

ON SIMPLE MODULES OVER CONFORMAL GALILEI ALGEBRAS

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ABSTRACT. We study irreducible representations of two classes of conformal Galilei algebras in 1-spatial dimension. We construct a functor which transforms simple modules with nonzero central charge over the Heisenberg subalgebra into simple modules over the conformal Galilei algebras. This can be viewed as an analogue of oscillator representations. We use oscillator representations to describe the structure of simple highest weight modules over conformal Galilei algebras. We classify simple weight modules with finite dimensional weight spaces over finite dimensional Heisenberg algebras and use this classification and properties of oscillator representations to classify simple weight modules with finite dimensional weight spaces over conformal Galilei algebras.

Keywords: Lie algebra; Heisenberg algebra; weight module; conformal Galilei algebra; Schrödinger algebra

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1. INTRODUCTION AND DESCRIPTION OF RESULTS

Galilei groups and their Lie algebras are important objects in theoretical physics and attract a lot of attention in related mathematical areas, see for example [AIK, CZ, Gil, GI2, HP, NOR]. The conformal extension $\widetilde{\mathfrak{g}}^{(l)}$ of the Galilei algebra is parameterized by $l \in \frac{1}{2}\mathbb{N}$ and is called the *l-conformal Galilei algebra*, see [NOR]. The instance of $l = 1/2$, known in the literature as the Schrödinger algebra, turns up in a variety of physical systems, see for example [ADD, DDM, HU]. Various classes of representations of the Schrödinger algebra were studied and classified in [DDM, WZ, Du]. Motivated by recent investigations of the non-relativistic version of the AdS/CFT correspondence, conformal Galilei algebras with $l > 1/2$ have recently attracted considerable attention, see [AIK, LSW1, LSW2, BG, DH]. In this paper, we study representations of the conformal Galilei algebras $\mathfrak{g}^{(l)}$ for $l \in \frac{1}{2}\mathbb{N}$ and $\widetilde{\mathfrak{g}}^{(l)}$ for $l \in \mathbb{N} - \frac{1}{2}$. Let us start by defining these algebras.

We denote by \mathbb{Z} , \mathbb{Z}_+ , \mathbb{N} , and \mathbb{C} the sets of all integers, nonnegative integers, positive integers, and complex numbers, respectively. For a Lie algebra \mathfrak{a} we denote by $U(\mathfrak{a})$ the universal enveloping algebra of \mathfrak{a} .

For $l \in \mathbb{N} - \frac{1}{2}$, the conformal centrally extended Galilei algebra $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}^{(l)}$ is the Lie algebra with the basis $\{e, h, f, p_k, z \mid k = 0, \dots, 2l\}$ and the Lie

bracket given by:

$$\begin{aligned} [h, e] &:= 2e, \quad [h, f] := -2f, \quad [e, f] := h, \\ [h, p_k] &:= 2(l-k)p_k, \quad [e, p_k] := kp_{k-1}, \quad [f, p_k] := (2l-k)p_{k+1}, \\ [z, \mathfrak{g}^{(l)}] &:= 0, \\ [p_k, p_{k'}] &:= \delta_{k+k', 2l}(-1)^{k+l+\frac{1}{2}}k!(2l-k)!z, \quad k, k' = 0, 1, \dots, 2l. \end{aligned}$$

Denote by $\mathfrak{g} = \mathfrak{g}^{(l)}$ the quotient Lie algebra $\widetilde{\mathfrak{g}}^{(l)}/\mathbb{C}z$ and note that the Lie algebras $\mathfrak{g}^{(l)}$ can be defined for all $l \in \mathbb{N}$ by setting $z = 0$ in the above. The Lie algebras $\mathfrak{g}^{(l)}$ for $l \in \mathbb{N}$ do not have nontrivial central extensions. The algebras $\mathfrak{g}^{(l)}$ for $l \in \frac{1}{2}\mathbb{N}$ are called *centerless conformal Galilei algebras*. In this paper we study both $\mathfrak{g}^{(l)}$ for $l \in \frac{1}{2}\mathbb{N}$ and $\widetilde{\mathfrak{g}}^{(l)}$ for $l \in \mathbb{N} - \frac{1}{2}$.

The conformal Galilei algebra $\widetilde{\mathfrak{g}}^{(l)}$ has the finite dimensional Heisenberg subalgebra:

$$\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}^{(l)} := \text{span}\{p_k, z \mid k = 0, 1, \dots, 2l\}.$$

The algebra $\widetilde{\mathfrak{g}}^{(l)}$ is \mathbb{Z} -graded with respect to the adjoint action of h . For $i \in \mathbb{Z}$ we set

$$\widetilde{\mathfrak{g}}_i := \{x \in \widetilde{\mathfrak{g}} \mid [h, x] = ix\}, \quad \widetilde{\mathfrak{g}}_0 := \mathbb{C}h + \mathbb{C}z, \quad \widetilde{\mathfrak{g}}_{\pm} := \bigoplus_{i \in \pm\mathbb{N}} \widetilde{\mathfrak{g}}_i.$$

For $l \in \mathbb{N}$ we have a Cartan subalgebra $\mathfrak{g}_0 := \mathbb{C}h + \mathbb{C}p_l$ in $\mathfrak{g}^{(l)}$ and an abelian subalgebra $\mathcal{H} = \text{span}\{p_k \mid k = 0, 1, \dots, 2l\}$ in $\mathfrak{g}^{(l)}$.

Both $\widetilde{\mathfrak{g}}$ and \mathfrak{g} have an \mathfrak{sl}_2 -subalgebra spanned by e, f, h . Let \mathfrak{a} be any of the following Lie algebras: \mathfrak{sl}_2 , $\widetilde{\mathfrak{g}}$, \mathfrak{g} , or $\mathbb{C}h + \widetilde{\mathcal{H}}$. An \mathfrak{a} -module V is called a weight module if h acts diagonally on V , i.e.

$$V = \bigoplus_{\lambda \in \mathbb{C}} V_{\lambda},$$

where $V_{\lambda} := \{v \in V \mid hv = \lambda v\}$. Denote $\text{supp}(V) := \{\lambda \in \mathbb{C} \mid V_{\lambda} \neq 0\}$.

For any $(\dot{z}, \dot{h}) \in \mathbb{C}^2$, we have the one dimensional $\widetilde{\mathfrak{g}}_+ \oplus \widetilde{\mathfrak{g}}_0$ -module $\mathbb{C}w$ with $dw = \dot{z}w$, $zw = \dot{z}w$ and $\widetilde{\mathfrak{g}}_+w = 0$. Using it we define, as usual, the Verma module

$$M_{\widetilde{\mathfrak{g}}}(\dot{z}, \dot{h}) = \text{Ind}_{\widetilde{\mathfrak{g}}_0 + \widetilde{\mathfrak{g}}_+}^{\widetilde{\mathfrak{g}}} \mathbb{C}w.$$

The module $M_{\widetilde{\mathfrak{g}}}(\dot{z}, \dot{h})$ is generated by a highest weight vector (the lift of w) of highest weight \dot{h} . Denote by $\widetilde{M}_{\widetilde{\mathfrak{g}}}(\dot{z}, \dot{h})$ the unique simple quotient of $M_{\widetilde{\mathfrak{g}}}(\dot{z}, \dot{h})$. Similarly we may define the Verma modules $M_{\mathfrak{g}}(\dot{p}_l, \dot{h})$ for $l \in \mathbb{N}$, $M_{\mathfrak{sl}_2}(\dot{h})$, $M_{\widetilde{\mathcal{H}}}(\dot{z})$ for $l \in \mathbb{N} - 1/2$, and the corresponding simple quotients $\widetilde{M}_{\mathfrak{g}}(\dot{p}_l, \dot{h})$, $\widetilde{M}_{\mathfrak{sl}_2}(\dot{h})$ and $\widetilde{M}_{\widetilde{\mathcal{H}}}(\dot{z})$. Verma modules have the usual universal property that each module generated by a highest weight vector is a quotient of some Verma module. Similarly one defines lowest weight module.

