Generalized Verma modules induced from $sl(2, \mathbb{C})$ and associated Verma modules

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Abstract

With each generalized Verma module induced from a "well-embedded" parabolic subalgebra of a Lie algebra with triangular decomposition we associate a Verma module over the same algebra in a natural way. In the case, when the semi-simple part of the Levi factor of the parabolic subalgebra is isomorphic to $sl(2, \mathbb{C})$ and the generalized Verma module is induced from an infinite-dimensional simple module, we prove that the associated Verma module is simple if and only if the original generalized Verma module is simple.

1 Introduction and Setup

Let $\mathfrak{g}$ be a semi-simple complex finite-dimensional Lie algebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and $\mathfrak{p} \supset \mathfrak{h} \oplus \mathfrak{n}_+$ a parabolic subalgebra of $\mathfrak{g}$ with the Levi decomposition $\mathfrak{p} = (\mathfrak{a} \oplus \mathfrak{h}_\mathfrak{a}) \oplus \mathfrak{n}$, where $\mathfrak{n}$ is nilpotent, $\mathfrak{a}' = \mathfrak{a} \oplus \mathfrak{h}_\mathfrak{a}$ is reductive, $\mathfrak{a}$ is semi-simple and $\mathfrak{h}_\mathfrak{a} \subset \mathfrak{h}$ is abelian and central in $\mathfrak{a}'$. A *Generalized Verma module* (GVM) is a module of the form

$$M_\mathfrak{p}(V) = U(\mathfrak{g}) \bigotimes_{U(\mathfrak{p})} V, \tag{1}$$

where V is a simple $\mathfrak{a}'$-module and $\mathfrak{n}V = 0$. In the case of a non-simple module V, the module $M_\mathfrak{p}(V)$ in (1) will be called *induced module*. The structure of GVMs is a popular subject and has been studied by several authors (see, for example, [Ba, CF, DFO, FKM1, FKM2, FM, KM1, MO, Mc1, Mc2, MS1, MS2, R] and references therein). In particular, it has been shown that these modules play important role in several generalizations of the celebrated category $\mathcal{O}$ ([R, Ba, MS1, FKM1]) and that they are related to the classification of all simple weight modules with finite-dimensional weight spaces ([F, M]).

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One of the main questions about GVMs is their structure, i.e. reducibility, submodules etc. For example, two natural questions are: under which conditions a GVM is simple? and under which conditions one GVM is a submodule of another one? The corresponding classical result for Verma modules ([BGG, D]) provides a criterion for inclusion of Verma modules in terms of the Weyl group action on the space $\mathfrak{h}^*$, which parameterizes highest weights of Verma modules. This result was extended to certain classes of GVMs in [R, FM, Mc2, MS1, Ba, KM1, MO] (see also references therein). But only rather particular classes of GVMs were covered. They correspond to families of "well-behaved" simple modules V, e.g. finite-dimensional, weight, Whittaker, or Gelfand-Zetlin modules. Simplicity criteria follow immediately from the mentioned results. But the problem to say something in a general case, i.e. for arbitrary simple module V, remains open.

In [KM2] with each GVM the authors associate a Verma module in a natural way. Let $M_{\mathfrak{p}}(V)$ be a GVM. As V is a simple $\mathfrak{a}'$-module, it possesses a central character. Let $M(\lambda)$ be a Verma module over $\mathfrak{a}'$ with the same central character as V. Assume that λ belongs to the closure of the anti-dominant Weyl chamber and consider the GVM $M_{\mathfrak{p}}(M(\lambda))$. In fact, the last one is a Verma module, say $M(\mu)$, over $\mathfrak{g}$. It was conjectured in [KM2] that the simplicity of $M(\mu)$ implies the simplicity of $M_{\mathfrak{p}}(V)$. Recently we have been informed by Wolfgang Soergel that [KM2, Theorem 1] is already sufficient to prove this conjecture using the classical BGG-criterion for simplicity of Verma modules and some standard properties of the Weyl group (see for example [J, Section 2.5]).

The original aim of this paper was to investigate the conjecture above in the case, when $\mathfrak{a}$ is isomorphic to $sl(2,\mathbb{C})$, in particular, to extend the conjecture to a criterion of simplicity for GVMs. There are two strong motivations to study this case. First of all this is the simplest case different from classical Verma modules, hence one should definitely start with it. Second, this is the only case, for which all simple $\mathfrak{a}$-modules are known ([Bl, B]), hence, theoretically, arbitrary GVM can be physically constructed. Several partial results in this case were already known. Namely, if V is a simple weight $sl(2,\mathbb{C})$-module or a Whittaker module, then the structure of $M_{\mathfrak{p}}(V)$ is relatively well understood ([FM, KM1, FKM2], resp. [Ba, Mc1, Mc2, MS1, MS2]). In particular, in the first case Mathieu's localization functor allows one to relate such $M_{\mathfrak{p}}(V)$ with Verma modules in a very natural way (see properties of the functor E in [FKM2]). In the present paper we will use this knowledge to study the structure of $M_{\mathfrak{p}}(V)$ for arbitrary, not necessarily weight, module V. Our approach will be based on description of (semi)primitive elements in GVMs. It is quite surprizing, but we will see that assuming V to be infinite-dimensional and simple, the statement in the conjecture above turns into a simplicity criterion for $M_{\mathfrak{p}}(V)$. In fact, our result will be much stronger than just a simplicity criterion, we will completely describe generalized highest weights of possible submodules in $M_{\mathfrak{p}}(V)$. Note that the case of finite-dimensional V is very well known (see [R]). We also remark that our result is really much more general than all known ones since "almost all" simple $sl(2,\mathbb{C})$-modules are not weight or Whittaker modules (see [Bl]).

