac{t}{\dot{z}} + (2\partial_x - 3)(\partial_t + \frac{1}{t^2(t-1)}) + 2\partial_t^2 + \frac{4}{t^2(t-1)}\partial_t - \right. \\ &\qquad \left. \frac{2+2t+6t^2}{t^4(t-1)^2} + \frac{(\partial_x - 1)(\partial_x - 2)}{2}\right), \end{aligned}$$

if $\dot{z} \neq 0$ (here $\partial_x = x\frac{\partial}{\partial x}$ and $\partial_t = t\frac{\partial}{\partial t}$).

4. Simple weight $\mathcal{S}$-modules having a simple $\mathfrak{age}(1)$-submodule

In this section we classify all simple weight $\mathcal{S}$-modules that have a simple $\mathfrak{age}(1)$-submodule.

Let V be a simple weight $\mathfrak{age}(1)$-module. We have the induced weight $\mathcal{S}$-module $H(V) := \mathrm{Ind}_{\mathfrak{age}(1)}^{\mathcal{S}} V$ which can be identified with $\mathbb{C}[f] \otimes V$ as a vector space. In this section we classify all simple quotient $\mathcal{S}$-modules of $H(V)$. From [DLMZ, Corollary 8] we know that the center $Z(U(\mathcal{S}))$ of $U(\mathcal{S})$ equals $\mathbb{C}[z, c]$, where

$$c := (fp^2 - eq^2 - hpq) - \frac{1}{2}(h^2 + h + 4fe)z. \tag{4.1}$$

Recall some simple $\mathcal{S}$-module constructed in [LMZ] and [D]. Let V be any simple module over $\mathcal{H}$ with nonzero central charge $\dot{z}$. The module V becomes an $\mathcal{S}$-module by setting

$$ev = \frac{1}{2\dot{z}}p^2 v, \quad fv = -\frac{1}{2\dot{z}}q^2 v, \quad hv = (-\frac{pq}{\dot{z}} + \frac{1}{2})v, \text{ where } v \in V. \tag{4.2}$$

The resulting module will be denoted by $V^{\mathcal{S}}$.

Any simple $\mathfrak{sl}_2$-module W becomes an $\mathcal{S}$-module by setting $\mathcal{H}W = 0$. The resulting module will also be denoted by $W^{\mathcal{S}}$.

Next let us define a class of simple weight $\mathcal{S}$-modules. Let N be an infinite dimensional simple $R_{\dot{z}}$-module. Let $\mathbf{A}_{(t)}$ denote the localization of $\mathbf{A}$ at powers of t. Then N is also naturally an $\mathbf{A}_{(t)}$-module since t acts bijectively on N, see Lemma 1(d). Let $\dot{c}, \dot{z}, \dot{h} \in \mathbb{C}$. Then we have the simple weight $\mathfrak{age}(1)$-module $N_{\dot{z},\dot{h}}$ given by Lemma 7. We extend this $\mathfrak{age}(1)$-module $N_{\dot{z},\dot{h}}$ to an $\mathcal{S}$-module $N_{\dot{c},\dot{z},\dot{h}}$ as follows: For $\dot{z} = 0$ we set:

$$\begin{array}{l} h(v \otimes x^i) = (\dot{h} - i)v \otimes x^i, \quad z(v \otimes x^i) = 0, \\ q(v \otimes x^i) = v \otimes x^{i+1}, \quad p(v \otimes x^i) = (tv)x^{i-1}, \\ e(v \otimes x^i) = ((t^2\frac{d}{dt} + (i-2)t)v) \otimes x^{i-2}, \\ f(v \otimes x^i) = ((\frac{d}{dt} + (\dot{h} - 2)t^{-1} + \dot{c}t^{-2})v) \otimes x^{i+2}. \end{array} \tag{4.3}$$

For $\dot{z} \neq 0$ we set:
(4.4)

$$\begin{array}{l} h(v \otimes x^i) = (\dot{h} - i)v \otimes x^i, \quad z(v \otimes x^i) = \dot{z}v \otimes x^i, \\ q(v \otimes x^i) = vx^{i+1}, \quad p(v \otimes x^i) = ((2\dot{z}t\frac{d}{dt} + (i-1)\dot{z})v) \otimes x^{i-1}, \\ e(v \otimes x^i) = ((t + (2i-3)\dot{z}t\frac{d}{dt} + 2\dot{z}(t\frac{d}{dt})^2 + \frac{(i-1)(i-2)}{2}\dot{z})v) \otimes x^{i-2}, \\ f(v \otimes x^i) = ((-\frac{1}{2\dot{z}} - (\dot{h} - \frac{1}{2})\frac{d}{dt} - t(\frac{d}{dt})^2 - (\frac{(\dot{h}-1)(\dot{h}-2)}{4} + \frac{\dot{c}}{2\dot{z}})t^{-1})v) \otimes x^{i+2}. \end{array}$$

Lemma 14. *Formulae* (4.3) *and* (4.4) *indeed define the structure of an $\mathcal{S}$-module, moreover, the element c acts on $N_{\dot{c},\dot{z},\dot{h}}$ as the scalar $\dot{c}$.*

Proof. If $\dot{z} = 0$, then from (3.4) we know that $p = tx^{-1}$ which acts bijectively on $N_{\dot{z},\dot{h}}$. From (4.1) we obtain that $f = (eq^2 + hpq + \dot{c})p^{-2}$, that is $f(v \otimes x^i) = ((\frac{d}{dt} + (\dot{h} - 2)t^{-1} + \dot{c}t^{-2})v) \otimes x^{i+2}$. This indicates the formulae for $\dot{z} = 0$ and they are checked by a (long but) straightforward computation.