We start with a brief overview of known results. It is well-known that the Verma module $M_{\mathfrak{sl}_2}(\dot{h})$ is simple if and only if $\dot{h} \notin \mathbb{Z}^+$ (see e.g. [Maz, Chapter 3]), while $M_{\widetilde{\mathcal{H}}}(\dot{z})$ is simple if and only if $\dot{z} \neq 0$ (see e.g. [KR]). All simple modules over \mathfrak{sl}_2 and $\widetilde{\mathcal{H}}^{(\frac{1}{2})}$ were classified in [BI] (up to description of irreducible elements of some non-commutative Euclidean rings). An

explicit description of simple lowest (or highest) weight $\widetilde{\mathfrak{g}}^{(l)}$ -modules for small l was given in [AI, DDM, Mr]. Furthermore, an explicit description of simple lowest (or highest) weight $\widetilde{\mathfrak{g}}^{(l)}$ -modules over all $\widetilde{\mathfrak{g}}^{(l)}$ was obtained in [AIK]. Very recently, a classification of simple weight modules with finite dimensional weight spaces over the Shrödinger algebra $\widetilde{\mathfrak{g}}^{(1/2)}$ was given in [Du].

Next we briefly describe the results of this paper. Our first observation is the following construction of what is natural to call *oscillator representations* for $\widetilde{\mathfrak{g}}^{(l)}$. For $l \in \mathbb{N} - \frac{1}{2}$ denote by $U(\widetilde{\mathcal{H}}^{(l)})_{(z)}$ the localization of $U(\widetilde{\mathcal{H}}^{(l)})$ with respect to the multiplicative subset $\{z^i | i \in \mathbb{Z}_0\}$.

Theorem 1. *Let $l \in \mathbb{N} - \frac{1}{2}$.*

- (i) *There is a unique algebra homomorphism $\Phi_l : U(\widetilde{\mathfrak{g}}^{(l)}) \rightarrow U(\widetilde{\mathcal{H}}^{(l)})_{(z)}$ which is the identity on $U(\widetilde{\mathcal{H}}^{(l)})$ and such that*

$$\begin{aligned} \Phi_l(e) = E &:= \frac{1}{z} \left(\sum_{k=1}^{2l} (-1)^{k+l+\frac{1}{2}} \frac{(l-k)}{(k-1)!(2l-k)!} p_{k-1} p_{2l-k} \right), \\ \Phi_l(f) = F &:= \frac{1}{z} \left(\sum_{k=1}^{2l} (-1)^{k+l-\frac{1}{2}} \frac{(l-k)}{(k-1)!(2l-k)!} p_k p_{2l-k+1} \right), \\ \Phi_l(h) = H &:= \frac{2}{z} \left(\sum_{k=0}^{l-\frac{1}{2}} (-1)^{k+l-\frac{1}{2}} \frac{l-k}{k!(2l-k)!} p_{2l-k} p_k \right) - \frac{(l+\frac{1}{2})^2}{2}. \end{aligned}$$

- (ii) *Composition of induction from $U(\widetilde{\mathcal{H}}^{(l)})$ to $U(\widetilde{\mathcal{H}}^{(l)})_{(z)}$ with pulling back along Φ defines a functor from the category of $\widetilde{\mathcal{H}}^{(l)}$ -modules to the category of $\widetilde{\mathfrak{g}}^{(l)}$ -modules which annihilates all simple $\widetilde{\mathcal{H}}^{(l)}$ -modules with zero central charge and sends simple $\widetilde{\mathcal{H}}^{(l)}$ -modules with nonzero central charge to simple $\widetilde{\mathfrak{g}}^{(l)}$ -modules.*

Theorem 1 is inspired by generalized oscillator representations for the Heisenberg-Virasoro algebra constructed in [LZ, Section 3.2]. If V is a simple $\widetilde{\mathcal{H}}^{(l)}$ -module with nonzero action of z , we will denote the image of V under the functor described in Theorem 1(ii) by $V^{\widetilde{\mathfrak{g}}}$. Any \mathfrak{sl}_2 -module V becomes a $\widetilde{\mathfrak{g}}^{(l)}$ -module by setting $\mathcal{H} \cdot V = 0$. The resulting $\widetilde{\mathfrak{g}}^{(l)}$ -module will be also denoted by $V^{\widetilde{\mathfrak{g}}}$ (we will make sure that this abuse of notation will not create any confusion).

Consider $\widetilde{\mathfrak{g}}^{(l)}$ for $l \in \mathbb{N} - \frac{1}{2}$. Then for every $\dot{h} \in \mathbb{C}$ we have (using the uniqueness of simple highest weight modules) that $\widetilde{M}_{\widetilde{\mathfrak{g}}^{(l)}}(0, \dot{h}) \cong \widetilde{M}_{\mathfrak{sl}_2}(\dot{h})^{\widetilde{\mathfrak{g}}}$, in particular, $\widetilde{\mathcal{H}} \widetilde{M}_{\widetilde{\mathfrak{g}}^{(l)}}(0, \dot{h}) = 0$. All highest weight $\widetilde{\mathfrak{g}}^{(l)}$ -modules with nonzero central charge were explicitly described in [AIK]. These modules can now be realized in a totally different way as tensor products of certain \mathfrak{sl}_2 -modules and oscillator modules as follows:

Theorem 2. *Let $l \in \mathbb{N} - \frac{1}{2}$ and $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}^{(l)}$. For any $\dot{h} \in \mathbb{C}$ and $\dot{z} \in \mathbb{C}^*$ we have*

$$M_{\widetilde{\mathfrak{g}}}(\dot{z}, \dot{h}) \cong M_{\widetilde{\mathcal{H}}}(\dot{z})^{\widetilde{\mathfrak{g}}} \otimes M_{\mathfrak{sl}_2}(\dot{h} + \frac{1}{2}(l + \frac{1}{2})^2)^{\widetilde{\mathfrak{g}}},$$

$$\bar{M}_{\bar{\mathfrak{g}}}(z, \hbar) \cong M_{\tilde{\mathcal{H}}}(z)^{\bar{\mathfrak{g}}} \otimes \bar{M}_{\mathfrak{sl}_2}(\hbar + \frac{1}{2}(l + \frac{1}{2})^2)^{\bar{\mathfrak{g}}}.$$

Hence the Verma module $M_{\bar{\mathfrak{g}}}(z, \hbar)$ is simple if and only if $\hbar + \frac{(l+\frac{1}{2})^2}{2} \notin \mathbb{Z}_+$.

Theorem 2 provides a very clear description of simple highest weight $\bar{\mathfrak{g}}^{(l)}$ -modules for $l \in \mathbb{N} - \frac{1}{2}$. Furthermore, Theorem 2 generalizes to the following result which even covers non-weight modules:

Theorem 3. *Let $l \in \mathbb{N} - \frac{1}{2}$ and $\tilde{\mathfrak{g}} = \bar{\mathfrak{g}}^{(l)}$.*

- (i) *If M is a simple $\tilde{\mathcal{H}}^{(l)}$ -module with nonzero central charge, and N is a simple \mathfrak{sl}_2 -module, then $M^{\bar{\mathfrak{g}}} \otimes N^{\bar{\mathfrak{g}}}$ is a simple $\tilde{\mathfrak{g}}$ -module.*
- (ii) *Suppose that V is a simple $\tilde{\mathfrak{g}}$ -module with nonzero central charge z . If $\text{Res}_{\tilde{\mathcal{H}}^{(l)}}^{\bar{\mathfrak{g}}} V$ contains a simple submodule M , then $V \cong M^{\bar{\mathfrak{g}}} \otimes N^{\bar{\mathfrak{g}}}$ for some simple \mathfrak{sl}_2 -module N .*

To describe simple highest weight $\mathfrak{g}^{(l)}$ -modules for $l \in \mathbb{N}$, we need more notation. Let $\dot{p}_l, \dot{h} \in \mathbb{C}$ and

$$\mathfrak{n} = \text{span}\{e, h, p_i \mid i = 0, \dots, 2l\}, \quad F_1 = \mathbb{C}[x].$$

Define an \mathfrak{n} -module structure on the Fock space F_1 as follows:

$$(1.1) \quad e = \frac{\partial}{\partial x}, \quad h = \dot{h} - \frac{2x\partial}{\partial x}, \quad p_l = \dot{p}_l, \quad p_0 = p_1 = \dots = p_{l-1} = 0,$$

$$(1.2) \quad p_{l+k} = \frac{(l+k)!x^k}{k!}, \quad \text{for } k = 1, 2, \dots, l.$$