One more generalization of the previous picture is that our arguments remain valid for arbitrary, not necessarily finite-dimensional, Lie algebra having a triangular decomposition (in the sense of [MP]), so we will work in the following general setup. We let $\mathfrak{g}$ be a fixed Lie algebra with a fixed triangular decomposition, $(\mathfrak{n}_+, \mathfrak{h}, \Delta_+, \sigma)$ ([MP]), where $\mathfrak{h}$ is a Cartan subalgebra, $\mathfrak{n}_+$ is a positive part, Δ_+ is the set of roots of $\mathfrak{n}_+$ and σ is an antiinvolution. We set $\mathfrak{n}_- = \sigma(\mathfrak{n}_+)$, $\Delta_- = -\Delta_+$ and fix a root basis $\{X_\beta\}$, $\{H_\beta\}$ (where each root is counted with the corresponding multiplicity) in $\mathfrak{g}$. We also fix a simple root, α, of multiplicity 1 in Δ_+, such that the corresponding subalgebra $\mathfrak{a} =< X_\alpha, X_{-\alpha} >$ is isomorphic to $sl(2,\mathbb{C})$, and assume that $\mathfrak{g}$ is an integrable $\mathfrak{a}$-module under the adjoint action. Then we choose $\mathfrak{p} = \mathfrak{a} + \mathfrak{h} + \mathfrak{n}_+$ and decompose it as $\mathfrak{p} = (\mathfrak{a} \oplus \mathfrak{h}_\mathfrak{a}) \oplus \mathfrak{n}$, where $\mathfrak{h}_\mathfrak{a} \subset \mathfrak{h}$ is central in $\mathfrak{a}' = \mathfrak{a} + \mathfrak{h}_\mathfrak{a} = \mathfrak{a} \oplus \mathfrak{h}_\mathfrak{a} = \mathfrak{a} + \mathfrak{h}$ and $\mathfrak{n}$ is generated by all X_β, $\beta \in \Delta_+ \setminus \{\alpha\}$ (again all roots are taken with multiplicities) and is quasi-nilpotent. Set $\mathfrak{n}^- = \sigma(\mathfrak{n})$ and $\mathfrak{h}(\mathfrak{a}) = \mathfrak{h} \cap \mathfrak{a} = \langle H_\alpha \rangle$. In what follows we will often refer to [FKM2], where the case of finite-dimensional $\mathfrak{g}$ is considered. However, it is clear that all results have natural analogs for Lie algebras with triangular decomposition, e.g. the reader can consult [FKM3].

The paper is organized as follows: in Section 2 we introduce our main technical tool – a universal GVM. In Section 3 we use a description of (semi)primitive elements in a GVM to get a new proof for the conjecture from [KM2] in the case $\mathfrak{a} \simeq sl(2,\mathbb{C})$ (modulo the so-called lifting property – Proposition 2, which will be proved in Section 4). Finally, in Section 5 we analyze our previous proofs and show that for infinite-dimensional V our results extend to a simplicity criterion. We also formulate two conjectures about some sufficient conditions and one conjecture about a simplicity criterion for $M_\mathfrak{p}(V)$ in the case of arbitrary $\mathfrak{a}$.

2 Universal GVM and its properties

Our main idea is to consider a big enough induced module, which surjects on arbitrary GVM with given central character. We will call this module a *universal GVM* and will construct it as follows. Let $\chi : Z(\mathfrak{a}') \to \mathbb{C}$ be a central character of $\mathfrak{a}'$ and $I_\chi = \mathrm{Ker}(\chi)$ be the corresponding maximal ideal of $Z(\mathfrak{a}')$. Set $U_\chi^\alpha = U(\mathfrak{a}')/(U(\mathfrak{a}')I_\chi)$ (see also [B] for more information about the structure of U_χ^α as a generalized Weyl algebra), which we can consider as a left $\mathfrak{a}'$-module in an obvious way. Quillen's lemma guarantees that for any simple $\mathfrak{a}'$-module V there exists χ such that V is a homomorphic image of U_χ^α. Applying the exactness of the induction functor (from $\mathfrak{p}$ to $\mathfrak{g}$), we get that $M_\mathfrak{p}(V)$ is a homomorphic image of $M_\mathfrak{p}(U_\chi^\alpha)$. In this Section we will study the universal GVM $M_\mathfrak{p}(U_\chi^\alpha)$ in more detail.

Lemma 1. *$M_\mathfrak{p}(U_\chi^\alpha)$ is locally $Z(\mathfrak{a})$-finite, i.e. $Z(\mathfrak{a})v$ is finite-dimensional for any $v \in M_\mathfrak{p}(U_\chi^\alpha)$.*

Proof. First, the $\mathfrak{a}$-submodule $M_0 = 1 \otimes U_\chi^\alpha$ of $M_\mathfrak{p}(U_\chi^\alpha)$ is locally $Z(\mathfrak{a})$-finite by definition. Recall that $\mathfrak{g}$ is an integrable $\mathfrak{a}$-module by our assumptions. Then, as an $\mathfrak{a}$-module, $M_\mathfrak{p}(U_\chi^\alpha)$ can be decomposed into a direct sum of modules of the form $M_F \simeq M_0 \otimes F$, where F is a finite-dimensional $\mathfrak{a}$-module (see, for example [FKM1, Proposition 2]). Now, by [BG, Section 2.6] or [K, Theorem 5.1], each M_F is locally $Z(\mathfrak{a})$-finite and the statement follows. □

By the construction, $M_{\mathfrak{p}}(U_\chi^\alpha) \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} U_\chi^\alpha$ and hence it has obvious left $U(\mathfrak{g})$- and right $U(\mathfrak{a})$- module structures. The next Lemma describes the structure of $M_{\mathfrak{p}}(U_\chi^\alpha)$ as a left and as a right $U(\mathfrak{h}(\mathfrak{a}))$-module.

Lemma 2. *$M_{\mathfrak{p}}(U_\chi^\alpha)$ is free as a left and as a right $U(\mathfrak{h}(\mathfrak{a}))$-module.*

Proof. First, the $\mathfrak{a}$-bimodule $M_0 = 1 \otimes U_\chi^\alpha$ is isomorphic to U_χ^α as a left and a right $\mathfrak{a}$-module. Consider the associative algebra U_χ^α. It is a generalized Weyl algebra ([B, Section 1.2.(3)]) and hence, U_χ^α is a free $U(\mathfrak{h}(\mathfrak{a}))$-module from any side by [B, Section 1.1]. We also have that $M_{\mathfrak{p}}(U_\chi^\alpha)$ is $U(\mathfrak{n}^-)$-free ([FKM1, Proposition 1]). Let $\{v_i\}$ (resp. $\{v_i'\}$) be any basis of M_0 as a left (resp. right) $U(\mathfrak{h}(\mathfrak{a}))$-module and $\{u_j\}$ be a weight PBW-basis of $U(\mathfrak{n}^-)$. We immediately obtain that $\{u_j v_i\}$ (resp. $\{u_j v_i'\}$) is a basis of $M_{\mathfrak{p}}(U_\chi^\alpha)$ as a left (resp. right) $U(\mathfrak{h}(\mathfrak{a}))$-module. □

An immediate corollary of this Lemma is the following.

