On multiplicities of simple subquotients in generalized Verma modules

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Abstract

We reduce the problem on multiplicities of simple subquotients in an α-stratified generalized Verma module to the analogous problem for classical Verma modules.

1 Introduction

The study of α-stratified modules over a simple complex finite-dimensional Lie algebra was originated in [CF] where several basic properties of such modules were obtained. The class of α-stratified modules contains the so-called generalized Verma modules (GVM). These modules are completely different from another family of GVMs introduced and studied in [R]. The α-stratified GVM were investigated in [FM, KM, Ma] where a BGG-like criterion for the existence of a non-trivial homomorphism between two α-stratified GVMs was established.

One of the most important results about classical Verma modules is the so-called Kazhdan-Lusztig theorem describing the multiplicities of simple subquotients in a Verma module (see for example [BK] and references therein). An analogous result for GVM in the sense of [R] was obtained in [CC]. It happened that the answer obtained in [CC] is different from the classical Kazhdan-Lusztig theorem. The latter means that the multiplicities of simple subquotients in a GVM (in the sense of [R]) cannot be obtained directly from the analogous multiplicities in the corresponding Verma module.

In the present paper we calculate the multiplicities of simple subquotients in an α-stratified GVM. In fact, with an arbitrary α-stratified GVM we associate a certain Verma module and prove that the required multiplicities coincide with the multiplicities of simple subquotients in this Verma module. This analogy with Verma modules provides one more difference between α-stratified GVMs and GVMs in the sense of [R].

We have to note that one related question for α-stratified modules was solved in [M, Theorem 13.4] in a full generality. In fact, for any simple complex finite-dimensional Lie algebra $\mathfrak{G}$ and its "well-embedded" subalgebra $\mathfrak{G}_1$ of type A_n or C_n, the character of the unique simple quotient of GVM induced from a homogeneous $\mathfrak{G}_1$-module was calculated.

Here homogeneous means that this module is weight, dense and has weight subspaces of the same dimension (see [M] for details). In the case when $\mathfrak{G}_1$ is of type A_1, simple homogeneous means the same as simple α-stratified. Thus, using the above mentioned result one can calculate the character of the unique simple quotient of an α-stratified GVM.

The paper is organized as follows: in Section 2 we collect all necessary preliminaries. In Section 3 we formulate our main result — Theorem 1, which is proved in Section 4.

2 Preliminaries

Let $\mathbb{C}$ denote the complex numbers, $\mathbb{Z}$ the set of integers and $\mathbb{N}$ the set of all positive integers. For a Lie algebra $\mathfrak{A}$ we will denote by $U(\mathfrak{A})$ its universal enveloping algebra.

Let $\mathfrak{G}$ be a simple complex finite-dimensional Lie algebra and $\mathfrak{H}$ its Cartan subalgebra. Denote by Δ the corresponding root system and choose a base, π, in Δ. This defines a partition of Δ into two sets of positive (Δ^+) and negative (Δ^-) roots. We will write P for the abelian subgroup in $\mathfrak{H}^*$ generated by the elements from Δ. For $\beta \in \Delta$ let $\mathfrak{G}_\beta$ denote the corresponding root subspace in $\mathfrak{G}$. Fix a Weyl-Chevalley basis, X_α, $\alpha \in \Delta$, H_α, $\alpha \in \pi$. Set

$$\rho = \frac{1}{2} \sum_{\beta \in \Delta^+} \beta.$$

Fix $\alpha \in \pi$. Let $\mathfrak{G}^\alpha$ denote the $sl(2)$-subalgebra of $\mathfrak{G}$ corresponding to the root α. Set $\mathfrak{N}^\alpha_\pm = \sum_{\beta \in \Delta^+ \setminus \{\alpha\}} \mathfrak{G}_{\pm\beta}$, $\mathfrak{H}^\alpha = \{h \in \mathfrak{H} \,|\, \alpha(h) = 0\}$, $\pi_\alpha = \pi \setminus \{\alpha\}$. Then we have the following decomposition: $\mathfrak{G} = \mathfrak{G}^\alpha \oplus \mathfrak{N}^\alpha_- \oplus \mathfrak{H}^\alpha \oplus \mathfrak{N}^\alpha_+$. For $\mathfrak{H}_\alpha = \mathfrak{G}^\alpha \cap \mathfrak{H}$ one obtains $\mathfrak{G}^\alpha = \mathfrak{G}_\alpha \oplus \mathfrak{H}_\alpha \oplus \mathfrak{G}_{-\alpha}$.