Now we can describe all simple highest weight $\mathfrak{g}^{(l)}$ -modules (note that in the case $l = 1$ this statement is proved in [Wi]):

Theorem 4. *Let $l \in \mathbb{N}$, and $\mathfrak{g} = \mathfrak{g}^{(l)}$.*

- (i) *If $\dot{p}_l = 0$, then $\bar{M}_{\mathfrak{g}}(0, \dot{h}) \cong \bar{M}_{\mathfrak{sl}_2}(\dot{h})^{\mathfrak{g}}$.*
- (ii) *If $\dot{p}_l \neq 0$, then $\bar{M}_{\mathfrak{g}}(\dot{p}_l, \dot{h})_{\hbar-2i} \cong \text{Ind}_{\mathfrak{n}}^{\mathfrak{g}^{(l)}} F_1$.*

For any $z \in \mathbb{C}^*$ and $a \in \mathbb{C} \setminus \mathbb{Z}$, the $\tilde{\mathcal{H}}^{(\frac{1}{2})}$ -module structure $D(a, z)$ on $\mathbb{C}[x, x^{-1}]$ is defined as follows: for $i \in \mathbb{Z}$ we set

$$(1.3) \quad p_1 x^i = x^{i+1}, \quad p_0 x^i = -z(a+i)x^{i-1}, \quad z x^i = \dot{z} x^i.$$

The next two theorems give a complete classification of simple weight modules which have a nonzero finite dimensional weight space over the algebras $\mathfrak{g}^{(l)}$ for $l \in \frac{1}{2}\mathbb{N}$ and $\bar{\mathfrak{g}}^{(l)}$ for $l \in \mathbb{N} - \frac{1}{2}$, respectively.

Theorem 5. *Let $l \in \mathbb{N}$, $\mathfrak{g} = \mathfrak{g}^{(l)}$, and V be any simple weight \mathfrak{g} -module with a finite dimensional nonzero weight space. Then one of the following holds:*

- (i) *V is isomorphic to $N^{\mathfrak{g}}$ for some simple weight \mathfrak{sl}_2 module N ;*
- (ii) *V is a highest or lowest weight module.*

Theorem 6. *Let $l \in \mathbb{N} - \frac{1}{2}$, $\tilde{\mathfrak{g}} = \bar{\mathfrak{g}}^{(l)}$, and V be any simple weight $\tilde{\mathfrak{g}}$ -module with a finite dimensional nonzero weight space.*

- (i) If V has zero central charge, the V is isomorphic to $N^{\bar{\mathfrak{g}}}$ for some simple weight \mathfrak{sl}_2 -module N ;
- (ii) If V has a nonzero central charge, say $\dot{z} \in \mathbb{C}^*$, then one of the following holds:
 - (a) V is a simple highest or lowest weight module;
 - (b) $l = \frac{1}{2}$ and V is isomorphic to $D(a, \dot{z})^{\bar{\mathfrak{g}}^{(1/2)}} \otimes N^{\bar{\mathfrak{g}}^{(1/2)}}$ for some $a \in \mathbb{C} \setminus \mathbb{Z}$ and a finite dimensional simple \mathfrak{sl}_2 -module N .

In the case $l = 1/2$ a disguised version of Theorem 6 is proved in [Du].

The paper is organized as follows. In Section 2, we prove Theorems 1, 2 and 3. Using Theorem 1, we construct three examples of families of simple $\bar{\mathfrak{g}}^{(l)}$ -modules: the first one containing weight modules with finite dimensional weight spaces, the second one containing weight modules with infinite dimensional weight spaces, and the third one containing non-weight modules. Given simple $\tilde{\mathcal{H}}^{(l)}$ -modules with nonzero central charge and simple \mathfrak{sl}_2 -modules, Theorem 3 provides a lot new examples of simple $\bar{\mathfrak{g}}$ -modules.

In Section 3 we prove Theorems 4, 5 and 6. We use our oscillator representations of $\bar{\mathfrak{g}}^{(l)}$ and classification of all simple graded modules with a finite dimensional homogeneous spaces over finite dimensional Heisenberg algebras. We would like to mention that our results on simple graded modules over finite Heisenberg algebras are somewhat similar to those for infinite Heisenberg algebras but our proofs here are quite different (compare e.g. with [Fu]).

2. OSCILLATOR REPRESENTATIONS

In this section, we will prove Theorems 1, 2 and 3.

2.1. Proof of Theorem 1. Claim (ii) follows directly from claim (i). To prove claim (i) we need to verify the defining relations (i.e. that Lie bracket) for the images (under Φ) of the generators. If both generators belong to the Heisenberg subalgebra, there is nothing to check. For the remaining generators the check is a direct but lengthy computation which we organize in a series of lemmata below.

Lemma 7. *We have $[H, p_i] = 2(l - i)p_i$ for all $i = 0, 1, \dots, 2l$.*

Proof. Using the fact that $2l$ is odd, we have

$$\begin{aligned}
 [H, p_i] &= \left[\frac{2}{z} \left(\sum_{k=0}^{l-\frac{1}{2}} (-1)^{k+l-\frac{1}{2}} \frac{l-k}{k!(2l-k)!} p_{2l-k} p_k \right) - \frac{(l+\frac{1}{2})^2}{2}, p_i \right] \\
 &= \left[\frac{1}{z} \left(\sum_{k=0}^{2l} (-1)^{k+l-\frac{1}{2}} \frac{l-k}{k!(2l-k)!} p_{2l-k} p_k \right), p_i \right]
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{z} \left(\sum_{k=0}^{2l} (-1)^{k+l-\frac{1}{2}} \frac{l-k}{k!(2l-k)!} [p_{2l-k} p_k, p_i] \right) \\
&= \frac{1}{z} \left(\sum_{k=0}^{2l} (-1)^{k+l-\frac{1}{2}} \frac{l-k}{k!(2l-k)!} [p_{2l-k}, p_i] p_k \right) \\
&\quad + \frac{1}{z} \left(\sum_{k=0}^{2l} (-1)^{k+l-\frac{1}{2}} \frac{l-k}{k!(2l-k)!} p_{2l-k} [p_k, p_i] \right) \\
&= \frac{1}{z} (-1)^{i+l-\frac{1}{2}} \frac{l-i}{i!(2l-i)!} (-1)^{2l-i+l+\frac{1}{2}} i!(2l-i)! z p_i \\
&\quad + \frac{1}{z} (-1)^{2l-i+l-\frac{1}{2}} \frac{l-(2l-i)}{i!(2l-i)!} (-1)^{2l-i+l+\frac{1}{2}} i!(2l-i)! z p_i \\
&= 2(l-i)p_i.
\end{aligned}$$

□

Lemma 8. We have $[E, p_i] = ip_{i-1}$ and $[F, p_i] = (2l-i)p_{i+1}$ for all $i = 0, 1, \dots, 2l$.

Proof. We first verify the relation $[E, p_i] = ip_{i-1}$:

$$\begin{aligned}
[E, p_i] &= \left[\frac{1}{z} \left(\sum_{k=1}^{2l} (-1)^{k+l+\frac{1}{2}} \frac{(l-k)}{(k-1)!(2l-k)!} p_{k-1} p_{2l-k} \right), p_i \right] \\
&= \frac{1}{z} \left(\sum_{k=1}^{2l} (-1)^{k+l+\frac{1}{2}} \frac{(l-k)}{(k-1)!(2l-k)!} [p_{k-1} p_{2l-k}, p_i] \right) \\
&= \frac{1}{z} \left(\sum_{k=1}^{2l} (-1)^{k+l+\frac{1}{2}} \frac{(l-k)}{(k-1)!(2l-k)!} [p_{k-1}, p_i] p_{2l-k} \right) \\
&\quad + \frac{1}{z} \left(\sum_{k=1}^{2l} (-1)^{k+l+\frac{1}{2}} \frac{(l-k)}{(k-1)!(2l-k)!} p_{k-1} [p_{2l-k}, p_i] \right) \\
&= \frac{(-1)^{2l-i+1+l+\frac{1}{2}} (l-(2l-i+1))}{z(2l-i+1-1)!(2l-(2l-i+1))!} (-1)^{l+2l-i+\frac{1}{2}} i!(2l-i)! z p_{i-1} \\
&\quad + \frac{1}{z} (-1)^{i+l+\frac{1}{2}} \frac{(l-i)}{(i-1)!(2l-i)!} (-1)^{l+2l-i+\frac{1}{2}} i!(2l-i)! z p_{i-1} \\
&= -(i-l-1)i - i(l-i) p_{i-1} \\
&= ip_{i-1}.
\end{aligned}$$

