Parabolic decomposition for properly stratified algebras

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Abstract

We generalize the machinery of exact Borel subalgebras of quasi-hereditary algebras on properly stratified algebras.

1 Introduction

The representation theories of a block of category $\mathcal{O}$, [BGG], and of the degree r polynomial representations of the algebraic groups correspond to representation theories of quasi-hereditary algebras, as defined in [CPS1]. In these cases the so-called *standard modules* (Verma, resp. Weyl modules) arise via induction from Borel subalgebras. A natural question is whether this induction works at the finite-dimensional algebra level; that is, can we find a subalgebra of the quasi-hereditary algebra such that tensor induction of one-dimensional representations gives the standard modules. Such considerations led S. König to study *exact Borel subalgebras* in quasi-hereditary algebras ([K1, K2, K3]).

In recent years, the study of representations of complex Lie algebras has extended to the study of representations induced from simple (not necessarily finite-dimensional) modules over parabolic subalgebras (see, for example [Fu, FM1] and references therein) and the corresponding parabolic generalizations $\mathcal{O}(\mathcal{P},\Lambda)$ of $\mathcal{O}$ (see, for example [FKM1] and references therein). In particular, some of these categories are closely related to the recent Mathieu's classification of simple dense modules over simple finite-dimensional Lie algebras, [M]. The connection is especially transparent in the case of parabolic induction from an $sl(2,\mathbb{C})$-subalgebra, in which all objects of $\mathcal{O}(\mathcal{P},\Lambda)$ are weight modules with finite-dimensional weight spaces. The corresponding example is considered in Section 7.

Similar to what occurs in category $\mathcal{O}$, there is a block decomposition and the representation theory of each block corresponds to the representation theory of a finite-dimensional algebra. Here the algebras, in general, are no longer quasi-hereditary, but are often so-called *standardly stratified* algebras in the sense of [CPS2] (this was obtained in [FKM1, FKM2, FKM3]) or even *fully standardly stratified* or *properly stratified*, as defined in [ADL1, D2]. Such algebras have been intensively studied in [ADL1, ADL2, ADL3, AHLU, D1, D2]; see also the bibliographies therein. Being so closely related to the quasi-hereditary algebras,

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the properly stratified algebras presumably have many of the properties. We explore some of these in the current paper.

Our goal in this paper is to generalize the results of König on triangular decomposition (or Cartan decomposition) in quasi-hereditary algebras ([K1, K2, K3]). Central to the study of such decomposition of quasi-hereditary algebras is a semisimple algebra contained in the exact Borel and Δ-subalgebras. In the current case, this must be replaced by a so-called quasi-local algebra which is a direct sum of local algebras; that is, of algebras with a unique simple module. After presenting some preliminaries in Section 2, we study in Section 3 quasi-directed algebras (algebras whose simple modules can be ordered so that $\mathrm{Ext}(L_i, L_j) \neq 0$ implies $L_i \geqslant L_j$) which are projective as left and right modules over their maximal quasi-local algebras. It turns out that this is a convenient condition for the exact Borel and Δ-subalgebras. The next section establishes that the notions of exact Borel and Δ-subalgebras are dual to each other. Section 6 proves the fact that parabolic decomposition, that is, the existence of an exact Borel subalgebra B and a Δ-subalgebra C such that $A \simeq C \otimes_S B$ as $C - B$ bimodule and S is the maximal quasi-local subalgebra in both B and C, is necessarily properly stratified and gives a partial converse. In Section 7, we give a method for constructing new properly stratified algebras from old ones. We finish, in Section 7, with an example from parabolic induction in complex Lie algebras.

2 Preliminaries

We let A be a finite dimensional algebra over $\mathbf{k}$, an algebraically closed field. When we want to make clear over which algebra we are taking a module we will give an indication via subscripts. Let $J = AeA$ be a two-sided ideal in A, generated by a primitive idempotent e. J is called *left properly* (resp. *properly*) *stratifying* if it is a projective left (resp. left and right) A-module. If we can order the equivalence classes $e_1, \dots, e_n$ of primitive idempotents of A such that for each l the idempotent e_l generates a left properly stratifying (resp. properly stratifying) ideal in the quotient algebra $A/ < e_n, \dots, e_{l+1} >$, then A is called *left properly stratified (resp. properly stratified)* (compare with [FKM1, Section 5]). We will indicate the (left) properly stratified structure on an algebra A by the pair $(A, \leqslant)$ with $\leqslant$ the above order on the idempotents. Left properly stratified algebras are stratified algebras in the sense of Cline, Parshall and Scott ([CPS2]) and have been studied by many authors, e.g. [CPS2, ADL1, ADL2, ADL3, AHLU, D1, D2]. Properly stratified algebras have been introduced and studied by V. Dlab in [D2]. The paper [D2] appeared after the first draft of the present paper was completed and we decided to change the original notation to avoid multiple names for the same objects (in the first draft the algebras above were called projectively stratified). Our definition is different from that given in [D2], but it is a direct corollary of the main theorem in [D2] that these two definitions are equivalent.

Remark 1. The referee noted the following interesting feature of the definition above: if in the definition of properly stratifying ideal we ask J instead to be projective as (A, A)-bimodule, it will automatically give us that J is a heredity ideal, i.e. a two-sided idempotent ideal, which is projective as a right A-module and satisfies $JNJ = 0$, where N is the

Jacobson radical of A. His arguments go as follows: If J is projective as (A, A)-bimodule, then the surjection $Ae \otimes_{\mathbf{k}} eA \to J$, defined by $(ae, ea') \mapsto aea'$ is split as a map of (A, A)-bimodules. Let $\varphi : J \to Ae \otimes eA$ be a splitting and observe that the image of $eAe \subset J$ under φ is contained in $e(Ae \otimes eA)e = eAe \otimes eAe$ (since φ is a map of (A, A)-bimodules). Thus the multiplication map $eAe \otimes eAe \to eAe$ is split as a map of bimodules and eAe is separable. In particular it is semi-simple. Now $JNJ = AeNeA$, and since eNe is a nilpotent ideal in eAe it follows that $JNJ = 0$ and J is a heredity ideal.

In particular, this clarifies the connection between the left properly stratified algebras and quasi-hereditary algebras. The change from two-sided modules to bimodules, while apparently small, is what distinguishes between the two types of algebras.

Two lemmas follow immediately from this definition.

Lemma 1. *Let $(A, \leqslant)$ be a properly stratified algebra. Then $(A^{op}, \leqslant)$ is properly stratified (with the same order on the isoclasses of primitive idempotents).*

Lemma 2. *Let A be an algebra with an order $\leqslant$ on the isomorphism classes of primitive idempotents. Then $(A, \leqslant)$ is properly stratified if and only if both $(A, \leqslant)$ and $(A^{op}, \leqslant)$ are left properly stratified.*

It is easy to see that the classes of left properly stratified algebras and properly stratified algebras are different. As an example of a left properly stratified algebra which is not properly stratified, consider the algebra of the quiver Γ with two vertices $\{a, b\}$, three arrows $\{\alpha : a \to b,\ \beta : b \to a,\ \gamma : a \to a\}$ and relations $\gamma^2 = 0$, $\gamma\alpha = 0$, $\beta\gamma = 0$, and $\beta\alpha\beta = 0$.

