GRADED SIMPLE LIE ALGEBRAS AND GRADED SIMPLE REPRESENTATIONS

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Abstract. For any finitely generated abelian group Q, we reduce the problem of classification of Q-graded simple Lie algebras over an algebraically closed field of "good" characteristic to the problem of classification of gradings on simple Lie algebras. In particular, we obtain the full classification of finite-dimensional Q-graded simple Lie algebras over any algebraically closed field of characteristic 0 based on the recent classification of gradings on finite dimensional simple Lie algebras.

We also reduce classification of simple graded modules over any Q-graded Lie algebra (not necessarily simple) to classification of gradings on simple modules. For finite-dimensional Q-graded semisimple algebras we obtain a graded analogue of the Weyl Theorem.

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1. Introduction

1.1. **General overview.** The present paper addresses classification of Q-graded simple Lie algebras for any finitely generated abelian group Q and classification of Q-graded simple modules over Q-graded Lie algebras in the case when the characteristic of the ground field $\Bbbk$ does not divide the order of the torsion subgroup of Q.

Study of gradings on Lie algebras goes back at least as far as to the paper [PZ] which started a systematic approach to understanding of gradings by abelian groups on simple finite dimensional Lie algebras over algebraically closed fields of characteristic 0. In the past two decades, there was a significant interest to the study of gradings on simple Lie algebras by arbitrary groups, see the recent monograph [EK2] and references therein. In particular, there is an essentially complete classification of fine gradings (up to equivalence) on all finite-dimensional simple Lie algebras over an algebraically closed field of characteristic 0, see [EK2, El, Yu]. Some properties of simple $\mathbb{Z}_2$-graded Lie algebras were obtained in [Zu]. For a given abelian group Q, the classification of Q-gradings (up to isomorphism) on classical simple Lie algebras over an algebraically closed field of characteristic different from 2 was done in [BK, El] (see also [EK2]).

1.2. **Notation and setup.** Throughout this paper, $\Bbbk$ denotes an algebraically closed field. If not explicitly stated otherwise, we do not put any restrictions on the characteristic of $\Bbbk$. Similarly, if not explicitly stated otherwise, all vector spaces, algebras and tensor products are assumed to be over $\Bbbk$. As usual, we denote by $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}_+$ and $\mathbb{C}$ the sets of integers, positive integers, nonnegative integers and complex numbers, respectively.

Let Q be an additive abelian group. A *Q-graded Lie algebra over $\Bbbk$* is a Lie algebra $\mathfrak{g}$ over $\Bbbk$ endowed with a decomposition

$$\mathfrak{g} = \bigoplus_{\alpha\in Q} \mathfrak{g}_\alpha \quad \text{such that} \quad [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta} \quad \text{for all} \quad \alpha, \beta \in Q.$$

Recall that a graded Lie algebra is called *graded simple* if $\mathfrak{g}$ is not commutative and does not contain any non-trivial graded ideal.

Let Q' be another abelian group, $\mathfrak{g}$ a Q-graded Lie algebra over a field $\Bbbk$ and $\mathfrak{g}'$ be a Q'-graded Lie algebra over $\Bbbk$. We say that graded Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ are *graded isomorphic* if there is a group isomorphism $\tau : Q \to Q'$ and a Lie algebra isomorphism $\sigma : \mathfrak{g} \to \mathfrak{g}'$ such that $\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}'_{\tau(\alpha)}$ for all $\alpha \in Q$.

A *Q-graded module* V over a Q-graded Lie algebra $\mathfrak{g}$ is a $\mathfrak{g}$-module endowed with a decomposition

$$V = \bigoplus_{\alpha\in Q} V_\alpha \quad \text{such that} \quad \mathfrak{g}_\alpha \cdot V_\beta \subset V_{\alpha+\beta} \quad \text{for all} \quad \alpha, \beta \in Q.$$

A graded module V is called *graded simple* if V is not zero and does not contain any non-trivial graded submodule.

Let $\mathfrak{g}$ be a Q-graded Lie algebra over $\Bbbk$ and

$$W' = \bigoplus_{\alpha\in Q} W_\alpha, \quad W = \bigoplus_{\alpha\in Q} W'_\alpha$$

be two Q-graded $\mathfrak{g}$-module. We say that the graded modules $\mathfrak{g}$ and W and W' are *graded isomorphic* if there is a $\mathfrak{g}$-module isomorphism $\sigma : W \to W'$ such that $\sigma(W_\alpha) = W'_\alpha$ for all $\alpha \in Q$.

1.3. **Results and structure of the paper.** In Section 2, for a subgroup P of an abelian group Q and a simple Lie algebra $\mathfrak{a}$ with a fixed Q/P-grading, we construct a Q-graded Lie algebra $\mathfrak{g}(Q, P, \mathfrak{a})$ and prove that $\mathfrak{g}(Q, P, \mathfrak{a})$ is always Q-graded simple. If P is finite and $\mathrm{char}(\Bbbk)$ does not divide $|P|$, then the algebra $\mathfrak{g}(Q, P, \mathfrak{a})$ is a direct sum of $|P|$

ideals that are isomorphic to $\mathfrak{a}$. If $\mathrm{char}(\Bbbk)$ does divide $|P|$, then the algebra $\mathfrak{g}(Q, P, \mathfrak{a})$ is not semisimple as an ungraded algebra.

In Section 3, for finite Q whose order is not a multiple of $\mathrm{char}(\Bbbk)$ we prove that any Q-graded simple Lie algebra has to be of the form $\mathfrak{g}(Q, P, \mathfrak{a})$ for some simple Lie algebra $\mathfrak{a}$ with a Q/P-grading, see Corollary 13 (we actually prove a more general result in Theorem 12). Our main tool is to analyze properties of the character group of Q. Due to the recent classification of all gradings on finite dimensional simple Lie algebras, see [EK2, El, Yu], we actually obtain a full classification of all finite-dimensional Q-graded simple Lie algebras over any algebraically closed field of characteristic 0.

In Section 4 we establish a graded analogue of Schur's Lemma. It is frequently used in the remainder of the paper. In Section 5, for a finitely generated additive abelian group Q, we prove that any Q-graded simple Lie algebra with $\dim \mathfrak{g} < |\Bbbk|$ has to be of the form $\mathfrak{g}(Q, P, \mathfrak{a})$ for some simple Lie algebra $\mathfrak{a}$ with a Q/P-grading, see Theorem 20. We also obtain some necessary and sufficient conditions for two Q-graded simple Lie algebras to be isomorphic in Theorem 25.

In Section 6, using our classification of Q-graded simple Lie algebras, we prove a graded analogue of the Weyl's Theorem, see Theorem 26. Namely, we show that any Q-graded finite dimensional module over a Q-graded semi-simple finite dimensional Lie algebra over an algebraically closed field of characteristic 0 is completely reducible. Here the crucial observation is that any finite dimensional Q-graded simple Lie algebra over an algebraically closed field $\Bbbk$ of characteristic 0 is semi-simple after forgetting the grading, which is a consequence of Theorem 20.

Finally, in the last section of the paper, we reduce classification of all Q-graded simple modules over any Q-graded Lie algebras $\mathfrak{g}$ for any finitely generated abelian group Q to the study of gradings on simple $\mathfrak{g}$-modules. For any simple $\mathfrak{g}$-module V with a Q/P-grading, we first construct a Q-graded $\mathfrak{g}$-module $M(Q, P, V)$ and then show that $M(Q, P, V)$ is Q-graded simple if and only if the Q/P-grading of V is fine, see Theorem 28. We prove that any Q-graded simple $\mathfrak{g}$-module W with $\dim W < |\Bbbk|$ has to be of the form $M(Q, P, V)$ in Theorem 30. In Theorem 32 we give some necessary and sufficient conditions for two Q-graded simple $\mathfrak{g}$-modules to be isomorphic.

2. Construction of graded simple Lie algebras

2.1. **Construction.** Let Q be an abelian group and P a subgroup of Q. Assume we are given a simple Lie algebra $\mathfrak{a}$ over $\Bbbk$ with a fixed

Q/P-grading

$$\mathfrak{a} = \bigoplus_{\bar{\alpha}\in Q/P} \mathfrak{a}_{\bar{\alpha}}.$$

Consider the group algebra $\Bbbk Q$ with the standard basis $\{t^\alpha : \alpha \in Q\}$ and multiplication $t^\alpha t^\beta = t^{\alpha+\beta}$ for all $\alpha, \beta \in Q$. Then we can form the Lie algebra $\mathfrak{a} \otimes \Bbbk Q$. For $x, y \in \mathfrak{a}$ and $\alpha, \beta \in Q$, we have

$$[x \otimes t^\alpha, y \otimes t^\beta] = [x, y] \otimes t^{\alpha+\beta}.$$

Define the Q-graded Lie algebra

$$\mathfrak{g}(Q, P, \mathfrak{a}) := \bigoplus_{\alpha\in Q} \mathfrak{g}(Q, P, \mathfrak{a})_\alpha, \quad \text{where} \quad \mathfrak{g}(Q, P, \mathfrak{a})_\alpha := \mathfrak{a}_{\bar{\alpha}} \otimes t^\alpha.$$

For example, $\mathfrak{g}(Q, Q, \mathfrak{a}) = \mathfrak{a} \otimes \Bbbk Q$ (with the obvious Q-grading) while $\mathfrak{g}(Q, \{0\}, \mathfrak{a}) = \mathfrak{a}$ (with the original Q-grading). From the definition it follows that $\dim \mathfrak{g}(Q, P, \mathfrak{a}) = \dim(\mathfrak{a})|P|$ if $\mathfrak{a}$ is finite-dimensional and P is finite.

Lemma 1. *If Q is an abelian group, then the algebra $\mathfrak{g}(Q, P, \mathfrak{a})$ is a Q-graded simple Lie algebra.*

Proof. Consider the ideal I in $\mathbb{C}Q$ generated by $\{t^\alpha - 1 : \alpha \in P\}$, and let

$$\mathfrak{b} = \mathfrak{g}(Q, P, \mathfrak{a})/(\mathfrak{g}(Q, P, \mathfrak{a}) \cap (\mathfrak{a} \otimes I)).$$

The algebra $\mathfrak{b}$ is naturally Q/P-graded and is, in fact, isomorphic to the Q/P-graded Lie algebra $\mathfrak{a}$ with the original grading. We have that

$$\mathfrak{b} = \bigoplus_{\bar{\alpha}\in Q/P} \mathfrak{a}_{\bar{\alpha}} \otimes t^{\bar{\alpha}},$$

where $\{t^{\bar{\alpha}} : \bar{\alpha} \in Q/P\}$ is a basis for the group algebra of Q/P. Let

$$\pi : \mathfrak{g}(Q, P, \mathfrak{a}) \twoheadrightarrow \mathfrak{b},$$

be the canonical epimorphism.