Now we verify the relation $[F, p_i] = (2l-i)p_{i+1}$:

$$\begin{aligned}
[F, p_i] &= \frac{1}{z} \left(\sum_{k=1}^{2l} (-1)^{k+l-\frac{1}{2}} \frac{(l-k)}{(k-1)!(2l-k)!} [p_k p_{2l-k+1}, p_i] \right) \\
&= \frac{1}{z} \left(\sum_{k=1}^{2l} (-1)^{k+l-\frac{1}{2}} \frac{(l-k)}{(k-1)!(2l-k)!} [p_k, p_i] p_{2l-k+1} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{z} \left(\sum_{k=1}^{2l} (-1)^{k+l-\frac{1}{2}} \frac{(l-k)}{(k-1)!(2l-k)!} [p_{2l-k+1}, p_i] p_k \right) \\
& = \frac{(-1)^{2l-i+l-\frac{1}{2}} (l-(2l-i))}{z(2l-i-1)!(2l-(2l-i))!} (-1)^{l+2l-i+\frac{1}{2}} i!(2l-i)! z p_{i+1} \\
& \quad + \frac{(-1)^{i+1+l-\frac{1}{2}} (l-i-1)}{z(i+1-1)!(2l-i-1)!} (-1)^{l+2l-i+\frac{1}{2}} i!(2l-i)! z p_{i+1} \\
& = (2l-i) p_{i+1}.
\end{aligned}$$

□

Lemma 9. *We have $[E, F] = H$.*

Proof. We will use Lemma 8 in the following computation:

$$\begin{aligned}
[E, F] & = [E, \frac{1}{z} \sum_{k=1}^{2l} (-1)^{k+l-\frac{1}{2}} \frac{(l-k)}{(k-1)!(2l-k)!} p_k p_{2l-k+1}] \\
& = \frac{1}{z} \sum_{k=1}^{2l} (-1)^{k+l-\frac{1}{2}} \frac{(l-k)}{(k-1)!(2l-k)!} [E, p_k] p_{2l-k+1} \\
& \quad + \frac{1}{z} \sum_{k=1}^{2l} (-1)^{k+l-\frac{1}{2}} \frac{(l-k)}{(k-1)!(2l-k)!} p_k [E, p_{2l-k+1}] \\
& = \frac{1}{z} \sum_{k=1}^{2l} (-1)^{k+l-\frac{1}{2}} \frac{(l-k)k}{(k-1)!(2l-k)!} p_{k-1} p_{2l-k+1} \\
& \quad + \frac{1}{z} \sum_{k=1}^{2l} (-1)^{k+l-\frac{1}{2}} \frac{(l-k)(2l-k+1)}{(k-1)!(2l-k)!} p_k p_{2l-k} \\
& = \frac{1}{z} \sum_{k'=0}^{2l-1} (-1)^{k'+1+l-\frac{1}{2}} \frac{(l-k'-1)(k'+1)}{(k')!(2l-k'-1)!} p_{k'} p_{2l-k'} \\
& \quad + \frac{1}{z} \sum_{k=1}^{2l} (-1)^{k+l-\frac{1}{2}} \frac{(l-k)(2l-k+1)}{(k-1)!(2l-k)!} p_k p_{2l-k} \\
& = \frac{1}{z} \sum_{k'=0}^{2l-1} (-1)^{k'+1+l-\frac{1}{2}} \frac{(l-k'-1)(k'+1)(2l-k')}{(k')!(2l-k')!} p_{k'} p_{2l-k'} \\
& \quad + \frac{1}{z} \sum_{k=0}^{2l} (-1)^{k+l-\frac{1}{2}} \frac{(l-k)(2l-k+1)k}{k!(2l-k)!} p_k p_{2l-k} \\
& = \frac{1}{z} \sum_{k=0}^{2l} (-1)^{k+l+\frac{1}{2}} \frac{(k+1)(l-k-1)(2l-k) + (l-k)(-2l+k-1)k}{k!(2l-k)!} p_k p_{2l-k} \\
& = \frac{1}{z} \sum_{k=0}^{l-\frac{1}{2}} (-1)^{k+l+\frac{1}{2}} \frac{3k^2 + 2l^2 - 6kl - 2l + k}{k!(2l-k)!} p_k p_{2l-k}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{z} \sum_{k=l+\frac{1}{2}}^{2l} (-1)^{k+l+\frac{1}{2}} \frac{3k^2 + 2l^2 - 6kl - 2l + k}{k!(2l-k)!} p_k p_{2l-k} \\
& = \frac{1}{z} \sum_{k=0}^{l-\frac{1}{2}} (-1)^{k+l+\frac{1}{2}} \frac{3k^2 + 2l^2 - 6kl - 2l + k}{k!(2l-k)!} p_{2l-k} p_k \\
& \quad + \frac{1}{z} \sum_{k=0}^{l-\frac{1}{2}} (-1)^{k+l+\frac{1}{2}} \frac{3k^2 + 2l^2 - 6kl - 2l + k}{k!(2l-k)!} (-1)^{l+k+\frac{1}{2}} k!(2l-k)! z \\
& \quad + \frac{1}{z} \sum_{k=l+\frac{1}{2}}^{2l} (-1)^{k+l+\frac{1}{2}} \frac{3k^2 + 2l^2 - 6kl - 2l + k}{k!(2l-k)!} p_k p_{2l-k} \\
& = \frac{1}{z} \sum_{k=0}^{l-\frac{1}{2}} (-1)^{k+l+\frac{1}{2}} \frac{3k^2 + 2l^2 - 6kl - 2l + k}{k!(2l-k)!} p_{2l-k} p_k \\
& \quad + \sum_{k=0}^{l-\frac{1}{2}} (3k^2 + 2l^2 - 6kl - 2l + k) \\
& \quad + \frac{1}{z} \sum_{k=l+\frac{1}{2}}^{2l} (-1)^{k+l+\frac{1}{2}} \frac{3k^2 + 2l^2 - 6kl - 2l + k}{k!(2l-k)!} p_k p_{2l-k} \\
& = \frac{1}{z} \sum_{k=0}^{l-\frac{1}{2}} (-1)^{k+l+\frac{1}{2}} \frac{3k^2 + 2l^2 - 6kl - 2l + k}{k!(2l-k)!} p_{2l-k} p_k - \frac{(l+\frac{1}{2})^2}{2} \\
& \quad + \frac{1}{z} \sum_{k=0}^{l-\frac{1}{2}} (-1)^{2l-k+l+\frac{1}{2}} \frac{-6kl + 3k^2 - k + 2l^2}{k!(2l-k)!} p_{2l-k} p_k \\
& = \frac{1}{z} \sum_{k=0}^{l-\frac{1}{2}} (-1)^{k+l+\frac{1}{2}} \frac{2(k-l)}{k!(2l-k)!} p_{2l-k} p_k - \frac{(l+\frac{1}{2})^2}{2} \\
& = \frac{1}{z} \sum_{k=0}^{l-\frac{1}{2}} (-1)^{k+l+\frac{1}{2}} \frac{2(k-l)}{k!(2l-k)!} p_{2l-k} p_k - \frac{(l+\frac{1}{2})^2}{2} = H.
\end{aligned}$$

□

Finally, the relations $[H, E] = 2E$ and $[H, F] = -2F$ follow easily from Lemma 7. Theorem 1 follows.