For a $\mathfrak{G}$-module, V, and $\lambda \in \mathfrak{H}^*$ let V_λ denote the weight space with respect to λ. A $\mathfrak{G}$-module, V, will be called a *weight module* if it decomposes into a direct sum of its weight spaces. A weight $\mathfrak{G}$-module, V, is called *α-stratified* ([CF]) if the actions of X_α and $X_{-\alpha}$ are injective on V. All modules considered in this paper are supposed to be weight modules with finite-dimensional weight spaces.

Consider the quadratic Casimir operator $c = (H_\alpha + 1)^2 + 4X_{-\alpha}X_\alpha$ in $U(\mathfrak{G}_\alpha)$. Any pair $a, b \in \mathbb{C}$ defines a unique indecomposable $\mathfrak{G}^\alpha$-module $N(a, b)$ such that $X_{-\alpha}$ acts bijectively on $N(a, b)$, all non-trivial weight spaces of $N(a, b)$ are one-dimensional, a is an eigenvalue of H_α and b is the (unique!) eigenvalue of c. One has $N(a, b) \simeq N(a + 2l, b)$ for any $l \in \mathbb{Z}$

Since $\mathfrak{H} = \mathfrak{H}_\alpha \oplus \mathfrak{H}^\alpha$ we can rewrite an arbitrary $\lambda \in \mathfrak{H}^*$ as $\lambda = \lambda_\alpha + \lambda^\alpha$, where $\lambda_\alpha \in \mathfrak{H}_\alpha$ and $\lambda^\alpha \in \mathfrak{H}^\alpha$. Let $a, b \in \mathbb{C}$ and let $\lambda \in \mathfrak{H}^*$ be such that $(\lambda - \rho)(H_\alpha) = (\lambda_\alpha - \rho)(H_\alpha) = a$. We can define the structure of an $\mathfrak{H}$-module on $N(a, b)$ by setting $hv = (\lambda - \rho)^\alpha(h)v$ for all $h \in \mathfrak{H}^\alpha$ and all $v \in N(a, b)$. Further, we can consider $N(a, b)$ as $D = \mathfrak{H} + \mathfrak{G}^\alpha \oplus \mathfrak{N}^\alpha_+$-module by setting $\mathfrak{N}^\alpha_+ N(a, b) = 0$.

The $\mathfrak{G}$-module

$$M_\alpha(\lambda, b) = U(\mathfrak{G}) \bigotimes_{U(D)} N(a, b)$$

is called the *generalized Verma module* associated with $\mathfrak{G}, \mathfrak{H}, \pi, \alpha, \lambda, b$. One can easily prove that $M_\alpha(\lambda, b)$ is α-stratified if and only if $b \neq (a+1+2l)^2$ for all $l \in \mathbb{Z}$ (see also [CF, Theorem 2.1]). We will denote by $L_\alpha(\lambda, b)$ the unique simple quotient of $M_\alpha(\lambda, b)$. It is well-known that $M_\alpha(\lambda, b)$ has a composition series ([CF, Theorem 2.8(i)]). For $\lambda \in \mathfrak{H}^*$ we will write $M(\lambda)$ for the Verma module with the highest weight $\lambda - \rho$ ([D, 7.1.4]) and $L(\lambda)$ for its unique simple quotient.

3 Main Theorem

Fix an analytic branch of the square root function satisfying the condition $\sqrt{1} = 1$. For arbitrary $\lambda \in \mathfrak{H}^*$ and $b \in \mathbb{C}$ set

$$f(\lambda, b) = \lambda - \frac{\lambda(H_\alpha) + \sqrt{b}}{\alpha(H_\alpha)}\alpha.$$

Theorem 1. *Suppose that $M_\alpha(\lambda, b)$ is α-stratified. Then the multiplicity of $L_\alpha(\mu, d)$ as a simple subquotient in a composition series of $M_\alpha(\lambda, b)$ equals the multiplicity of $L(f(\mu, d))$ as a simple subquotient in a composition series of $M(f(\lambda, b))$.*

4 Proof of the Main Theorem

For $u \in \mathbb{C}$ consider the $\mathfrak{G}^\alpha$-module

$$T(u) = \bigoplus_{a \in \mathbb{C}/2\mathbb{Z}} N(a, u)$$

and the corresponding induced module

$$M_T(\lambda, u) = U(\mathfrak{G}) \bigotimes_{U(D)} T(u).$$

A weight $\mathfrak{G}$-module, V, will be called *normal* provided $X_{-\alpha}$ acts bijectively on V. It follows from the definition of $N(a, b)$ that $M_T(\lambda, b)$ is normal.