From $[\bar{\mathfrak{a}}, \bar{\mathfrak{a}}] = \bar{\mathfrak{a}}$ it follows that for every $\bar{\alpha} \in Q/P$ we have

$$\mathfrak{a}_{\bar{\alpha}} = \sum_{\bar{\beta}\in Q/P} [\bar{\mathfrak{a}}_{\bar{\beta}}, \bar{\mathfrak{a}}_{\bar{\alpha}-\bar{\beta}}]. \tag{1}$$

We use this property to prove graded simplicity of the Lie algebra $\mathfrak{g}(Q, P, \mathfrak{a})$. Take a nonzero homogeneous element

$$x_{\bar{\beta}} \otimes t^\beta \in \mathfrak{g}_\beta$$

and denote by K the graded ideal of $\mathfrak{g}(Q, P, \mathfrak{a})$ which this element generates. Then $\pi(K) = \mathfrak{b}$, that is, for every $\xi \in Q/P$ we fix one representative $\alpha \in \xi$, thus we have $\xi = \bar{\alpha}$. Then there is a basis $\{x^{(i)}_{\bar{\alpha}} : i \in N(\bar{\alpha})\}$ in $\mathfrak{a}_{\bar{\alpha}}$, and there exists $\beta^{(i)}_\alpha \in P$ such that

$$x^{(i)}_{\bar{\alpha}} \otimes t^{\alpha+\beta^{(i)}_\alpha} \in \mathfrak{g}_{\alpha+\beta^{(i)}_\alpha} \cap K.$$

For any $\gamma \in Q$, we see that

$$\begin{aligned}
&\sum_{\bar\alpha\in Q/P}\sum_{i\in N(\bar\alpha)}[x_{\bar\alpha}^{(i)}\otimes t^{\alpha+\beta_\alpha^{(i)}},\mathfrak{g}_{\gamma-\alpha-\beta_\alpha^{(i)}}]\\
=&\sum_{\bar\alpha\in Q/P}\sum_{i\in N(\bar\alpha)}[x_{\bar\alpha}^{(i)}\otimes t^{\alpha+\beta_\alpha^{(i)}},\mathfrak{a}_{\bar\gamma-\bar\alpha}\otimes t^{\gamma-\alpha-\beta_\alpha^{(i)}}]\\
=&\sum_{\bar\alpha\in Q/P}\sum_{i\in N(\bar\alpha)}[x_{\bar\alpha}^{(i)},\mathfrak{a}_{\bar\gamma-\bar\alpha}]\otimes t^{\gamma}\\
=&\sum_{\bar\alpha\in Q/P}[\mathfrak{a}_{\bar\alpha},\mathfrak{a}_{\bar\gamma-\bar\alpha}]\otimes t^{\gamma}\\
=&\mathfrak{a}_{\bar\gamma}\otimes t^{\gamma}\subset K,
\end{aligned}$$

where the third step is justified by (1). Thus $K = \mathfrak{g}(Q, P, \mathfrak{a})$ and hence $\mathfrak{g}(Q, P, \mathfrak{a})$ is graded simple. □

Later on we will prove that the above graded simple Lie algebras exhaust all Q-graded simple Lie algebras. Making a parallel with affine Kac-Moody algebras [Ka, MP], it is natural to divide these algebras into two classes. The algebras $\mathfrak{g}(Q, Q, \mathfrak{a})$ will be called *untwisted graded simple Lie algebra* while all other algebras will be called *twisted graded simple Lie algebra*.

2.2. **Properties of $\mathfrak{g}(Q, P, \mathfrak{a})$ in the case of finite Q.** Now we need to establish some properties of the graded simple Lie algebras $\mathfrak{g}(Q, P, \mathfrak{a})$ for finite groups Q. So in the rest of this subsection we assume that Q is finite and $\mathrm{char}(\mathbb{k})$ does not divide $|Q|$.

Let $\hat{Q}$ denote the character group of Q, that is the group of all group homomorphisms $Q \to \mathbb{k}^*$ under the operation of pointwise multiplication. Note that $\hat{Q} \cong Q$ because of our assumption on $\mathrm{char}(\mathbb{k})$. For any $f \in \hat{Q}$, we define the associative algebra automorphism

$$\tau_f : \mathbb{k}Q \to \mathbb{k}Q \quad \text{via} \quad \tau_f(t^\alpha) = f(\alpha)t^\alpha \quad \text{for all } \alpha \in Q.$$

This induces the Lie algebra automorphism

$$\begin{array}{rrcl}\tau_f: & \mathfrak{g}(Q,P,\mathfrak{a}) & \to & \mathfrak{g}(Q,P,\mathfrak{a})\\ & x_{\bar\alpha}(\alpha) & \mapsto & f(\alpha)x_{\bar\alpha}(\alpha)\end{array} \tag{2}$$

for all $\alpha \in Q$ and $x_{\bar\alpha} \in \mathfrak{a}_{\bar\alpha}$. Note that $\tau_{fg} = \tau_f\tau_g$ for all $f, g \in \hat{Q}$, in other words, $\hat{Q}$ acts on $\mathfrak{g}(Q, P, \mathfrak{a})$ via automorphisms τ_f. We will use the following:

Remark 2. If Q is a finite abelian group, P a subgroup of Q, $\hat{Q}$ the group of characters of Q over a field $\mathbb{k}$ such that $\mathrm{char}(\mathbb{k})$ does not divide $|Q|$ and $P^\perp := \{f \in \hat{Q} : f(\alpha) = 1 \text{ for all } \alpha \in P\}$, then $|\hat{Q}/P^\perp| = |P|$. Indeed, because of our assumption on $\mathbb{k}$, we know that

$|\hat{P}| = |P|$. Therefore it is enough to prove that each character of P can be extended to a character of Q. The latter follows directly from the Frobenius reciprocity.

Lemma 3. *If Q is finite and* $\mathrm{char}(\Bbbk)$ *does not divide $|Q|$, then the algebra $\mathfrak{g}(Q,P,\mathfrak{a})$ is a direct sum of $|P|$ ideals. Each of these ideals is Q/P-graded and, moreover, isomorphic to $\mathfrak{a}$ as Q/P-graded Lie algebras.*

Proof. For $\alpha \in Q$ set

$$\underline{t}^{\alpha} := t^{\alpha} \sum_{\beta \in P} t^{\beta}.$$

Then, for any $\alpha, \alpha' \in Q$, we have $\underline{t}^{\alpha} = \underline{t}^{\alpha'}$ if and only if $\alpha - \alpha' \in P$. Consider the vector space

$$I = \bigoplus_{\bar{\alpha} \in Q/P} \mathfrak{a}_{\bar{\alpha}} \otimes \underline{t}^{\alpha}, \tag{3}$$

which is well-defined because of the observation in the previous sentence. Since $\underline{t}^{\alpha} t^{\beta} = \underline{t}^{\alpha+\beta}$ for all $\alpha, \beta \in Q$, the space I is an ideal. Note that $[I, I] \neq 0$ since $\mathfrak{a}$ is a simple Lie algebra and $\mathrm{char}(\Bbbk)$ does not divide $|Q|$ (and thus it does not divide $|P|$ either, which implies $\underline{t}^{\alpha}\underline{t}^{\alpha} = |P|\underline{t}^{\alpha} \neq 0$). It follows that $I \cong \mathfrak{a}$ as a Q/P-graded Lie algebra. Simplicity of $\mathfrak{a}$ even implies that I is a minimal ideal.

Define the *invariant subgroup* $\mathrm{Inv}(I)$ *of* I as

$$\mathrm{Inv}(I) = \{f \in \hat{Q} \,|\, \tau_f(I) = I\},$$

which is a subgroup of $\hat{Q}$. Then the set $\mathbf{I} := \{\tau^{\alpha}(I) \,:\, \alpha \in \hat{Q}\}$ consist of $|\hat{Q}/\mathrm{Inv}(I)|$ different minimal ideals of $\mathfrak{g}(Q,P,\mathfrak{a})$. From the definitions it follows that

$$\mathrm{Inv}(I) \subset P^{\perp} := \{f \in \hat{Q} \,:\, f(\alpha) = 1 \text{ for all } \alpha \in P\}$$

and hence $|\hat{Q}/\mathrm{Inv}(I)| \geq |\hat{Q}/P^{\perp}| = |P|$, see Remark 2 for the latter equality. Comparing the number of non-zero homogeneous components in $\mathfrak{g}(Q,P,\mathfrak{a})$ and in the subspace

$$\bigoplus_{J \in \mathbf{I}} J \subset \mathfrak{g}(Q,P,\mathfrak{a}),$$

we deduce that these two algebras coincide. The statement of the lemma follows. □

Example 4. In case Q is finite and $\mathrm{char}(\Bbbk)$ does divide $|Q|$, the algebra $\mathfrak{g}(Q,P,\mathfrak{a})$ is not a direct sum of simple ideals in general. For example, let us consider the case that $\mathrm{char}(\Bbbk) = |Q|$, $Q = \mathbb{Z}_p$ and $P = 0$. Since $(t^{\bar{1}})^p - 1 = (t^{\bar{1}} - 1)^p$, we have that

$$\mathfrak{g}(Q,P,\mathfrak{a}) \simeq \mathfrak{a} \otimes (\mathbb{C}[x]/\langle x^p \rangle),$$

where $x = t^{\bar{1}} - 1$. The latter algebra has an abelian ideal $\mathfrak{a} \otimes x^{p-1}$ and a nilpotent ideal $\mathfrak{a} \otimes x\mathbb{C}[x]$ which is, in fact, a maximal ideal.

Example 5. Let $Q = \mathbb{Z}_2 \times \mathbb{Z}_{16}$ and $\mathfrak{a} = \mathfrak{sl}_2$ with standard basis $\{e, h, f\}$. Let $P = \langle(1,4)\rangle$. Then $Q/P \cong \mathbb{Z}_8$ with generator $\overline{(0,1)}$. Consider the following $\mathbb{Z}_8$-grading on $\mathfrak{a}$:

$$\deg(h) = \bar{0}, \quad \deg(e) = \bar{1}, \quad \deg(f) = \bar{7}.$$

Then the non-zero homogeneous components of $\mathfrak{g}(Q, P, \mathfrak{a})$ are:

$$\begin{gathered}
\mathfrak{a}_{\overline{(0,0)}} \otimes 1, \quad \mathfrak{a}_{\overline{(0,1)}} \otimes t^{(0,1)}, \quad \mathfrak{a}_{\overline{(0,7)}} \otimes t^{(0,7)}; \\
\mathfrak{a}_{\overline{(1,4)}} \otimes t^{(1,4)}, \quad \mathfrak{a}_{\overline{(1,5)}} \otimes t^{(1,5)}, \quad \mathfrak{a}_{\overline{(1,11)}} \otimes t^{(1,11)}; \\
\mathfrak{a}_{\overline{(0,8)}} \otimes t^{(0,8)}, \quad \mathfrak{a}_{\overline{(0,9)}} \otimes t^{(0,9)}, \quad \mathfrak{a}_{\overline{(0,15)}} \otimes t^{(0,15)}; \\
\mathfrak{a}_{\overline{(1,12)}} \otimes t^{(1,12)}, \quad \mathfrak{a}_{\overline{(1,13)}} \otimes t^{(1,13)}, \quad \mathfrak{a}_{\overline{(1,3)}} \otimes t^{(1,3)}.
\end{gathered}$$

3. Classification of graded simple Lie algebras: the case of finite Q

3.1. **Preliminaries.** In this section, for a finite abelian group Q, we will classify all Q-graded simple Lie algebras in the case when $\mathrm{char}(\Bbbk)$ does not divide $|Q|$. In fact, we even obtain a more general result. We first assume that Q is an additive abelian group and $\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$ a Q-graded simple Lie algebra.

Every $x \in \mathfrak{g}$ can be written in the form $x = \sum_{\alpha \in Q} x_\alpha$ where $x_\alpha \in \mathfrak{g}_\alpha$. In what follows the notation x_α for some $\alpha \in Q$ always means $x_\alpha \in \mathfrak{g}_\alpha$. We define the *support* of x as

$$\mathrm{supp}(x) := \{\alpha \in Q \mid x_\alpha \neq 0\}.$$

Similarly, we can define $\mathrm{supp}(X)$ for any nonempty subset $X \subset \mathfrak{g}$. Without loss of generality, we may assume that the grading on $\mathfrak{g}$ is *minimal* in the sense that $\mathrm{supp}(\mathfrak{g})$ generates Q.

Classification of graded simple Lie algebras for which the underlying Lie algebra $\mathfrak{g}$ is simple reduces to classification of gradings on simple Lie algebras. Grading on simple Lie algebras are, to some extent, well-studied, see [EK1, EK2], and we will not study this problem in the present paper. Instead, we assume that $\mathfrak{g}$ is not simple.

