Complete Reducibility of Torsion Free C_n-Modules of Finite Degree

Daniel Britten, Oleksandr Khomenko, Frank Lemire
and Volodymyr Mazorchuk

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Abstract

We show that every torsion free weight module with finite dimensional weight spaces over a symplectic complex Lie algebra, which is different from $\mathfrak{sp}(2,\mathbb{C})$, is completely reducible.

1 Introduction and the main result

Let $\mathfrak{g}$ be a simple complex finite dimensional Lie algebra and $\mathfrak{h}$ be a fixed Cartan subalgebra of $\mathfrak{g}$. A $\mathfrak{g}$-module, V, is called a *weight*-module provided that the action of $\mathfrak{h}$ on V is diagonalizable. Alternatively, if for $\lambda \in \mathfrak{h}^*$ one defines $V_\lambda = \{v \in V : h(v) = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$, then V is a weight module if and only if $V = \oplus_{\lambda\in\mathfrak{h}^*} V_\lambda$. A weight $\mathfrak{g}$-module, V, is said to be *torsion free* provided that the action of any non-zero element in $\mathfrak{g} \setminus \mathfrak{h}$ is bijective. Throughout this paper $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{C})$ for some $n > 1$. The main result of the present paper is the following theorem.

Theorem 1. *Assume that $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{C})$, $n > 1$, is a symplectic Lie algebra and V is a weight torsion free $\mathfrak{g}$-module with finite dimensional weight spaces. Then V is completely reducible. Equivalently, for $n > 1$ the category $\mathcal{T}(\mathfrak{sp}(2n,\mathbb{C}))$ of all weight torsion free $\mathfrak{sp}(2n,\mathbb{C})$-modules with finite dimensional weight spaces is semi-simple.*

Theorem 1 was conjectured in 1994 by the first and third authors in presentations at conferences in Banff and Detroit. Good evidence for this conjecture was obtained in [BL3], where it was shown that the tensor product of arbitrary simple torsion free $\mathfrak{sp}(2n,\mathbb{C})$-modules with finite dimensional weight spaces and any finite dimensional $\mathfrak{sp}(2n,\mathbb{C})$-module is completely reducible.

We remark that the classification of the simple objects in the category $\mathcal{T}(\mathfrak{sp}(2n,\mathbb{C}))$ is known [M]. The algebra $\mathfrak{sp}(2n,\mathbb{C})$ is the Lie algebra of type C_n, which explains the title of the paper. And finally, the condition $n>1$ is necessary. For $n=1$ the algebra $\mathfrak{sp}(2,\mathbb{C})$ is of type A_1 and is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. It is well-known that in this case weight torsion free modules with finite dimensional weight spaces can have self-extensions. Actually, it can be easily derived from the results of [DFO] that the indecomposable blocks of the category of weight torsion free $\mathfrak{sl}(2,\mathbb{C})$-module with finite dimensional weight spaces are equivalent to the category of nilpotent representations of the polynomial algebra $\mathbb{C}[x]$. An explicit example of a non-split self-extension of a simple torsion free A_1-module is provided in [BL2]. We further remark that the statement of Theorem 1 is, in general, false for torsion free $\mathfrak{sl}(n,\mathbb{C})$-modules, $n\geqslant 1$, on the other hand, the result trivially extends to any direct sum of symplectic algebras. For simple algebras, which are not of type A_n or C_n, simple weight torsion-free modules with finite-dimensional weight spaces do not exist, see [F].

Our approach to the proof of Theorem 1 can be split into three steps. In the first step we use an equivalence of certain categories from [BG] to reduce the question to the case of the so-called *completely pointed* modules, i.e. those weight modules V for which $\dim(V_\lambda)\leqslant 1$ for all $\lambda\in\mathfrak{h}^*$. In the second step we use Mathieu's twisting functor, [M], and some specific features of the root system of $\mathfrak{g}$ to reduce the study of self-extensions of completely pointed $\mathfrak{sp}(2n,\mathbb{C})$-modules to the study of completely pointed $\mathfrak{sp}(4,\mathbb{C})$-modules. These two steps form Section 2. In section 3 we use a direct computational approach to show that completely pointed torsion free $\mathfrak{sp}(4,\mathbb{C})$-modules do not have self-extensions, and we continue this approach in Section 4 to obtain an alternative computational proof of Theorem 1. In the last section we derive a corollary of Theorem 1 for some parabolic generalizations of the category $\mathcal{O}$.

2 Proof of Theorem 1

Our proof of Theorem 1, which we present in this section will use the following lemma which will be proven in Section 3.

Lemma 1. *Assume that $\mathfrak{g} = \mathfrak{sp}(4, \mathbb{C})$, and V is a simple, torsion free, completely pointed weight $\mathfrak{g}$-module. Then $\mathrm{Ext}^1_{\mathcal{W}(\mathfrak{g})}(V, V) = 0$, where $\mathcal{W}(\mathfrak{g})$ denotes the category of all weight $\mathfrak{g}$-modules. In particular, the action of an arbitrary element from the centralizer $U_0(\mathfrak{g})$ of the Cartan subalgebra of $\mathfrak{sp}(4, \mathbb{C})$ on any weight space of an arbitrary extension of completely pointed torsion free simple modules is a multiple of the identity.*

Since every module $V \in \mathcal{T}_n = \mathcal{T}(\mathfrak{sp}(2n, \mathbb{C}))$ has finite dimensional weight spaces, the action of the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ on V is locally finite. Hence, we have the decomposition $\mathcal{T}_n = \oplus_{\chi \in Z(\mathfrak{g})^*} \mathcal{T}_n(\chi)$, where $\mathcal{T}_n(\chi)$ is the full subcategory of $\mathcal{T}_n$, which consists of all modules M such that there exists $k \in \mathbb{N}$ with $(z - \chi(z))^k M = 0$ for all $z \in Z(\mathfrak{g})$. We can further decompose the categories $\mathcal{T}_n(\chi)$ as follows: for $\lambda \in \mathfrak{h}^*$ we denote by $\mathcal{T}_n(\chi, \lambda)$ the full subcategory of $\mathcal{T}_n(\chi)$, consisting of all M, whose *support* $\mathrm{supp}(M) = \{\mu \in \mathfrak{h}^* : M_\mu \neq (0)\} \subseteq \lambda + \mathbb{Z}\Delta$, where Δ is the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. It is obvious that $\mathcal{T}_n(\chi, \lambda)$ is a direct summand of $\mathcal{T}_n(\chi)$. We remark that $\mathcal{T}_n(\chi, \lambda) = \mathcal{T}_n(\chi, \mu)$ if and only if $\mu \in \lambda + \mathbb{Z}\Delta$. According to Mathieu's classification and using [M, Section 9] one gets that every $\mathcal{T}_n(\chi, \lambda)$ is either zero or contains exactly one simple module.