Corollary 1. *Let $v = 1 \otimes \hat{1}$ be the canonical generator of $M_{\mathfrak{p}}(U_\chi^\alpha)$. Fix a total order, $\leqslant$, on the set $\Delta_- \setminus \{-\alpha\}$. Then the set $\mathcal{B}v$, where*

$$\mathcal{B} = \left\{ X_{\beta_1}^{m_1} X_{\beta_2}^{m_2} \dots X_{\beta_k}^{m_k} X_{\pm\alpha}^{m} \,|\, \beta_1 \leqslant \beta_2 \leqslant \dots \leqslant \beta_k \right\},$$

is a basis of $M_{\mathfrak{p}}(U_\chi^\alpha)$ as a left and as a right $U(\mathfrak{h}(\mathfrak{a}))$-module.

In principal, the module $M_{\mathfrak{p}}(U_\chi^\alpha)$ is too big, so we will need certain quotients of this module. For $a \in \mathbb{C}$ let M^a denote the $\mathfrak{a}'$-submodule of U_χ^α generated by $H_\alpha - a$. Set

$$N(a, \chi) = U_\chi^\alpha / M^a.$$

Again, using the exactness of the induction, $M_{\mathfrak{p}}(N(a,\chi))$ is isomorphic to $M_{\mathfrak{p}}(U_\chi^\alpha)/M_{\mathfrak{p}}(M^a)$. Moreover, the classical $sl(2,\mathbb{C})$-theory makes it possible to describe $M_{\mathfrak{p}}(N(a,\chi))$ much better, in fact, to show that the structure of this module is very well known. Indeed, by the arguments from general nonsense (see, for example [FKM1, Section 10]), $N(a,\chi)$, as an $\mathfrak{a}$-module, can be characterized as the unique $\mathfrak{a}$-module, satisfying the following conditions (compare, for example, with [KM1, Section 2]):

(i) for $\mathfrak{c} = (H_\alpha + 1)^2 + 4X_{-\alpha}X_\alpha \in Z(\mathfrak{a}) \subset Z(\mathfrak{a}')$, $\chi(\mathfrak{c})$ is the eigenvalue of $\mathfrak{c}$ on $N(a,\chi)$;

(ii) $\operatorname{supp} N(a,\chi) = a + 2\mathbb{Z}$ and all the weight spaces are one-dimensional;

(iii) $N(a,\chi)$ is generated by $N(a,\chi)_a$.

Lemma 3. *The canonical projection $\varphi : M_{\mathfrak{p}}(U_\chi^\alpha) \to M_{\mathfrak{p}}(N(a,\chi))$ sends $\mathcal{B}v$ to a $\mathbb{C}$-basis of $M_{\mathfrak{p}}(N(a,\chi))$ coordinated with the $\mathfrak{h}$-weight structure of $M_{\mathfrak{p}}(N(a,\chi))$, i.e. the set of elements sent to a fixed weight subspace of $M_{\mathfrak{p}}(N(a,\chi))$ forms there a basis.*

Proof. Follows from Corollary 1. □

Each $M_{\mathfrak{p}}(N(a,\chi))$ is $\mathfrak{h}$-diagonalizable (because the module $N(a,\chi)$ is weight). But, in general, $M_{\mathfrak{p}}(V)$ is not $\mathfrak{h}$-diagonalizable and is only $\mathfrak{h}_{\mathfrak{a}}$-diagonalizable. For an $\mathfrak{h}_{\mathfrak{a}}$-diagonalizable module, M, and $\mu\in\mathfrak{h}_{\mathfrak{a}}^*$ we will denote by M_μ the corresponding $\mathfrak{h}_{\mathfrak{a}}$-weight space.

Further we will also need modules, which differ a little bit from the modules $N(a,\chi)$ above. Denote by $V(a,\chi)$ the unique $\mathfrak{a}'$-module which has the same simple subquotients as $N(a,\chi)$ and on which the element $X_{-\alpha}$ acts bijectively (see [FM, Section 2]). We emphasize that, by definition, $V(a,\chi)\simeq N(a,\chi)$ if one of them is simple. Recall that an element, v, of a weight $\mathfrak{g}$-module, V, is called $\mathfrak{p}$-*primitive* provided $\mathfrak{n}v=0$. The motivation to study $\mathfrak{p}$-primitive elements is the following standard fact ([MO, Proposition 1]):

Lemma 4. *Assume that V is simple. Then the module $M_{\mathfrak{p}}(V)$ is simple if and only if any $\mathfrak{p}$-primitive element of $M_{\mathfrak{p}}(V)$ has the form $1\otimes v$, $v\in V$.*

Now we show that the structures of $M_{\mathfrak{p}}(N(a,\chi))$ and $M_{\mathfrak{p}}(V(a,\chi))$ are closely related.

Lemma 5. *For any $\lambda\in\mathfrak{h}^*$ there is a natural bijection between $\mathfrak{p}$-primitive elements in $M_{\mathfrak{p}}(N(a,\chi))_{\lambda-k\alpha}$ and $M_{\mathfrak{p}}(V(a,\chi))_{\lambda-k\alpha}$ for all k big enough.*

Proof. From the definition of $V(a,\chi)$ and $N(a,\chi)$ it follows that there is a homomorphism $\varphi:N(a,\chi)\to V(a,\chi)$ whose kernel (resp. cokernel) is either zero (for example this is the case provided $V(a,\chi)\simeq N(a,\chi)$) or is a lowest weight (resp. weight dual to a lowest weight) module. This map naturally extends to a homomorphism $\hat{\varphi}:M_{\mathfrak{p}}(N(a,\chi))\to M_{\mathfrak{p}}(V(a,\chi))$, which is a vector-space isomorphism, restricted to all $M_{\mathfrak{p}}(N(a,\chi))_{\lambda-k\alpha}$ with k big enough. As $\hat{\varphi}$ sends $\mathfrak{p}$-primitive elements to $\mathfrak{p}$-primitive elements, our statement follows for all such $M_{\mathfrak{p}}(N(a,\chi))_{\lambda-k\alpha}$. □

From the PBW-Theorem it follows that for any Verma module V (over $\mathfrak{a}'$), the corresponding induced module $M_{\mathfrak{p}}(V)$ is again a Verma module (over $\mathfrak{g}$). For $\chi\in Z(\mathfrak{a}')$ let M^χ denote the Verma module (over $\mathfrak{a}'$) with the central character χ, whose highest weight lies in the closure of the antidominant Weyl chamber.