For $\alpha \in Q$, define $\pi_\alpha : \mathfrak{g} \to \mathfrak{g}_\alpha$ as the projection with respect to the graded decomposition. Take a non-homogeneous non-zero proper ideal I of $\mathfrak{g}$. Define the *size* of I as

$$\mathrm{size}(I) = \min\big\{|\mathrm{supp}(x)| \,:\, x \in I \setminus \{0\}\big\}.$$

Lemma 6.

(*a*) *For any nonzero* $x \in I$ *we have* $|\mathrm{supp}(x)| > 1$.

(*b*) *We have* $\pi_\alpha(I) = \mathfrak{g}_\alpha$ *for each* $\alpha \in Q$.

Proof. Claim (a) follows from the observation that the set

$$J = \{x \in I \,:\, |\mathrm{supp}(x)| \leq 1\}$$

is a nontrivial graded ideal of $\mathfrak{g}$ which has to be zero as $\mathfrak{g}$ is graded simple.

To prove claim (b), let $\bar{I} = \sum_{\alpha \in Q} \pi_\alpha(I)$. It is easy to see that $\bar{I}$ is a nonzero graded ideal of $\mathfrak{g}$ which has to be $\mathfrak{g}$ itself since $\mathfrak{g}$ is graded simple. This completes the proof. □

3.2. **Auxiliary lemmata.** Now we assume that $Q = Q_0 \times Q_1$ where Q_0 if finite and any ideal of $\mathfrak{g}$ is Q_1-graded. Then we have the graded isomorphisms $\tau_f : \mathfrak{g} \to \mathfrak{g}$ for any $f \in \hat{Q}_0$ defined as in (2) (with the convention that $f(\alpha) = 1$ for all $\alpha \in Q_1$).

Lemma 7.

(*a*) *An ideal* I *of* $\mathfrak{g}$ *is* Q*-graded if and only if* $\tau_f(I) \subset I$ *for all* $f \in \hat{Q}_0$.

(*b*) *The center of* $\mathfrak{g}$ *is zero.*

Proof. Claim (a) is clear. Claim (b) follows from claim (a) since the center is an ideal and is invariant under all automorphisms. □

For convenience, we redefine

$$\mathrm{Inv}(I) = \{f \in \hat{Q}_0 \,|\, \tau_f(I) = I\},$$

and set

$$P_0 = \mathrm{Inv}(I)^\perp := \{\alpha \in Q_0 : f(\alpha) = 1, \text{ for all } f \in \mathrm{Inv}(I)\}.$$

From Remark 2 it follows that $|P_0| = |\hat{Q}_0/\mathrm{Inv}(I)|$.

For $\mathbf{I} := \{\tau_f(I) \,:\, f \in \hat{Q}_0\}$, we have that $|\mathbf{I}| = |\hat{Q}_0/\mathrm{Inv}(I)|$.

Lemma 8. *If* $J \cap J' = 0$ *for any* $J \neq J'$ *in* $\mathbf{I}$*, then we have:*

(*a*) $[J, J'] = 0$ *for any* $J \neq J'$ *in* $\mathbf{I}$*;*

(*b*) $\mathfrak{g} = \bigoplus_{J \in \mathbf{I}} J$*;*

(*c*) I *is a simple Lie algebra.*

In particular, if I *is a minimal non-homogeneous non-zero proper ideal of* $\mathfrak{g}$*, then all the above statements hold.*

Proof. Claim (a) follows from the fact that $[J, J'] = J \cap J' = 0$.

To prove claim (b), we first note that $\sum_{J \in \mathbf{I}} J = \mathfrak{g}$ as the left hand side, being closed under the action of $\hat{Q}_0$, is a homogeneous ideal of $\mathfrak{g}$ (and hence coincides with $\mathfrak{g}$ as the latter is graded simple). Let us prove that this sum is direct. For $J \in \mathbf{I}$, consider

$$X_J = J \cap \sum_{J' \in \mathbf{I} \setminus \{J\}} J',$$

which is an ideal of $\mathfrak{g}$. We have $[X_J, \mathfrak{g}] = 0$ by claim (a). Hence $X_J = 0$ by Lemma 7(b). Claim (b) follows.

Finally, suppose I is not simple as a Lie algebra. If $[I, I] = 0$, then from claims (a) and (b) we have $[\mathfrak{g}, \mathfrak{g}] = 0$, which contradicts Lemma 7(b). If $[I, I] \neq 0$, then we take a non-zero proper ideal I_1 of I. From claims (a) and (b) it follows that

$$\bigoplus_{f \in \hat{Q}_0/\mathrm{Inv}(I)} \tau_f(I_1)$$

is a homogeneous non-zero proper ideal of $\mathfrak{g}$ which contradicts graded simplicity of $\mathfrak{g}$. This completes the proof. □

Now we can consider $\mathfrak{g}$ as a Q/P_0-graded Lie algebra. The homogeneous spaces in this graded Lie algebra are indexed by $\bar{\alpha} = \alpha + P_0$, where $\alpha \in Q$. Note that all these homogeneous components are eigenspaces for each τ_f, where $f \in \mathrm{Inv}(I)$. For $\alpha \in Q$ we thus have

$$\mathfrak{g}_{\bar{\alpha}} = \bigoplus_{\beta \in P_0} \mathfrak{g}_{\alpha+\beta}. \tag{4}$$

The decomposition

$$\mathfrak{g} = \bigoplus_{\bar{\alpha} \in Q/P_0} \mathfrak{g}_{\bar{\alpha}}$$

is the decomposition of $\mathfrak{g}$ into a direct sum of common eigenvectors with respect to the action of all τ_f, where $f \in \mathrm{Inv}(I)$. Since I is preserved by all such τ_f, we obtain

$$I_{\bar{\alpha}} = I \cap \mathfrak{g}_{\bar{\alpha}}.$$

Now we can prove existence of a minimal non-homogeneous non-zero proper ideal of $\mathfrak{g}$.

Lemma 9. *Assume that $\mathfrak{g}$ is not simple. Then it has a minimal non-homogeneous non-zero proper ideal.*

Proof. Take a non-homogeneous non-zero proper ideal I of $\mathfrak{g}$. Then I is a Q/P_0-graded Lie algebra

$$I = \bigoplus_{\bar{\alpha}\in Q/P_0} I_{\bar{\alpha}}.$$

From (4) it follows that $\mathrm{size}(I) \leq |P_0|$. Directly from the definitions we also have $\mathrm{Inv}(I) \subset Q_0$.

If $J \cap J' = 0$ for any $J \neq J'$ in $\mathbf{I}$, then from Lemma 8 it follows that I is a minimal ideal of $\mathfrak{g}$. Therefore we assume that $I \cap \tau_f(I) = I_1 \neq 0$ for some $f \in \hat{Q}_0 \setminus \mathrm{Inv}(I)$. We see that $\mathrm{Inv}(I) \subseteq \mathrm{Inv}(I_1) \subset \hat{Q}_0$.

Let $r = \mathrm{size}(I)$ and $r_1 = \mathrm{size}(I_1)$. Note that $r \leq |P_0|$ and $r_1 \leq |P_0|$. Let I_2 be the subideal of I_1 generated by all $x \in I_1$ with $|\mathrm{supp}(x)| = r_1$. Take a nonzero $x \in I_1$ with $|\mathrm{supp}(x)| = r_1$. Then $x = \tau_f(y)$ for some $y \in I$. If $x \notin \Bbbk y$, then I contains a linear combination of x and y which has strictly smaller support. This means that $r < r_1$. Consequently, in the case $r = r_1$ the previous argument shows that $\tau_f^{-1}(x) \in I_1$ for any $x \in I_1$ with $|\mathrm{supp}(x)| = r_1$. This means that $\tau_f(I_2) \subset I_2$ and thus either $\mathrm{Inv}(I_2)$ properly contains $\mathrm{Inv}(I)$ or $r_1 > r$.

Now we change our original ideal I to I_2. In this way we either increase the size of the ideal or the cardinality of the invariant subgroup and start all over again. Since both the size and the cardinality of the invariant subgroup are uniformly bounded, the process will terminate in a finite number of steps resulting in a minimal ideal of $\mathfrak{g}$. □

Because of Lemma 9, from now on we may assume that I is a minimal non-homogeneous non-zero proper ideal of $\mathfrak{g}$.

Let $\alpha \in Q$ and $x \in I_{\bar{\alpha}} \setminus \{0\}$. Since

$$I_{\bar{\alpha}} = I \bigcap \bigoplus_{\beta\in P_0} \mathfrak{g}_{\alpha+\beta},$$

there are unique vectors $x_{\alpha+\beta} \in \mathfrak{g}_{\alpha+\beta}$, where $\beta \in P_0$, such that

$$x = \sum_{\beta\in P_0} x_{\alpha+\beta}. \tag{5}$$

Applying τ_f, where $f \in \hat{Q}_0/\mathrm{Inv}(I)$, to both side of (5), we obtain $|P_0|$ identities:

$$\sum_{\beta\in P_0} f(\alpha+\beta)x_{\alpha+\beta} = \tau_f(x), \quad f \in \hat{Q}_0/\mathrm{Inv}(I).$$

From Lemma 8 (b), we have that $\{\tau_f(x) : f \in \hat{Q}_0/\mathrm{Inv}(I)\}$ is a set of linearly independent elements. Since $|P_0| = |\hat{Q}_0/\mathrm{Inv}(I)|$, we see that $\{x_{\alpha+\beta} : \beta \in P_0\}$ is a set of linearly independent elements and

$$\mathrm{span}\{\tau_f(x) : f \in \hat{Q}_0/\mathrm{Inv}(I)\} = \mathrm{span}\{x_{\alpha+\beta} : \beta \in P_0\}.$$

Thus $\{x_{\alpha+\beta} : \beta \in P_0\}$ can be uniquely determined from the above $|Q_0|$ identities in terms of $\{\tau_f(x) : f \in \hat{Q}_0/\mathrm{Inv}(I)\}$. Therefore the coefficient matrix $\big(f(\alpha+\beta)\big)$, where $f \in \hat{Q}_0/\mathrm{Inv}(I)$ and $\beta \in P_0$, is invertible. The above argument yields the following linear algebra result:

Lemma 10. *Let $\alpha \in Q$ and*

$$x = \sum_{\beta \in P} x_{\alpha+\beta} \in I_{\bar{\alpha}} \setminus \{0\},$$

where $x_{\alpha+\beta} \in \mathfrak{g}_{\alpha+\beta}$ for $\beta \in P_0$. Then $\{x_{\alpha+\beta} : \beta \in P_0\}$ is a set of linearly independent elements and each $x_{\alpha+\beta}$ can be uniquely expressed in terms of elements in $\{\tau_f(x) : f \in \hat{Q}_0/\mathrm{Inv}(I)\}$ and the invertible matrix $\big(f(\alpha+\beta)\big)$, where $f \in \hat{Q}_0/\mathrm{Inv}(I)$ and $\beta \in P_0$. Consequently, $\mathrm{size}(I) = |P_0|$.

We will also need the following recognition result.