2.2. Proof of Theorem 2. Theorem 2 can be proved by a direct computation. We give here a more conceptual argument.

Let w be a highest weight vector of $M_{\tilde{\mathfrak{g}}}(\tilde{z}, \tilde{h})$ and assume $\tilde{z} \neq 0$. Then $U(\tilde{\mathcal{H}})w = M_{\tilde{\mathcal{H}}}(\tilde{z})$, which is a simple $\tilde{\mathcal{H}}$ -module. From Theorem 1, we have the simple highest weight $\tilde{\mathfrak{g}}$ module $(U(\tilde{\mathcal{H}})w)^{\tilde{\mathfrak{g}}}$ with $h \cdot w = -\frac{(l+\frac{1}{2})^2}{2}w$ and $e \cdot w = 0$, where we use \cdot to denote the new action. Hence we have

the decomposition $U(\tilde{\mathcal{H}})w \cong (U(\mathcal{H})w)^{\tilde{\mathfrak{g}}} \otimes (\mathbb{C}u)$ of $\mathbb{C}h + \tilde{\mathcal{H}} + \mathbb{C}e$ -modules, where $\mathbb{C}u$ is regarded as a $\mathbb{C}h + \mathbb{C}e + \tilde{\mathcal{H}}$ -module via $hu = (\dot{h} + \frac{(l+\frac{1}{2})^2}{2})u$ and $(\mathbb{C}e + \tilde{\mathcal{H}})u = 0$. Using [LZ, Lemma 8], we have

$$\begin{aligned} M_{\tilde{\mathfrak{g}}}(\dot{z}, \dot{h}) &\cong \text{Ind}_{\mathbb{C}h+\mathbb{C}e+\tilde{\mathcal{H}}}^{\tilde{\mathfrak{g}}} \text{Ind}_{\tilde{\mathfrak{g}}+\tilde{\mathfrak{g}}_0}^{\mathbb{C}h+\mathbb{C}e+\tilde{\mathcal{H}}} \mathbb{C}w \cong \text{Ind}_{\mathbb{C}h+\mathbb{C}e+\tilde{\mathcal{H}}}^{\tilde{\mathfrak{g}}} (U(\tilde{\mathcal{H}})w) \\ &\cong \text{Ind}_{\mathbb{C}h+\mathbb{C}e+\tilde{\mathcal{H}}}^{\tilde{\mathfrak{g}}} ((M_{\tilde{\mathcal{H}}}(\dot{z}))^{\tilde{\mathfrak{g}}} \otimes \mathbb{C}u) \cong (M_{\tilde{\mathcal{H}}}(\dot{z}))^{\tilde{\mathfrak{g}}} \otimes \text{Ind}_{\mathbb{C}h+\mathbb{C}e+\tilde{\mathcal{H}}}^{\tilde{\mathfrak{g}}} \mathbb{C}u \\ &\cong M_{\tilde{\mathcal{H}}}(\dot{z})^{\tilde{\mathfrak{g}}} \otimes M_{\mathfrak{sl}_2}(\dot{h} + \frac{1}{2}(l + \frac{1}{2})^2)^{\tilde{\mathfrak{g}}}. \end{aligned}$$

The second part of Theorem 2 follows from this and [LZ, Theorem 7].

2.3. Proof of Theorem 3. Claim (i) follows from [LZ, Theorem 7]. To prove claim (ii), let $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}^{(l)}$. We have an isomorphism $M \cong M^{\tilde{\mathfrak{g}}} \otimes \mathbb{C}$ of $\tilde{\mathcal{H}}$ -modules, where \mathbb{C} is the trivial $\tilde{\mathcal{H}}$ -module. Applying [LZ, Lemma 8], we get

$$\text{Ind}_{\tilde{\mathcal{H}}}^{\tilde{\mathfrak{g}}} M \cong \text{Ind}_{\tilde{\mathcal{H}}}^{\tilde{\mathfrak{g}}} (M^{\tilde{\mathfrak{g}}} \otimes \mathbb{C}) \cong M^{\tilde{\mathfrak{g}}} \otimes \text{Ind}_{\tilde{\mathcal{H}}}^{\tilde{\mathfrak{g}}} \mathbb{C} \cong M^{\tilde{\mathfrak{g}}} \otimes U(\mathfrak{sl}_2)^{\tilde{\mathfrak{g}}}.$$

Now V is isomorphic to a simple quotient of $M^{\tilde{\mathfrak{g}}} \otimes U(\mathfrak{sl}_2)^{\tilde{\mathfrak{g}}}$, which, by [LZ, Theorem 7], has to be of the form $M^{\tilde{\mathfrak{g}}} \otimes N^{\tilde{\mathfrak{g}}}$ for some simple \mathfrak{sl}_2 -module N .

2.4. Examples. Here we give three concrete examples of simple $\tilde{\mathfrak{g}}^{(l)}$ -modules with different properties.

Example 10. Let $\dot{z} \in \mathbb{C}^*$, $l \in N - \frac{1}{2}$, $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}^{(l)}$ and $F = \mathbb{C}[x_1, x_3, \dots, x_{2l}]$ be the usual Fock space. Then we have the classical oscillator representation of $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}^{(l)}$ on F with the action given as follows (here \rightarrow means ‘‘acts as’’):

$$\begin{aligned} p_k &\rightarrow x_{2(k-l)}, \quad k = l + \frac{1}{2}, \dots, 2l, \\ p_k &\rightarrow (-1)^{k+l+\frac{1}{2}} \dot{z} k! (2l-k)! \frac{\partial}{\partial x_{2(l-k)}}, \quad k = 0, 1, \dots, l - \frac{1}{2}, \\ e &\rightarrow -\frac{\dot{z}}{2} \left((l + \frac{1}{2})! \frac{\partial}{\partial x_1} \right)^2 + \sum_{k=1}^{l-\frac{1}{2}} (2l-k+1) x_{2(l-k)} \frac{\partial}{\partial x_{2(l-k+1)}}, \\ f &\rightarrow \frac{1}{2\dot{z}} \left(\frac{x_1}{(l-\frac{1}{2})!} \right)^2 + \sum_{k=0}^{l-\frac{1}{2}} k x_{2(l-k+1)} \frac{\partial}{\partial x_{2(l-k)}}, \\ h &\rightarrow -d - \frac{(l + \frac{1}{2})^2}{2}, \quad z \rightarrow \dot{z}; \end{aligned}$$

where $d = \sum_{k=0}^{l-\frac{1}{2}} 2(l-k) x_{2(l-k)} \frac{\partial}{\partial x_{2(l-k)}}$ is the degree derivation. It is easy to check that $F \cong \bar{M}_{\tilde{\mathfrak{g}}}(\dot{z}, -\frac{(l+\frac{1}{2})^2}{2})$.

Now we construct simple $\tilde{\mathfrak{g}}$ -module from simple Whittaker modules over $\tilde{\mathcal{H}}$ (in the sense of [BM, Ch]).

Example 11. Let $\dot{z} \in \mathbb{C}^*$, $l \in N - \frac{1}{2}$, $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}^{(l)}$ and $F = \mathbb{C}[x_1, x_3, \dots, x_{2l}]$. Then for any $\mu = (\mu_0, \mu_1, \dots, \mu_{l-\frac{1}{2}}) \in \mathbb{C}^{l+\frac{1}{2}}$ we have the oscillator representation of $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}^{(l)}$ on F with the action given by $z \rightarrow \dot{z}$,

$$\begin{aligned} p_k &\rightarrow x_{2(k-l)}, \quad k = l + \frac{1}{2}, \dots, 2l, \\ p_k &\rightarrow (-1)^{k+l+\frac{1}{2}} \dot{z} k! (2l-k)! \left(\frac{\partial}{\partial x_{2(l-k)}} + \mu_k \right), \quad k = 0, 1, \dots, l - \frac{1}{2}, \\ e &\rightarrow -\frac{\dot{z}}{2} \left((l + \frac{1}{2})! \left(\frac{\partial}{\partial x_1} + \mu_{l-\frac{1}{2}} \right)^2 + \sum_{k=1}^{l-\frac{1}{2}} (2l-k+1) x_{2(l-k)} \left(\frac{\partial}{\partial x_{2(l-k+1)}} + \mu_{k-1} \right) \right), \\ f &\rightarrow \frac{1}{2\dot{z}} \left(\frac{x_1}{(l-\frac{1}{2})!} \right)^2 + \sum_{k=0}^{l-\frac{1}{2}} k x_{2(l-k+1)} \left(\frac{\partial}{\partial x_{2(l-k)}} + \mu_k \right), \\ h &\rightarrow -d - \left(\sum_{k=0}^{l-\frac{1}{2}} 2(l-k) x_{2(l-k)} \mu_k \right) - \frac{(l+\frac{1}{2})^2}{2}, \end{aligned}$$

where d is the degree derivation from the previous example. This simple $\tilde{\mathfrak{g}}$ -module on F is isomorphic to the module $(\text{Ind}_{\tilde{\mathcal{H}}^+ + \mathbb{C}z}^{\tilde{\mathcal{H}}^l} \mathbb{C}w)^{\tilde{\mathfrak{g}}}$ where $zw = \dot{z}w$ and $p_i w = (-1)^{i+l+1/2} i! (2l-i) \mu_i w$ for $i = 0, 1, \dots, l-1/2$.