If $\dot{z}\neq 0$, then from (3.5) we know that

$$\begin{aligned} p^2-2ze =&(x^{-1}(2\dot{z}\partial_t+(\partial_x-1)\dot{z}))^2- \\ &-2x^{-2}\dot{z}\left(t+\dot{z}(2\partial_x-3))\partial_t+2\dot{z}\partial_t^2+\frac{(\partial_x-1)(\partial_x-2)}{2}\dot{z}\right) \\ =&-2\dot{z}tx^{-2} \end{aligned}$$

which is bijective on $A_{\dot{z},\dot{h}}$. From (4.1) we obtain that

$$f=(eq^2+hpq+z(h^2+h)/2+\dot{c})(p^2-2ze)^{-1},$$

that is $f(v\otimes x^i)$ equals

$$\left(\left(-\frac{1}{2\dot{z}}-(\dot{h}-\frac{1}{2})\frac{d}{dt}-t(\frac{d}{dt})^2-(\frac{(\dot{h}-1)(\dot{h}-2)}{4}+\frac{\dot{c}}{2\dot{z}})t^{-1}\right)v\right)\otimes x^{i+2}.$$

This indicates the formulae for $\dot{z}\neq 0$ and they are checked by a (long but) straightforward computation. □

Let $\dot{z}\in\mathbb{C}^*$ and $\lambda\in\mathbb{C}\backslash\mathbb{Z}$. For convenience, we define the simple $\mathcal{H}$-module $G(\dot{z},\lambda)=\mathbb{C}[x,x^{-1}]$ as follows (see [LMZ]):

$$qx^i=x^{i+1},\quad px^i=\dot{z}(\lambda+i)x^{i-1},\quad zx^i=\dot{z}x^i.$$

Note that the $\mathcal{H}$-module $G(\dot{z},\lambda)$ is an $\mathcal{H}$-submodule of the simple $\mathfrak{age}(1)$-module $Q'(\dot{z},\dot{h},\lambda)$ for any $\dot{h}\in\mathbb{C}$.

Now we can formulate the main result of this section.

Theorem 15. *Let V be a simple weight $\mathfrak{age}(1)$-module.*

(*a*) *If V is a highest weight module over $\mathfrak{age}(1)$, then $H(V)$ is a highest weight $\mathcal{S}$-module, which has a unique simple quotient.*

(*b*) *If $V\cong\bar{M}^-_{\mathfrak{age}(1)}(\dot{z},\dot{h})$ with $\dot{z}\neq 0$, then*

$$H(V)\cong M^-_{\mathcal{H}}(\dot{z})^{\mathcal{S}}\otimes M_{\mathfrak{sl}_2}(\dot{h}-\frac{1}{2})^{\mathcal{S}}.$$

The latter module has a unique simple quotient and this simple quotient is isomorphic to $M^-_{\mathcal{H}}(\dot{z})^{\mathcal{S}}\otimes\bar{M}_{\mathfrak{sl}_2}(\dot{h}-\frac{1}{2})^{\mathcal{S}}$.

(*c*) *If $V\cong Q(\lambda,\dot{h})$ for some $\dot{h}\in\mathbb{C}$ and $\lambda\in\mathbb{C}^*$, then $H(V)$ is simple.*

(*d*) *If $V\cong Q'(\dot{z},\dot{h},\lambda)$ for some $\dot{h}\in\mathbb{C},\dot{z}\in\mathbb{C}^*,\lambda\in\mathbb{C}\backslash\mathbb{Z}$, then*

$$H(V)\cong G(\dot{z},\lambda)^{\mathcal{S}}\otimes M_{\mathfrak{sl}_2}(\lambda+\dot{h}+\frac{1}{2})^{\mathcal{S}}.$$

The latter module has a unique simple quotient and this simple quotient is isomorphic to $G(\dot{z},\lambda)^{\mathcal{S}}\otimes\bar{M}_{\mathfrak{sl}_2}(\lambda+\dot{h}+\frac{1}{2})^{\mathcal{S}}$.