We also note that a left properly stratified algebra is a stratifying endomorphism algebra in the sense of Cline, Parshall and Scott [CPS2] and all quasi-hereditary algebras are properly stratified. Also a left properly stratified algebra is quasi-hereditary if and only if it has finite global dimension [ADL1, ADL2, AHLU, CPS2].

Remark 2. The referee also noted a confusion in the comparison of the definition of left properly stratified algebras and that of standardly stratified algebras given in [CPS2], which seems to persist throughout many of the references, in particular, in [ADL1, AHLU, FKM1, FKM2, FKM3]. The confusion consists in the assumption that, in the definition given in [CPS2], one can always use a complete set of representatives of all isoclasses of primitive idempotents in A. This is not possible in the general case of the definition in [CPS2], because this one uses the notion of quasi-partial order and not that of partial order on the set of representatives of idempotents. For example, in the sense of [CPS2], any finite dimensional algebra A has a standard stratification of length 1 with $A = J_1$ as the stratifying ideal. On the other hand, left properly stratified algebras are standardly stratified.

Let $(A, \leqslant)$ be a (left) properly stratified algebra. In what follows we will denote by $L(\lambda)$ the simple A-module which corresponds to e_λ, and will call λ a *weight*. We will also denote by $P(\lambda)$ (resp. $I(\lambda)$) the corresponding projective cover (resp. injective envelope).

Following [CPS2], for a simple A-module L corresponding to the idempotent e_λ we define the *standard* module $\Delta(\lambda)$ as $A/_{\langle e_n,\dots,e_{\lambda+1}\rangle}e_\lambda$ and *costandard* module $\nabla(\lambda)$ as the largest submodule of $I(\lambda)$ having factors $L(\mu)$ with $\mu \leqslant \lambda$. We note that each projective has a *standard flag*, i.e. a filtration whose quotients are standard modules.

Let $(B,\preceq)$ be a finite-dimensional algebra with $\preceq$ a partial order on the set of equivalence classes of simple modules. $(B,\preceq)$ is called *quasi-directed* if $\mathrm{Ext}^k_B(L,L') \neq 0$ for some k implies $L' \preceq L$. By a *quasi-local algebra* we will mean a direct sum of local algebras.

For a quasi-directed algebra B call an indecomposable module M *local* if all its simple composition factors are isomorphic. Call it *projectively local* if it is projective over the maximal quasi-local subalgebra of B (whose existence will be proved in Section 3). If M is a projectively local module, then the *weight* of M is the weight of its unique composition factor.

Let S be a quasi-local subalgebra of an algebra B. We say B is *S-diagonalizable* if B is projective as left and right S-module. A quasi-directed algebra diagonalizable over its maximal quasi-local subalgebra will be called *pyramidal*.

Definition 1. *Let $(A,\leqslant)$ be a properly stratified algebra and B and C subalgebras of A.*

1. *We will call B an* exact Borel subalgebra *of A if*
   - *there is a one-to-one correspondence between the simples of B and the simples of A;*
   - *$(B,\succeq)$ is pyramidal with the opposite order induced from the simples of A;*
   - *the tensor induction functor $A\otimes_B \_$ is exact;*
   - *$A\otimes_B \_$ sends the projectively local B-module V to the standard A-module of the same weight;*
   - *$[A\otimes_B L_B(i) : L_A(i)] = 1$.*
2. *We will call C a* Δ-subalgebra *of A if*
   - *there is a one-to-one correspondence between the simples of C and the simples of A;*
   - *$(C,\preceq)$ is pyramidal with the order induced from the simples of A;*
   - *for each weight i the indecomposable projective Ae_i occurs exactly once in the decomposition of the projective A-module $A\otimes_C Ce_i$;*
   - *fixing epimorphisms $\kappa(i) : A\otimes_C Ce_i \to \Delta(i)$, one has isomorphisms $\kappa(i)|_{Ce_i} : 1\otimes Ce_i \xrightarrow{\cong} \Delta(i)$.*

The importance of the last condition on exact Borel subalgebras can be seen from the example $A = \mathbf{k}[x]/(x^4)$ and $B = \mathbf{k}[x^2]/(x^4) \subset A$. In this example the unique standard objects are the algebras and induction clearly doubles the length of modules.

In the remainder of the paper, set $i \preceq j$ if and only if $i \leqslant j$. The first symbol will indicate quasi-directedness and the second will indicate a properly stratified structure.

Definition 2. *Let A be an algebra and $\leqslant$ be a total order on the set of equivalence classes of simple A-modules. Let B and C be subalgebras of A such that $B \cap C = S$ is a quasi-local subalgebra of A containing at least one representative from each isomorphism class of primitive idempotent, maximal in both B and C. Assume that $(B, \succeq)$ and $(C, \preceq)$ are pyramidal. Call (B, C) a* parabolic decomposition *of A provided that the multiplication in A induces isomorphisms $C \otimes_S B \simeq A$ as left C-modules and right B-modules.*

We have chosen this terminology to reflect the fact that the major example we know comes from tensor induction from a parabolic subalgebra of a Lie algebra.

A quasi-hereditary algebra $(A, \leqslant)$ can be characterized by the existence of standard modules $\Delta(i)$ having simple subquotients $L(k)$ with $k \leqslant i$ and $L(i)$ occurring once, and such that the projective $P(j)$ has a standard flag with sections $\Delta(i)$ with $j \leqslant i$ among which $\Delta(j)$ occurs exactly once. Left properly stratified algebras have the same module-theoretic characterization except that the multiplicity of $L(i)$ in $\Delta(i)$ may exceed one (see [CPS2, Section 2.2] for details).

Theorem 1. *$(A, \leqslant)$ is left properly stratified if and only if there exist modules $\Delta(i)$ such that*

(i) there is a surjection $\varphi_i : \Delta(i) \to L(i)$ whose kernel has composition factors $L(j)$, $j \leqslant i$.

(ii) $P(i)$ surjects onto $\Delta(i)$ and the kernel of this map has a standard flag with sections $\Delta(j)$, $j \nleqslant i$.

Proof. This theorem is a special case of [CPS2, Section 2.2]. $\square$

Corollary 1. *Let $(A, \leqslant)$ be an algebra. Then $(A, \leqslant)$ is quasi-directed if and only if $(A, \leqslant)$ is left properly stratified with projective standard modules.*

Proof. If $(A, \leqslant)$ is left properly stratified with projective standard modules then $(A, \leqslant)$ is quasi-directed by definition. Now assume that $(A, \leqslant)$ is quasi-directed. Define $\Delta(i) = P(i)$. Condition (ii) is trivial and condition (i) follows by directedness. $\square$

One can clearly obtain the notion of *right* properly stratified algebras by requiring right projectivity of the stratifying ideals. Then the dual version of Theorem 1 will look as follows.