The next example is constructed using simple weight $\tilde{\mathcal{H}}$ -modules with infinite dimensional weight spaces.

Example 12. Let $\dot{z} \in \mathbb{C}^*$, $l \in N - \frac{1}{2}$, $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}^{(l)}$ and $\hat{F} = \mathbb{C}[x_1^{\pm 1}, x_3^{\pm 1}, \dots, x_{2l}^{\pm 1}]$. Then for any $\mu = (\mu_0, \mu_1, \dots, \mu_{l-\frac{1}{2}}) \in \mathbb{C}^{l+\frac{1}{2}}$, we have the oscillator representation of $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}^{(l)}$ on \hat{F} with the action given by $z \rightarrow \dot{z}$,

$$\begin{aligned} p_k &\rightarrow x_{2(k-l)}, \quad k = l + \frac{1}{2}, \dots, 2l, \\ p_k &\rightarrow (-1)^{k+l+\frac{1}{2}} \dot{z} k! (2l-k)! \left(\frac{\partial}{\partial x_{2(l-k)}} + \mu_k x_{2(l-k)}^{-1} \right), \quad k = 0, 1, \dots, l - \frac{1}{2}, \\ e &\rightarrow -\frac{\dot{z}}{2} \left((l + \frac{1}{2})! \left(\frac{\partial}{\partial x_1} + \mu_{l-\frac{1}{2}} x_1^{-1} \right)^2 + \sum_{k=1}^{l-\frac{1}{2}} (2l-k+1) x_{2(l-k)} \left(\frac{\partial}{\partial x_{2(l-k+1)}} + \mu_{k-1} x_{2(l-k+1)}^{-1} \right) \right), \\ f &\rightarrow \frac{1}{2\dot{z}} \left(\frac{x_1}{(l-\frac{1}{2})!} \right)^2 + \sum_{k=0}^{l-\frac{1}{2}} k x_{2(l-k+1)} \left(\frac{\partial}{\partial x_{2(l-k)}} + \mu_k x_{2(l-k)}^{-1} \right), \\ h &\rightarrow -d - \left(\sum_{k=0}^{l-\frac{1}{2}} 2(l-k) \mu_k \right) - \frac{(l+\frac{1}{2})^2}{2}, \end{aligned}$$

where d is the degree derivation from the previous examples. It is clear that \hat{F} is a weight $\tilde{\mathfrak{g}}$ -module and all nonzero weight spaces of \hat{F} are infinite dimensional for $l > \frac{1}{2}$. It is easy to check that \hat{F} is simple if and only if $\mu_i \notin \mathbb{Z}$ for all $i = 0, 1, \dots, l - \frac{1}{2}$.

Similarly to the above examples, using Theorem 3 one constructs a lot of simple $\tilde{\mathfrak{g}}^{(l)}$ -modules from simple \mathfrak{sl}_2 -modules and simple $\tilde{\mathcal{H}}^{(l)}$ -modules with nonzero central charge. All simple modules over \mathfrak{sl}_2 and $\tilde{\mathcal{H}}^{(1/2)}$ are described in [Bl]. The problem to classify all simple modules over $\tilde{\mathcal{H}}^{(l)}$ with $l \in \frac{1}{2} + \mathbb{N}$ is still open.

3. SIMPLE WEIGHT MODULES WITH A FINITE DIMENSIONAL NONZERO WEIGHT SPACE

In this section, we will classify all simple $\tilde{\mathfrak{g}}^{(l)}$ -modules and all simple $\mathfrak{g}^{(l)}$ -modules which are weight modules and have a finite dimensional nonzero weight space.

3.1. Proof of Theorem 4. Claim (i) is straightforward. To prove claim (ii) assume that $p_l \neq 0$. It is easy to check that the \mathfrak{n} -module F_1 defined by (1.1) and (1.2) is a simple highest weight \mathfrak{n} -module. Now claim (ii) would follow if we could prove that $\text{Ind}_{\mathfrak{n}}^{\mathfrak{g}} F_1$ is a simple \mathfrak{g} -module.

Assume that $\text{Ind}_{\mathfrak{n}}^{\mathfrak{g}} F_1$ is not simple and therefore there exists some nonzero $v = \sum_{i=0}^k f^i \otimes v_i$, where $v_i \in F_1$ with $U(\mathfrak{g})v \neq \text{Ind}_{\mathfrak{n}}^{\mathfrak{g}} F_1$. We have $k > 0$ and we may assume that k is minimal. Recall that $p_i F_1 = 0$ for all $i = 0, \dots, l-1$. It is now easy to see that $0 \neq p_{l-1}v = \sum_{i=0}^k ([p_{l-1}, f^i]) \otimes v_i$ has degree $k-1$ with respect to f , a contradiction.

3.2. Preliminary results on weight modules. In this subsection we extend the techniques and methods from [Du] to all conformal Galilei algebras.

Lemma 13. *Let $l \in \frac{1}{2}\mathbb{N}$ and $s \in \{e, f, p_k \mid k = 0, \dots, 2l\} \subset \mathfrak{g} = \mathfrak{g}^{(l)}$.*

- (i) *The action of $\text{ad } s$ on $U(\mathfrak{g})$ is locally nilpotent.*
- (ii) *If V is a \mathfrak{g} module, then the set $\{v \in V \mid s^n v = 0 \text{ for some } n \in \mathbb{N}\}$ is a submodule of V .*
- (iii) *Let V be a simple \mathfrak{g} module. Then s acts locally nilpotently on V if and only if there exists some $0 \neq v \in V$ such that $sv = 0$.*

Proof. Follows mutatis mutandis the proof of [Du, Lemma 3]. □

Lemma 14. *Let $l \in \frac{1}{2}\mathbb{N}$, and V be a simple weight $\mathfrak{g}^{(l)}$ -module. Suppose that $\text{supp}(V) \subseteq \lambda + \mathbb{Z}$ for some $\lambda \in \mathbb{C}$, that $\dim V_\lambda < \infty$ and that $\mathcal{H}V \neq 0$.*

- (i) *If V is not a highest (resp. lowest) weight module, then e (resp. f) acts injectively on V .*
- (ii) *If V is neither a highest nor a lowest weight module, then both e and f act bijectively on V . Consequently, for any $\mu \in \text{supp}(V)$, $\dim V_{\mu+2i} = \dim V_\mu$ for all $i \in \mathbb{Z}$. If $l \in \mathbb{N} - \frac{1}{2}$, then $\dim V_{\mu+i} = \dim V_\mu$ for all $i \in \mathbb{Z}$.*

Proof. We start with claim (i). We prove the claim for e (and then the claim for f follows by symmetry). Assume that V is not a highest weight module and that the kernel of e on V is nonzero. Consider first the situation when the kernel of each p_i , $i = 0, 1, \dots, [l-1/2]$, on V is nonzero. Using Lemma 13 and nilpotency of the Lie algebra $\text{span}\{e, p_i \mid i = 0, \dots, [l-1/2]\}$, we may find a common eigenvector for all these elements. This implies that

V is a highest weight module, a contradiction. Therefore such situation is not possible. Hence there is a minimal $k \in \{0, 1, \dots, \lfloor l - 1/2 \rfloor\}$ such that the element p_k acts injectively on V . Again, using Lemma 13 and nilpotency of the Lie algebra $\text{span}\{e, p_0, \dots, p_{k-1}\}$, we may find a weight vector $0 \neq w \in V_{\lambda+k_0}$ with $ew = p_i w = 0$ for all $i = 0, 1, \dots, k-1$. Now it is straightforward to verify that for each $j \in \mathbb{N}$ the vector $p_k^j w$ is a nonzero highest weight vector for the \mathfrak{sl}_2 -subalgebra. This implies that all weight spaces of V are infinite dimensional, a contradiction. Claim (i) follows.

