$*$-Representations of twisted generalized Weyl constructions

Volodymyr Mazorchuk and Lyudmyla Turowska

[1]

Abstract

We study bounded and unbounded $*$-representations of Twisted Generalized Weyl Algebras and algebras similar to them for different choices of involutions.

1 Introduction

Generalized Weyl Algebras (GWA) were first introduced by Bavula as some natural generalization of the Weyl algebra A_1 (see [B2] and references therein). Since then, GWA have become objects of much interest (see for example [B1, B2, KMP, Sm, Sk, DGO]). Many known algebras such as $U(sl(2,\mathbb{C}))$, $U_q(sl(2,\mathbb{C}))$, down-up algebras and others can be viewed as generalized Weyl algebras and thus can be studied from some unifying point of view.

In [MT] we introduce a non-trivial higher rank generalization of GWA, which we call Twisted Generalized Weyl Algebras. We study simple weight modules over twisted GWA in a special (torsion-free) case and show that there arise new effects which might be of interest for further investigations. We also note, that Twisted GWA are not isomorphic to the higher rank GWA, considered in [B1] in general.

For such algebras it is natural to study their unitarizable modules, i.e. $*$-representations in a Hilbert space. The purpose of this paper is to introduce natural $*$-structures over twisted generalized Weyl algebras and some their non-commutative ("quantum") deformations and to study Hilbert space representations of the corresponding $*$-algebras (real forms) by bounded and unbounded operators. The class of $*$-algebras considered in the paper contains a number of known $*$-algebras such as $U(su(2))$, $U(sl(2,\mathbb{R}))$, $U_q(su(2))$, $U_q(su(1,1))$, $SU_q(2)$ as well as $*$-algebras generated by Q_{ij}-CCR ([Jor]), twisted canonical (anti)-commutation relations ([Pus, PW]) and others.

The technique of study of $*$-representations used in the paper is based on the study of structure and properties of some dynamical systems. This approach goes back to the classical papers [M1, M2, EH, Kir1, Ped] and turns out to be a useful tool for investigation

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of representations of many $*$-algebras ([V1, V2, OS1, OT]). Using this method we obtain a complete classification of irreducible representations of the introduced real forms of twisted GWA (the case of commutative ground $*$-algebra R, see section 2 for the precise definitions) provided that the corresponding dynamical system is simple, i.e., it possesses a measurable section. Any such representations is related to an orbit of the dynamical system. Otherwise, the problem of unitary classification of their representations can be problematic. Namely, if the dynamical system does not have a measurable section there might exist non-atomic quasi-invariant measures which generate factor-representations which are not of type I ([MN]). For a non-commutative $*$-algebra R of type I, we show that there is a one-to-one correspondence between weight irreducible representations of the corresponding $*$-algebras and projective unitary irreducible representations of some groups isomorphic to $\mathbb{Z}^l$. We study bounded and unbounded $*$-representations of our algebras. Note that the first problem that arises when one deals with representations by unbounded operators is to select the "well-behaved" representations like the integrable representations of Lie algebras. In the paper we define a class of "well-behaved" representations for our $*$-algebras and study them up to unitary equivalence.

The paper is organised in the following way: in section 2 we introduce a deformation of twisted GWA and define $*$-structures on it. In section 3 we study bounded representations of the corresponding $*$-algebras (real forms). After discussing some properties of representations, we describe irreducible ones in terms of two models. As a results of our classification all irreducible weight $*$-representations of real forms for twisted GWA are listed in Theorem 4. In section 4 the results obtained in the previous section are generalized on a class of unbounded representations.

2 Twisted generalized Weyl construction and its $*$-structures

2.1 Definition of the algebras and $*$-structures

Throughout the paper $\mathbb{C}$ is the complex field, $\mathbb{R}$ is the field of real numbers, $\mathbb{Z}$ is the ring of integers, $\mathbb{N}$ is the set of all positive integers.