Lemma 11. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two Q-graded simple Lie algebras with minimal non-homogeneous non-zero proper ideals I and I', respectively. If $\mathrm{Inv}(I) = \mathrm{Inv}(I')$ and $I \simeq I'$ as Q/P_0-graded Lie algebras, then $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic as Q-graded Lie algebras.*

Proof. From the discussion above we know that both I and I' are simple Lie algebras. Let $\varphi_0 : I \to I'$ be an isomorphism of Q/P_0-graded Lie algebras. We know that, for each $\alpha \in Q$, we have the decomposition

$$I_{\bar{\alpha}} \subset \bigoplus_{\beta \in P_0} \mathfrak{g}_{\alpha+\beta}.$$

For any $\bar{f} \in \hat{Q}_0/\mathrm{Inv}(I)$, set

$$\varphi_f := \tau_f \circ \varphi_0 \circ \tau_f^{-1} : \tau_f(I) \to \tau_f(I').$$

By taking the direct sum, we obtain an isomorphism of Q/P_0-graded Lie algebras as follows:

$$\Phi := \bigoplus_{\bar{f} \in \hat{Q}_0/\mathrm{Inv}(I)} \varphi_f : \mathfrak{g} = \bigoplus_{\bar{f} \in \hat{Q}_0/\mathrm{Inv}(I)} \tau_f(I) \longrightarrow \mathfrak{g}' = \bigoplus_{\bar{f} \in \hat{Q}_0/\mathrm{Inv}(I)} \tau_f(I').$$

The isomorphism Φ commutes with all τ_f by construction. Therefore, Φ is even an isomorphism of Q-graded Lie algebras. □

3.3. **Classification.** The following theorem is the main result of this section.

Theorem 12. *Let $Q = Q_0 \times Q_1$ be an additive abelian group where Q_0 is finite and $\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$ be a Q-graded simple Lie algebra over a field $\Bbbk$ such that* $\mathrm{char}(\Bbbk)$ *does not divide $|Q_0|$. Assume that any ideal of $\mathfrak{g}$*

is Q_1-graded. Then there exists a subgroup $P_0 \subset Q_0$ and a simple Lie algebra $\mathfrak{a}$ with a Q/P_0-grading such that $\mathfrak{g} \simeq \mathfrak{g}(Q, P_0, \mathfrak{a})$.

Proof. We may assume that $\mathfrak{g}$ is not simple. Using Lemma 9, fix a minimal nontrivial non-graded ideal $\mathfrak{a}$ of $\mathfrak{g}$. Then from the lemmata above it follows that $\mathfrak{a}$ is a simple Lie algebra. The Q/P_0-grading for $\mathfrak{a}$ is given by

$$\mathfrak{a} = \bigoplus_{\bar{\alpha} \in Q/P_0} \mathfrak{a}_{\bar{\alpha}}, \tag{6}$$

where

$$\mathfrak{a}_{\bar{\alpha}} = \mathfrak{a} \bigcap \bigoplus_{\beta \in \alpha + P_0} \mathfrak{g}_{\beta}.$$

From the definition of P_0 it follows that $\mathrm{Inv}(\mathfrak{a}) = P_0^{\perp}$. Now the claim follows from Lemma 11 applied to the graded Lie algebras $\mathfrak{g}$ and $\mathfrak{g}(Q, P_0, \mathfrak{a})$, where in both cases the distinguished Q/P_0-graded ideal is $\mathfrak{a}$. □

The following result is a direct consequence of Theorem 12.

Corollary 13. *Let Q be a finite additive abelian group and $\mathfrak{g}$ be a Q-graded simple Lie algebra over $\Bbbk$ such that $\mathrm{char}(\Bbbk)$ does not divide $|Q|$. Then there exists a subgroup $P \subset Q$ and a simple Lie algebra $\mathfrak{a}$ with a Q/P-grading such that $\mathfrak{g} \simeq \mathfrak{g}(Q, P, \mathfrak{a})$.*

Classification of Q-graded simple Lie algebras over a field $\Bbbk$ for a finitely generated additive abelian group Q in the general case has to be dealt with by different methods. In what follows we approach this problem using a graded version of Schur's lemma. At the same time, this will allow us to remove the restriction that $\mathrm{char}(\Bbbk)$ does not divide the cardinality of the torsion subgroup of Q.

4. Graded Schur's Lemma

In this section we prove a graded version of Schur's lemma which we will frequently use in the rest of the paper. This is a standard statement, but we could not find a proper reference for the generality we need.

Let Q be an abelian group, $\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$ a Q-graded Lie algebra over a field $\Bbbk$ and $W = \bigoplus_{\bar{\alpha} \in Q} W_\alpha$ a Q-graded module over $\mathfrak{g}$. For $\alpha \in Q$, we call a module homomorphism $\sigma : W \to W$ *homogeneous of degree α* provided that $\sigma(W_\beta) \subset W_{\beta+\alpha}$.

Theorem 14 (Graded Schur's Lemma)**.** *Let Q be an abelian group and $\mathfrak{g}$ a Q-graded Lie algebra over an algebraically closed field $\Bbbk$. Let W be a Q-graded simple module over $\mathfrak{g}$ with $\dim W < |\Bbbk|$. Then, for any fixed $\alpha \in Q$, any two degree α automorphisms of W differ by a scalar factor only.*

Proof. Let $\mathrm{End}_0(W)$ be the algebra of all homogeneous degree zero endomorphisms of W. It is enough to show that $\mathrm{End}_0(W) = \Bbbk$. The usual arguments give that $\mathrm{End}_0(W)$ is a division algebra over $\Bbbk$. Then W, viewed as an $\mathrm{End}_0(W)$-module, is a sum of copies of $\mathrm{End}_0(W)$. In particular,

$$\dim \mathrm{End}_0(W) \leq \dim W < |\Bbbk|.$$

Since $\Bbbk$ is algebraically closed, if $\mathrm{End}_0(W)$ were strictly larger than $\Bbbk$, then $\mathrm{End}_0(W)$ would contain some σ which is transcendental over $\Bbbk$. Then the fraction field $\Bbbk(\sigma)$ would be contained in $\mathrm{End}_0(W)$. However, we have the elements $\frac{1}{\sigma - a} \in \Bbbk(\sigma)$, where $a \in \Bbbk$, which are linearly independent. Therefore

$$\dim \mathrm{End}_0(W) \geq \dim \Bbbk(\sigma) \geq |\Bbbk|,$$

contradicting the fact that $\dim \mathrm{End}_0(W) < |\Bbbk|$. Thus we conclude that $\mathrm{End}_0(W) = \Bbbk$. $\square$

5. Classification of graded simple Lie algebras: general case

5.1. **Preliminaries.** Let Q be a finitely generated additive abelian group and $\mathfrak{g}$ a Q-graded simple Lie algebra such that $\dim \mathfrak{g} < |\Bbbk|$.

As before, we assume that $\mathfrak{g}$ is not simple. Let I be a nonzero non-homogeneous ideal in $\mathfrak{g}$. Set $r := \mathrm{size}(I) > 1$ and define

$$R(I) := \mathrm{span}\{x \in I : \mathrm{size}(x) = r\}.$$

Then $R(I)$ is an ungraded non-zero proper ideal of $\mathfrak{g}$. We will say that an ideal J of $\mathfrak{g}$ is *pure* of size r if $\mathrm{size}(J) = r$ and, moreover, $R(J) = J$.

Take a nonzero $y = y_\gamma + y_{\gamma+\alpha_1} + y_{\gamma+\alpha_2} + \cdots + y_{\gamma+\alpha_{r-1}} \in I$. Denote by $I_{\alpha_1,\alpha_2,\ldots,\alpha_{r-1}}$ the set of all $x \in I$ for which there exists $\beta \in Q$ such that

$$x = x_\beta + x_{\beta+\alpha_1} + x_{\beta+\alpha_2} + \cdots + x_{\beta+\alpha_{r-1}}.$$

Then the linear span of $I_{\alpha_1,\alpha_2,\ldots,\alpha_{r-1}}$ is an ideal in $\mathfrak{g}$, so we may assume that it coincides with I.

Thus, for any $x_\beta \in \mathfrak{g}_\beta$, where $\beta \in Q$, there are unique

$$x_{\beta+\alpha_1} \in \mathfrak{g}_{\beta+\alpha_1},\ x_{\beta+\alpha_2} \in \mathfrak{g}_{\beta+\alpha_2},\ \ldots,\ x_{\beta+\alpha_{r-1}} \in \mathfrak{g}_{\beta+\alpha_{r-1}}$$

such that

$$x = x_\beta + x_{\beta+\alpha_1} + x_{\beta+\alpha_2} + \cdots + x_{\beta+\alpha_{r-1}} \in I. \tag{7}$$

This allows us to define, for every $i = 1, 2, \dots, r-1$, the map

$$\Lambda_{I,\alpha_i} : \mathfrak{g} \to \mathfrak{g} \quad \text{which sends} \quad x_\beta \mapsto x_{\beta+\alpha_i}.$$

For any $y_\gamma \in \mathfrak{g}_\gamma$, from

$$[y_\gamma, x] = [y_\gamma, x_\beta] + [y_\gamma, x_{\beta+\alpha_1}] + \cdots + [y_\gamma, x_{\beta+\alpha_{r-1}}] \in I$$

we have that $\Lambda_{I,\alpha_i}([y_\gamma, x_\beta]) = [y_\gamma, \Lambda_{I,\alpha_i}(x_\beta)]$. Thus, Λ_{I,α_i} is a Q-graded $\mathfrak{g}$-automorphism of the adjoint $\mathfrak{g}$-module $\mathfrak{g}$ which is, moreover, homogeneous of degree α_i. Since the adjoint representation is simple, these Q-graded $\mathfrak{g}$-module automorphisms are independent of I up to a nonzero scalar multiple, due to Theorem 14. Denote by $\mathbf{G}$ the group generated by all such Λ_{I,α_i} (taken over all ideals I as above). Note that, for each $\alpha \in Q$, all elements of the group $\mathbf{G}$ which are homogeneous of degree α differ only by a scalar multiple. Let P' denote the set of all $\alpha \in Q$ for which $\mathbf{G}$ contains an element which is homogeneous of degree α. For each $\alpha \in P'$ we now fix some $\Lambda_\alpha \in \mathbf{G}$ which is homogeneous of degree α. Set

$$D' = \mathrm{span}\{\Lambda_\alpha : \alpha \in P'\}.$$

Then the following lemma is obvious.

Lemma 15.

(*a*) *The set P' is a subgroup of Q.*

(*b*) *The vector space D' has the structure of an associative division algebra induced by composition of endomorphisms. Moreover, D' is naturally Q-graded with* $\deg(\Lambda_\alpha) = \alpha$.

Now we need the following lemma.

Lemma 16. *The Q-graded associative division algebra D' has a maximal Q-graded commutative subalgebra D.*

Proof. Define the support $\mathrm{supp}(X)$ for any set $X \in D'$ in the obvious way. For any Q-graded commutative subalgebra A of D', it is clear that $\mathrm{supp}(A)$ is a subgroup of Q. Then, for any ascending chain of Q-graded commutative subalgebras $A_1 \subset A_2, \subset \cdots \subset A_n \subset \cdots$, we have the corresponding ascending chain of subgroups

$$\mathrm{supp}(A_1) \subset \mathrm{supp}(A_2) \subset \cdots \subset \mathrm{supp}(A_n) \subset \cdots.$$

Since Q is a finitely generated abelian group, any ascending chain of subgroups of Q stabilizes. Thus, there is $n \in \mathbb{N}$ such that $\mathrm{supp}(A_n) = \mathrm{supp}(A_{n+i})$ for all $i \in \mathbb{N}$. This means that $A_n = A_{n+i}$ for all $i \in \mathbb{N}$. Therefore D' has a maximal Q-graded commutative subalgebra D by Zorn's lemma. $\square$

Set $P = \operatorname{supp}(D)$. Then P is a subgroup of Q. Later on we will show that $D = D'$.