Now we prove claim (ii). From claim (i), both e and f act injectively on V . This implies that $\dim V_{\lambda+i} = \dim V_{\lambda+j}$ for all $i - j \in 2\mathbb{Z}$, in particular, both e and f act bijectively on V . If $l \in \mathbb{N} - \frac{1}{2}$ and $\dim V_\lambda \neq \dim V_{\lambda+1}$, then for all $i = 0, 1, \dots, 2l$ the kernel of p_i is nonzero. Similarly to the above it follows that $\mathcal{H}w = 0$ for some $w \in V$, a contradiction. Hence $\dim V_{\lambda+i} = \dim V_{\lambda+j} < \infty$ for all $i, j \in \mathbb{Z}$. This completes the proof. \square

For $l \in \frac{1}{2}\mathbb{N}$ and $\mathfrak{g} = \mathfrak{g}^{(l)}$ we denote by $U(\mathfrak{g})^{(f)}$ the localization of $U(\mathfrak{g})$ with respect to the multiplicative set $\{f^i | i \in \mathbb{Z}_+\}$.

Lemma 15. *Let $l \in \frac{1}{2}\mathbb{N}$ and $\mathfrak{g} = \mathfrak{g}^{(l)}$. For any $x \in \mathbb{C}$ there is a unique automorphism θ_x of $U(\mathfrak{g})^{(f)}$ such that*

$$\theta_x(f) = f, \quad \theta_x(h) = h - 2x, \quad \theta_x(e) = e + x(h - 1 - x)f^{-1}$$

and

$$(3.1) \quad \theta_x(p_j) = \sum_{k=0}^{2l-j} (-1)^k \binom{x}{k} \frac{(2l-j)!}{(2l-j-k)!} f^{-k} p_{j+k}, \quad j = 0, 1, \dots, 2l.$$

Proof. The proof follows mutatis mutandis the proof of [Mat, Lemma 4.3] (see also [Du, Proposition 8] or [Maz, Proposition 3.45]). \square

Let $\mathfrak{g} = \mathfrak{g}^{(l)}$ and V be a \mathfrak{g} -module on which f act bijectively. Let σ be an automorphism of $U(\mathfrak{g})^{(f)}$. Setting $g \cdot v = \sigma(g)v$ for all $g \in \mathfrak{g}$ and $v \in V$ defines on V a new \mathfrak{g} -module structure which we will denote by V^σ .

Proposition 16. *Let $l \in \frac{1}{2}\mathbb{N}$, and V be a simple weight $\mathfrak{g}^{(l)}$ -module with a nonzero finite dimensional weight space. If V is not a highest or a lowest weight module, then $\mathcal{H}V = 0$.*

Proof. Suppose that V is neither a highest nor a lowest weight module and $\mathcal{H}V \neq 0$. Let $\lambda \in \text{supp}(V)$ with $0 \neq \dim V_\lambda = n < \infty$. Then from Lemma 14 we see that e and f acts bijectively on V and V is uniformly bounded. Take an eigenvector $v_\lambda \in V_\lambda$ of ef . Then it is easy to see that there exists some $x \in \mathbb{C}$ such that $\theta_x(e)f v_\lambda = (e + x(h - 1 - x)f^{-1})f v_\lambda = 0$. Now we consider the \mathfrak{g} -module V^{θ_x} which is uniformly bounded weight module. From Lemma 13(ii) we have that

$$M = \{v \in V^{\theta_x} | e \text{ acts locally nilpotently on } v\}$$

is a nonzero \mathfrak{g} -submodule of V^{θ_x} . Moreover, being a uniformly bounded module, M has finite length by [Mat, Lemma 3.3] and hence it has a simple submodule M' (which is uniformly bounded as well). If $l \in \mathbb{N}$, from Proposition 4 it follows that $\theta_x(\mathcal{H})M' = 0$. For $l \in \mathbb{N} - 1/2$ the fact that $\theta_x(\mathcal{H})M' = 0$ is obvious. Using induction and 3.1 we obtain $p_{2l-i}M' = 0$ for $i = 0, 1, \dots, 2l$. Since the subspace annihilated by \mathcal{H} is a $\mathfrak{g}^{(l)}$ -submodule of V , we have $\mathcal{H}V = 0$, a contradiction. This proves the proposition. \square

Next we will first establish some properties for weight $\mathbb{C}h + \tilde{\mathcal{H}}^{(l)}$ -modules with nonzero central charge. Choose a new basis of $\mathbb{C}h + \tilde{\mathcal{H}}^{(l)}$ as follows:

$$t_{2(l-k)} := \begin{cases} (-1)^{k+l+\frac{1}{2}} 2(l-k) \frac{p_k}{k!}, & k = 0, 1, \dots, l - \frac{1}{2}; \\ \frac{p_k}{k!}, & k = l + \frac{1}{2}, \dots, 2l. \end{cases}$$

Then $\mathbb{C}h + \tilde{\mathcal{H}}^{(l)} = \text{span}\{h, t_i, z | i = \pm 1, \pm 3, \dots, \pm 2l\}$ with

$$[t_i, t_j] = i\delta_{i+j,0}z, \quad [h, t_i] = it_i, \quad i, j = \pm 1, \dots, \pm 2l.$$

We may naturally regard $\mathbb{C}h + \tilde{\mathcal{H}}^{(l)}$ as a subalgebra of $\mathbb{C}h + \tilde{\mathcal{H}}^{(l+1)}$.

Lemma 17. *Let $l \in \mathbb{N} - \frac{1}{2}$, V be a simple weight $\mathbb{C}h + \tilde{\mathcal{H}}^{(l)}$ -module with nonzero central charge \dot{z} and $\text{supp}(V) \subset \lambda + \mathbb{Z}$ with $\dim V_\lambda < \infty$ for some $\lambda \in \mathbb{C}$.*

- (i) *If $l = \frac{1}{2}$, then V is a highest (or a lowest) weight module or isomorphic to $D(a, \dot{z})$ for some $a \in \mathbb{C}$.*
- (ii) *If $l \in \mathbb{N} + \frac{1}{2}$, then V is either a highest or a lowest weight module.*

Proof. We start with claim (i). Let $l = \frac{1}{2}$, $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}^{(\frac{1}{2})}$ and M be a simple weight $\mathbb{C}h + \tilde{\mathcal{H}}^{(l)}$ -module with central charge \dot{z} . For $\lambda \in \text{supp}(M)$ the space M_λ is a simple $\mathbb{C}(t_1 t_{-1}, h, z)/(h - \lambda, z - \dot{z}) \cong \mathbb{C}[t_1 t_{-1}]$ -module. As $\mathbb{C}[t_1 t_{-1}]$ is commutative, it follows that $\dim V_\lambda = 1$. As usual for weight modules (see e.g. [DFO]), there is exactly one (up to isomorphism) simple weight $\mathbb{C}h + \tilde{\mathcal{H}}^{(l)}$ -module N such that $N_\lambda \cong M_\lambda$ as $\mathbb{C}(t_1 t_{-1}, h, z)$ -modules (this module is, in fact, M). It is easy to check that in the case $\dot{z} \neq 0$ all simple $\mathbb{C}(t_1 t_{-1}, h, z)$ -modules can be obtained restricting simple highest weight $\mathbb{C}h + \tilde{\mathcal{H}}^{(l)}$ -modules, simple lowest weight $\mathbb{C}h + \tilde{\mathcal{H}}^{(l)}$ -modules and modules $D(a, \dot{z})$ to some nonzero weight spaces. Therefore, if t_1 does not act injectively on H , then H is a highest weight module; if t_{-1} does not act injectively on H , then H is a lowest weight module; if both t_1 and t_{-1} act injectively and $\dot{z} \neq 0$, we have $H \cong D(a, \dot{z})$. Claim (i) follows.