Lemma 2. *Let $\chi, \chi' \in Z(\mathfrak{g})^*$ and $\lambda, \lambda' \in \mathfrak{h}^*$ be such that both $\mathcal{T}_n(\chi, \lambda)$ and $\mathcal{T}_n(\chi', \lambda')$ are non-zero. Then $\mathcal{T}_n(\chi, \lambda)$ and $\mathcal{T}_n(\chi', \lambda')$ are equivalent.*

Proof. If $\chi = \chi'$, then, according to [M, Section 9], the simple modules L from $\mathcal{T}_n(\chi, \lambda)$ and L' from $\mathcal{T}_n(\chi, \lambda')$ belong to the same coherent family. Hence, these modules are related by the so-called *Mathieu's twisting functor* briefly described as follows. Let $X_1, \dots, X_n$ be the set of pairwise commuting root elements of $\mathfrak{g}$, which correspond to linearly independent roots. Denote by U' the localization of $U(\mathfrak{g})$ with respect to the Ore multiplicative subset, generated by $X_1, \dots, X_n$. Then the algebra U' has an n-parameter family of automorphisms $\Theta_{(t_1,\dots,t_n)}$ such that $\Theta_{(t_1,\dots,t_n)}(r) = X_1^{t_1} \dots X_n^{t_n} r X_n^{-t_n} \dots X_1^{-t_1}$ provided that all t_i's are integers and the map $(t_1, \dots, t_n) \mapsto \Theta_{(t_1,\dots,t_n)}(r)$ is polynomial in $(t_1, \dots, t_n)$ for every $r \in U'$. This polynomial nature of $\Theta_{(t_1,\dots,t_n)}$ allows one to extend the class of automorphisms so that $\Theta_{(t_1,\dots,t_n)}$ is defined for all $(t_1,\dots,t_n)\in\mathbb{C}^n$. Denote by $\hat{\mathcal{F}}_{(t_1,\dots,t_n)}$ the composition of the following functors: $U'\otimes_{U(\mathfrak{g})}-$, twisting by $\Theta_{(t_1,\dots,t_n)}$, and restriction to $U(\mathfrak{g})$. Clearly, this is an endofunctor on the category of all $\mathfrak{g}$-modules on which the X_i's act injectively. According to [M, Section 9], there exist $t_1,\dots,t_n\in\mathbb{C}$ such that $\hat{\mathcal{F}}_{(t_1,\dots,t_n)}(L)=L'$. Then it is obvious that the functors $\hat{\mathcal{F}}_{(t_1,\dots,t_n)}:\mathcal{T}_n(\chi,\lambda)\to\mathcal{T}_n(\chi,\lambda')$ and $\hat{\mathcal{F}}_{(-t_1,\dots,-t_n)}:\mathcal{T}_n(\chi,\lambda')\to\mathcal{T}_n(\chi,\lambda)$ are mutually inverse equivalences of categories.

To complete the proof it is now enough to show that the statement of the lemma is true when $\lambda=\lambda'$. In this case we again can use the classification in [M] and state that there exists a finite dimensional $\mathfrak{g}$-module, F, such that tensoring with F and projecting on $\mathcal{T}(\chi')$ defines an exact functor from $\mathcal{T}_n(\chi,\lambda)$ to $\mathcal{T}_n(\chi',\lambda)$. In fact, this functor is a translation functor. Since, according to [M, Lemma 9.1], the highest weight μ of every simple highest weight $\mathfrak{g}$-module with uniformly bounded dimensions of the weight spaces satisfies either $(\mu,\alpha)\geqslant 0$ or $(\mu,\alpha)\in\frac{1}{2}+\mathbb{Z}$ for every simple root, this translation does not cross the walls. Hence, by [BG, Theorem 4.1] or [BeGi, Proposition 3.1], it is an equivalence of $\mathcal{T}(\chi)$ and $\mathcal{T}(\chi')$. As tensoring with finite dimensional modules preserves cosets with respect to the weight lattice, we conclude that the categories $\mathcal{T}_n(\chi,\lambda)$ and $\mathcal{T}_n(\chi',\lambda)$ are equivalent as well. This completes the proof. □

By Lemma 2, in order to prove Theorem 1, it is now enough to show that completely pointed torsion free $\mathfrak{g}$-modules do not have self-extensions. Now we are going to simplify the situation even more, reducing all the questions to the algebra $\mathfrak{sp}(4,\mathbb{C})$ (alternatively, one can also use computational arguments given in the end of Section 3).

Lemma 3. *Assume that we have chosen a basis, π, of Δ. Let V be a completely pointed simple highest weight (with respect to π) $\mathfrak{g}$-module. If $\alpha\in\Delta$ is simple and short then every element from $\mathfrak{g}_\alpha$ acts locally nilpotent on V. If $\alpha\in\Delta$ is long and negative then every non-zero element from $\mathfrak{g}_\alpha$ acts injectively on V.*

Proof. This is an immediate consequence of [M, Lemma 9.1]. □

Corollary 1. *Let V be as in Lemma 3 and let $\beta_1,\dots,\beta_n$ be the list of all positive long roots. Then there exists $\lambda\in\mathfrak{h}^*$ such that the support of V belongs to the set $\{\lambda-\sum_{i=1}^n a_i\beta_i\,:\,a_i\in\mathbb{R}_+\}$.*

Proof. Let $\pi = \{\alpha_1, \dots \alpha_n\}$ where α_n is the long simple root. Let μ be the highest weight of V. Then, according to Lemma 3, the set $A = \mathrm{supp}(V) \cap \{\mu - \sum_{i=1}^{n-1} a_i\alpha_i : a_i \in \mathbb{Z}_+\}$ is finite since all roots α_i, $i = 1, \dots, n-1$, are short. It follows now from the PBW Theorem that $\mathrm{supp}(V) \subset \cup_{\nu \in A}\{\nu - \sum_{i=1}^{n} a_i\beta_i : a_i \in \mathbb{R}_+\}$. Since this is a finite union of cones, we can find a weight $\lambda \in \mathfrak{h}^*$, which generates the cone, containing all these cones. This completes the proof. $\square$

Proposition 1. *Let V be as in Lemma 3. Denote by U' Mathieu's localization of $U(\mathfrak{g})$ with respect to the negative long root vectors. Then the following statements hold:*

1. *The module $V' = U' \otimes_{U(\mathfrak{g})} V$ has length 2^n.*
2. *Every simple subquotient of this module is a simple completely pointed highest weight module with respect to some choice of a basis in Δ.*
3. *There exists $\lambda \in \mathfrak{h}^*$ such that for every simple subquotient W of V' there exist $\varepsilon_i \in \{\pm 1\}$, $i = 1, \dots, n$, such that* $\mathrm{supp}(W) \subset \{\lambda - \sum_{i=1}^{n} \varepsilon_i a_i \beta_i : a_i \in \mathbb{R}_+\}$.