(*e*) *If $V\cong N_{\dot{z},\dot{h}}$, where N is an infinite dimensional simple $R_{\dot{z}}$-module, then any simple quotient of $H(V)$ is isomorphic to $N_{\dot{c},\dot{z},\dot{h}}$ for some $\dot{c}\in\mathbb{C}$.*

Proof. Claim (a) is clear. To prove claim (b), we note that the module $V = \bar{M}^-_{\mathfrak{age}(1)}(\dot{z}, \dot{h})$ has a simple $\mathcal{H}$-submodule $M^-_{\mathcal{H}}(\dot{z})$. Then, using (4.2), we can extend the action of $\mathcal{H}$ on $M^-_{\mathcal{H}}(\dot{z})$ to a lowest weight $\mathfrak{age}(1)$-module $\bar{M}^-_{\mathfrak{age}(1)}(\dot{z}, -1/2)$. Let $\mathbb{C}v$ be the 1-dimensional $\mathfrak{age}(1)$-module given by $ev = pv = qv = zv = 0$ and $hv = (\dot{h} - \frac{1}{2})v$. Then we have $V \cong \mathbb{C}v \otimes \bar{M}^-_{\mathfrak{age}(1)}(\dot{z}, -1/2)$. From [LMZ, Theorem 3] we know that $H(V) \cong M^-_{\mathcal{H}}(\dot{z})^{\mathcal{S}} \otimes \bar{M}_{\mathfrak{sl}_2}(\dot{h} - \frac{1}{2})^{\mathcal{S}}$. The rest of claim (b) now follows easily.

Let M be a nonzero submodule of $H(V)$. Then M is a weight module. Choose $0 \neq v_n = \sum_{i=0}^{k} c_i f^i \otimes x^{n-2i} \in M$ such that k is minimal. If $k > 0$, then $0 \neq pv_n = -\sum_{i=1}^{k} ic_i f^{i-1} \otimes x^{n-2i-1} \in M$ which contradicts our choice of v_n. Hence $k = 0$ and $v_n \in 1 \otimes V$. Now M has to coincide with $H(V)$ since V is a simple $\mathfrak{age}(1)$-module. Claim (c) follows.

To prove claim (d), we note that $V = Q'(\dot{z}, \dot{h}, \lambda)$ has a simple $\mathcal{H}$-submodule $G(\dot{z}, \lambda)$. Then we apply (4.2) to make $G(\dot{z}, \lambda)$ into an $\mathfrak{age}(1)$-module $G(\dot{z}, \lambda)^{\mathfrak{age}(1)}$ with $h(x^i) = -(\lambda + i + 1/2)x^i$. Let $\mathbb{C}v$ be the one-dimensional $\mathfrak{age}(1)$-module given by $ev = pv = qv = zv = 0$ and $hv = (\dot{h} + \lambda + \frac{1}{2})v$. Then $V \cong \mathbb{C}v \otimes G(\dot{z}, \lambda)_{\mathfrak{age}(1)}$. From [LMZ, Theorem 3] we know that $H(V) \cong C(\dot{z}, \lambda)^{\mathcal{S}} \otimes M_{\mathfrak{sl}_2}(\lambda + \dot{h} + \frac{1}{2})^{\mathcal{S}}$. The rest of claim (d) now follows easily.

Finally, we prove claim (e). During the proof of Lemma 14 we computed that in the module $N_{\dot{z},\dot{h}}$ we have $p^2(v \otimes x^i) = (t^2 v) \otimes x^{i-2}$ if $\dot{z} = 0$ and $(p^2 - 2ez)(v \otimes x^i) = (-2\dot{z}tv) \otimes x^{i-2}$ if $\dot{z} \neq 0$. Recall that t acts bijectively on N since N is infinite dimensional. Thus $p^2 - 2ez$, which is the coefficient at f in c, acts bijectively on $N_{\dot{z},\dot{h}}$ in this case. Let M be any simple quotient of $H(V)$. Then c acts as the scalar $\dot{c}$ on M. Now we have $f(1 \otimes V) \subset 1 \otimes V$ in M, and the action of f is uniquely determined by $\dot{c}$. Thus we have $M \cong N_{\dot{c},\dot{z},\dot{h}}$. This completes the proof. □

Let $\mathfrak{age}(1)^{op}$ be the parabolic subalgebra of $\mathcal{S}$ spanned by $\{f, h, p, q, z\}$, which is isomorphic to $\mathfrak{age}(1)$. There are actually many simple weight $\mathcal{S}$-modules that do not contain any simple $\mathfrak{age}(1)$-submodule or any simple $\mathfrak{age}(1)^{op}$-submodule. For example, if we take V to be a simple weight $\mathfrak{sl}_2$-module that is not highest or lowest weight module and $\dot{c} \neq 0$, then the simple $\mathcal{S}$-module $V^{\mathcal{S}} \otimes M_{\mathcal{H}}(\dot{z})^{\mathcal{S}}$ contains neither simple $\mathfrak{age}(1)$-submodules nor simple $\mathfrak{age}(1)^{op}$-submodules.

Taking into account the results of this paper it is natural to ask whether one can classify all simple weight $\mathcal{S}$-modules or all simple $\mathfrak{age}(1)$-module.

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