Now we prove claim (ii). Assume $l \in \mathbb{N} + \frac{1}{2}$ and V is a simple weight $\mathbb{C}h + \tilde{\mathcal{H}}^{(l)}$ -module with nonzero central charge \dot{z} . Further, let $\lambda \in \mathbb{C}$ be such that $\text{supp}(V) \subset \lambda + \mathbb{Z}$ and $\dim V_\lambda < \infty$. We claim that V contains a simple submodule over the algebra $\mathfrak{a} := \mathbb{C}h + \mathbb{C}t_{2l} + \mathbb{C}t_{-2l} + \mathbb{C}z$.

Indeed, if $V_\lambda = 0$, it is clear that V contains a highest or lowest weight simple $\mathbb{C}h + \mathbb{C}t_1 + \mathbb{C}t_{-1} + \mathbb{C}z$ -submodule. Therefore, without loss of generality, we may assume that either $V_\lambda = 0$ and $\lambda - \mathbb{N} \subset \text{supp}(V)$, in which case

V contains a highest weight α -submodule; or $V_\lambda = 0$ and $\lambda + \mathbb{N} \subset \text{supp}(V)$, in which case V contains a lowest weight α -submodule.

If $V_\lambda \neq 0$, we either have that $U(\alpha)V_\lambda$ is uniformly bounded or that it has a highest or a lowest weight vector as a α -module. In both cases, V has a simple α -submodule.

Now we are going to prove claim (ii) of the lemma by induction on l (we use claim (i) as the basis of the induction).

Define the Lie algebra $\mathfrak{t} = \mathfrak{t}^{(l)} = \text{span}\{h, t_{\pm i}, z_1, z_2 | i = 1, 3, \dots, 2l\}$ with the following Lie bracket:

$$\begin{aligned} [z_1, \mathfrak{t}] &= [z_2, \mathfrak{t}] = 0; \\ [h, t_i] &= it_i, \quad i = \pm 1, \dots, \pm 2l; \\ [t_i, t_j] &= i\delta_{i+j,0}z_1, [h, t_i] = it_i, \quad i, j = \pm 1, \dots, \pm 2(l-1); \\ [t_{2l}, t_j] &= 2l\delta_{j-2l}z_2, \quad j = \pm 1, \dots, \pm 2l. \end{aligned}$$

The algebra \mathfrak{t} has a subalgebra $\text{span}\{h, t_{\pm i}, z_1 | i = 1, 3, \dots, 2(l-1)\}$ which is isomorphic to $\tilde{\mathcal{H}}^{(l-1)}$. Moreover, we have $\mathfrak{t}/\mathbb{C}(z_1 - z_2) \cong \tilde{\mathcal{H}}$. Hence we may regard V as a \mathfrak{t} -module with the action $z_1 = z_2 = \dot{z}$. Let $\tilde{\mathfrak{t}} = \mathbb{C}t_{2l} + \mathbb{C}h + \mathbb{C}t_{-2l} + \mathbb{C}z_2$. Then V contains a simple $\tilde{\mathfrak{t}}$ -submodule M . Extend M to a \mathfrak{t} -module by $t_i M = z_1 M = 0$ for all $i = \pm 1, \dots, \pm 2(l-1)$. Denote the resulting module by $M^{\mathfrak{t}}$. Let $\mathbb{C}w$ be the one dimensional module over $\tilde{\mathfrak{t}} + \mathbb{C}z_1$ with $\tilde{\mathfrak{t}}w = 0$ and $z_1 w = \dot{z}w$. Then $M \cong M^{\mathfrak{t}} \otimes \mathbb{C}w$ as $\tilde{\mathfrak{t}} + \mathbb{C}z_1$ module. Therefore $V = U(\mathfrak{t})M = U(\mathcal{H})M$ is a quotient of

$$\text{Ind}_{\tilde{\mathfrak{t}} + \mathbb{C}z_1}^{\mathfrak{t}} M \cong \text{Ind}_{\tilde{\mathfrak{t}} + \mathbb{C}z_1}^{\mathfrak{t}} (M^{\mathfrak{t}} \otimes \mathbb{C}w) \cong M^{\mathfrak{t}} \otimes \text{Ind}_{\tilde{\mathfrak{t}} + \mathbb{C}z_1}^{\mathfrak{t}} \mathbb{C}w.$$

From [LZ, Theorem 7] it follows that any simple quotient of $M^{\mathfrak{t}} \otimes \text{Ind}_{\tilde{\mathfrak{t}} + \mathbb{C}z_1}^{\mathfrak{t}} \mathbb{C}w$ is of the form $M^{\mathfrak{t}} \otimes N$, where N is a simple quotient module of $\text{Ind}_{\tilde{\mathfrak{t}} + \mathbb{C}z_1}^{\mathfrak{t}} \mathbb{C}w$. From $\dim V_\lambda < \infty$ and the induction hypothesis, we have either M and N are both highest weight modules or M and N are both lowest weight modules (note that, in particular, the condition $\dim V_\lambda < \infty$ excludes the case $N \cong D(a, \dot{z})$). Then V is a highest or a lowest weight module, which completes the proof. \square

Lemma 18. *Suppose that $l \in \mathbb{N} - \frac{1}{2}$ and V is a weight module over $\mathbb{C}h + \tilde{\mathcal{H}}^{(l)}$ with a finite dimensional nonzero weight space and z acts as a nonzero scalar. Then V has a simple $\tilde{\mathcal{H}}^{(l)}$ -submodule with all weight spaces finite dimensional.*

Proof. Let $\lambda \in \text{supp}(V)$ with $0 < \dim V_\lambda < \infty$. Let W_λ be a simple $U(\tilde{\mathcal{H}})_0$ -submodule of V_λ and denote $W = U(\tilde{\mathcal{H}})W_\lambda$, which is a $\mathbb{C}h + \tilde{\mathcal{H}}$ -submodule of V . It is clear that any proper $\mathbb{C}h + \tilde{\mathcal{H}}$ -submodule of W has trivial intersection with W_λ . Hence W has a unique maximal submodule W' with $W'_\lambda = 0$. If $W' = 0$, then W is a simple $\mathbb{C}h + \tilde{\mathcal{H}}$ submodule with a finite dimensional nonzero weight space. Then from Lemma 17 we have that W is also a simple $\tilde{\mathcal{H}}$ module with finite dimensional weight space. If $W' \neq 0$, then, without losing of generality, we may assume that $\text{supp}(W') \cap (\lambda - \mathbb{N}) \neq \emptyset$. Denote $W'' = U(\mathbb{C}h + \tilde{\mathcal{H}})(\oplus_{i \in \mathbb{N}} W'_{\lambda-i})$. By Lemma 17, any simple subquotient

of W'' is a highest weight module with highest weight contained in $\lambda - \mathbb{N}$, i.e., $\bigoplus_{i \in \mathbb{N}} W'_{\lambda-i}$ is a submodule. Therefore W' contains a highest weight vector, which generates a simple $\tilde{\mathcal{H}}^{(l)}$ submodule. \square

3.3. Proof of Theorems 5 and 6. Theorem 5 and claim (i) of Theorem 6 follow from Proposition 16. Claim (ii) of Theorem 6 follows from Lemmata 17 and 18 and Theorem 3.

3.4. Simple weight modules with an infinite dimensional weight space. From Theorems 5 and 6 and our discussion above we immediately get the following:

Corollary 19. (i) *Let $l \in \mathbb{N}$, $\mathfrak{g} = \mathfrak{g}^{(l)}$ and V be a simple weight \mathfrak{g} -module. Assume that $\dim V_\lambda = \infty$ for some $\lambda \in \mathbb{C}$. Then $\text{supp}(V) = \lambda + 2\mathbb{Z}$ and $\dim V_{\lambda+2i} = \infty$ for all $i \in \mathbb{Z}$.*

(ii) *Let $l \in \mathbb{N} - \frac{1}{2}$, $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}^{(l)}$ and V be a simple weight \mathfrak{g} -module. Assume that $\dim V_\lambda = \infty$ for some $\lambda \in \mathbb{C}$. Then $\text{supp}(V) = \lambda + \mathbb{Z}$ and $\dim V_{\lambda+i} = \infty$ for all $i \in \mathbb{Z}$.*

Note that in the case $l = \frac{1}{2}$ this statement is proved in [WZ].

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