Proof. Clearly every simple subquotient of V' is a completely pointed highest weight module for some choice of the basis in Δ.

Let B_r be the ball of radius r in $\mathbb{R}^n$ and b_n denote the number of integer points in $B_b \cap \mathbb{Z}^n$. It is well-known that b_n has a polynomial growth, moreover, this growth is exactly n. Let C be the leading coefficient of the growth polynomial. According to Lemma 3 all negative long root vectors act injectively on V and the same (for the corresponding negative long root vectors) can be stated for all simple subquotients of V'. Hence, the growth of the support of every simple subquotient of V' equals n and the leading coefficient of the corresponding polynomial is not less than $C2^{-n}$. Since the support of V' can be identified with $\mathbb{Z}^n$, it has growth n and leading coefficient C. Hence the length of V' does not exceed 2^n.

Take a positive long root, β, and denote by U'' the Mathieu's localization with respect to $\mathfrak{g}_{-\beta}$. By the same arguments as above one can see that the support of the module $V'' = U'' \otimes_{U(\mathfrak{g})} V$ has growth n and the leading coefficient $C2^{-n+1}$. By the arguments above its length is at most 2. However, it can not be 1 since V is a submodule, and the growth of $\mathrm{supp}(V)$ is strictly smaller than the growth of $\mathrm{supp}(V'')$. This means that V'' has length exactly 2. Denote by $\tilde{V}$ the second simple subquotient of V''. From this construction and using Corollary 1 one easily gets that there exists $\lambda \in \mathfrak{h}^*$ such that $\mathrm{supp}(V) \subset \{\lambda - \sum_{i=1}^{n} a_i\beta_i \,:\, a_i \in \mathbb{R}_+\}$ and, taking λ the minimal possible with respect to the natural order on $\mathfrak{h}^*$, we get $\mathrm{supp}(\tilde{V}) \subset \{\lambda - \sum_{i=1}^{n} \varepsilon_i a_i\beta_i \,:\, a_i \in \mathbb{R}_+\}$, where $\varepsilon_i = -1$ if and only if $\beta_i = \beta$.

Now let $D \subset \{-\beta_1, \dots, -\beta_n\}$ be a subset of long negative roots. Applying the previous construction to all roots from D we construct simple subquotients of V', whose support is included into the set $\{\lambda - \sum_{i=1}^{n} \varepsilon_i a_i\beta_i \,:\, a_i \in \mathbb{R}_+\}$, where $\varepsilon_i = -1$ if and only if $-\beta_i \in D$. In particular, this gives us 2^n non-isomorphic simple subquotients of V'. This means that the length of V' is exactly 2^n and the modules constructed above are all simple subquotients of V'. The statement about the supports now follows directly from the construction. $\square$

For V as above and for $\varepsilon_i \in \{\pm 1\}$, $i = 1, \dots, n$, we denote by $V(\varepsilon_1, \dots, \varepsilon_n)$ the simple subquotient of V', whose support is contained in the set $\{\lambda - \sum_{i=1}^{n} \varepsilon_i a_i\beta_i \,:\, a_i \in \mathbb{R}_+\}$ (here V' and λ as in Proposition 1). We remark that $V = V(1, 1, \dots, 1)$.

Lemma 4. *Let $\varepsilon_i \in \{\pm 1\}$, $i = 1, \dots, n$. Then the following statements are equivalent:*

1. $\mathrm{Ext}^1_{\mathcal{W}(\mathfrak{g})}(V, V(\varepsilon_1, \dots, \varepsilon_n)) \neq 0$.
2. $\mathrm{Ext}^1_{\mathcal{W}(\mathfrak{g})}(V(\varepsilon_1, \dots, \varepsilon_n), V) \neq 0$.
3. *Exactly one of ε_i, $i = 1, \dots, n$, equals -1.*

Proof. The equivalence of the first two statements follows by the standard duality arguments, using the Chevalley anti involution on $\mathfrak{g}$, see for example [FM, Section 5.5].

The fact that V does not have self-extensions is obvious. The existence of a non-split extension of V by $V(\varepsilon_1, \dots, \varepsilon_n)$, where exactly one ε_i equals -1 follows from the construction of the module V'' in the proof of Proposition 1 (that V'' is indecomposable follows from the fact that non-zero elements from $\mathfrak{g}_{-\beta}$ act, by construction, bijectively on V''). Hence we have only to prove that for example $\mathrm{Ext}^1_{\mathcal{W}(\mathfrak{g})}(V(\varepsilon_1, \dots, \varepsilon_n), V) = 0$ provided that at least two of ε_i are equal to -1. Assume that W is a non-split extension with submodule V and subquotient $V(\varepsilon_1, \dots, \varepsilon_n)$.