Proposition 1. *Assume that $M_{\mathfrak{p}}(N(a,\chi))_{\lambda-k\alpha}$ contains a non-zero $\mathfrak{p}$-primitive element for all k big enough. Then there exists $x\in\mathbb{C}$ such that $M_{\mathfrak{p}}(M^\chi)_{\lambda-x\alpha}$ contains a non-trivial $\mathfrak{p}$-primitive element.*

Proof. Using Lemma 5, we get a non-trivial $\mathfrak{p}$-primitive element in $M_{\mathfrak{p}}(V(a,\chi))_{\lambda-k\alpha}$ for all k big enough, and hence there is a non-trivial $\mathfrak{p}$-primitive element in $M_{\mathfrak{p}}(V(a',\chi))_{\lambda-k(a')\alpha}$ for all $a'\in\mathbb{C}$ and all $k(a')$ big enough ([FKM2, Theorem 1], resp. [FKM3, Theorem 4]). Now the statement follows from [FKM2, Lemma 4] (resp. [FKM3, Lemma 7]). □

3 Sufficient condition for simplicity of $M_{\mathfrak{p}}(V)$

In this Section we develop a new approach to [KM2, Cojecture 1], which then will be used to derive a simplicity criterion for GVMs. Recall that we are working in the setup fixed in the end of Section 1.

Theorem 1. *Let $M_{\mathfrak{p}}(V)$ be a generalized Verma module and $M(\mu)$ be the Verma module over $\mathfrak{g}$ associated with $M_{\mathfrak{p}}(V)$ as in Section 1. Assume that $M(\mu)$ is simple. Then $M_{\mathfrak{p}}(V)$ is also simple.*

In fact, we are going to prove that the reducibility of $M_{\mathfrak{p}}(V)$ implies the reducibility of $M(\mu)$. First we note that, in the case of a weight $\mathfrak{a}$-module V, the statement is quite clear. Indeed, if V is a highest or a lowest weight module (this includes the case when V is finite-dimensional), this follows directly from the BGG criterion ([D, Theorem 7.6.23]). Otherwise V is isomorphic to some simple $V(a,\chi)$ (see e.g. [FKM1, Section 10]) and, by Lemma 4, $M_{\mathfrak{p}}(V)$ has a non-zero $\mathfrak{p}$-primitive element, say w, different from $1\otimes v$, $v\in V$. Assume that this element is a weight element of weight $\lambda\in\mathfrak{h}^*$. As V is simple, $X_{-\alpha}$ acts bijectively on $M_{\mathfrak{p}}(V)$, and we get that there is a non-zero $\mathfrak{p}$-primitive element in all $M_{\mathfrak{p}}(V)_{\lambda-k\alpha}$, $k\in\mathbb{Z}$. Applying now Lemma 5 and Proposition 1 we obtain that there is a non-zero $\mathfrak{p}$-primitive element in $M_{\mathfrak{p}}(M^{\chi})_{\lambda-k\alpha}$ for all k big enough. By the construction, we have $M_{\mathfrak{p}}(M^{\chi})=M(\mu)$. The inequality $w\neq 1\otimes v$, $v\in V$, gives us that $\lambda+k\alpha$ is not the highest weight of $M(\mu)$ for any $k\in\mathbb{Z}$. Now, as $M(\mu)_{\lambda-k\alpha}$ contains a non-zero $\mathfrak{p}$-primitive element and $M(\mu)$ is X_{α}-locally finite, $M(\mu)$ contains a non-zero primitive element of the weight $\lambda+k\alpha$ for some $k\in\mathbb{Z}$. This element can not coincide with the generator of $M(\mu)$, because it has another weight, and hence $M(\mu)$ contains a proper highest weight submodule.

The observation above allows us to assume from now on that V is not a weight $\mathfrak{a}$-module. The main ingredient of our proof is the following *lifting property*, which we will prove in the next Section.

Proposition 2. *Assume that V is not weight and $v\in M_{\mathfrak{p}}(V)$ is a non-zero $\mathfrak{p}$-primitive element of $\mathfrak{h}_{\mathfrak{a}}$-weight λ. Let χ be the central character of V. Then $M_{\mathfrak{p}}(U_{\chi}^{\alpha})$ has a non-zero $\mathfrak{p}$-primitive element of $\mathfrak{h}_{\mathfrak{a}}$-weight λ.*

Using the lifting property one completes the proof of Theorem 1 as follows.

Proof of Theorem 1. We have only to consider the case, when the module V is not weight. Assume that $M_{\mathfrak{p}}(V)$ is not simple. Then, by Lemma 4, it has a non-zero $\mathfrak{p}$-primitive element of, say, $\mathfrak{h}_{\mathfrak{a}}$-weight λ, which is not of the form $1\otimes v$, $v\in V$. By the lifting property, the corresponding universal GVM $M_{\mathfrak{p}}(U_{\chi}^{\alpha})$ has a non-zero $\mathfrak{p}$-primitive element, w, of the same $\mathfrak{h}_{\mathfrak{a}}$-weight λ. Using Corollary 1, write $w=\sum_{b\in\mathcal{B}}bh_b$, where $h_b\in U(\mathfrak{h}(\mathfrak{a}))$ and only finite number of them are non-zero. Choose $a\in\mathbb{C}$ such that $h_b(a)\neq 0$ for all non-zero h_b. Then the image of w in $M_{\mathfrak{p}}(N(a,\chi))$ equals $\sum_{b\in\mathcal{B}}bh_b(a)$ and hence is non-zero by Lemma 3. This means that $M_{\mathfrak{p}}(N(a,\chi))$ contains a non-zero $\mathfrak{p}$-primitive element of the weight λ. Applying Proposition 1 and the argument as above we get that $M(\mu)$ contains a proper highest weight submodule and hence is not simple. □

4 Proof of the lifting property

Conceptually, this is the main part of the paper and the main ingredient in the proof of Theorem 1. The argument used above are relatively standard and mostly based on the results already known from [D, MP, CF, FM, FKM2]. The content of this Section is quite different, mainly, since it considers the general situation, which has not been studied yet.

To prove the lifting property we need several Lemmas and some notation. Write $U_\chi^\alpha = \oplus_{j\in\mathbb{Z}} U^{(j\alpha)}$, where $U^{(j\alpha)} = \{v \in U_\chi^\alpha \,|\, [H_\alpha, v] = j\alpha(H_\alpha)v\}$. As the centralizer of $U(\mathfrak{h}(\mathfrak{a}))$ in $U(\mathfrak{a})$ coincides with $\mathbb{C}[H_\alpha, \mathfrak{c}]$, one has $U^{(0)} \simeq U(H_\alpha)$ from the definition of U_χ^α. Any element $f \in U^{(j\alpha)}$ will be called *graded* of *degree* $\deg(f) = j$. Denote by $\mathcal{I}$ the kernel of a (fixed) projection $U_\chi^\alpha \to V$ and by $\hat{\mathcal{I}} \simeq M_\mathfrak{p}(\mathcal{I})$ the kernel of the induced projection $M_\mathfrak{p}(U_\chi^\alpha) \to M_\mathfrak{p}(V)$. Recall that we assume V to be a simple non-weight $\mathfrak{a}$-module.