Corollary 2. *$(A, \leqslant)$ is right properly stratified if and only if there exists a choice of co-standard modules $\nabla(i)$ such that*

(i) there is an injection $\varphi_i : L(i) \to \nabla(i)$ whose cokernel has composition factors $L(j)$, $j \leqslant i$.

(ii) $\nabla(i)$ injects into $I(i)$ and the cokernel of this map has a costandard flag with sections $\nabla(j)$, $j \nleqslant i$.

Corollary 3. *$(A, \leqslant)$ is properly stratified if and only if there exists a choice of standard modules $\Delta(i)$ and costandard modules $\nabla(i)$ satisfying the necessary conditions from Theorem 1 and Corollary 2.*

For properly stratified algebras there is an appropriate generalization of the classical Brauer-Humphreys reciprocities obtained in [ADL1, Theorem 2.5] for a more general situation. Here we present this result in the form which we will use later on and refer the reader to [ADL1, Theorem 2.5] for more details. As usual, $[M : \Delta(i)]$ (resp. $[M : \nabla(i)]$) denotes the number of occurrences of $\Delta(i)$ (resp. $\nabla(i)$) in a standard (resp. costandard) flag of M, should it exist. Similarly, for a simple L, $[M : L]$ means the corresponding composition multiplicity.

Theorem 2. *Let $(A, \leqslant)$ be a properly stratified algebra. Then for any pair of weights i and j we have*

$$\dim_{\mathbf{k}}(\mathrm{End}(\Delta(j)))[P(i) : \Delta(j)] = [\nabla(j) : L(i)],$$
$$\dim_{\mathbf{k}}(\mathrm{End}(\nabla(j)))[I(i) : \nabla(j)] = [\Delta(j) : L(i)].$$

Proof. Although this is a partial case of [ADL1, Theorem 2.5], it is formulated in a slightly different form, so we give a short proof.

By duality it is sufficient to prove the first. By induction it is sufficient to prove it for maximal $j = n$. In this case $P(i) = Ae_i$, $P(n) = \Delta(n) = Ae_n$ and $I(n) = \nabla(n)$. Set $l = [P(i) : \Delta(n)]$. It is easy to see that the last is equal to

$$l = \frac{\dim_{\mathbf{k}} \mathrm{Hom}_A(P(n), P(i))}{\dim_{\mathbf{k}} \mathrm{End}_A(P(n))};$$

that is $Ae_i \cap Ae_nA = (Ae_n)^l$. So, it remains to show that $\dim_{\mathbf{k}} \mathrm{Hom}_A(P(n), P(i)) = [I(n) : L(i)]$. Passing to the opposite algebra we have

$$\begin{aligned}\dim_{\mathbf{k}} \mathrm{Hom}_A(P(n), P(i)) &= \dim_{\mathbf{k}}(e_nAe_i) = \dim_{\mathbf{k}}(e_iA^{op}e_n)\\ &= \dim_{\mathbf{k}} \mathrm{Hom}_A(A^{op}e_i, A^{op}e_n) = [A^{op}e_n : L(i)] = [I(n) : L(i)].\end{aligned}$$

□

In particular, one has the following corollary:

Corollary 4. *Assume that A is properly stratified and has a duality (i.e. a contravariant exact equivalence on the category of A-modules which preserves simples). Then*

$$\dim_{\mathbf{k}}(\mathrm{End}(\Delta(j)))[P(i) : \Delta(j)] = [\Delta(j) : L(i)].$$

3 Pyramidal algebras as properly stratified algebras

In the theory of quasi-hereditary algebras we see that an algebra, $(A, \leqslant)$, is *directed* (that is $\mathrm{Ext}(L, L') \neq 0$ implies $L \nleqslant L'$) if and only if it is quasi-hereditary with projective standard modules. We have already seen (Corollary 1) that the same relationship holds between quasi-directed and left properly stratified algebras. In this section, we examine pyramidal algebras; in fact, we prove that all pyramidal algebras are properly stratified.

Lemma 3. *Let $(B, \preceq)$ be a quasi-directed algebra. Then the maximal quasi-local subalgebra S of B is isomorphic to $\oplus_i e_i B e_i$.*

Proof. Clearly, $S' = \oplus_i e_i B e_i$ is a subalgebra of B. Because of the directedness of B no endomorphisms of Be_i can factor through a non-isomorphic projective, so each $e_i B e_i$ is local, and hence S' is quasi-local. Now let P be an indecomposable summand of S. Then P equals Se_i is local and so $e_i S e_i = Se_i$ is a subalgebra of $e_i B e_i$. Hence $S \subset S'$. □

Lemma 4. *Let $(B, \preceq)$ be quasi-directed. Then the projectively local module $K(i)$ is isomorphic to Be_i/N, where N is the trace of all $P(j)$ with $j \preceq i$.*

Proof. Because of the directedness of B we have

$$Be_i = e_i B e_i \oplus \sum_{j<i} e_j B e_i$$

as a vectorspace. For each element of $e_j B e_i$ there is a map for $P(j)$ to $P(i)$ covering it, so $\sum_{j<i} e_j B e_i \subset N$. But $[Be_i : L(i)] = [e_i B e_i : L(i)]$ by directedness of B. This completes the proof. □

Proposition 1. *Let $(B, \preceq)$ be a pyramidal algebra. Then $(B, \leqslant)$ is properly stratified.*

Proof. $(B, \leqslant)$ is left properly stratified by Corollary 1. Now consider B^{op}. Then the algebra $(B^{op}, \succeq)$ is quasi-directed. This implies that a left projective B^{op}-module $P(i)$ has only $L(j)$ with $i \leqslant j$ as composition subquotients. Since B^{op} is pyramidal each $P(i)$ has a projectively local flag whose subquotients satisfy the same order condition. Let S be the maximal quasi-local subalgebra of B. For B^{op} choose $\Delta(i) = e_i S^{op} e_i$ as the standard objects. By Lemma 4 these are the projectively local modules. Then these standards clearly satisfy conditions (i) and (ii) of Theorem 1, and so $(B^{op}, \leqslant)$ is left properly stratified. And now, by Lemma 2, $(B, \leqslant)$ is properly stratified. □

Thus, a pyramidal algebra $(B, \preceq)$ has both $(B, \leqslant)$ and $(B, \geqslant)$ properly stratified structures. To finish this section we give necessary and sufficient conditions for a quasi-directed algebra to be properly stratified. We begin with the following lemma.