Fix a positive integer n and set $\mathbb{N}_n = \{1, 2, \ldots, n\}$. Let R be a unital algebra over $\mathbb{C}$, $\{\sigma_i \,|\, 1 \le i \le n\}$ a set of pairwise commuting automorphisms of R and M a matrix $(\mu_{ij})_{i,j\in\mathbb{N}_n}$ with complex non-zero entries $\mu_{ij} \in \mathbb{C}$, $i, j \in \mathbb{N}_n$. Fix central elements $t_i \in R$, $i \in \mathbb{N}_n$, satisfying the following relations:

$$t_i t_j = \mu_{ij}\mu_{ji}\sigma_i^{-1}(t_j)\sigma_j^{-1}(t_i), \quad i, j \in \mathbb{N}_n, i \neq j. \tag{1}$$

We define $\mathfrak{A}$ to be an R-algebra generated over R by indetermines X_i, Y_i, $i \in \mathbb{N}_n$, subject to the relations

- $X_i r = \sigma_i(r) X_i$ for any $r \in R$, $i \in \mathbb{N}_n$;

- $Y_i r = \sigma_i^{-1}(r) Y_i$ for any $r \in R$, $i \in \mathbb{N}_n$;
- $X_i Y_j = \mu_{ij} Y_j X_i$ for any $i, j \in \mathbb{N}_n$, $i \neq j$;
- $Y_i X_i = t_i$, $i \in \mathbb{N}_n$;
- $X_i Y_i = \sigma_i(t_i)$, $i \in \mathbb{N}_n$.

We will say that $\mathfrak{A}$ is obtained from R, M, σ_i, t_i, $i \in \mathbb{N}_n$ by *twisted generalized Weyl construction.*

One can easily show that the elements X_i, X_j, Y_i and Y_j satisfy additionally the relations of the form:

$$\begin{array}{lcl} X_i X_j Y_i X_i = \mu_{ji} X_i Y_i X_j X_i & \Leftrightarrow & X_i X_j t_i = \mu_{ji} X_j X_i \sigma_j^{-1}(t_i), \\ Y_i Y_j X_i Y_i = \mu_{ij}^{-1} Y_i X_i Y_j Y_i & \Leftrightarrow & Y_i Y_j \sigma_i(t_i) = \mu_{ij}^{-1} Y_j Y_i \sigma_i(\sigma_j(t_i)). \end{array} \quad (2)$$

Algebra $\mathfrak{A}$ possesses a natural structure of $\mathbb{Z}^n$-graded algebra by setting $\deg R = 0$, $\deg X_i = g_i$, $\deg Y_i = -g_i$, $i \in \mathbb{N}_n$, where g_i, $i \in \mathbb{N}_n$ are the standard generators of $\mathbb{Z}^n$.

Remark 1. If R is commutative and $\mu_{ij} = 1$, $i, j \in \mathbb{N}_n$ then $\mathfrak{A}$ coincides with the algebra $\mathfrak{A}'$ defined in [MT]. *Twisted GWA* $\mathfrak{A}(R, \sigma_1, \dots, \sigma_n, t_1, \dots, t_n)$ of rank n can be obtained as the quotient ring $\mathfrak{A}'/I$, where I is the maximal graded two-sided ideal of $\mathfrak{A}'$ intersecting R trivially. By [MT, lemma 2], this ideal is unique. It is worth to point out that the requirement for μ_{ij} to be equal 1 is not important. All the results obtained in [MT] can be reformulated easily for the case $\mu_{ij} \neq 1$. In particular, given a commutative ring R, its automorphisms σ_i, $i \in \mathbb{N}_n$, a matrix $M = (\mu_{ij})_{i,j \in \mathbb{N}_n}$, $\mu_{ij} \neq 0$ and elements $t_i \in R$, $i \in \mathbb{N}_n$ satisfying the relations $t_i t_j = \mu_{ij} \mu_{ji} \sigma_i^{-1}(t_j) \sigma_j^{-1}(t_i)$, $i, j \in \mathbb{N}_n$, $i \neq j$, we can define twisted GWA as follows: $\mathfrak{A}(R, \sigma_1, \dots, \sigma_n, t_1, \dots, t_n, M) = \mathfrak{A}'/I$, where I is the ideal defined above. This class of algebras contains beside the algebras $U(sl(2))$, $U_q(sl(2))$, the algebras of skew differential operators on the quantum n-space known as the quantized Weyl algebras [DJ, Jord] and some other coordinate rings of quantum symplectic and euclidean spaces.