Lemma 1. *Let V be a normal weight $\mathfrak{G}$-module and W a normal submodule of V. Then the module V/W is normal.*

Proof. Since V is normal it follows that $X_{-\alpha}$ acts surjectively on V/W. Moreover, since W is normal it follows that the pre-image of any element from W is contained in W and thus $X_{-\alpha}$ acts injectively on V/W. Combining these results we obtain that V/W is normal. □

Consider a normal $\mathfrak{G}$-module, V. Let $U(\alpha)$ denote the localization of $U(\mathfrak{G})$ with respect to the multiplicative set $\{X_{-a}^n \mid n \in \mathbb{N}\}$. $U(\alpha)$ is well-defined by [M, Lemma 4.2]. Since V is normal, we can define the $U(\alpha)$-module $V(\alpha) = U(\alpha) \otimes_{U(\mathfrak{G})} V$. By [M, Lemma 4.3] there exists a unique polynomial extension, $\{\theta_x \mid x \in \mathbb{C}\}$, of the family of automorphisms $\theta_x : U(\alpha) \to U(\alpha)$, $x \in \mathbb{Z}$ such that $\theta_x(v) = X_{-\alpha}^{x} v X_{-\alpha}^{-x}$, $x \in \mathbb{Z}$. For a $U(\alpha)$-module, W, and $x \in \mathbb{C}$ we will denote by $\theta_x(W)$ the $U(\alpha)$-module which is equal to W as a vector space and $v \cdot w = \theta(v)w$ for all $v \in U(\alpha)$, $w \in W$. Clearly, one can consider any $U(\alpha)$-module as a $U(\mathfrak{G})$-module by restriction.

Set $P_\alpha = \{\sum_{\beta \in \pi_\alpha} z_\beta \beta \mid z_\beta \in \mathbb{C}\}$ and $P(\alpha) = P + P_\alpha$. Let V be a weight $\mathfrak{G}$-module and $\lambda \in \mathfrak{H}^*$. We will denote by $V(\lambda)$ the direct summand $\sum_{\mu \in \lambda + P(\alpha)} V_\mu$ of V. For $\lambda_1, \lambda_2 \in \mathfrak{H}^*$ let $x(\lambda_1, \lambda_2)$ denote the unique complex number such that $\lambda_2 - (\lambda_1 + x(\lambda_1, \lambda_2)\alpha)$ belongs to P_α. A weight $\mathfrak{G}$-module, V, will be called *α-homogeneous* provided $V(\lambda_2) \simeq \theta_{x(\lambda_1,\lambda_2)} V(\lambda_1)$ for all $\lambda_1, \lambda_2 \in \mathfrak{H}^*$. It follows immediately from the definition that $M_T(\lambda, b)$ is α-homogeneous. One can easily see that the quotient of an α-homogeneous module by an α-homogeneous submodule is again α-homogeneous.

Let V be an α-homogeneous $\mathfrak{G}$-module. By a *solid structure* on V we will mean a family of linear maps, $\psi(y) = \theta^{-1}_{x(y\alpha,0)} \circ \varphi(y) : V(0) \to V(y\alpha)$, $y \in \mathbb{C}$, where $\varphi(y)$, $y \in \mathbb{C}$, are isomorphisms of $V(0)$, which can be chosen in an arbitrary way. If a solid structure on V is given, V will be called a *solid* module. We will say that an α-homogeneous submodule, W, of V is *solid* provided

$$W(\lambda_1) = \psi(x(\lambda_1, 0)) \circ \psi^{-1}(x(\lambda_2, 0))(W(\lambda_2)).$$

It follows immediately from the definition that $M_T(\lambda, b)$ can be viewed as a solid α-homogeneous module (remark that the only automorphisms of the zero part of $M_T(\lambda, b)$ are scalars by [CF], hence all $\varphi(y)$ are scalars). One can easily see that the quotient of a solid α-homogeneous module by a solid α-homogeneous submodule (whose solid structure is inherited from the big module) is again solid α-homogeneous.

Lemma 2. *Let V be a solid α-homogeneous module and let W be a normal submodule in V. Then the submodule $\hat{W}$ of V defined by*

$$\hat{W}(\mu) = \sum_{\mu' \in \mathfrak{H}^*} \psi(x(\mu, 0)) \circ \psi^{-1}(x(\mu', 0))(W(\mu')),$$

$\mu \in \mathfrak{H}^*$ is the unique minimal solid normal α-homogeneous submodule containing W.