Lemma 5. *Let e be a primitive idempotent in a quasi-directed algebra $(B, \preceq)$ and X an eAe-module. Then $X \otimes_{eBe} eB$ is a right B-projective if and only if a X is right eBe-projective.*

Proof. If X_{eBe} is projective, then X is free over eBe and hence $X \otimes_{eBe} eB$ is right B-projective. On the other hand, suppose $X \otimes_{eBe} eB$ is right B-projective. Let L be the maximal local top of eB. X has top $(\hat{L})^n$, where $\hat{L}$ is the unique simple eBe-module. We have an exact sequence

$$0 \to T \xrightarrow{\varphi} (eBe)^n \to X \to 0.$$

Inducing to B we have

$$T \otimes_{eBe} eB \xrightarrow{\beta} (eBe)^n \otimes_{eBe} eB \xrightarrow{\alpha} X \otimes_{eBe} eB \to 0.$$

Since X surjects on $(\hat{L})^n$, we have the following chain of surjections

$$X \otimes_{eBe} eB \to (\hat{L} \otimes_{eBe} eB)^n \to (\hat{L} \otimes_{eBe} L)^n \to L^n.$$

$X \otimes_{eBe} eB$ is projective and must contain, as a direct summand, the projective cover $(eB)^n$ of L^n. This implies that α is an isomorphism, $\beta = 0$ and, last, $\beta \cdot e = 0$. But $\beta \cdot e = \varphi : T \otimes_{eBe} eBe \to (eBe)^n \otimes_{eBe} eBe$ is non-zero if X is not projective. This contradiction proves the lemma. $\square$

And this lemma allows us to give the following characterization of which quasi-directed algebras are properly stratified.

Proposition 2. *Let $(B, \preceq)$ be quasi-directed. Then the following conditions are equivalent.*

(i) B is properly stratified,

(ii) the maximal quasi-local subalgebra S of B is an exact Borel subalgebra,

(iii) Be is eBe-projective for every primitive idempotent e.

Proof. ($(i) \Leftrightarrow (iii)$) By [DR, Statement 7] we have $Be \otimes_{eSe} eB \simeq BeB$, where e is the maximal primitive idempotent. So BeB is right B-projective if and only if Be is right eSe-projective (Lemma 5).

($(ii) \Rightarrow (iii)$) Since $B \otimes_S \_$ is exact, B is a right flat, hence right projective ([F, 11.31]), S-module. Then Be is a projective S module and hence a projective eSe-module.

($(iii) \Rightarrow (ii)$) We have $B \otimes_S \_ = \oplus_i Be_i \otimes_{e_iSe_i} \_$ and hence exact since each Be_i is e_iSe_i-projective. Now we just need to prove that this functor carries $\Delta_S(i) = e_iSe_i$ to $\Delta_B(i)$. Set $e = e_i$. We have $B \otimes_S eSe = B \otimes_{eSe} eSe = Be \otimes_{eSe} eSe = Be = \Delta_B(i)$. $\square$

4 Duality between exact Borel and Δ-subalgebras

In this section we explore the left-right symmetry of properly stratified algebras and their exact Borel and Δ-subalgebras. For the remainder of the paper we will assume that the properly stratified structures on exact Borel and Δ-subalgebras are given by the same order on the idempotents of the algebra. With this properly stratified structure the fourth condition of an exact Borel subalgebra can be rephrased as: tensor induction carries standard modules to standard modules.

Lemma 6. *Let $(A, \leqslant)$ be a properly stratified algebra. Then*

$$\mathrm{Ext}_A^k(\Delta(i), \nabla(j)) = 0$$

unless $k = 0$ and $i = j$.

Proof. Assume $k > 0$. Let m be the maximum of i and j. Put $e = \sum_{s=m+1}^{n} e_s$ and put $A' = A/AeA$. Over this algebra $\Delta(m)$ is projective and $\nabla(m)$ injective. Thus we have for $k \neq 0$ $\mathrm{Ext}_A^k(\Delta(i), \nabla(j)) = \mathrm{Ext}_{A'}^k(\Delta(i), \nabla(j)) = 0$ ([CPS2, 2.1.2]). If $k = 0$, $i \neq j$ then the image of any map $\varphi : \Delta(i) \to \nabla(j)$ must have $L(i)$ as top and other composition factors $L(j)$, $j \leqslant i$, since it is a quotient of $\Delta(i)$. If this map is non-zero then $L(i)$ is a composition factor of $\nabla(j)$; that is, $i < j$. But then every submodule of $\nabla(j)$ has composition factor $L(j)$, which forces $j < i$. □

Let $(B, \succeq)$ be an exact Borel subalgebra of a properly stratified algebra $(A, \leqslant)$. If A is quasi-hereditary then the standard objects are induced from simple B-modules. In general, this no longer holds; however, these modules $\tilde{\Delta}_A(i) = A \otimes_B L_B(i)$, which we will call *proper standard modules*, continue to play an important role (see also [ADL1, ADL2, ADL3, AHLU, D1, D2]). In particular, $\Delta(i)$ has a $\tilde{\Delta}(i)$-flag.

Lemma 7. *Let $(A, \leqslant)$ be a properly stratified algebra with an exact Borel subalgebra B. Then*

$$\mathrm{Ext}_A^k(\tilde{\Delta}(i), \nabla(j)) = 0,$$

unless $k = 0$ and $i = j$.

Proof. For $k = 0$, $\tilde{\Delta}(i)$ is an image of $\Delta(i)$ and the statement follows from Lemma 6. Consider $k \neq 0$. Let l be the maximal of i and j. Then $\tilde{\Delta}(i)$ and $\nabla(j)$ are modules over $A/\langle e_{k+1}, \ldots, e_n \rangle$ and so (by [CPS2, 2.1.2]) we may assume that $l = n$. If $j = n$, then $\nabla(j)$ is injective and the current lemma clearly holds. So now assume $i = n$ and $j < i$ and consider the exact sequence

$$0 \to N \to \Delta(i) \to \tilde{\Delta}(i) \to 0.$$

Apply $\mathrm{Hom}_A(\_, \nabla(j))$ and pass to the long exact sequence. We get $\mathrm{Ext}_A^1(\tilde{\Delta}(i), \nabla(j)) = 0$ and $\mathrm{Ext}_A^{l+1}(\tilde{\Delta}(i), \nabla(j)) \simeq \mathrm{Ext}_A^l(N, \nabla(j))$. The lemma now follows from standard dimension shift arguments. □

Lemma 8. *Let $(B, \succeq)$ be an exact Borel subalgebra of a properly stratified algebra $(A, \leqslant)$. Then for all weights,* $\dim \mathrm{End}_A(\Delta_A(\lambda)) = \dim \mathrm{End}_B(\Delta_B(\lambda))$.