Lemma 6. *Let w (resp. f) be a non-zero element (resp. non-zero graded element) from U_χ^α. Then $fw \neq 0$, moreover $fw \in \mathcal{I}$ if and only if $w \in \mathcal{I}$.*

Proof. The first statement follows easily from [B] in a bigger generality since U_χ^α is a generalized Weyl algebra which does not have zero divisors. But we present here another proof, which then will be used to prove the second statement.

First assume that $f \in U^{(0)}$, from which we have $f = F(H_\alpha)$ for some $F \in \mathbb{C}[t]$ and the statement follows from the proof of Lemma 2. If $f \in U^{(j\alpha)}$ then, according to Corollary 1, $f = X_{\pm\alpha}^{|j|} f'$ for some $f' \in U^{(0)}$ and hence it is enough to show that $X_{\pm\alpha} w \neq 0$. Consider X_α (the case of $X_{-\alpha}$ can be handled analogously). From $X_\alpha w = 0$ and $(\mathfrak{c} - \chi(\mathfrak{c}))w = 0$ we get $((H_\alpha + 1)^2 - \chi(\mathfrak{c}))w = 0$, which is impossible as $(H_\alpha + 1)^2 - \chi(\mathfrak{c})$ is a non-zero element of grade 0.

For the second statement we have only to prove that $fw \in \mathcal{I}$ implies $w \in \mathcal{I}$. And, by induction, it is enough to prove this for $f = F(H_\alpha)$, $F \in \mathbb{C}[t]$, and for $f = X_{\pm\alpha}$. If $w \notin \mathcal{I}$ such that $fw \in \mathcal{I}$, then, projecting on V, we get a non-zero element $\hat{w} \in V$ such that $f\hat{w} = 0$. If $f = F(H_\alpha)$ we decompose F into a product of linear polynomials and find a non-zero element of V which will be annihilated by some $H_\alpha - c$. This means that V contains a weight element. As V is simple, it is generated by any non-zero element, and hence should be a weight $\mathfrak{a}$-module. This contradicts our assumptions. If, say $f = X_\alpha$, by the same argument as above we get $((H_\alpha + 1)^2 - \chi(\mathfrak{c}))\hat{w} = 0$ and again the module V should be weight. The case $f = X_{-\alpha}$ can be treated in the same way. This completes the proof. □

We recall that $M_\mathfrak{p}(V)$ contains a non-zero $\mathfrak{p}$-primitive element, v, of $\mathfrak{h}_\mathfrak{a}$-weight λ. Let λ' be the $\mathfrak{h}_\mathfrak{a}$-highest weight of $M_\mathfrak{p}(V)$. Let $\mathcal{B}_\lambda$ (resp. $\mathcal{B}'_\lambda$) denote the set of all $\prod_\beta X_\beta^{n_\beta}$ having the $\mathfrak{h}_\mathfrak{a}$-weight $\lambda - \lambda'$ (resp. having the $\mathfrak{h}_\mathfrak{a}$-weight greater than $\lambda - \lambda'$), where the product is taken over all negative roots $\beta \neq -\alpha$ counted with multiplicities. Again, from the assumption that $\mathfrak{g}$ is an integrable $\mathfrak{a}$-module it follows that both $\mathcal{B}_\lambda$ and $\mathcal{B}'_\lambda$ are finite and we can order their elements in an arbitrary way, say $\mathcal{B}_\lambda = \{B_i \,|\, i \in I\}$, $\mathcal{B}'_\lambda = \{B'_j \,|\, j \in J\}$. By PBW-Theorem, any element $w \in M_\mathfrak{p}(U_\chi^\alpha)_\lambda$ can be written as $w = \sum_{i\in I} B_i a_i^w$ for some uniquely determined $a_i^w \in U_\chi^\alpha$. We set $[w] = (a_i^w)_{i\in I}$. Analogously, for $M_\mathfrak{p}(U_\chi^\alpha) \ni w = \sum_{j\in J} B'_j a_j^w$, $a_j^w \in U_\chi^\alpha$, we set $[w] = (a_j^w)_{j\in J}$. Remark that $w \in \hat{\mathcal{I}}$ if and only if $[w] \in \mathcal{I}^{|I|}$ (resp. $\mathcal{I}^{|J|}$), and for $x \in U_\chi^\alpha$ and $w \in M_\mathfrak{p}(U_\chi^\alpha)_\lambda$ the equality $[xw] = x[w]$ is not true in general.

For all $\beta \in \Delta_+ \setminus \{\alpha\}$ we have

$$X_\beta B_i = \sum_{j \in J} B'_j G_{i,j,\beta} \mod U(\mathfrak{n})\mathfrak{n},$$

for some $G_{i,j,\beta} \in U_\chi^\alpha$.

Lemma 7. *$G_{i,j,\beta}$ are graded for all i, j, β. Moreover, the equality*

$$\deg(G_{i,j,\beta}) - \deg(G_{i',j,\beta}) = \deg(G_{i,j',\gamma}) - \deg(G_{i',j',\gamma})$$

holds for all $i, i' \in I$, $j, j' \in J$ and $\beta, \gamma \in \Delta_+ \setminus \{\alpha\}$.