Lemma 17. *The P-graded associative division algebra D is isomorphic to the group algebra $\Bbbk P$.*

Proof. Since P is a subgroup of Q, the group P is also a finitely generated abelian group. There exist $\alpha_1, \alpha_2, \cdots, \alpha_n \in P$ such that

$$P = \mathbb{Z}\alpha_1 \times \mathbb{Z}\alpha_2 \times \cdots \times \mathbb{Z}\alpha_m.$$

Denote by m_i the order of α_i in Q (possibly infinite). Since $\Bbbk$ is algebraically closed, we may replace each Λ_{α_i}, if necessary, by its scalar multiple such that $(\Lambda_{\alpha_i})^{m_i} = 1$. Redefine now

$$\Lambda_{k_1\alpha_1 + k_2\alpha_2 + \cdots + k_m\alpha_m} := (\Lambda_{\alpha_1})^{k_i} (\Lambda_{\alpha_2})^{k_2} \cdots (\Lambda_{\alpha_n})^{k_n},$$

where $k_i = 1, 2, \ldots, m_i$ if $m_i < \infty$ and $k_i \in \mathbb{Z}$ otherwise, for all $i = 1, 2, \ldots, n$. It is easy to see that the set $\mathbf{\Lambda} := \{\Lambda_{k_1\alpha_1 + k_2\alpha_2 + \cdots + k_m\alpha_m}\}$ obtained in this way is a basis of D. The set $\mathbf{\Lambda}$ is also a commutative group with respect to composition and is isomorphic to P. Thus D is isomorphic to the group algebra $\Bbbk P$. $\square$

From the proof of Lemma 17 we have that elementS Λ_α, for $\alpha \in P$, now satisfy

$$\Lambda_\alpha \Lambda_\beta = \Lambda_{\alpha+\beta} \quad \text{for all} \quad \alpha, \beta \in P.$$

Let $\mathcal{I}$ be the span of the set

$$\{x_\beta - \Lambda_\alpha(x_\beta) : \beta \in Q, x_\beta \in \mathfrak{g}_\beta \text{ and } \alpha \in P\}.$$

Then $\mathcal{I}$ is an ideal of $\mathfrak{g}$.

Lemma 18. *The ideal $\mathcal{I}$ is a proper ideal of $\mathfrak{g}$.*

Proof. Let $\{\beta_j : j \in B\}$ be a set of representatives for cosets in Q/P, where B is an index set. Let $\{x_{\beta_j}^{(k)} : k \in B_j\}$ be a basis of $\mathfrak{g}_{\beta_j}$, where B_j is an index set. Then the set

$$\{\mathbf{\Lambda}(x_{\beta_j}^{(k)}) \,:\, j \in B, k \in B_j\}$$

is a basis for $\mathfrak{g}$. For $j \in B$ and $k \in B_j$, set

$$S_{jk} := \{x_{\beta_j}^{(k)} - \Lambda(x_{\beta_j}^{(k)}) \,:\, \Lambda \in \mathbf{\Lambda}\}$$

and

$$S = \bigcup_{j \in B} \bigcup_{k \in B_j} S_{jk}.$$

Comparing supports of involved elements, it is easy to see that the sum

$$\mathcal{I} = \sum_{j \in B} \sum_{k \in B_j} \operatorname{span}(S_{jk})$$

is direct. Define the linear map

$$\sigma : \mathfrak{g} \to \Bbbk \text{ such that } \Lambda(x_{\beta_j}^{(k)}) \mapsto 1 \text{ for all } \Lambda \in \mathbf{\Lambda}. \tag{8}$$

Clearly σ is onto and $\sigma(\mathcal{I}) = 0$. Therefore $\mathcal{I} \neq \mathfrak{g}$ and thus $\mathcal{I}$ is a proper ideal. By construction, this ideal is pure of size two. □

Lemma 19. *The ideal $\mathcal{I}$ is a maximal ideal of $\mathfrak{g}$.*

Proof. Assume that $\mathcal{I}$ is not maximal. Then there is a proper ideal $\mathcal{I}'$ of $\mathfrak{g}$ which properly contains $\mathcal{I}$. There must exist different elements $\beta, \beta+\gamma_1, \beta+\gamma_2, \dots$ in Q and non-zero elements $x_\beta \in \mathfrak{g}_\beta$, $x_{\beta+\gamma_1} \in \mathfrak{g}_{\beta+\gamma_1}$, $x_{\beta+\gamma_2} \in \mathfrak{g}_{\beta+\gamma_2}, \dots, x_{\beta+\gamma_r} \in \mathfrak{g}_{\beta+\gamma_r}$ such that

$$x := x_\beta + x_{\beta+\gamma_1} + x_{\beta+\gamma_2} + \dots + x_{\beta+\gamma_r} \in \mathcal{I}' \setminus \mathcal{I}. \tag{9}$$

Assume that the support of the latter element has the minimal possible size among all elements in $\mathcal{I}' \setminus \mathcal{I}$. Then all $\gamma_i \in P'$ and we have

$$x := x_\beta + \Lambda_{\gamma_1}(x_\beta) + \dots + \Lambda_{\gamma_r}(x_\beta).$$

For any $j = 1, 2, \dots, m$, we have

$$\begin{aligned} \Lambda_{\alpha_j}(x) &= x - (x - \Lambda_{\alpha_j}(x)) \\ &= \Lambda_{\alpha_j}(x_\beta) + \Lambda_{\alpha_j}\Lambda_{\gamma_1}(x_\beta) + \dots + \Lambda_{\alpha_j}\Lambda_{\gamma_2}(x_\beta) \in \mathcal{I}' \setminus \mathcal{I}. \end{aligned}$$

At the same time, we have

$$y = \Lambda_{\alpha_j}(x_\beta) + \Lambda_{\gamma_1}\Lambda_{\alpha_j}(x_\beta) + \dots + \Lambda_{\gamma_r}\Lambda_{\alpha_j}(x_\beta) \in \mathcal{I}',$$

as this element has the form (9) with x_β replaced by $\Lambda_{\alpha_j}(x_\beta)$.

Since $\Lambda_{\alpha_j}(x)$ and y are both in $\mathcal{I}'$ and have the same minimal support, they must differ by a scalar factor only. Since their $\beta + \alpha_j$-terms coincide, it follows that $y = \Lambda_{\alpha_j}(x)$. Hence $\Lambda_{\alpha_j}\Lambda_{\gamma_i} = \Lambda_{\gamma_i}\Lambda_{\alpha_j}$ for all $j = 1, 2, \dots, m$ and $i = 1, 2, \dots, n$. Thus all $\gamma_i \in P$ and hence $\mathcal{I} = \mathfrak{g}$ or $\mathcal{I}' = \mathcal{I}$, a contradiction. The claim follows. □

5.2. **The main result.**

Theorem 20. *Let Q be a finitely generated additive abelian group and $\mathfrak{g}$ be a Q-graded simple Lie algebra over a field $\Bbbk$ such that $\dim \mathfrak{g} < |\Bbbk|$. Then there exists a subgroup $P \subset Q$ and a simple Lie algebra $\mathfrak{a}$ with a Q/P-grading such that $\mathfrak{g} \simeq \mathfrak{g}(Q, P, \mathfrak{a})$.*

Proof. Let $\mathcal{I}$ be the ideal constructed in Lemma 18. Then from Lemma 19 it follows that $\mathfrak{a} = \mathfrak{g}/\mathcal{I}$ is a simple Lie algebra with a (Q/P)-grading.

Now, the Q-graded canonical map

$$\begin{aligned} \mathfrak{g} &\to \mathfrak{g}(Q, P, \mathfrak{a}), \\ x_\alpha &\mapsto (x_\alpha + \mathcal{I}) \otimes t^\alpha \end{aligned}$$

is, clearly, bijective (note that $\mathcal{I}$ is not mapped to zero as it is not homogeneous). Moreover, this map is a homomorphism of Lie algebras since

$$[x_\alpha, x_\beta] \mapsto \big([x_\alpha, x_\beta] + \mathcal{I}\big) \otimes t^{\alpha+\beta} = [(x_\alpha + \mathcal{I}) \otimes t^\alpha, (x_\beta + \mathcal{I}) \otimes t^\beta].$$

Therefore, this map is an isomorphism of Lie algebras. Consequently, $\mathfrak{g} \simeq \mathfrak{g}(Q, P, \mathfrak{a})$. $\square$

Combining Theorems 12 and 20, we obtain the following:

Corollary 21. *Let Q be an abelian group and $\mathfrak{g}$ a finite dimensional Q-graded simple Lie algebra over an algebraically closed field $\mathbb{k}$. Then there exists a subgroup $P \subset Q$ and a simple Lie algebra $\mathfrak{a}$ with a Q/P-grading such that $\mathfrak{g} \simeq \mathfrak{g}(Q, P, \mathfrak{a})$.*

Combining Lemma 3 with Theorems 12 and 20, we obtain:

Corollary 22. *Let Q be a finite additive abelian group and $\mathfrak{g}$ a Q-graded simple Lie algebra over a field $\mathbb{k}$ such that* $\operatorname{char}(\mathbb{k})$ *does not divide $|Q|$ or* $\dim \mathfrak{g} < |\mathbb{k}|$. *Then $\mathfrak{g}$ is a direct sum of at most $|Q|$ copies of isomorphic ideals, each of which is a simple Lie algebra.*

The following result is a generalization of [Ma, Main Theorem (a)] which follows directly from Theorem 20.

Corollary 23. *Let $Q = Q_0 \times Q_1$ be an additive abelian group where Q_0 is torsion subgroup of Q. Let $\mathfrak{g}$ be a finite dimensional Q-graded simple Lie algebra. Then $\mathfrak{g}$ is a Q_0-graded simple Lie algebra.*

Next we determine some necessary and sufficient conditions for two graded simple Lie algebras to be isomorphic. To do this, we need some properties for the Lie algebras $\mathfrak{g}(Q, P, \mathfrak{a})$ constructed in Section 2.

Lemma 24. *Let $\mathfrak{g}(Q, P, \mathfrak{a})$ be as in Theorem 20 with* $\dim \mathfrak{a} < |\mathbb{k}|$ *and $\alpha \in Q$. Then there is a degree α homogeneous $\mathfrak{g}(Q, P, \mathfrak{a})$-module automorphism of the adjoint module $\mathfrak{g}(Q, P, \mathfrak{a})$ if and only if $\alpha \in P$.*

Proof. Suppose $\alpha \in P$. By construction, $\mathfrak{a}_{\bar{\alpha}+\bar{\beta}} = \mathfrak{a}_{\bar{\beta}}$ for any $\bar{\beta} \in Q/P$. It is easy to verify that the linearization of following map is a degree α homogeneous $\mathfrak{g}(Q, P, \mathfrak{a})$-module automorphism of the adjoint module $\mathfrak{g}(Q, P, \mathfrak{a})$:

$$\begin{array}{rcl} \mathfrak{g}(Q, P, \mathfrak{a}) & \to & \mathfrak{g}(Q, P, \mathfrak{a}), \\ x_{\bar{\beta}} \otimes t^\beta & \mapsto & x_{\bar{\beta}} \times t^{\alpha+\beta}, \end{array}$$

where $\beta \in Q$.