By Proposition 1 we have that $\mathrm{supp}(V) \subset S_1 = \{\lambda - \sum_{i=1}^{n} a_i\beta_i \,:\, a_i \in \mathbb{R}_+\}$ and $\mathrm{supp}(V(\varepsilon_1, \dots, \varepsilon_n)) \subset S_2 = \{\lambda - \sum_{i=1}^{n} \varepsilon_i a_i\beta_i \,:\, a_i \in \mathbb{R}_+\}$. Since at least two ε_i are equal to -1, the intersection of the cones S_1 and S_2 has codimension at least 2 (considering the n-dimensional cube it is easy to see that this codimension is just the number of ε_i, which are equal to -1). Let $-\beta$ be some long negative root (with respect to π), which acts locally nilpotent on $V(\varepsilon_1, \dots, \varepsilon_n)$. Since the intersection of S_1 and S_2 has codimension at least 2 and the set of all $\nu \in \mathrm{supp}(V(\varepsilon_1, \dots, \varepsilon_n))$, such that $\nu - \beta \notin \mathrm{supp}(V(\varepsilon_1, \dots, \varepsilon_n))$ has codimension 1, we can find some weight $\nu \in \mathrm{supp}(W)$ such that $\mathfrak{g}_{-\beta} W_\nu = 0$, moreover, this ν certainly belongs to $\mathrm{supp}(V(\varepsilon_1, \dots, \varepsilon_n))$. But the module W is of course generated by W_ν, since W is indecomposable and $W_\nu = V(\varepsilon_1, \dots, \varepsilon_n)_\nu$, the latter being in the top of W. This implies that $\mathfrak{g}_{-\beta}$ must act locally nilpotent on W since $W = U(\mathfrak{g})W_\mu$ and $\mathfrak{g}_{-\beta}$ acts locally nilpotent on $U(\mathfrak{g})$ and W_μ. But this contradicts the fact that $\mathfrak{g}_{-\beta}$ acts injectively on V by Lemma 3. □

Now we are ready to prove our main result.

Proof of Theorem 1. From Lemma 1 it suffices to prove the statement for completely pointed modules. For $n = 2$ the result is given by Lemma 1. Now assume that $n > 2$ and let M be an arbitrary completely pointed torsion free $\mathfrak{sp}(2n, \mathbb{C})$-module. Assume that M' is a non-split self-extension of M. Choose some basis $\pi \subset \Delta$ and consider Mathieu's localization U' of $U(\mathfrak{g})$ with respect to the set of all negative long roots. Choose $t_1, \dots, t_n$ such that $\tilde{M} = \Theta_{(t_1,\dots,t_n)}(M')$ contains a simple highest weight submodule, say V (existence is given by Mathieu's classification of coherent families, [M]). Since Mathieu's twisting is invertible, the module $\tilde{M}$ will be a non-split self-extension of the module $\hat{M} = \Theta_{(t_1,\dots,t_n)}(M)$.

Now consider the centralizer U_0 of $\mathfrak{h}$ in $U(\mathfrak{g})$. Since $\tilde{M}$ is a non-split self-extension of $\hat{M}$, we immediately get that every weight subspace of $\tilde{M}$ is an indecomposable module over U_0. The quotient $\tilde{M}/\hat{M}$ is isomorphic to $\hat{M}$ and hence contains V in the socle. Consider the minimal submodule N of $\tilde{M}$ such that $[N : V] = 2$. This module N will then contain one copy of V in the socle (this one comes from the socle of $\tilde{M}$) and one copy of V in the top. Since N is minimal, V will be the simple top of N. From the construction of $\hat{M}$ one easily gets that V is the simple socle of $\hat{M}$ and thus N has V as the simple socle as well. The radical of N belongs to $\hat{M}$. By Proposition 1, the module $\hat{M}$ has length 2^n and all its simple subquotients are $V(\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_i \in \{\pm 1\}$, $i = 1, \dots, n$. From Lemma 4 it follows that the second socle of $\hat{M}$ can contain only those $V(\varepsilon_1, \dots, \varepsilon_n)$ for which exactly one ε_i equals -1.

Denote V_i that $V(\varepsilon_1,\dots,\varepsilon_n)$, for which $\varepsilon_i=-1$ and $\varepsilon_j=1$, $j\neq i$. Moreover, since all $\mathfrak{g}_{-\beta}$, where β is a positive long root, act bijectively on $\hat{M}$ we get that the second socle of $\hat{M}$ contains all such V_i. In particular, these modules do not have self-extensions again by Lemma 4. Using Lemma 4 once more, we see that the top copy of V can extend only some of the modules from the second radical of $\hat{M}$. Recall that the module $\hat{M}$ does not depend on our choice of π but rather on the choice of the long positive roots. Hence it will not change if we choose a different set of simple roots π which preserves the set of long positive roots. Hence similar arguments imply that the top copy of V must extend all modules V_i, which are in the second socle. This implies that the module N has Loewy length exactly 3 and the quotients of its (unique) Loewy filtration, which coincides with the radical filtration, are the following:

$$\begin{array}{c} V \\ V_1\oplus V_2\oplus\dots\oplus V_n \\ V \end{array}.$$

Now consider an arbitrary monomial, $u\in U_0$, which is written such that all positive roots are collected to the right hand side. And let μ be the highest weight of V. Applying u to an arbitrary element from N_μ we either immediately go to 0 or arrive to some subquotient V_i and further can proceed only to the socle copy of V. Hence the calculation of the action of u on N_μ takes place in the part of the module N, consisting of the top copy of V, the module V_i and the socle copy of V. Now consider all possible u, which go "through" the module V_n (we recall that α_n is the only long basis root). Now let us determine for which positive roots α it is possible that $\mu+\alpha\in\mathrm{supp}(V_n)$. Certainly it is possible for $\alpha=\alpha_n$. Recall that $\mathrm{supp}(V_n)\subset\{\mu-\sum_{i=1}^{n-1}a_i\beta_i+a_n\beta_n\ :\ a_i\in\mathbb{R}_+\}$. Hence, if α is a positive root such that $\mu+\alpha\in\mathrm{supp}(V_n)$ it must be possible to write α in the form $-\sum_{i=1}^{n-1}a_i\beta_i+a_n\beta_n$, where $a_i\in\mathbb{R}_+$. Clearly the only positive root α, satisfying $\mu+\alpha\in\mathrm{supp}(V_n)$ is actually α_n. Hence, the only element $u\in U_0$ that we need to consider is the element $X_{-\alpha_n}X_{\alpha_n}$, where $X_{\pm\alpha_n}\in\mathfrak{g}_{\pm\alpha_n}$.

Let $\mathfrak{a}=\mathfrak{sp}(4,\mathbb{C})$ denote the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_{\pm\alpha_n}$ and $\mathfrak{g}_{\pm\alpha_{n-1}}$. The restriction of N to $\mathfrak{a}$ decomposes into a direct sum of completely pointed modules and their extensions. Moreover, applying the inverse of $\Theta_{(t_1,\dots,t_n)}$ we can make the picture torsion free. Now Lemma 1 implies that the action of the centralizer of the Cartan subalgebra of $\mathfrak{a}$ on N is diagonalizable. In particular, the action of $X_{-\alpha_n}X_{\alpha_n}$ on N, and hence on N_μ, is diagonalizable.