Proof. We prove the lemma by induction. Let n be the maximal weight. We have

$$\dim \mathrm{End}_X(\Delta_X(n)) = \dim \mathrm{Hom}_X(P_X(n), \Delta_X(n)) = [\Delta_X(n) : L_X(n)]$$

for both $X = A$ or $X = B$. By the last condition for an exact Borel subalgebra we have $[\Delta_A(n) : L_A(n)] = [\Delta_B(n) : L_B(n)]$, and the statement follows for the maximal weight. Induction is clear. □

Proposition 3. *Let $(A, \leqslant)$ be a properly stratified algebra and $(B, \succeq)$ a pyramidal subalgebra of A with the same poset of isoclasses of primitive idempotents. Then B is an exact Borel subalgebra of A if and only if for each weight i restriction from A to B induces an isomorphism $\nabla_A(i) \simeq \nabla_B(i)$ as B-modules.*

Proof. Assume that B is an exact Borel subalgebra of A. We compare the dimensions of $\nabla_A(i)$ and $\nabla_B(i)$. Using Theorem 2 we have

$$\begin{aligned}
\dim_{\mathbf{k}} \nabla_X(i) &= \sum_j \dim_{\mathbf{k}} L_X(j)[\nabla_X(i) : L_X(j)] && (1)\\
&= \sum_j \dim \operatorname{End}(\Delta_X(i)) \dim_{\mathbf{k}} L_X(j)[P_X(j) : \Delta_X(i)] && (2)\\
&= \dim \operatorname{End}(\Delta_X(i))[X : \Delta_X(i)] && (3)
\end{aligned}$$

for both $X = A$ and $X = B$. By the exactness of induction, $[A : \Delta_A(i)] = [B : \Delta_B(i)]$. By Lemma 8, $\dim \operatorname{End}(\Delta_A(i)) = \dim \operatorname{End}(\Delta_B(i))$ and so $\dim_{\mathbf{k}} \nabla_A(i) = \dim_{\mathbf{k}} \nabla_B(i)$.

From Lemma 7 it follows that the functor $\operatorname{Hom}_A(\_, \nabla_A(j))$ is exact on A-modules having a proper standard flag. Thus $\operatorname{Hom}_B(\_, \nabla_A(j))$ is exact on the category of B-modules. So $\nabla_A(j)$ containing $\nabla_B(j)$ is an injective B-module. The previous dimension count says that they are, in fact, equal.

Now assume that for each weight i, restriction from A to B induces an isomorphism $\nabla_A(i) \simeq \nabla_B(i)$ as B-modules. Since $(B, \succeq)$ is pyramidal, $(B, \leqslant)$ is properly stratified by Lemma 1 with injective costandard modules (Corollary 1 and Corollary 3). We want to prove that A is right projective over B implying $A \otimes_B \_$ is exact. The right standard modules for A (and both right standard and projective over B) are $\nabla_A(i)^*$, so, as a right projective A-module, A has a $\nabla_A(i)^*$-flag, the last being a direct sum decomposition over B.

We are finished when we show that $A \otimes_B \_$ sends projectively local B-modules to standard A-modules. Let $K(i)$ be a projectively local B-module corresponding to the weight i. First we show $\dim_{\mathbf{k}}(A \otimes_B K(i)) = \dim_{\mathbf{k}} \Delta_A(i)$. We have $d := \dim_{\mathbf{k}}(A \otimes_B K(i)) = \dim_{\mathbf{k}}(\operatorname{Hom}_B(K(i), A^*))^*$. Since A is a right projective B-module, A^* is left injective and as such decomposes into a direct sum of injective B-modules, $\nabla_B(i)$. On each summand we have

$$\dim_{\mathbf{k}} \operatorname{Hom}_B(K(i), \nabla_B(j)) = \begin{cases} 0, & i \neq j; \\ \dim_{\mathbf{k}} \operatorname{End}_B(K(i)), & i = j; \end{cases}$$

this follows from the fact that $\nabla_B(i) = (e_i B)^*$ implies $[\nabla_B(i) : L_B(i)] = [K(i) : L_B(i)]$. Now $\dim \operatorname{End}_B(K(i)) = [K(i) : L(i)]$ because $K(i)$ is projective in the category of B-modules filtered by $L_B(i)$. Thus d equals $\dim_{\mathbf{k}} \operatorname{End}_B(K(i)) \cdot [A^* : \nabla_B(i)] = \dim_{\mathbf{k}} \operatorname{End}_B(K(i)) \cdot [A^* : \nabla_A(i)]$. Further, $\dim_{\mathbf{k}} \operatorname{End}_A(\nabla_A(i)) = \dim_{\mathbf{k}} \operatorname{End}_{A^{op}}(\Delta_{A^{op}}(i)) = \dim_{\mathbf{k}} \operatorname{End}_{B^{op}}(K(i)^*) = \dim_{\mathbf{k}} \operatorname{End}_B(K(i))$ by Lemma 8. Applying Brauer-Humphreys reciprocity we get

$$
\begin{aligned}
d &= \dim_{\mathbf{k}} \operatorname{End}_B(K(i)) \cdot [A^* : \nabla_A(i)] = & (4)\\
&= \sum_j \dim_{\mathbf{k}} \operatorname{End}_A(\nabla_A(i)) \cdot [I_A(j) : \nabla_A(i)] \cdot \operatorname{mult}_{A^*}(I_A(j)) & (5)\\
&= \sum_j [\Delta_A(i) : L_A(j)] \cdot \operatorname{mult}_A(P_A(j)) = \dim_{\mathbf{k}}(\Delta_A(i)). & (6)
\end{aligned}
$$

From the quasi-directedness of B and adjunction we have that $A \otimes_B P_B(i)$ has $P_A(i)$ as a direct summand exactly once and all other direct summands (if any) are of the form $P_A(j)$, $j > i$. So, for the largest weight we have $\Delta_A(n) = P_A(n) = A \otimes_B P_B(n) = A \otimes_B K(n)$. Now we proceed by induction. We have an exact sequence:

$$0 \to V \to P_B(i) \to K(i) \to 0$$

with V filtered by $K(j)$, $j > i$, because $\Delta_B(j) = K(j)$. By exactness of $A \otimes_B \_$ we obtain

$$0 \to A \otimes_B V \to A \otimes_B P_B(i) \to A \otimes_B K(i) \to 0,$$

and by the inductive hypothesis, $A \otimes_B V$ is filtered by $\Delta_A(j)$ with $j > i$. Now, since $P_A(i)$ occurs as a summand of $A \otimes_B P_B(i)$, there is a surjection of $A \otimes_B P_B(i)$ onto $\Delta_A(i)$ and the kernel V' is the the sum of the images of all possible maps from $P_A(j)$ for $j > i$. Hence $V \subset V'$ and so $A \otimes_B K(i)$ surjects onto $\Delta_A(i)$ and the isomorphism follows from the dimension count. □

Corollary 5. *B is an exact Borel subalgebra of a properly stratified algebra $(A, \leqslant)$ if and only if B^{op} is a Δ-subalgebra of $(A^{op}, \leqslant)$.*

Proof. Follows from Proposition 3, its proof, the duality of the conditions for a Δ-subalgebra, and the equivalent conditions for an exact Borel subalgebra. □

5 Properly stratified structure of algebras with parabolic decomposition

In this section we prove a theorem relating parabolic decomposition to properly stratified algebras. It generalizes the corresponding result for quasi-hereditary algebras [K2, Theorem 4.1]. The proof closely follows the ideas of the proof there.

Theorem 3. *Let A be a finite-dimensional algebra and $\leqslant$ be a total order on the set of isomorphism classes of simple A-modules. Assume that $(B, \succeq)$ and $(C, \preceq)$ are pyramidal basic subalgebras whose intersection $B \cap C = S$ is the maximal quasi-local subalgebra of both B and C. The following statements are equivalent.*

(i) The algebra A is properly stratified with an exact Borel subalgebra B and a Δ-subalgebra C.