Now suppose $\alpha \in Q \setminus P$. Assume that the following is a degree α homogeneous $\mathfrak{g}(Q, P, \mathfrak{a})$-module automorphism of the adjoint module

$\mathfrak{g}(Q,P,\mathfrak{a})$:

$$\begin{array}{rcl}\tau:\mathfrak{g}(Q,P,\mathfrak{a}) & \to & \mathfrak{g}(Q,P,\mathfrak{a}),\\ x_{\bar\beta}\otimes t^{\beta} & \mapsto & \mu(x_{\bar\beta})\times t^{\alpha+\beta},\end{array}$$

where $\beta\in Q$ and where μ is a degree $\bar\alpha$ linear automorphism of the Q/P-graded vector space $\mathfrak{a}$. For any $x_{\bar\gamma}\otimes t^{\gamma}\in\mathfrak{g}(Q,P,\mathfrak{a})_{\gamma}$, we have

$$\begin{array}{rcl}\mu([x_{\bar\gamma},x_{\bar\beta}])\otimes t^{\gamma+\beta} & = & \tau([x_{\bar\gamma}\otimes t^{\gamma},x_{\bar\beta}\otimes t^{\beta}])\\ & = & [x_{\bar\gamma}\otimes t^{\gamma},\tau(x_{\bar\beta}\otimes t^{\beta})]\\ & = & [x_{\bar\gamma}\otimes t^{\gamma},\mu(x_{\bar\beta})\otimes t^{\beta}]\\ & = & [x_{\bar\gamma},\mu(x_{\bar\beta})]\otimes t^{\gamma+\beta},\end{array}$$

where for the second equality we use that τ is a homomorphism of the adjoint module $\mathfrak{g}(Q,P,\mathfrak{a})$. This implies $\mu([x_{\gamma},x_{\bar\beta}])=[x_{\gamma},\mu(x_{\bar\beta})]$ for all $\beta,\gamma\in Q$, that is, μ is an automorphism of the adjoint module $\mathfrak{a}$. Then μ is a scalar by Schur's lemma. Therefore $\mathfrak{a}_{\bar\beta}=\mathfrak{a}_{\bar\beta+\bar\alpha}$, which contradics our assumption that $\alpha\notin P$. □

We note that from Lemma 24 it follows that the division algebra D' in Lemma 16 is actually commutative, that is, $D'=D$. Now we can formulate some necessary and sufficient conditions for two graded simple Lie algebras to be isomorphic.

Theorem 25. *Let Q, Q' be abelian groups, $\mathfrak{g}(Q,P,\mathfrak{a})$ be a Q-graded simple Lie algebra over $\Bbbk$ with $\dim\mathfrak{a}<|\Bbbk|$ and $\mathfrak{g}(Q',P',\mathfrak{a}')$ be a Q'-graded Lie algebra over $\Bbbk$ with minimal gradings. Then $\mathfrak{g}(Q,P,\mathfrak{a})$ is graded isomorphic to $\mathfrak{g}(Q',P',\mathfrak{a}')$ if and only if there is a group isomorphism $\tau:Q\to Q'$ such that $\tau(P)=P'$ and the simple Lie algebras $\mathfrak{a}$ and $\mathfrak{a}'$ are graded isomorphic.*

Proof. The "if" part is clear. Now suppose that $\mathfrak{g}=\mathfrak{g}(Q,P,\mathfrak{a})$ is graded isomorphic to $\mathfrak{g}'=\mathfrak{g}(Q',P',\mathfrak{a}')$. There is a group isomorphism $\tau:Q\to Q'$ and a Lie algebra isomorphism

$$\sigma:\mathfrak{g}\to\mathfrak{g}',\ \ x_{\bar\beta}\otimes t^{\beta}\mapsto x'_{\overline{\tau(\beta)}}\otimes t^{\tau(\beta)},$$

where $\bar\beta\in Q/P$, $\overline{\tau(\beta)}\in Q'/P'$, and the map $\mu:x_{\bar\beta}\mapsto x'_{\overline{\tau(\beta)}}$ is a vector space isomorphism from $\mathfrak{a}\to\mathfrak{a}'$. Suppose that $\alpha\in P$. Then

$$\theta:\mathfrak{g}\to\mathfrak{g},\ \ x_{\bar\beta}\otimes t^{\beta}\mapsto x_{\bar\beta}\otimes t^{\beta+\alpha}$$

is a Q-graded $\mathfrak{g}$-module automorphism of degree α. Then there is a unique linear map $\theta':\mathfrak{g}'\to\mathfrak{g}'$ which makes the following diagram commutative:

$$\begin{tikzcd}\mathfrak{g} \arrow[r, "\theta"] \arrow[d, "\sigma"'] & \mathfrak{g} \arrow[d, "\sigma"]\\ \mathfrak{g}' \arrow[r, "\theta'"'] & \mathfrak{g}'\end{tikzcd}$$

This means that the following diagram is commutative:

$$\begin{tikzcd}
x_{\bar\beta} \otimes t^{\beta} \arrow[r, "\theta"] \arrow[d, "\sigma"'] & x_{\bar\beta} \otimes t^{\alpha+\beta} \arrow[d, "\sigma"] \\
x'_{\overline{\tau(\beta)}} \otimes t^{\tau(\beta)} \arrow[r, "\theta'"'] & x'_{\overline{\tau(\beta)}} \otimes t^{\tau(\alpha)+\tau(\beta)}
\end{tikzcd}$$

We claim that θ' is a homomorphism of Q'-graded $\mathfrak{g}'$-modules. Indeed,

$$\begin{aligned}
\theta'([x'_{\overline{\tau(\gamma)}} \otimes t^{\tau(\gamma)}, x'_{\overline{\tau(\beta)}} \otimes t^{\tau(\beta)}]) &= \theta'([\sigma(x_{\bar\gamma} \otimes t^{\gamma}), \sigma(x_{\bar\beta} \otimes t^{\beta})]) \\
&= \theta'\sigma([x_{\bar\gamma} \otimes t^{\gamma}, x_{\bar\beta} \otimes t^{\beta}]) \\
&= \sigma\theta([x_{\bar\gamma} \otimes t^{\gamma}, x_{\bar\beta} \otimes t^{\beta}]) \\
&= \sigma([x_{\bar\gamma} \otimes t^{\gamma}, \theta(x_{\bar\beta} \otimes t^{\beta})]) \\
&= [\sigma(x_{\bar\gamma} \otimes t^{\gamma}), \sigma\theta(x_{\bar\beta} \otimes t^{\beta})]) \\
&= [\sigma(x_{\bar\gamma} \otimes t^{\gamma}), \theta'\sigma(x_{\bar\beta} \otimes t^{\beta})]) \\
&= [x'_{\overline{\tau(\gamma)}} \otimes t^{\tau(\gamma)}, \theta'(x'_{\overline{\tau(\beta)}} \otimes t^{\tau(\beta)})],
\end{aligned}$$

for any $\beta, \gamma \in Q$. Here we use that σ is an algebra isomorphism and θ is an automorphism of the adjoint module $\mathfrak{g}(Q, P, \mathfrak{a})$. By construction, θ' is homogeneous and has degree $\tau(\alpha)$.

By Lemma 24, $\tau(\alpha) \in P'$ and thus $\tau(P) \subset P'$. By symmetry, we obtain that $P' \subset \tau(P)$ and thus $\tau(P) = P'$. Then

$$\begin{aligned}
\mu([x_{\overline{\gamma}}, x_{\overline{\beta}}]) \otimes t^{\tau(\gamma)+\tau(\beta)} &= \sigma([x_{\overline{\gamma}} \otimes t^{\gamma}, x_{\overline{\beta}} \otimes t^{\beta}]) \\
&= [\sigma(x_{\overline{\gamma}} \otimes t^{\gamma}), \sigma(x_{\overline{\beta}} \otimes t^{\beta})] = [\mu(x_{\overline{\gamma}}), \mu(x_{\overline{\beta}})] \otimes t^{\tau(\gamma)+\tau(\beta)}),
\end{aligned}$$

and thus $\mu([x_{\overline{\gamma}}, x_{\overline{\beta}}]) = [\mu(x_{\overline{\gamma}}), \mu(x_{\overline{\beta}})]$ for all $\beta, \gamma \in Q$. It follows that μ is a graded isomorphism from the Q/P-graded Lie algebra $\mathfrak{a}$ to the Q'/P'-graded Lie algebra $\mathfrak{a}'$. The theorem follows. □

From Corollary 21, we actually obtain a full classification of all finite-dimensional Q-graded simple Lie algebras over any algebraically closed field of characteristic 0 due to the recent classification of all gradings on finite dimensional simple Lie algebras, see [EK2, El, Yu]. For a similar classification over an algebraically closed field of characteristic $p > 0$ it remains only to determine all gradings on finite dimensional simple Lie algebras. Some partial results in this direction can be found in [EK2], see also references therein.

6. Graded Weyl Theorem

One consequence of our classification in Theorem 20 is that any finite dimensional Q-graded simple Lie algebra over an algebraically closed field $\mathbb{k}$ of characteristic 0 is semi-simple after forgetting the grading (note that this property is not true in positive characteristic). This allows us to prove a graded version of the Weyl Theorem.

Theorem 26 (Graded Weyl Theorem)**.** *Let Q be an abelian group and $\mathfrak{g}$ a finite dimensional Q-graded semi-simple Lie algebra over an algebraically closed field $\Bbbk$ of characteristic 0. Then any finite dimensional Q-graded module V over $\mathfrak{g}$ is completely reducible as a graded module, that is, V is a direct sum of Q-graded simple submodules of V.*

Proof. Since $\mathfrak{g}$ is finite-dimensional, the minimal grading of $\mathfrak{g}$ is by a finitely generated subgroup of Q. From Theorem 20 and Lemma 3, it follows that, as an ungraded Lie algebra, $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra.

We need to show that any Q-graded submodule X of a Q-graded finite dimensional $\mathfrak{g}$-module W has a Q-graded complement. By Weyl Theorem, we have an ungraded $\mathfrak{g}$-submodule Y_1 such that $W = X \oplus Y_1$. By [EK1, Lemma 1.1] (see also [CM, Theorem 2.3']), there is a Q-graded submodule Y of W such that $W = X \oplus Y$. The theorem follows. $\square$

7. Graded simple modules over graded Lie algebras

7.1. **Construction.** Let Q be an abelian group and P a subgroup of Q. Let, further, $\mathfrak{g}$ be a Q-graded Lie algebra over $\Bbbk$. Consider $\mathfrak{g} = \bigoplus_{\bar{\alpha}\in Q/P} \mathfrak{g}_{\bar{\alpha}}$ as a Q/P-graded Lie algebra with $\mathfrak{g}_{\bar{\alpha}} = \bigoplus_{\beta\in P} \mathfrak{g}_{\alpha+\beta}$. Let $V = \bigoplus_{\bar{\alpha}\in Q/P} V_{\bar{\alpha}}$ be a simple $\mathfrak{g}$-module with a fixed Q/P-grading.

Then we can form the $\mathfrak{g}$-module $V \otimes \Bbbk Q$ as follows: for $x \in \mathfrak{g}_\alpha$, $v \in V$ and $\beta \in Q$, define

$$x \cdot \big(v \otimes t^\beta\big) = (xv) \otimes t^{\alpha+\beta}.$$

Define the Q-graded $\mathfrak{g}$-module

$$M(Q,P,V) := \bigoplus_{\alpha\in Q} M(Q,P,V)_\alpha, \qquad \text{where} \qquad M(Q,P,V)_\alpha := V_{\bar{\alpha}}(\alpha).$$

For example, $M(Q,Q,V) = V \otimes \Bbbk Q$ (with the obvious Q-grading) while $M(Q,\{0\},V) = V$ (with the original Q-grading). From the definition it follows that $\dim M(Q,P,V) = \dim(V)|P|$ if V is finite-dimensional and P is finite.

We say that a simple $\mathfrak{g}$-module V with a Q/P-grading is *grading extendable* if there is a decomposition

$$V = \sum_{\alpha\in Q} X_\alpha \quad \text{with} \quad V_{\bar{\alpha}} = \sum_{\beta\in P} X_{\alpha+\beta}$$

for any $\alpha \in Q$ (here both sums are not necessarily direct) such that $\mathfrak{g}_\beta X_\alpha \subset X_{\alpha+\beta}$ for all $\beta \in Q$ and at least one $X_\alpha \neq V_{\bar{\alpha}}$.

Now we can obtain some necessary and sufficient conditions for the Q-graded module $M(Q,P,V)$ to be Q-graded simple.