Since this is the only generating monomial of U_0, "connecting" V with V_n (that is for all other generating monomials of U_0, which have X_{α_n} on the right, their action on V_μ is trivial since they send V_μ outside the support of the module N), we conclude that the submodule, generated by V does not contain V_n, hence N does not contain V_n. This contradiction completes the proof. $\square$

We remark, that in the proof of Theorem 1 we really need to treat the case $\mathfrak{g} = \mathfrak{sp}(4, \mathbb{C})$ separately. Analogous reduction arguments do not work for the algebra $\mathfrak{sp}(4, \mathbb{C})$ since in this case one would be forced to go down to the algebra of type A_1, for which the statement of Theorem 1, as we have already mentioned in the introduction, is not true.

3 Proof of Lemma 1

In this section we prove Lemma 1 as well as provide an alternate computational approach to proving that completely pointed torsion free C_n-modules do not admit non-split self-extensions. The authors acknowledge that some of the computational results relating to the algebra C_2 appeared in [C].

Proof of Lemma 1. For computational purposes we fix a Chevalley basis of $\mathfrak{sp}(4, \mathbb{C}) = C_2$:

$$\begin{array}{lcl@{\qquad\qquad}rcl}
H_\alpha & = & E_{11} - E_{22} - E_{33} + E_{44} & H_\beta & = & E_{33} - E_{11} \\
X_\alpha & = & E_{12} - E_{43} & Y_\alpha & = & E_{21} - E_{34} \\
X_\beta & = & E_{31} & Y_\beta & = & E_{13} \\
X_{\alpha+\beta} & = & -(E_{32} + E_{41}) & Y_{\alpha+\beta} & = & -(E_{23} + E_{14}) \\
X_{2\alpha+\beta} & = & 2E_{42} & Y_{2\alpha+\beta} & = & 2E_{24}.
\end{array}$$

We also recall from [BL1] that the centralizer U_0 of the Cartan subalgebra $\mathfrak{h}$ in the universal enveloping algebra is generated by the following elements

$$\begin{array}{ll}
H_\alpha & H_\beta \\
D_1 = Y_\alpha X_\alpha & D_2 = Y_\beta X_\beta \\
D_3 = Y_{\alpha+\beta} X_{\alpha+\beta} & D_4 = Y_{2\alpha+\beta} X_{2\alpha+\beta} \\
D_5 = Y_{\alpha+\beta} X_\beta X_\alpha & D_6 = Y_\alpha Y_\beta X_{\alpha+\beta} \\
D_7 = Y_{2\alpha+\beta} X_\alpha X_{\alpha+\beta} & D_8 = Y_{\alpha+\beta} Y_\alpha X_{2\alpha+\beta} \\
D_9 = Y_{2\alpha+\beta} X_\beta X_\alpha^2 & D_{10} = Y_\alpha^2 Y_\beta X_{2\alpha+\beta} \\
D_{11} = Y_{2\alpha+\beta} Y_\beta X_{\alpha+\beta}^2 & D_{12} = Y_{\alpha+\beta}^2 X_\beta X_{2\alpha+\beta}.
\end{array}$$

By direct computation we obtain the following identities in U_0

$$
\begin{aligned}
[D_1, D_2] &= D_6 - D_5 &(3.1)\\
[D_1, D_4] &= 2D_7 - 2D_8 &(3.2)\\
[D_1, D_5] &= D_3D_1 - 2D_1D_2 + 2D_6 - D_9 - D_7 + D_4 - D_5H_\alpha &(3.3)
\end{aligned}
$$

$$
\begin{aligned}
[D_1, D_6] &= -D_3D_1 + 2D_1D_2 - 2D_5 + D_{10} + D_7 - D_4 + D_6H_\alpha &(3.4)\\
[D_1, D_7] &= -2D_3D_1 + D_4D_1 + 2D_9 + 4D_7 - 2D_4 - 4D_5 &\\
&\qquad -D_7H_\alpha + C_4H_\alpha - 2D_8 &(3.5)\\
[D_2, D_5] &= D_1D_2 - D_3H_\beta - D_3D_2 + D_5 - D_6 - D_5H_\beta &(3.6)
\end{aligned}
$$

$$
\begin{aligned}
[D_2, D_7] &= -D_9 - 2D_7 + D_4 - D_{11} &(3.7)\\
[D_2, D_8] &= D_{12} + D_{10} + 2D_8 - D_4 &(3.8)\\
[D_4, D_{11}] &= -4D_6D_4 - 4D_4D_3 + 8D_7 + 8D_8 + 4D_4D_2 + 8D_{10} &\\
&\qquad -8D_4 + 4D_{11}(H_\alpha + H_\beta) + 8D_{11} &(3.9)
\end{aligned}
$$

Assume now that V is a completely pointed torsion free C_2-module and W is a self-extension of V. Fix a weight λ of W and select a basis $\mathcal{B} = \{v_1, v_2\}$ of W_λ such that v_1 generates a submodule W_1 of W where W_1 and W/W_1 are isomorphic to V. We claim that W is a completely reducible C_2-module provided W_λ is a completely reducible U_0-module. Assume to the contrary that $W_\lambda \simeq U_0v_1 \oplus U_0v_2$ as U_0-modules and there exists a nonzero vector $v \in Uv_1 \cap Uv_2$. Without loss of generality, we may assume that v is a weight vector of weight μ and $v = u_1v_1 = u_2v_2$ where $u_1, u_2 \in U$. Since W is torsion free we can select an element $u \in U$ such that u acts injectively on W and $uu_1, uu_2 \in U_0$. Then $0 \neq uv = uu_1v_1 = uu_2v_2 \in U_0v_1 \cap U_0v_2 = (0)$. This contradiction implies that $W \simeq Uv_1 \oplus Uv_2$.

For convenience we denote the matrix representations of $H_\alpha \downarrow W_\lambda, H_\beta \downarrow W_\lambda$ and $D_i \downarrow W_\lambda$ with respect to the basis $\mathcal{B}$ by $\Lambda_\alpha, \Lambda_\beta$ and Z_i respectively. Clearly $\Lambda_\alpha = \lambda(H_\alpha)I_2$, $\Lambda_\beta = \lambda(H_\beta)I_2$ and the Z_i's are each 2×2 upper triangular matrices with equal diagonal entries.