Proof. Clearly, $\hat{W}$ is solid, α-homogeneous and contains W. It is normal by the definition of θ_x. Its minimality follows directly from the construction. The uniqueness follows from the solidness. □

The submodule $\hat{W}$ constructed in Lemma 2 will be called the *α-homogeneous hat* of W. A $\mathfrak{G}$-module, V, is said to be *simple normal* if there are no non-trivial normal $\mathfrak{G}$-submodules in V.

Lemma 3. *Let V be a solid α-homogeneous $\mathfrak{G}$-module and let W be its simple normal submodule. Let $\hat{W}$ be the α-homogeneous hat of W. Then $\hat{W}(\mu)$ is simple normal for any $\mu \in \mathfrak{H}^*$.*

Proof. Since W is simple normal it follows that $W = W(\mu')$ for some $\mu' \in \mathfrak{H}^*$. Thus $W(\mu') = \psi(x(\mu', 0)) \circ \psi^{-1}(x(\mu, 0))(\hat{W}(\mu))$. Suppose that $\hat{W}(\mu)$ is not simple normal and contains a non-trivial normal submodule, say N. Then $\psi(x(\mu', 0)) \circ \psi^{-1}(x(\mu, 0))(N)$ is a non-trivial normal submodule in $W(\mu')$, which contradicts our assumptions. □

Lemma 4. *Let V be solid α-homogeneous and let $W = W(\mu)$, $\mu \in \mathfrak{H}^*$, be a normal submodule in V. Suppose that W has a composition series,*

$$W = W_0 \supset W_1 \supset \cdots \supset W_k = 0,$$

such that all simple quotients $W^i = W_i/W_{i+1}$, $0 \leqslant i \leqslant k$, are normal. Let $\hat{W}$ be the α-homogeneous hat of W. Then $\hat{W}$ has a filtration,

$$\hat{W} = \hat{W}_0 \supset \hat{W}_1 \supset \cdots \supset \hat{W}_k = 0,$$

such that each $\hat{W}_i$ is the α-homogeneous hat of W_i for all $0 \leqslant i \leqslant k$. Moreover, $\hat{W}^i = \hat{W}_i/\hat{W}_{i+1}$ is the α-homogeneous hat of W^i in $V/\hat{W}_{i+1}$ and $\hat{W}^i(\xi)$ is simple normal for all $\xi \in \mathfrak{H}^*$.

Proof. Follows from Lemma 3 and Lemma 1 by trivial induction in k. □

Lemma 5. *Suppose that W is simple normal. Then W contains the unique subquotient N such that $X_{-\alpha}$ acts injectively on N. Moreover, this subquotient is a submodule of W.*

Proof. As any simple subquotient of W on which X_{-a} acts injectively defines some normal subquotient of W, the first statement follows from the assumption that W is simple normal. The second statement follows from the bijectivity of $X_{-\alpha}$. □

Now we are ready to prove our main theorem.

Consider the module $M_T(\lambda, b)$. Clearly, it is normal and we can view it as a solid α-homogeneous module with respect to an arbitrary solid structure. Consider its normal submodule $M_\alpha(\lambda, b)$. One can see that $M_T(\lambda, b)$ is the α-homogeneous hat of $M_\alpha(\lambda, b)$. Let $N = (M_T(\lambda, b))(f(\lambda, b))$. By Lemma 4 any composition series of $M_\alpha(\lambda, b)$ leads to a filtration of N with simple normal subquotients. By Lemma 5 each simple normal subquotient of N has a unique simple submodule on which $X_{-\alpha}$ acts injectively. Clearly, this correspondence is a bijection between the set of all simple subquotients of $M_\alpha(\lambda, b)$ and all simple subquotients of $M(f(\lambda, b))$ on which $X_{-\alpha}$ acts injectively. The rest follows from the trivial observation that the module corresponding to $L_\alpha(\mu, d)$ is exactly $L(f(\mu, d))$. Theorem 1 is proved.

5 Acknowledgments

We thank Professor O.Mathieu for helpful discussions and Professor V.Futorny who informed us about the existence of [M]. The paper was completed during the visit of the second author to Bielefeld University as an Alexander von Humboldt fellow. The financial support of Humboldt foundation and the hospitality of Bielefeld University are greatfully acknowledged. We also thank the referee for helpful comments.

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