(ii) (B, C) *is a parabolic decomposition of* A.

Proof. Assume we have listed the idempotents in A (and hence in B and C) with respect to the natural total order.

($(i) \Rightarrow (ii)$) We proceed by induction on the number of direct summands in S (the number of weights). If S is local, then $A = B = C = S$ and we are done. Assume S is not local and $e = e_n$ is the maximal primitive idempotent in A. Since A is properly stratified, AeA is a properly stratifying ideal and hence is projective as left A-module. In particular, by [DR, Statement 7], we have that the multiplication in A induces a bijection

$$Ae \otimes_{eAe} eA \to AeA.$$

We also have $eAe = eSe$, since e is the maximal primitive idempotent. We have the identifications

$$\Delta_A(n) \simeq \Delta_C(n) \simeq Ce$$

by the definition of Δ-subalgebra and Corollary 1, and

$$\nabla_A(n)^* \simeq \nabla_B(n)^* \simeq eB$$

by dual arguments (Corollary 5). We get a bijection

$$Ce \otimes_{eSe} eB \simeq Ae \otimes_{eAe} eA \simeq AeA,$$

compatible with left C and right B multiplication. Continuing by induction (see arguments in [K2, Theorem 4.1]) we see that the $\mathbf{k}$-dimensions of both sides of

$$C \otimes_S B \to A$$

are equal and we are done.

($(ii) \Rightarrow (i)$) Let $e = e_n$ be the maximal primitive idempotent. We wish to show that AeA is properly stratifying. Since B and C are quasi-directed, we have $eC = eCe = eSe$ and $Be = eBe = eSe$ are projectively local modules. First we show that $eSe = eAe$ (this says $\mathrm{End}_A(\Delta_A(n)) \simeq \mathrm{End}_C(\Delta_C(n))$). We have

$$eAe \simeq eC \otimes_S Be \simeq eSe \otimes_S eSe \simeq eSe \otimes_{eSe} eSe \simeq eSe.$$

Analogously,

$$Ae \simeq C \otimes_S Be \simeq C \otimes_S eSe \simeq C \otimes_{eSe} eSe \simeq Ce$$

and

$$eA \simeq eC \otimes_S B \simeq eSe \otimes_S B \simeq eSe \otimes_{eSe} B \simeq eB.$$

Now, since B and C are pyramidal, they are projective over S both as left and right modules. So $Ae = Ce$ (resp. $eB = eA$) is a right (resp. left) projective eSe-module. Thus they are free over eSe and so $Ae \otimes_{eSe} eA$ is projective as left and right A-module. The theorem now follows from standard induction. $\square$

6 Construction of algebras with given parabolic decomposition

In this section we give a general construction of a properly stratified algebra having given exact Borel and Δ-subalgebras. The central theorem allows us to construct a properly stratified algebra as an extension of a properly stratified algebra by a pyramidal algebra. If the properly stratified algebra has a parabolic decomposition, then the extension will as well.

To state the theorem we assume the following set-up: If we are given a semi-local algebra S an *S-algebra* will mean an algebra $T_S(M)/I$, where M is an S-bimodule, finite-dimensional over $\mathbf{k}$. Let $(A, \leqslant)$ be a basic properly stratified algebra with an exact Borel subalgebra B. Let $(D, \preceq)$ be a basic pyramidal algebra. Assume that B and D have isomorphic maximal quasi-local subalgebras S (in particular, they have the same set of idempotents). To fix notation, then $A \simeq T_S(M_A)/I_A$, $B \simeq T_S(M_B)/I_B$ and $D \simeq T_S(M_D)/I_D$. Let $A' = T_S(M_A \oplus M_D)/(I_A + I_D + \langle a \otimes_S d \,|\, a \in M_A, d \in M_D\rangle)$.

Theorem 4. *Let A, B, D and A' be defined as above. A' is properly stratified and isomorphic to $D \otimes_S A$ as left D-module and right A-module. B is an exact Borel subalgebra of A' via the embedding $b \mapsto 1 \otimes b$. If $C \simeq T_S(M_C)$ is a Δ-subalgebra of A containing S, then A' has a Δ-subalgebra $C' \simeq T_S(M_C \oplus M_D)/(I_C + I_D + \langle c \otimes_S d \,|\, c \in M_C, d \in M_D\rangle)$. Last, $C' \simeq D \otimes_S C$ as left D-module and right C-module.*

Proof. By construction, we have appropriate module isomorphisms: $A' \simeq D \otimes_S A$ (and when relevant $C' \simeq D \otimes_S C$). Let e be the maximal primitive idempotent. Then $e = e \otimes_S e$ $(= 1 \otimes_S e)$. So, $J = A'eA' = D \otimes_S C \cdot e \otimes_S e \cdot D \otimes_S C = D \otimes_S AeA$. Since D (resp. AeA) is a two-sided projective D (resp. A)-module, J is a two-sided projective A'-module and A' is properly stratified by induction.

We prove B is an exact Borel subalgebra of A'. To begin, $D \otimes_S A \otimes_B \_$ is exact, since $A \otimes_B \_$ is exact and D is pyramidal and hence flat over S. By induction, $D \otimes_S A \otimes_B \_$ sends standard B-modules to standard A'-modules. Indeed, let e be the maximal primitive idempotent. $D \otimes_S A \otimes_B Be \simeq D \otimes_S Ae \simeq (D \otimes_S A)(1 \otimes_S e)$, which is A'-standard. It remains to show that $[A' \otimes_B L_B(i) : L_{A'}(i)] = 1$. Now $A' \otimes_B L_B(i) = D \otimes_S A \otimes_B L_B(i)$. Since B is an exact Borel subalgebra of A, we have $[A \otimes_B L_B(i) : L_A(i)] = 1$ and $[A \otimes_B L_B(i) : L_A(j)] \neq 0$ implies $j \leqslant i$. We are done, if we show $[D \otimes_S L_A(j) : L_{A'}(j)] = 1$ and $[D \otimes_S L_A(j) : L_{A'}(k)] \neq 0$ implies $k \leqslant j$, for this would clearly imply that $[A' \otimes_B L_B(i) : L_{A'}(i)] = 1$. But $D \otimes_S L_A(j) \simeq \sum_{k \leqslant j} e_k D e_j \otimes_S L_A(j)$ and $[\sum_{k \leqslant j} e_k D e_j \otimes_S L_A(j) : L_S(m)] \neq 0$, $m \leqslant j$ and so $[D \otimes_S L_A(j) : L_{A'}(k)] \neq 0$ implies $k \leqslant j$. Further, $[\sum_{k \leqslant j} e_k D e_j \otimes_S L_A(j) : L_S(j)] = 1$ implies $[D \otimes_S L_A(j) : L_{A'}(j)] = 1$. The remaining statements follow arguments already seen. □

Corollary 6. *When $C = A$ we have that $A' \simeq C \otimes_S B$ is a parabolic decomposition.*

7 Parabolic decomposition of properly stratified algebras attached to blocks of $\mathcal{O}(\mathcal{P},\Lambda)$

We have mentioned that our motivation stems from the representation theory of complex Lie algebras and algebraic groups. In this section we give a parabolic decomposition for algebras arising from generalizations of classical category $\mathcal{O}$. We will work with properly stratified algebras whose module category is equivalent to a block of the category $\mathcal{O}(\mathcal{P},\Lambda)$, studied in [FKM2]. Let us give an overview of the set-up.