Lemma 27. *The Q-graded module $M(Q,P,V)$ is Q-graded simple if and only if the simple module V is not grading extendable.*

Proof. If V is grading extendable, there is a decomposition $V = \sum_{\alpha \in Q} X_\alpha$ with $V_{\bar{\alpha}} = \sum_{\beta \in P} X_{\alpha+\beta}$, for any $\alpha \in Q$, such that $\mathfrak{g}_\alpha X_\beta \subset X_{\alpha+\beta}$ and at least one $X_{\alpha+\beta} \neq V_{\bar{\alpha}}$. Then the module $M(Q,P,V)$ has a nonzero proper Q-graded submodule

$$\bigoplus_{\alpha \in Q} X_\alpha \otimes t^\alpha.$$

Thus $M(Q,P,V)$ is not Q-graded simple.

Now suppose that $M(Q,P,V)$ is not Q-graded simple. Consider the ideal I in $\mathbb{C}Q$ generated by $\{t^\alpha - 1 : \alpha \in P\}$ and let

$$N = M(Q,P,V)/(M(Q,P,V)) \cap (V \otimes I)).$$

The module N is naturally Q/P-graded and is, in fact, isomorphic to the Q/P-graded module V with the original grading. We have that

$$N = \bigoplus_{\bar{\alpha} \in Q/P} V_{\bar{\alpha}} \otimes t^{\bar{\alpha}},$$

where $\{t^{\bar{\alpha}} : \bar{\alpha} \in Q/P\}$ is a basis for the group algebra of Q/P. Let

$$\pi : M(Q,P,V) \twoheadrightarrow N,$$

be the canonical epimorphism.

Take a proper Q-graded submodule $X = \bigoplus_{\alpha \in Q} X_\alpha \otimes t^\alpha$ of $M(Q,P,V)$. Since $\pi(X) = N$, we have $V_{\bar{\alpha}} = \sum_{\beta \in P} X_{\alpha+\beta}$ for any $\alpha \in Q$ and, also, $\mathfrak{g}_\alpha X_\beta \subset X_{\alpha+\beta}$. Since W is proper, we have $X_\alpha \neq V_{\bar{\alpha}}$. Thus V is grading extendable. $\square$

Before giving some better necessary and sufficient conditions for the Q-graded module $M(Q,P,V)$ to be Q-graded simple in the case of a finitely generated P, we need to introduce some concepts on grading refinements in parallel to the corresponding concepts for Lie algebras (cf. [EK2]).

Let $R \subset P$ be subgroups of Q, let $\mathfrak{g}$ be a Q-graded Lie algebra. Assume that

$$V = \bigoplus_{\bar{\alpha} \in Q/P} V_{\bar{\alpha}} \tag{10}$$

is a Q/P-graded $\mathfrak{g}$-module. We call a Q/R-grading

$$V = \bigoplus_{\bar{\bar{\alpha}} \in Q/R} V_{\bar{\bar{\alpha}}} \tag{11}$$

a *refinement* of the grading (10) if for any $\alpha \in Q$ we have

$$V_{\bar{\alpha}} = \bigoplus_{\bar{\bar{\beta}} \in P/R} V_{\bar{\bar{\alpha}} + \bar{\bar{\beta}}}.$$

The grading (10) is then said to be a *coarsening* of the grading (11). The refinement (11) is *proper* if $V_{\bar{\bar{\alpha}}} \neq V_{\bar{\alpha}}$ for some $\alpha \in Q$. The grading (10) is called *fine* if it does not admit any proper refinement.

Now we can obtain some better necessary and sufficient conditions for the Q-graded module $M(Q, P, V)$ to be Q-graded simple in the case when P is finitely generated.

Lemma 28. *Let P be a finitely generated subgroup of the abelian group Q. Assume that* $\dim V < |\Bbbk|$*. Then the Q-graded module $M(Q, P, V)$ is Q-graded simple if and only if the Q/P-grading of the simple module V is fine.*

Proof. First suppose that the Q/P-grading of V is not fine, i.e., it has a refinement of the form (11) for some subgroup $R \subsetneq P$ with $V_{\bar{\bar{\alpha}}} \neq V_{\bar{\alpha}}$ for some $\alpha \in Q$. Then $M(Q, P, V)$ has a nonzero proper Q-graded submodule

$$\bigoplus_{\alpha \in Q} V_{\bar{\bar{\alpha}}} \otimes t^{\alpha}$$

and hence is not Q-graded simple.

Now suppose that $M(Q, P, V)$ is not Q-graded simple. Consider the ideal I in $\mathbb{C}Q$ generated by $\{t^{\alpha} - 1 : \alpha \in P\}$ and let

$$N = M(Q, P, V)/(M(Q, P, V)) \cap (V \otimes I)).$$

The module N is naturally Q/P-graded and is, in fact, isomorphic to the Q/P-graded module V with the original grading. We have that

$$N = \bigoplus_{\bar{\alpha} \in Q/P} V_{\bar{\alpha}} \otimes t^{\bar{\alpha}},$$

where $\{t^{\bar{\alpha}} : \bar{\alpha} \in Q/P\}$ forms a basis for the group algebra of Q/P. Let

$$\pi : M(Q, P, V) \twoheadrightarrow N,$$

be the canonical epimorphism.

For any nonzero proper Q-graded submodule $X = \bigoplus_{\alpha \in Q} X_\alpha \otimes t^\alpha$ of $M(Q, P, V)$, where $X_\alpha \subset V_{\bar\alpha}$, we have that $\pi(X) = N$ since N is simple. Consequently,

$$V_{\bar\alpha} = \sum_{\beta \in P} X_{\alpha+\beta} \tag{12}$$

for any $\alpha \in Q$ and $\mathfrak{g}_\alpha X_\gamma \subset X_{\alpha+\gamma}$ for any $\gamma \in Q$. Since X is a proper submodule, we have $X_\alpha \neq V_{\bar\alpha}$ for some $\alpha \in Q$.

Since P is finitely generated, in particular, noetherian, there is a (not necessarily unique) maximal subgroup R of P such that the Q/R-graded module $M(Q/R, P/R, V)$ is not simple. In this case for any subgroup R_1 of Q such that $R \subsetneq R_1$, the corresponding Q/R_1-graded module $M(Q/R_1, P/R_1, V)$ is simple.

Without loss of generality we may assume that $R = 0$. Take a nonzero proper Q-graded submodule X. We will show that the sum (12) is direct.

For any $\beta \in P$, the map

$$\begin{array}{rcl} \Lambda_\beta : M(Q, P, V) & \to & M(Q, P, V), \\ v_{\bar\alpha} \otimes t^\alpha & \mapsto & v_{\bar\alpha} \otimes t^{\alpha+\beta}, \end{array}$$

is a homogeneous automorphism of $M(Q, P, V)$ of degree β. Moreover, we have $\Lambda_{\beta_1}\Lambda_{\beta_2} = \Lambda_{\beta_1+\beta_2}$ for any $\beta_1, \beta_2 \in P$.

If $X_{\alpha+\beta} \cap X_\alpha \neq 0$ for some nonzero $\beta \in P$ and some $\alpha \in Q$, the submodule $X^{(1)} := \Lambda_\beta(X) \cap X \neq 0$. Clearly $\Lambda_\beta(X^{(1)}) = X^{(1)}$. Thus $X^{(1)}_\gamma = X^{(1)}_{\gamma+\beta}$ for any $\gamma \in Q$, that is

$$X^{(1)}_\gamma = X^{(1)}_{\gamma+k\beta}, \text{ for any } \gamma \in Q, k \in \mathbb{Z}.$$

Let $R_1 = \mathbb{Z}\beta$ which is a subgroup of P. Then

$$\bigoplus_{\bar{\bar\alpha} \in Q/R_1} X^{(1)}_{\bar{\bar\alpha}} \otimes t^{\bar{\bar\alpha}}, \text{ where } X^{(1)}_{\bar{\bar\alpha}} = X^{(1)}_\alpha,$$

is a nonzero proper submodule of $M(Q/R_1, P/R_1, V)$. This contradicts our assumption on P. Thus

$$X_{\alpha+\beta} \cap X_\alpha = 0, \text{ for all } \alpha \in Q \text{ and } \beta \in P \setminus \{0\}. \tag{13}$$

Let U be the universal enveloping algebra of $\mathfrak{g}$. We have the obvious decomposition

$$U = \bigoplus_{\gamma \in Q} U_\gamma$$

corresponding to the Q-grading of $\mathfrak{g}$.

Assume that the sum in (12) is not direct and

$$v_\alpha + v_{\alpha+\beta_1} + v_{\alpha+\beta_2} + \cdots + v_{\alpha+\beta_r} = 0$$

where $v_\alpha \in X_\alpha$, $v_{\alpha+\beta_1} \in X_{\alpha+\beta_1}, \dots,$ $v_{\alpha+\beta_r} \in X_{\alpha+\beta_r}$, $v_\alpha \neq 0$ and r is minimal possible. Let $X^{(2)}$ be the Q/P-graded submodule of X generated by the element

$$v = v_\alpha \otimes t^\alpha + v_{\alpha+\beta_1} \otimes t^{\alpha+\beta_1} + v_{\alpha+\beta_2} \otimes t^{\alpha+\beta_2} + \cdots + v_{\alpha+\beta_r} \otimes t^{\alpha+\beta_r}.$$

Elements in $X^{(2)}$ are linear combinations of vectors of the form

$$u_\gamma v_\alpha \otimes t^{\alpha+\gamma} + u_\gamma v_{\alpha+\beta_1} \otimes t^{\alpha+\beta_1+\gamma} + \cdots + u_\gamma v_{\alpha+\beta_r} \otimes t^{\alpha+\beta_r+\gamma},$$

where $u_\gamma \in U_\gamma$. By construction, $X^{(2)}$ is a pure submodule of size $r+1$ satisfying (13). By varying γ and u_γ, the element $u_\gamma v_\alpha$ will cover the whole of V since V is a simple $\mathfrak{g}$-module. Define the vector space homomorphisms $\Phi_i : V \to V$, for $i = 1, 2, \dots, r$, such that

$$\begin{aligned} &u_\gamma v_\alpha \otimes t^{\alpha+\gamma} + u_\gamma v_{\alpha+\beta_1} \otimes t^{\alpha+\beta_1+\gamma} + \cdots + u_\gamma v_{\alpha+\beta_r} \otimes t^{\alpha+\beta_r+\gamma} = \\ &\quad = u_\gamma v_\alpha \otimes t^{\alpha+\gamma} + \Phi_1(u_\gamma v_\alpha) \otimes t^{\alpha+\beta_1+\gamma} + \cdots + \Phi_r(u_\gamma v_\alpha) \otimes t^{\alpha+\beta_r+\gamma}. \end{aligned}$$

From the minimality of r it follows that that each Φ_i is a nonzero $\mathfrak{g}$-module homomorphism (for otherwise, taking a linear combination, we get an element in $X^{(2)}$ of smaller size). From Theorem 14 it thus follows that that each Φ_i is a nonzero scalar. In particular, $v_{\alpha+\beta_1} = a v_\alpha$ for some nonzero $a \in \Bbbk$. So $X_\alpha \cap X_{\alpha+\beta_1} \neq 0$, which contradicts (13). Thus the sums (12) are direct. Hence the Q/P-grading of V is not fine. This completes the proof. □

7.2. **Classification of graded simple modules.** Let Q be a finitely generated additive abelian group and $\mathfrak{g} = \bigoplus_{\alpha\in Q} \mathfrak{g}_\alpha$ a Q-graded Lie algebra over a field $\Bbbk$. Let $W = \bigoplus_{\alpha\in Q} W_\alpha$ be a Q-graded simple $\mathfrak{g}$-module such that $\dim(W) < |\Bbbk|$. Similarly to the above, we define $\mathrm{supp}(v)$ for any $v = \sum_{\alpha\in Q} v_\alpha \in W$, where $v_\alpha \in W_\alpha$, and $\mathrm{size}(N)$ for any subset N of the module W.