Since W is torsion free we have that $\mathcal{B}_1 = \{X_{\alpha+\beta}v_1, X_{\alpha+\beta}v_2\}$ and $\mathcal{B}_2 = \{X_\beta X_\alpha v_1, X_\beta X_\alpha v_2\}$ are bases for the weight space $W_{\lambda+\alpha+\beta}$. Let K denote the change of coordinate matrix, i.e. formally we have

$$(X_{\alpha+\beta}v_1, X_{\alpha+\beta}v_2)K = (X_\beta X_\alpha v_1, X_\beta X_\alpha v_2)$$

Multiplying this equation by $Y_{\alpha+\beta}, Y_\alpha Y_\beta, Y_{2\alpha+\beta}X_\alpha$ and $Y_{2\alpha+\beta}Y_\beta X_{\alpha+\beta}$ respectively we obtain the following equations

$$\begin{aligned} Z_5 &= Z_3K & (3.10)\\ Z_6K + Z_6 &= Z_1Z_2 & (3.11)\\ Z_9 &= Z_7K - Z_7 + Z_4 & (3.12)\\ Z_{11}K &= Z_7Z_2 - Z_4Z_2 - Z_9 - 2Z_7 + Z_4 - Z_{11} & (3.13)\end{aligned}$$

Equation 3.10 implies that K is a 2×2 upper triangular matrix with equal diagonal entries. Therefore we have that the matrices $\Lambda_\alpha, \Lambda_\beta, Z_i (i = 1, \ldots, 12)$ and K are pairwise commuting matrices.

The strategy is to use the identities 3.1 through 3.9 applied to W_λ and equations 3.10 through 3.13 to express each Z_i in terms of $\Lambda_\alpha, \Lambda_\beta$ and K and then to show that K is a diagonal matrix which would complete the proof of Lemma 1.

To begin we observe that Eqn 3.1 implies that $Z_6 = Z_5$, Eqn 3.2 implies that $Z_8 = Z_7$, Eqn 3.3+Eqn 3.4 implies that $Z_{10} = Z_9$, and Eqn 3.7+Eqn 3.8 implies that $Z_{12} = Z_{11}$.

Substituting $Z_1Z_2 = Z_6(K + I) = Z_3K(K + I)$ and $Z_5 = Z_3K$ from Eqns 3.11 and 3.10 into Eqn 3.6 yields

$$0 = Z_3[K(K + I) - Z_2 - (K + I)\Lambda_\beta].$$

Since Z_3 is invertible we conclude that

$$Z_2 = (K + I)(K - \Lambda_\beta) \tag{3.14}$$

In particular, we observe that $K + I$ and $K - \Lambda_\beta$ are invertible.

Substituting $Z_9 = Z_7(K - I) + Z_4$ from Eqn 3.12 into Eqn 3.7 yields

$$Z_{11} = -Z_7(K + I). \tag{3.15}$$

Multiply by $-K$ and substitute $Z_{11}K = Z_7Z_2 - Z_4Z_2$ from Eqn 3.13 and $Z_2 = (K + I)(K - \Lambda_\beta)$ from Eqn 3.14 yields

$$(K + I)Z_7(2K - \Lambda_\beta) = (K + I)Z_4(K - \Lambda_\beta).$$

Since $K+I, Z_7, Z_4$ and $K-\Lambda_\beta$ are invertible we conclude that $2K-\Lambda_\beta$ is invertible and

$$Z_7 = Z_4(K-\Lambda_\beta)(2K-\Lambda_\beta)^{-1}. \tag{3.16}$$

Substituting Eqn 3.16 into Eqn 3.15 yields

$$Z_{11} = -Z_4(K+I)(K-\Lambda_\beta)(2K-\Lambda_\beta)^{-1}. \tag{3.17}$$

Substituting Eqn 3.16 into Eqn 3.12 yields

$$Z_9 = Z_4 K(K-\Lambda_\beta+I)(2K-\Lambda_\beta)^{-1}. \tag{3.18}$$

Substitute $Z_6 = Z_5 = Z_3 K$ from Eqn 3.10, $Z_8 = Z_7$, $Z_{10} = Z_9$ and $Z_9 = -2Z_7 + Z_4 - Z_{11}$ from Eqn 3.7 into Eqn 3.9 yields

$$-4Z_6Z_4 - Z_4Z_3 + 4Z_4Z_2 + 4Z_{11}(\Lambda_\alpha+\Lambda_\beta) = 0.$$

Therefore

$$Z_6 = -Z_3 + Z_2 + Z_4^{-1}Z_{11}(\Lambda_\alpha+\Lambda_\beta).$$

Since $Z_6 = Z_5 = Z_3K$ and $Z_2 = (K+I)(K-\Lambda_\beta)$ we have

$$Z_3 = (K-\Lambda_\beta)((2K-2\Lambda_\beta-\Lambda_\alpha)(2K-\Lambda_\beta)^{-1}. \tag{3.19}$$

and hence from Eqn 3.10 we have

$$Z_5 = K(K-\Lambda_\beta)((2K-2\Lambda_\beta-\Lambda_\alpha)(2K-\Lambda_\beta)^{-1}. \tag{3.20}$$

Substituting for Z_6 and Z_2 in Eqn 3.11 we obtain

$$Z_1 = K(2K-2\Lambda_\beta-\Lambda_\alpha)(2K-\Lambda_\beta)^{-1}. \tag{3.21}$$

Substituting for Z_1, Z_2, Z_5 and Z_7 into Eqn 3.5 yields

$$Z_4 = (2K-2\Lambda_\beta-\Lambda_\alpha)(2K-\Lambda_\alpha+2I) \tag{3.22}$$

At this stage we have expressed all Z_i in terms of $K, \Lambda_\alpha, \Lambda_\beta$. If we now substitute for all Z_i in Eqn 3.9 and simplify we obtain

$$(2K+\Lambda_\alpha)(4K-2\Lambda_\beta-I) = 0. \tag{3.23}$$

Finally we claim that $2K+\Lambda_\alpha$ is invertible. Suppose to the contrary that $2K+\Lambda_\alpha$ is not invertible and take a nonzero vector x such that $(2K+\Lambda_\alpha)x =$

0. It follows that $(2K - \Lambda_\beta)x = -(\Lambda_\alpha + \Lambda_\beta)x$. Since $2K - \Lambda_\beta$ is invertible so is $\Lambda_\alpha + \Lambda_\beta$. Using this we have that

$$Z_1 x = K(2K - 2\Lambda_\beta - \Lambda_\alpha)(2K - \Lambda_\beta)^{-1}x = K((2K - \Lambda_\beta) - (\Lambda_\alpha + \Lambda_\beta))x = 2Kx$$