Proof. $G_{i,j,\beta}$ is graded since all X_β, B_i and B'_j are H_α-diagonalizable with respect to the adjoint action. Now the common value in the equality above is $x/2 = x/\alpha(H_\alpha$, where x is the difference between H_α-eigenvalues of B_i and $B_{i'}$ under the adjoint action. □

For a positive $\beta \neq \alpha$ define a matrix, $\mathbf{A}^\beta$, by $\mathbf{A}^\beta_{j,i} = G_{i,j,\beta}$. This is a $|J| \times |I|$-matrix with entries being graded elements from U_χ^α. By the construction, it represents the action of X_β on $\mathcal{B}_\lambda$ (by multiplication from the left) with the result written in the (right) U_χ^α-basis $\mathcal{B}'_\lambda$. Define

$$\mathbf{A}' = \begin{pmatrix} \mathbf{A}^{\beta_1} \\ \mathbf{A}^{\beta_2} \\ \vdots \\ \mathbf{A}^{\beta_k} \end{pmatrix},$$

where $\beta_1, \dots, \beta_k$ is a finite generating system in $\Delta_+ \setminus \{\alpha\}$, each root counted with the corresponding multiplicity, whose elements are written in some order. The existence of such system follows directly from our assumption on $\mathfrak{g}$ to be an integrable $\mathfrak{a}$-module. We will call a set of rows *independent* if it is linearly independent with left coefficients being graded elements from U_χ^α, and *dependent* otherwise. Directly from the construction we have that $w \in (M(U_\chi^\alpha))_\lambda$ is $\mathfrak{p}$-primitive if and only if $\mathbf{A}'[w] = 0$. In particular, from this we get that the set of $\mathfrak{p}$-primitive elements in $M_\mathfrak{p}(U_\chi^\alpha)$ is a right $U(\mathfrak{a})$-submodule. Let $\{l_k \mid k \in K\}$ be a maximal set of independent rows of $\mathbf{A}'$ indexed by a finite set K. Choose any non-zero graded f_k, $k \in K$, such that the degree of the first component of $f_k l_k$ is 0 (such elements obviously exist) and denote by $\mathbf{A}$ the matrix with $|K|$ rows $f_k l_k$. Clearly, $|K| \leqslant |I|$.

Lemma 8. *For any $w \in (M(U_\chi^\alpha))_\lambda$ we have $\mathbf{A}'[w] = 0$ if and only if $\mathbf{A}[w] = 0$. In particular, $w \in (M(U_\chi^\alpha))_\lambda$ is $\mathfrak{p}$-primitive if and only if $\mathbf{A}[w] = 0$.*

Proof. If $\mathbf{A}'[w] = 0$ then $l_j[w] = 0$ for any j and hence $f_k l_k[w] = 0$. Conversely, if $\mathbf{A}[w] = 0$ then $f_k l_k[w] = 0$ and hence $l_k[w] = 0$ by Lemma 6. As $\{l_k\}$ is a maximal set of independent elements, for any j there exist graded f and f'_k, $k \in K$, such that $f l_j = \sum_{k \in K} f'_k l_k$. Hence $f l_j[w] = 0$ and again $l_j[w] = 0$ by Lemma 6 since f is graded. □

Any entry of $\mathbf{A}$ is graded, thus there are $g_{k,i} \in \mathbb{C}[H_\alpha]$, $k \in K$, $i \in I$, such that

$$\mathbf{A}_{k,i} = \begin{cases} g_{k,i} X_\alpha^{\deg(\mathbf{A}_{k,i})} & \deg(\mathbf{A}_{k,i}) \geqslant 0 \\ g_{k,i} X_{-\alpha}^{-\deg(\mathbf{A}_{k,i})} & \deg(\mathbf{A}_{k,i}) < 0 \end{cases}.$$

Moreover, from Lemma 7 it follows that $\deg(\mathbf{A}_{k,i}) = \deg(\mathbf{A}_{k',i})$ for all $k, k' \in K$ and $i \in I$. Define a $|K| \times |I|$-matrix, $\mathbf{H}$, via $\mathbf{H}_{k,i} = g_{k,i}$. It is important that the components of this matrix are polynomials in H_α. As the rows of $\mathbf{A}$ are independent and $\deg(\mathbf{A}_{k,i}) = \deg(\mathbf{A}_{k',i})$ for all $k, k' \in K$ and $i \in I$, one easily gets that the rows of $\mathbf{H}$ are also independent.

First we consider the case $|K| = |I|$. Since the rows of $\mathbf{H}$ are independent, we have that $\det(\mathbf{H})$ is a non-zero polynomial in $\mathbb{C}[H_\alpha]$, in particular, there exists $\hat{\mathbf{H}} \in \mathbb{C}[H_\alpha]$ such that $\hat{\mathbf{H}}\mathbf{H} = \det(\mathbf{H})\mathbf{1}$. Thus we obtain that $\hat{\mathbf{H}}\mathbf{A}$ is diagonal and equals

$$\det(\mathbf{H}) \operatorname{diag}(1, X_{\pm\alpha}^{\pm\deg(\mathbf{A}_{2,2})}, \dots, X_{\pm\alpha}^{\pm\deg(\mathbf{A}_{|I|,|I|})}).$$

We have $\mathbf{A}[v] \in \mathcal{I}^{|I|}$, hence $\hat{\mathbf{H}}\mathbf{A}[v] \in \mathcal{I}^{|I|}$. If $v = \sum_i B_i a_i^v$, this means $\det(\mathbf{H}) X_{\pm\alpha}^{\pm\deg(\mathbf{A}_{i,i})} a_i^v \in \mathcal{I}$. Thus, by Lemma 6, we get $a_i^v \in \mathcal{I}$ and $v = 0$ in $M_\mathfrak{p}(V)$.

Now we assume $|K| < |I|$. Our goal is to show that the system $\mathbf{A}[x] = 0$ has a non-zero solution in $\mathcal{I}^{|I|}$, since any such solution will give us a non-zero $\mathfrak{p}$-primitive element of $M_\mathfrak{p}(U_\chi^\alpha)_\lambda$. Without loss of generality we can assume that first $|K|$ columns of $\mathbf{H}$ are independent and, as in the previous paragraph, there exists a $|K| \times |K|$-matrix $\hat{\mathbf{H}}$ with entries in $\mathbb{C}[H_\alpha]$ such that $\hat{\mathbf{H}}A = (C_1|C_2)$, where $C_1 = G(H_\alpha)D$, with a non-zero $G(H_\alpha) \in \mathbb{C}[H_\alpha]$ and a diagonal $|K| \times |K|$-matrix

$$D = \operatorname{diag}(1, X_{\pm\alpha}^{\pm\deg(\mathbf{A}_{2,2})}, \dots, X_{\pm\alpha}^{\pm\deg(\mathbf{A}_{|K|,|K|})}),$$

and C_2 is a matrix with entries from U_χ^α. Consider now the system $\hat{\mathbf{H}}\mathbf{A}[x] = 0$. As $\det(\hat{\mathbf{H}})$ is non-zero and graded, by Lemma 6, $\mathbf{A}[x] = 0$ if and only if $\hat{\mathbf{H}}\mathbf{A}[x] = 0$. Multiply our system by

$$D' = \operatorname{diag}(1, X_{\mp\alpha}^{\pm\deg(\mathbf{A}_{2,2})}, \dots, X_{\mp\alpha}^{\pm\deg(\mathbf{A}_{|K|,|K|})})$$

and by the same arguments we obtain a new system, $(C_1'|C_2')[x] = 0$, where

$$C_1' = \operatorname{diag}(F_1(H_\alpha), F_2(H_\alpha), \dots, F_{|K|}(H_\alpha)),$$

and all F_i are non-zero polynomials. In particular, we have $(C_1', C_2')[x] = 0$ if and only if $\mathbf{A}[x] = 0$.