Let $\mathfrak{A}$ denote the Lie algebra $sl(2,\mathbb{C})$ with a fixed root basis $e = X_\alpha$, $f = X_{-\alpha}$, $h = H_\alpha$, where α is the root of $\mathfrak{A}$. For any $\gamma \in \mathbb{C}$ and $\lambda \in \mathbb{C}/2\mathbb{Z}$ there is a unique weight $\mathfrak{A}$-module $V(\lambda,\gamma)$ satisfying the following conditions ([FM1]):

1. λ is the support of $V(\lambda,\gamma)$ and all weight spaces of $V(\lambda,\gamma)$ are one-dimensional,
2. γ is the unique eigenvalue of the Casimir operator, $\mathfrak{c} = (h+1)^2 + 4fe$, on $V(\lambda,\gamma)$,
3. $V(\lambda,\gamma)$ is *admissible*, i.e. f acts bijectively.

Let $\tilde{\Lambda} = \tilde{\Lambda}(V(\lambda,\gamma))$ denote the full subcategory of $\mathfrak{A}$-modules having as objects all admissible submodules and quotients of modules of the form $V(\lambda,\gamma)\otimes F$, as F varies over all finite-dimensional $\mathfrak{A}$-modules. It has been shown ([FKM2, Section 2]) that $\tilde{\Lambda}$ is an abelian category.

Now let $\mathfrak{G}$ be a complex simple finite-dimensional Lie algebra and $\mathcal{P}$ a parabolic subalgebra of $\mathfrak{G}$ such that $\mathcal{P}$ has Levi factor $\mathfrak{A}\oplus\mathfrak{H}_{\mathfrak{A}}$ ($\mathfrak{A}$ as above) and nilpotent radical $\mathfrak{N}$. Construct the full subcategory $\Lambda = \Lambda(V(\lambda,\gamma))$ of the category of $\tilde{\mathfrak{A}} = \mathfrak{A}\oplus\mathfrak{H}_{\mathfrak{A}}$-modules satisfying:

1. any $M \in \Lambda$ belongs to $\tilde{\Lambda}$, when viewed as an $\mathfrak{A}$-module,
2. any $M \in \Lambda$ is $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable,
3. for any $M \in \Lambda$ and any $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable finite-dimensional $\tilde{\mathfrak{A}}$-module F the module $M\otimes F$ decomposes into a direct sum of indecomposable modules from Λ.

Define $\mathcal{O}(\mathcal{P},\Lambda)$ to be the full subcategory of the category of $\mathfrak{G}$-modules whose objects are finitely generated and $\mathfrak{N}$-finite $\mathfrak{G}$-modules that decompose into a direct sum of modules from Λ when viewed as $\tilde{\mathfrak{A}}$-modules. By [FKM1, Section 4], $\mathcal{O}(\mathcal{P},\Lambda)$ has a block decomposition (with finitely many simples in each block) and, by [FKM1, Section 10], this decomposition can be chosen such that each block is equivalent to the module category over a left properly stratified finite-dimensional algebra. By [FKM1, Section 12], $\mathcal{O}(\mathcal{P},\Lambda)$ has a duality (see Corollary 4). Thus by Corollary 2, these algebras are also right properly stratified and hence properly stratified. The main result of this section is the following.

Theorem 5. *For every block of $\mathcal{O}(\mathcal{P},\Lambda)$ there is an algebra A, with parabolic decomposition, whose module category is equivalent to this block.*

We will prove this by explicit construction of A and its exact Borel and Δ-subalgebras. We require more terminology. For the rest of the section we fix a block $\mathcal{O}_i$ of $\mathcal{O}(\mathcal{P},\Lambda)$ assumed to have finitely many simples.

Let $\mathcal{Q}_+$ be the set of positive roots of $\mathfrak{G}$. $\mathcal{P}$ uniquely identifies a copy of $\mathfrak{A}$ in $\mathfrak{G}$. Assume that the root α of $\mathfrak{A}$ is contained in $\mathcal{Q}_+$. A weight λ of a weight $\mathfrak{G}$-module V will be called an *α-highest weight* provided that $\lambda+\beta$ is not a weight of V for any $\beta\in\mathcal{Q}_+\setminus\{\alpha\}$. For a weight λ and a weight $\mathfrak{G}$-module V set $V[\lambda]=\oplus_{k\in\mathbb{Z}}V_{\lambda+k\alpha}$, which is closed under the action of $\mathfrak{A}$.

It is known that indecomposable modules from Λ have the form $V(a,b)$ or $\tilde{V}(a,b)$, the second being a self-extension of $V(a,b)$ (see [FKM1, Section 10]). For simple $V\in\Lambda$ denote by $\hat{V}$ its projective cover (in Λ); this is either V itself or its self-extension. Given $M\in\mathcal{O}_i$, a weight λ and a $b\in\mathbb{C}$ set

$$M_{\lambda,b}=\{v\in M_\lambda\,|\text{ there exists }k\in\mathbb{N}\text{ such that }(\mathfrak{c}-b)^k v=0\}.$$

Since, as an $\mathfrak{A}'$-module, M decomposes into a direct sum of objects from Λ, one has that $M_{\lambda,b}=\{v\in M_\lambda\,|\,(\mathfrak{c}-b)^2v=0\}$. For a simple module $L\in\mathcal{O}_i$ denote its α-highest weight by μ_L. Then $L[\mu_L]\simeq V(a_L,b_L)$ for some a_L and b_L. Put $V_L=V(a_L,b_L)$.

Lemma 9. *Let L be a simple module in $\mathcal{O}_i$. There exists a projective module P^L such that $\mathrm{Hom}_{\mathfrak{G}}(P^L,M)\simeq M_{\mu_L,b_L}$ for any $M\in\mathcal{O}_i$.*

Proof. Set $\mu=\mu_L$, $a=a_L$ and $b=b_L$. We can pick $k\in\mathbb{N}$ big enough such that $\mathfrak{N}^kM_\mu=0$ for any $M\in\mathcal{O}_i$ and consider the $\mathfrak{G}$-module

$$\hat{P}^L=U(\mathfrak{G})\bigotimes_{U(\mathcal{P})}\Big((U(\mathfrak{N})/(U(\mathfrak{N})\mathfrak{N}^k))\otimes\hat{V}_L\Big).$$

Now we can take P^L to be the $\mathcal{O}_i$-projection of $\hat{P}^L$. That $\mathrm{Hom}_{\mathfrak{G}}(P^L,M)\simeq M_{\mu,b}$ for any $M\in\mathcal{O}_i$ is identical to the classical argument in [BGG, Theorem 2]. □

Since the α-highest weight of L is unique up to shifts by α, P^L is independent of the choice of this α-highest weight. Now we take

$$A=\mathrm{End}_{\mathfrak{G}}\left(\bigoplus_{L\text{ simple in }\mathcal{O}_i}P^L\right).$$

Clearly, the category of A-modules is equivalent to $\mathcal{O}_i$.