Let $v \in W_\lambda$ have $\mathcal{B}$ coordinates x then the $\mathcal{B}$ coordinates of $X_\alpha Y_\alpha v = (H_\alpha + D_1)v$ are given by

$$(\Lambda_\alpha + Z_1)x = \Lambda_\alpha x + 2Kx = 0$$

This contradicts the assumption that W is torsion free and hence $X_\alpha Y_\alpha$ acts injectively on W_λ. It follows then that $K = \frac{1}{2}\Lambda_\alpha - \frac{1}{4}I$ and hence K as well as all Z_i are diagonal matrices thus completing the proof of Lemma 1. $\square$

Now that we have proven that there are no non-split self-extensions of simple, completely pointed, torsion free C_2-modules we can apply induction to provide a computational proof that the same result is true for C_n-modules. This approach has the advantage that it deals directly with torsion free modules and avoids the use of Mathieu's coherent family construction. This result together with Lemma 2 provides an alternate proof of Theorem 1.

4 An alternative proof of Theorem 1

Once again from Lemma 2 it suffices to prove the theorem for completely pointed modules. Assume that W is a self-extension of a completely pointed, torsion free $\mathfrak{sp}(2n, \mathbb{C})$-module V and let W_1 denote a submodule of W equivalent to V. For a fixed weight λ of W we let v_1 denote a basis of $W_{1\lambda}$ and extend to a basis $\mathcal{B} = \{v_1, v_2\}$ of W_λ. As in the C_2 case, it suffices to prove that there exists a weight space W_λ which is a completely reducible $U_0(C_n)$-module.

Recall that if $\{\epsilon_1, \dots, \epsilon_n\}$ is a standard basis of $\mathbb{R}^n$ then we can realize the root system Δ of C_n as

$$\Delta = \{\pm(\epsilon_i - \epsilon_j) \,:\, 1 \leqslant i < j \leqslant n\} \cup \{\pm(\epsilon_i + \epsilon_j) \,:\, 1 \leqslant i \leqslant j \leqslant n\}.$$

and a base of simple roots is given by

$$\pi = \{\epsilon_1 - \epsilon_2, \dots, \epsilon_{n-1} - \epsilon_n, 2\epsilon_n\}.$$

Fix a Chevalley basis of C_n given by $\{H_\alpha \,:\, \alpha \in \pi\} \cup \{X_\mu \,:\, \mu \in \Delta\}$. With this notation, $U_0(C_n)$ is generated by $\{H_\alpha \,:\, \alpha \in \pi\}$ together with all *basic* *cycles* – i.e. all elements $X_{\mu_1}\cdots X_{\mu_k}\in U(C_n)$ where $\mu_i\in\Delta$, $\sum_{i=1}^k\mu_i=0$ and no proper subsum is zero. Let $\mathcal{S}$ denote the set of all sequences $\{a_1,\ldots,a_k\}$ such that the $a_i\in\pm\{1,\ldots,n\}$, $a_i\neq a_j$ for $i\neq j$ and there does not exist a subsequence $1\leqslant i_1<i_2<i_3<i_4\leqslant n$ with $a_{i_1}=-a_{i_3}$ and $a_{i_2}=-a_{i_4}$. In [BL1], it is shown that for each sequence $\{a_1,\ldots,a_k\}\in\mathcal{S}$ the element

$$C(a_1,\ldots,a_k)=X_{\epsilon_{a_1}-\epsilon_{a_2}}\cdots X_{\epsilon_{a_{n-1}}-\epsilon_{a_n}}X_{\epsilon_{a_n}-\epsilon_{a_1}}$$

is a basic cycle and further that $U_0(C_n)$ is generated by $\{H_\alpha\ :\ \alpha\in\pi\}$ together with $\{C(a_1,\ldots,a_k)\ :\ (a_1,\ldots,a_k)\in\mathcal{S}\}$. In order to prove that W is completely reducible it suffices to show that for each $(a_1,\ldots,a_k)\in\mathcal{S}$ the matrix representation $Z(a_1,\ldots,a_k)$ of the action of $C(a_1,\ldots,a_k)$ restricted to W_λ with respect to the basis $\mathcal{B}$ is diagonal.

Observe that every basic cycle $C(a_1,a_2)$ is a basic cycle of a regular subalgebra of C_n which is isomorphic to C_2 and hence by Lemma 1 its matrix representation $Z(a_1,a_2)$ is diagonal. We also note that if $(a_1,a_2,a_3)\in\mathcal{S}$ where $a_1=-a_2$ or $a_1=-a_3$ or $a_2=-a_3$ then the basic cycle $C(a_1,a_2,a_3)$ is a basic cycle of a regular subalgebra of C_n which is isomorphic to C_2 hence again $Z(a_1,a_2,a_3)$ is diagonal by Lemma 1.

There exists one other type of basic cycle of length 3, namely $C(a_1,a_2,a_3)$ where $a_1\neq-a_2$ and $a_1\neq-a_3$ and $a_2\neq-a_3$. In this case, we have the following identity in $U_0(C_n)$

$$\begin{aligned}[][C(a_1,a_2,a_3),C(a_1,a_3)] &= (H_{\epsilon_{a_1}-\epsilon_{a_3}}+D)C(a_1,a_2,a_3)\\ &\quad +AC(a_1,a_2)C(a_1,a_3)+BC(a_3,a_2)C(a_1,a_3)\end{aligned}$$

where A,B,D are constants determined by structure constants of the fixed Chevalley basis for each choice of $(a_1,a_2,a_3)\in\mathcal{S}$. Applying this identity to W_λ and taking the matrix representations we have

$$0=(\lambda(H_{\epsilon_{a_3}-\epsilon_{a_1}})+D)Z(a_1,a_2,a_3)+AZ(a_1,a_2)Z(a_1,a_3)+BZ(a_3,a_2)Z(a_1,a_3).$$

Since $\mathcal{S}$ is a finite set, only finitely many constants can occur in such equations. Clearly then we can select a weight λ of W such that $\lambda(H_{\epsilon_{a_3}-\epsilon_{a_1}})-D$ is nonzero for all choices of $(a_1,a_2,a_3)\in\mathcal{S}$ and hence all such matrices are diagonal.