We assume that W is not simple as a $\mathfrak{g}$-module. Let N be a proper nonzero submodule of W. Set $r := \mathrm{size}(N) > 1$ and define

$$R(N) := \mathrm{span}\{v \in N : \mathrm{size}(v) = r\}.$$

Then $R(N)$ is a non-homogeneous non-zero proper submodule of $\mathfrak{g}$. We will say that a submodule N of W is *pure* of size r if $\mathrm{size}(N) = r$ and, moreover, $R(N) = N$.

Consider a nonzero element $v_\beta + v_{\beta+\alpha_1} + \cdots v_{\beta+\alpha_{r-1}} \in N$. Fix these $\alpha_1, \alpha_2, \cdots, \alpha_{r-1} \in Q_0$. Then the subset of N spanned by all elements of the form $w_\gamma + w_{\gamma+\alpha_1} + \cdots w_{\gamma+\alpha_{r-1}}$, where $\gamma \in Q$, which belong to N, forms a nonzero submodule of N which is pure. Now we replace our N with this pure submodule of size r. Since W is graded simple, we can define a linear map

$$\Lambda_{N,\alpha_i} : W \to W, \ \ w_\gamma \mapsto w_{\gamma+\alpha_i},$$

where $w_\gamma + w_{\gamma+\alpha_1} + \cdots + w_{\gamma+\alpha_{r-1}} \in N$. Similarly to Subsection 3.1, one shows that each Λ_{N,α_i} is a homogeneous degree α_i automorphism of the module W. From Theorem 14 it follows that Λ_{N,α_i} does not depend on the choice of N up to a scalar multiple. We thus simplify the notation Λ_{N,α_i} to Λ_{α_i}. Set

$$P' = \{\alpha \in Q : \text{there is a degree } \alpha \text{ module isomorphism of } W\},$$
$$D' = \operatorname{span}\{\Lambda_\alpha : \alpha \in P'\}.$$

It is easy to check that we have analogues of Lemmata 15, 16, and 17 for these P' and D'.

Let D be a maximal Q-graded commutative subalgebra of D' and let $P = \operatorname{supp}(D)$ which is a subgroup of Q. Using Lemmata 15, 16, and 17, from now on, we may take Λ_α for $\alpha \in P$ such that

$$\Lambda_\alpha \Lambda_\beta = \Lambda_{\alpha+\beta} \quad \text{for all} \quad \alpha, \beta \in P.$$

Let V' be the span of the set

$$\{v_\beta - \Lambda_\alpha(v_\beta) : \beta \in Q, v_\beta \in W_\beta \text{ and } \alpha \in P\}.$$

Then V' is a submodule of W.

Lemma 29. *The submodule V' is a proper maximal submodule of W.*

Proof. Mutatis mutandis the proof of Lemmata 18, 19. □

Now we have the following:

Theorem 30. *Let Q be a finitely generated additive abelian group and $\mathfrak{g}$ be a Q-graded Lie algebra over a field $\Bbbk$. Let W be a graded simple $\mathfrak{g}$-module such that $\dim(W) < |\Bbbk|$. Then there is a subgroup $P \subset Q$ and a simple $\mathfrak{g}$-module V with a fine Q/P-grading such that we have $W \simeq M(Q, P, V)$.*

Proof. From Lemma 29 we have that the module $V = W/V'$ is a simple $\mathfrak{g}$-module with a Q/P-grading. It is easy to verify that the Q-graded canonical map

$$\begin{array}{rcl} W & \to & M(Q, P, V), \\ v_\alpha & \mapsto & (v_\alpha + \mathcal{I}) \otimes t^\alpha \end{array}$$

is a degree 0 isomorphism of $\mathfrak{g}$-modules. Thus $W \simeq M(Q, P, V)$. The fact that the Q/P-grading V is fine follows from Theorem 28. □

Lemma 31. *Let $\mathfrak{g}$ be a Q-graded Lie algebra over $\Bbbk$ and $M(Q,P,V)$ a graded simple $\mathfrak{g}$-module, where P is a subgroup of Q, V is a simple $\mathfrak{g}$-module with a fine Q/P-grading and* $\dim(V) < |\Bbbk|$. *Then the module $M(Q,P,V)$ admits a homogeneous $\mathfrak{g}$-module automorphism of degree α if and only if $\alpha \in P$.*

Proof. For $\alpha \in P$, we have that $V_{\bar{\alpha}+\bar{\beta}} = V_{\bar{\beta}}$ for any $\bar{\beta} \in Q/P$. Then the map

$$\begin{array}{rcl} M(Q,P,V) & \to & M(Q,P,V), \\ v_{\bar{\beta}} \otimes t^{\beta} & \mapsto & v_{\bar{\beta}} \times t^{\alpha+\beta}, \end{array}$$

where $\beta \in Q$, is a homogeneous automorphism of $M(Q,P,V)$ of degree α.

Now suppose $\alpha \in Q \setminus P$ and we have a homogeneous automorphism τ of $M(Q,P,V)$ of degree α given by

$$\begin{array}{rcl} M(Q,P,V) & \to & M(Q,P,V), \\ v_{\bar{\beta}} \otimes t^{\beta} & \mapsto & \mu(v_{\bar{\beta}}) \times t^{\alpha+\beta}, \end{array}$$

where $\beta \in Q$ and μ is a degree $\bar{\alpha}$ linear automorphism of the Q/P-graded vector space V. For any $x_\gamma \in \mathfrak{g}_\gamma$, we have

$$\begin{array}{rcl} \mu(x_\gamma v_{\bar{\beta}}) \otimes t^{\alpha+\gamma+\beta} & = & \tau((x_\gamma v_{\bar{\beta}}) \otimes t^{\beta+\gamma}) \\ & = & \tau(x_\gamma(v_{\bar{\beta}} \otimes t^{\beta})) \\ & = & x_\gamma(\tau(v_{\bar{\beta}} \otimes t^{\beta})) \\ & = & (x_\gamma \mu(v_{\bar{\beta}})) \otimes t^{\alpha+\gamma+\beta}. \end{array}$$

This means that $\mu(x_\gamma v_{\bar{\beta}}) = x_\gamma \mu(v_{\bar{\beta}})$ for all $\beta, \gamma \in Q$ and hence μ is an automorphism of V. From Schur's lemma it follows that μ is scalar. Hence $V_{\bar{\beta}} = V_{\bar{\beta}+\bar{\alpha}}$, contradicting $\alpha \notin P$. The claim of the lemma follows. □

Now we can obtain some necessary and sufficient conditions for two graded simple modules over a Q-graded Lie algebra $\mathfrak{g}$ to be isomorphic as graded modules.

Theorem 32. *Let $\mathfrak{g}$ be a Q-graded Lie algebra. Let $M(Q,P,V)$ and $M(Q,P',V')$ be two graded simple $\mathfrak{g}$-modules where P,P' are subgroups of Q and V,V' are simple $\mathfrak{g}$-modules of dimension smaller than $|\Bbbk|$ with fine gradings over Q/P and Q/P', respectively. Then $M(Q,P,V)$ is graded isomorphic to $M(Q,P',V')$ if and only if $P = P'$ and the simple modules V and V' are graded isomorphic.*

Proof. The "if" part is clear. Now suppose that $M = M(Q,P,V)$ is graded isomorphic to $M' = M(Q,P',V')$. There is $\alpha_0 \in Q$ and a degree α_0 module isomorphism

$$\sigma : M \to M', \;\; v_{\bar{\beta}} \otimes t^{\beta} \mapsto v'_{\bar{\bar{\beta}}+\bar{\alpha}_0} \otimes t^{\alpha_0+\beta},$$

where $\bar{\beta} \in Q/P$, $\bar{\bar{\beta}}, \bar{\bar{\alpha}}_0 \in Q/P'$ and the map $\mu : v_{\bar{\beta}} \mapsto v'_{\bar{\bar{\beta}}+\bar{\bar{\alpha}}_0}$ is a vector space isomorphism from V to V' (because σ is an isomorphism). Let $\alpha \in P$ and

$$\theta : M \to M, \;\; v_{\bar{\beta}} \otimes t^{\beta} \mapsto v_{\bar{\beta}} \otimes t^{\beta+\alpha}$$

be the Q-graded $\mathfrak{g}$-module automorphism of degree α. Then there is a unique linear isomorphism θ' making the following diagram commutative:

$$\begin{tikzcd} M \arrow[r, "\theta"] \arrow[d, "\sigma"'] & M \arrow[d, "\sigma"] \\ M' \arrow[r, "\theta'"'] & M' \end{tikzcd}$$

This means that the following diagram is commutative:

$$\begin{tikzcd} v_{\bar{\beta}} \otimes t^{\beta} \arrow[r, "\theta"] \arrow[d, "\sigma"'] & v_{\bar{\beta}} \otimes t^{\alpha+\beta} \arrow[d, "\sigma"] \\ v'_{\bar{\bar{\beta}}+\bar{\bar{\alpha}}_0} \otimes t^{\beta+\alpha_0} \arrow[r, "\theta'"'] & v'_{\bar{\bar{\alpha}}+\bar{\bar{\beta}}+\bar{\bar{\alpha}}_0} \otimes t^{\alpha+\beta+\alpha_0} \end{tikzcd}$$

For any $x_\gamma \in \mathfrak{g}_\gamma$, we have

$$\begin{array}{rcl} \theta'(x_\gamma(v'_{\bar{\bar{\beta}}+\bar{\bar{\alpha}}_0} \otimes t^{\beta+\alpha_0})) & = & \theta'(x_\gamma \sigma(v_{\bar{\beta}} \otimes t^{\beta})) \\ & = & \theta'(\sigma(x_\gamma v_{\bar{\beta}} \otimes t^{\gamma+\beta})) \\ & = & \sigma\theta(x_\gamma v_{\bar{\beta}} \otimes t^{\gamma+\beta})) \\ & = & x_\gamma(\theta'\sigma(v_{\bar{\beta}} \otimes t^{\beta})) \\ & = & x_\gamma\theta'(v'_{\bar{\bar{\beta}}+\bar{\bar{\alpha}}_0} \otimes t^{\beta+\alpha_0}) \end{array}$$

for all $\beta, \gamma \in Q$. This shows that θ' is an automorphism of $\mathfrak{g}'$-modules. This means that $\alpha \in P'$ by Lemma 31. Thus $P \subset P'$. By symmetry, we even get $P' \subset P$ and hence $P = P'$. Further,

$$\begin{array}{rcl} \mu(x_\gamma v_{\bar{\beta}}) \otimes t^{\gamma+\beta+\alpha_0} & = & \sigma(x_\gamma v_{\bar{\beta}} \otimes t^{\beta+\gamma}) \\ & = & \sigma(x_\gamma(v_{\bar{\beta}} \otimes t^{\beta})) \\ & = & x_\gamma\sigma(v_{\bar{\beta}} \otimes t^{\beta}) \\ & = & x_\gamma(\mu(v_{\bar{\beta}}) \otimes t^{\beta+\alpha_0}) \\ & = & x_\gamma\mu(v_{\bar{\beta}}) \otimes t^{\gamma+\beta+\alpha_0}. \end{array}$$

Therefore, $\mu(x_\gamma v_{\bar{\beta}}) = x_\gamma \mu(v_{\bar{\beta}})$ and hence μ is a graded isomorphism from the Q/P-graded module V to the Q/P-graded module V'. The theorem follows. $\square$

Our Classification Theorem 30 reduces classification of Q-graded simple modules over a Q-graded Lie algebra $\mathfrak{g}$ to classification of fine Q/P-grading on all simple $\mathfrak{g}$-modules, for any subgroup P of Q. Some results in this direction can be found in [EK1].

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