Lemma 9. *Let $F \in \mathbb{C}[x]$. Then for any $m \times n$-matrix A with coefficients from U_χ^α there exists $F^A \in \mathbb{C}[x]$ satisfying the following condition: for any $[x] \in (U_\chi^\alpha)^n$ there exists $[y] \in (U_\chi^\alpha)^m$ such that $F(H_\alpha)[y] = A(F^A(H_\alpha)[x])$.*

Proof. Let $z \in U_\chi^\alpha$, $z = \sum_{l\in\mathbb{Z}} z_l$, where $z_l \in U^{(l\alpha)}$. Set $F^z(x) = \prod_{l\in\mathbb{Z}, z_l\neq 0} F(x + 2l)$. Then for all $x \in U_\chi^\alpha$ we have $zF^z(H_\alpha)x = F(H_\alpha)y$ for some y. Now the lemma will follow if we choose $F^A = \prod_{i,j} F^{A_{i,j}}$. □

Set $F = \prod_i F_i$ and $[x]_j = F^{C'_2}(H_\alpha)$, $j > |K|$. By Lemma 9, for any $i \in K$ there exists $y_i \in U_\chi^\alpha$ such that $\sum_{j>|K|}(C'_2)_{i,j}[x]_j = F(H_\alpha)y_i$. Now we can set

$$[x]_i = -\left(\prod_{1\leqslant k\leqslant |K|, k\neq i} F_k(H_\alpha)\right) y_i, \quad i \leqslant |K|$$

and obtain that $[x]$ is a non-zero solution of our system. This means that the element $\sum_i B_i[x]_i$ is a non-zero $\mathfrak{p}$-primitive element of $M_\mathfrak{p}(U_\chi^\alpha)$ and completes the proof of the lifting property and hence of Theorem 1.

5 Deriving a criterion

Let us now analyze, where in Section 4 we used the fact that the module V is not weight. This was used only in the proof of the second part of Lemma 6 (recall that the first part of this Lemma follows from [B] in a general case). Clearly, the statement of Proposition 2 is not true for arbitrary V because of problems arising in the case of finite-dimensional V (see [R]). But we may observe one more nice property of $sl(2,\mathbb{C})$-case: if V is a simple infinite-dimensional weight module then either $M_\mathfrak{p}(V) \simeq M(\mu)$ or for all $\lambda \in \mathfrak{h}_\mathfrak{a}^*$ $M(\mu)_\lambda$ contains a non-zero $\mathfrak{p}$-primitive element if and only if $M_\mathfrak{p}(V)_\lambda$ does (this follows from [M, Lemma A.1]). This tempt us to try to prove the following *advanced lifting property.*

Proposition 3. *Let V be an infinite-dimensional simple $\mathfrak{a}'$-module. If for some $\lambda \in \mathfrak{h}_\mathfrak{a}^*$ there is a non-zero $\mathfrak{p}$-primitive element in $M_\mathfrak{p}(V)_\lambda$, then there is a non-zero $\mathfrak{p}$-primitive element in $M_\mathfrak{p}(U_\chi^\alpha)_\lambda$*

Proof. Because of the known lifting property we have only to consider the case, when V is an infinite-dimensional simple weight module. We look at Section 4. As we have already mentioned, the only place where we used the fact that V was not weight is the proof of the second statement in Lemma 6. The last was used later only when the case $|K| = |I|$ was considered. Hence, in the case $|K| < |I|$ we have our statement for free. Now we are going to show that under our assumptions the case $|K| = |I|$ is not possible.

Indeed, in this case the system $\hat{\mathbf{H}}\mathbf{A}[x] = 0$ can have only trivial solution by the first part of Lemma 6. Now, we recall that $M_\mathfrak{p}(V)_\lambda$ is a weight module and contains a non-zero $\mathfrak{p}$-primitive element. And we know that V is either highest or lowest weight module or coincides with some $N(a',\chi)$. Using [FKM2, Theorem 1], we get that there exists a non-zero $\mathfrak{p}$-primitive element in $M_\mathfrak{p}(N(a,\chi))_\lambda$ for all $a \in \mathbb{C}$. Return to the proof of the lifting property in the case $|K| = |I|$ (with $V = N(a,\chi)$ for some a and v being a non-zero $\mathfrak{p}$-primitive element of $M_\mathfrak{p}(N(a,\chi))_\lambda$). We have $\det(H)X_{\pm\alpha}^{\pm\deg(\mathbf{A}_{i,i})}a_i^v \in \mathcal{I}$ and we want to deduce that $a_i^v \in \mathcal{I}$ obtaining a contradiction. Rewrite $\det(H)X_{\pm\alpha}^{\pm\deg(\mathbf{A}_{i,i})}$ as $X_{\pm\alpha}^{\pm\deg(\mathbf{A}_{i,i})}G_i(H_\alpha)$, where $G_i(H_\alpha) \in \mathbb{C}[H_\alpha]$ and choose a such that the following two conditions are satisfied:

1. $G_i(a+j) \neq 0$ for all i and all $j \in \mathbb{Z}$.

2. $N(a,\chi)$ is simple.

This is clearly possible. But in this case both $X_{\pm\alpha}$ act bijectively on $M_{\mathfrak{p}}(N(a,\chi))$ (see Section 2), hence on $U_\chi^\alpha/\mathcal{I}$, in particular, this means that $X_{\pm\alpha}x \in \mathcal{I}$ if and only if $x \in \mathcal{I}$. Thus we obtain that $G_i(H_\alpha)a_i^v \in \mathcal{I}$. Again, by the choice of a, the image of $G_i(H_\alpha)a_i^v = \sum_{j\in\mathbb{Z}}(a_i^v)_j G_i(H_\alpha \pm 2j)$ (here we use the graded decomposition of a_i^v) in $U_\chi^\alpha/\mathcal{I}$ equals $\sum_{j\in\mathbb{Z}} x_j G_i(a\pm 2j)$ and is non-zero if any only if $a_i^v \notin \mathcal{I}$. From this we finally get that all $a_i^v \in \mathcal{I}$ which means that $M_{\mathfrak{p}}(N(a,\chi))$ can not have any non-zero $\mathfrak{p}$-primitive elements. This contradiction completes the proof. □

Now we can easily extend Theorem 1 to a criterion for $M_{\mathfrak{p}}(V)$ to be simple, under the condition that V is infinite-dimensional.