Let $\mathfrak{G}=\mathfrak{N}_-\oplus\mathfrak{H}\oplus\mathfrak{N}_+$ be the standard triangular decomposition of $\mathfrak{G}$ and $\mathfrak{N}^-$ be the image of $\mathfrak{N}$ under the Chevalley involution. Consider an $\mathfrak{A}'$-submodule $V^L=1\otimes1\otimes\hat{V}_L$ in P^L and set $M_L=U(\mathfrak{N}_-)V^L$. Let $\Delta(L)$ denote the standard module associated with L ([FKM1, Section 3]). We have $\Delta(L)\simeq U(\mathfrak{G})\otimes_{U(\mathcal{P})}\hat{V}_L$.

For any simple $L\in\mathcal{O}_i$ fix the canonical generator $p(L)=1\otimes1\otimes v$ of P^L, where v is a canonical generator of $\hat{V}_L$; then the map $\varphi\mapsto\varphi(p(L))$ is a $\mathbb{C}$-isomorphism between $\mathrm{Hom}_{\mathfrak{G}}(P^L,M)$ and M_{μ_L,b_L} for any $M\in\mathcal{O}_i$.

Lemma 10. *Any surjection $P^L \to \Delta(L)$ induces an $U(\mathfrak{N}_-)$-isomorphism $M_L \to \Delta(L)$.*

Proof. Let $\varphi : P^L \to \Delta(L)$ be a surjection. It carries $p(L)$ to a generator of $\Delta(L)$ and hence it induces an $X_{-\alpha}$-isomorphism $M_L[\mu_L] \to \Delta(L)[\mu_L]$. Now the statement follows from the facts that $U(\mathfrak{N}_-) = U(\mathfrak{N}^-) \otimes_{\mathbb{C}} U(X_{-\alpha})$ and both M_L and $\Delta(L)$ are $U(\mathfrak{N}^-)$-free ([FM2, Proposition 1]). $\square$

Lemma 11. *M_L is an $\mathfrak{A}'$-module.*

Proof. Follows from the construction of P^L and the definition of M_L. $\square$

Lemma 12. *Assume that $\varphi : P^{L_j} \to P^{L_k}$ is a homomorphism and $\varphi(p(L_j)) \in M_{L_k}$. Then $\varphi(M_{L_j}) \subset M_{L_k}$.*

Proof. By definition, $p(L_j)$ generates V^{L_j} as $\mathfrak{A}'$-module. Since φ is an $\mathfrak{A}'$-homomorphism, Lemma 11 says that $\varphi(V^{L_j}) \subset M^{L_k}$ and the statement follows from the fact that $M_{L_j} = U(\mathfrak{N}_-)V^{L_j}$ and the fact that M_{L_k} is stable under left $U(\mathfrak{N}_-)$-multiplication. $\square$

Proof of Theorem 5. First we prove the existence of a Δ-subalgebra in A. Denote by I an indexing poset of simple modules in $\mathcal{O}_i$. Put

$$C = \bigoplus_{j,k\in I} \{\varphi \in \mathrm{Hom}_{\mathfrak{G}}(P^{L_j}, P^{L_k}) \mid \varphi(p(L_j)) \in M_{L_k}\}.$$

$C \subset A$ is a vectorspace, which is a subalgebra by Lemma 12. By Lemma 10, C has trivial intersection with the kernel of the projection $A \to \oplus_{j\in I}\Delta_A(i)$. Clearly, C is quasi-directed and contains a maximal quasi-local subalgebra which is isomorphic to $\oplus_{j\in I}\mathrm{End}(\Delta(j))$. Now, we have to prove that the vectorspace Ce_j is large enough, i.e. $\dim_{\mathbb{C}}(Ce_j) = \dim_{\mathbb{C}}(\Delta_A(j))$. Let $t = \dim_{\mathbb{C}}(\Delta_A(j))$. By the definition of A and by Lemma 9, we have

$$\begin{aligned} t = \sum_{k\in I} \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathfrak{G}}(P^{L_k}, \Delta(L_j))) &= \sum_{k\in I} \dim_{\mathbb{C}}(\Delta(L_j)_{\mu_{L_k}, b_{L_k}}) \\ &= \sum_{k\in I} \dim_{\mathbb{C}}((M_{L_j})_{\mu_{L_k}, b_{L_k}}) = \sum_{k\in I} \dim_{\mathbb{C}}(e_k C e_j) = \dim_{\mathbb{C}} Ce_j. \end{aligned}$$

So, we have only to show that C is pyramidal. The maximal quasi-local subalgebra of C is

$$S = \bigoplus_{j\in I} \{\varphi \in \mathrm{Hom}_{\mathfrak{G}}(P^{L_j}, P^{L_j}) \mid \varphi(p(L_j)) \in M_{L_j}\}.$$

We will show that C is right S-projective. Left projectivity can be proved analogously. In fact, we will show that for any $j, k \in I$

$$e_j C e_k = \{\varphi \in \mathrm{Hom}_{\mathfrak{G}}(P^{L_j}, P^{L_k}) \mid \varphi(p(L_j)) \in M_{L_k}\}$$

is a free right $e_k S e_k$-module. Recall that M_{L_k} maps bijectively onto $\Delta(L_k)$ for any surjection from P^{L_k} to $\Delta(L_k)$. Let $M_{\mathcal{P}}(V_{L_k})$ be the generalized Verma module associated with L_k

([FKM1, Section 2]). It follows from the description of Λ that either $M_{\mathcal{P}}(V_{L_k}) \simeq \Delta(L_k)$ or $\Delta(L_k)$ is a self-extension of $M_{\mathcal{P}}(V_{L_k})$. Let M^k denote a vectorsubspace of M_{L_k} that maps bijectively on $M_{\mathcal{P}}(V_{L_k})$ under any composition $P^{L_k} \to \Delta(L_k) \to M_{\mathcal{P}}(V_{L_k})$. Such an M^k clearly exists. Now an e_kSe_k basis of e_jCe_k is given by any linear basis of the vectorspace of all maps $\varphi \in \mathrm{Hom}_{\mathfrak{G}}(P^{L_j}, P^{L_k})$ such that $\varphi(p(L_j)) \in M^k$. Hence, e_jCe_k is e_kSe_k-free.

Since A has a Δ-subalgebra and there is a duality on $\mathcal{O}_i$ ([FKM1, Section 12]), one has that A has an exact Borel subalgebra; the statement follows. □

Acknowledgments

The paper was written during the visit of the second author to Bielefeld University as an Alexander von Humboldt fellow. Financial support of the Humboldt Foundation and the hospitality of Bielefeld University are gratefully acknowledged. We also thank S. König and V. Dlab for helpful discussions and comments. We are particularly indebted to the referee for a number of suggestions, remarks and comments that led to the improvements in this paper.

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