We now have that all basic cycle of length less than or equal to 3 have diagonal matrix representations. To complete the result we proceed by induction assuming that $k\geqslant 4$ and $Z(a_1,\ldots,a_p)$ is diagonal for all $p<k$.

Consider the basic cycle $C(a_1,\ldots,a_k)$ where $-a_1 \neq a_k \neq -a_{k-1}$. In this case we have

$$C(a_1,\ldots,a_k)C(a_{k-1},a_1) = C(a_1,\ldots,a_{k-1})C(a_{k-1},a_k,a_1).$$

Applying this identity to W_λ we obtain

$$Z(a_1,\ldots,a_k)Z(a_{k-1},a_1) = Z(a_1,\ldots,a_{k-1})Z(a_{k-1},a_k,a_1).$$

Since $Z(a_{k-1},a_1)$ is diagonal and invertible, the induction hypothesis implies that the matrix $Z(a_1,\ldots,a_k)$ is diagonal.

For any cyclic permutation $\sigma = (1,\ldots,k)$ of the indices it is clear that $C(a_{\sigma(1)},\ldots,a_{\sigma(k)})$ is equal to $C(a_1,\ldots,a_k)$ plus a sum of cycles of length less than k. It follows that the argument above handles all basic cycles except those of the form $C(a_1,-a_1,a_2,-a_2,\ldots,a_\ell,-a_\ell)$ where $|a_1|,\ldots,|a_\ell|$ are distinct elements from the set $\{1,\ldots,n\}$ and $\ell \geqslant 2$. In this case we observe that

$$\begin{aligned} C(a_1,-a_1,\ldots,a_\ell,-a_\ell)C(a_\ell,a_1) &= C(a_1,-a_1,\ldots,-a_{\ell-1},a_\ell)C(a_\ell,-a_\ell,a_1) \\ &\quad +AC(a_1,-a_1)C(-a_1,a_2,\ldots,a_\ell,-a_\ell) \end{aligned}$$

where A is a constant dependent on the sequence $\{a_1,-a_1,\ldots,a_\ell,-a_\ell\}$. Applying this identity to W_λ we obtain

$$\begin{aligned} Z(a_1,-a_1,\ldots,a_\ell,-a_\ell)Z(a_\ell,a_1) &= Z(a_1,-a_1,\ldots,a_{\ell-1},a_\ell)Z(a_\ell,-a_{\ell-1},a_1) \\ &\quad +AZ(a_1,-a_1)Z(-a_1,a_2,\ldots,a_\ell,-a_\ell) \end{aligned}$$

Since $Z(a_\ell,a_1)$ is diagonal and invertible, the inductive hypothesis implies that the matrix $Z(a_1,-a_1,\ldots,a_\ell,-a_\ell)$ is diagonal. Therefore $Z(a_1,\ldots,a_k)$ is diagonal for all $(a_1,\ldots,a_k) \in \mathcal{S}$, and hence there exists a weight λ such that W_λ is a completely reducible $U_0(C_n)$-module. This establishes that W is a completely reducible C_n-module as claimed.

5 An application

Let $\mathfrak{a}$ be a semi-simple complex Lie algebra with a fixed Cartan subalgebra $\mathfrak{h}_\mathfrak{a}$ and $\mathfrak{p} \supset \mathfrak{h}_\mathfrak{a}$ be a parabolic subalgebra of $\mathfrak{a}$. Let $\mathfrak{p} = \mathfrak{n} \oplus \mathfrak{a}' \oplus \mathfrak{h}^{\mathfrak{a}'}$ be the Levi decomposition of $\mathfrak{p}$, where $\mathfrak{n}$ is nilpotent, $\mathfrak{a}' \oplus \mathfrak{h}^{\mathfrak{a}'}$ reductive, $\mathfrak{a}'$ semi-simple and $\mathfrak{h}^{\mathfrak{a}'}$ is the abelian center of $\mathfrak{a}' \oplus \mathfrak{h}^{\mathfrak{a}'}$. Assume that $\mathfrak{a}' \simeq \mathfrak{g} = \mathfrak{sp}(2n,\mathbb{C})$ for

$n > 1$ and consider the category $\mathcal{O}(\mathfrak{p}, \mathcal{T}(\mathfrak{g}))$, which is a full subcategory in the category of all $\mathfrak{a}$-modules, consisting of all $\mathfrak{a}$-modules M, that satisfy the following conditions

1. finitely generated;

2. $\mathfrak{h}^{\mathfrak{a}'}$-diagonalizable;

3. locally $U(\mathfrak{n})$-finite (i.e. $\dim(U(\mathfrak{n})v) < \infty$ for all $v \in V$);

4. decompose into a (possibly infinite) direct sum of modules from $\mathcal{T}(\mathfrak{g})$, when viewed as $\mathfrak{g}$-modules.

Theorem 2. *The category $\mathcal{O}(\mathfrak{p}, \mathcal{T}(\mathfrak{g}))$ is a highest weight category. Equivalently, the category $\mathcal{O}(\mathfrak{p}, \mathcal{T}(\mathfrak{g}))$ decomposes into a direct sum of full subcategories, each of which is equivalent to the module category of a finite dimensional complex quasi-hereditary associative algebra.*

Proof. The category $\mathcal{T}(\mathfrak{g})$ is semi-simple and obviously closed under tensoring with finite dimensional $\mathfrak{g}$-modules. Moreover, every object in $\mathcal{T}(\mathfrak{g})$ has finite length (see e.g. [M, Lemma 3.3]). The standard arguments, as for example in [FKM, Section 4] show that with respect to the action of the center of $U(\mathfrak{a})$, the category $\mathcal{O}(\mathfrak{p}, \mathcal{T}(\mathfrak{g}))$ decomposes into blocks, each of which has only finitely many simple modules. The result then follows from [FM, Theorem 3]. $\square$

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D.B.: Department of Mathematics, University of Windsor, Windsor, Ontario N9B3P4, CANADA, e-mail: britten@uwindsor.ca

O.K.: Mathematisches Institut der Universität Freiburg, Eckestrasse 1, D-79104, Freiburg im Breisgau, FRG, e-mail: khomenko@math.uni-freiburg.de

F.L.: Department of Mathematics, University of Windsor, Windsor, Ontario N9B3P4, CANADA, e-mail: lemire@uwindsor.ca

V.M.: Department of Mathematics, Uppsala University, Box 480, SE-751 06, Uppsala, SWEDEN, e-mail: mazor@math.uu.se