Theorem 2. *Under the notation of Theorem 1, for a simple infinite-dimensional $\mathfrak{a}'$-module V, the module $M_{\mathfrak{p}}(V)$ is simple if and only if $M(\mu)$ is simple. Moreover, for any $\lambda \in \mathfrak{h}_{\mathfrak{a}}^*$, there is a non-zero $\mathfrak{p}$-primitive element in $M(\mu)_\lambda$ if and only if there is a non-zero $\mathfrak{p}$-primitive element in $M_{\mathfrak{p}}(V)_\lambda$.*

Proof. First we assume that V is a highest (or lowest) weight module. Then either $M_{\mathfrak{p}}(V) \simeq M(\mu)$ and our result is a tautology or the statement follows from [M, Lemma A.1] mentioned above.

If V is weight but do not have any highest (lowest) weights, then $V \simeq N(a,\chi)$ for some a and χ and the result follows from [FKM2, Theorem 1,Lemma 4].

Finally, we again have only to consider the case, when V is not weight. In this case the "if" part is just Theorem 1 together with the lifting property. Let us prove the "only if" part. Assume that $M(\mu)$ is not simple. As μ is chosen to be $\mathfrak{a}$-antidominant, we have that $M(\mu)$ has a non trivial primitive (and hence $\mathfrak{p}$-primitive) element of some $\mathfrak{h}_{\mathfrak{a}}$-weight λ, which is different from the highest $\mathfrak{h}_{\mathfrak{a}}$-weight of $M(\mu)$. Moreover, by construction, $M(\mu) \simeq M_{\mathfrak{p}}(V')$ for some simple highest weight infinite-dimensional $\mathfrak{a}'$-module V'. Using the advanced lifting property we get that $M_{\mathfrak{p}}(U_\chi^\alpha)_\lambda$ contains a non-zero $\mathfrak{p}$-primitive element, say x. Moreover, from the proof of the advanced lifting property it follows that this element can be chosen, for example, as one constructed in the last paragraph of Section 4. From this construction we get that several entries of $[x]$ are non-zero polynomials in H_α. Now, if V is not weight, $\mathbb{C}[H_\alpha] \cap \mathcal{I} = 0$ by Lemma 6 and we obtain that the image of $\sum_i B_i[x]_i$ in $M_{\mathfrak{p}}(V)$ is non-zero, and, of course, $\mathfrak{p}$-primitive. Thus $M_{\mathfrak{p}}(V)$ is not simple, moreover, it contains necessary $\mathfrak{p}$-primitive element. □

Theorem 2 completely answers the question about simplicity of arbitrary GVM in our setup, reducing it to the known situation of Verma module. The final result really seems to be quite surprizing. In particular, it shows that simplicity of $M_{\mathfrak{p}}(V)$ depends only on the central character χ of V and does not depend on V itself (of course assuming that V is infinite-dimensional). We would like to add the following remark to the Theorem 2: from the proof one can easily derive that the statement about $\mathfrak{p}$-primitive elements can be read even as follows: for any $\lambda \in \mathfrak{h}_{\mathfrak{a}}^*$ and for any $c \in \mathbb{C}$, there is a non-zero $\mathfrak{p}$-primitive element,

v, in $M(\mu)_\lambda$ such that $\mathfrak{c}v = cv$ if and only if there is a non-zero $\mathfrak{p}$-primitive element, w, in $M_\mathfrak{p}(V)_\lambda$ such that $\mathfrak{c}w = cw$.

Theorem 1 and Theorem 2 encourage us to formulate the following two conjectures about sufficient conditions for a GVM to be reducible in the case of arbitrary $\mathfrak{a}$.

Conjecture 1. *Assume that $\mathfrak{g}$ is as above, $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ associated with a standard triangular decomposition and $\mathfrak{a}'$ is reductive finite-dimensional such that $\mathfrak{g}$ is an integrable $\mathfrak{a}'$-module under adjoint action. Let $M_\mathfrak{p}(V)$ be a GVM over $\mathfrak{g}$ and L be the simple quotient of the Verma module $M(\lambda)$ (over $\mathfrak{a}'$) with dominant λ, which has the same central character as V. If $M_\mathfrak{p}(L)$ is reducible then $M_\mathfrak{p}(V)$ is also reducible.*

The following is just a stronger version of Conjecture 1.

Conjecture 2. *Under the notation of Conjecture 1, if for $\lambda \in \mathfrak{h}_\mathfrak{a}^*$ there is a non-zero $\mathfrak{p}$-primitive element in $M_\mathfrak{p}(L)_\lambda$ then there is a non-zero $\mathfrak{p}$-primitive element in $M_\mathfrak{p}(V)_\lambda$.*

First, we note that the GVM $M_\mathfrak{p}(L)$ used in the conjectures above is, in fact, a highest weight module, hence its study should be much easier than that of $M_\mathfrak{p}(V)$. Second, if $\mathfrak{g}$ is infinite-dimensional, e.g. affine Kac-Moody Lie algebra, there are several non-equivalent classes of parabolic subalgebras in $\mathfrak{g}$ (see [Fu]). Our conjectures are formulated in the simplest case (parabolic subalgebra of type I in [Fu]), associated with a standard triangular decomposition, which means, in particular, that in the case of weight L with finite-dimensional weight spaces, $M_\mathfrak{p}(L)$ will also have finite-dimensional weight spaces (or, using the notation from the paper, the sets $\mathcal{B}_\lambda$ and $\mathcal{B}'_\lambda$ are finite for any λ). In other cases the last statement is no longer true, and there may arise several strange effects which will make the picture much more complicated, see [Fu] for details and examples with classical Verma modules.

We finish with the following conjecture about simplicity criterion for a GVM in the case of arbitrary $\mathfrak{a}$.

Conjecture 3. *Assume that $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{a}'$ and $M_\mathfrak{p}(V)$ are as in Conjecture 1. Then there exists a simple highest weight $\mathfrak{a}'$-module L having the same central character as V such that for any $\lambda \in \mathfrak{h}_\mathfrak{a}^*$ the following holds: $M_\mathfrak{p}(V)_\lambda$ contains a non-zero $\mathfrak{p}$-primitive element if and only if $M_\mathfrak{p}(L)_\lambda$ does.*

According to Theorem 2, all the conjectures are true in the case $\mathfrak{a} \simeq sl(2, \mathbb{C})$.

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