Assume that $\mu_{ij} = \mu_{ji} \in \mathbb{R}$ and R is a $*$-algebra satisfying the condition $\sigma_i(r^*) = (\sigma_i(r))^*$ for any $r \in R$, $i \in \mathbb{N}_n$. Then the algebra $\mathfrak{A}$ possesses the following $*$-structures:

$$X_i^* = \varepsilon_i Y_i, \quad t_i^* = t_i, \quad \text{where } \varepsilon_i = \pm 1, \ i \in \mathbb{N}_n.$$

We will denote the corresponding $*$-algebras by $\mathfrak{A}_R^{\varepsilon_1, \dots, \varepsilon_n}$.

Remark 2. It is clear that any maximal graded two-sided ideal of the $*$-algebra $\mathfrak{A}_R^{\varepsilon_1, \dots, \varepsilon_n}$ is a $*$-ideal. Thus in the case of commutative R, the introduced $*$-structures generate $*$-structures on the corresponding twisted GWA.

2.2 Examples

1. **The universal enveloping algebra** $U(sl(2,\mathbb{C}))$**.** Let $R = \mathbb{C}[H,T]$ be the polynomial ring in two variables, $t = T$, $T^* = T$, $H^* = H$, $\sigma(H) = H - 1$, $\sigma(T) = T + H$. Then $\mathfrak{A}_R^1 \simeq U(su(2))$ and $\mathfrak{A}_R^{-1} \simeq U(sl(2,\mathbb{R}))$
2. **The quantum algebra** $U_q(sl(2,\mathbb{C}))$**.** Let $R = \mathbb{C}[T,k,k^{-1}]$ (polynomials in T and Laurent polynomials in k), $t = T$, $T^* = T$, $\sigma(k) = q^{-1}k$, $\sigma(T) = T + \frac{k-k^{-1}}{q-q^{-1}}$.

- If $q \in \mathbb{R}$ and $k = k^*$ then $\mathfrak{A}_R^1 \simeq su_q(2)$, $\mathfrak{A}_R^{-1} \simeq su_q(1,1)$.
- If $|q| = 1$ and $k^* = k^{-1}$ then $\mathfrak{A}_R^1 \simeq \mathfrak{A}_R^{-1} \simeq su_q(2)$.

Irreducible representations of the $*$-algebras $su_q(2)$, $su_q(1,1)$ were studied in [V3].
3. **Quantized Weyl algebras.** Let $\Lambda = (\lambda_{ij})$ be an $n \times n$ matrix with non-zero complex entries such that $\lambda_{ij} = \lambda_{ji}^{-1}$, let $\overline{q} = (q_1, \dots, q_n)$ be an n-tuple of elements from $\mathbb{C} \setminus \{0,1\}$. The n-th quantized Weyl algebra $A_n^{\overline{q},\Lambda}$ ([Jord]) is the $\mathbb{C}$-algebra with the generators x_i, y_i, $1 \le i \le n$, and the relations

$$\begin{aligned} x_i x_j &= q_i \lambda_{ij} x_j x_i, \\ y_i y_j &= \lambda_{ij} y_j y_i, \\ x_i y_j &= \lambda_{ji} y_j x_i, \\ x_j y_i &= q_i \lambda_{ij} y_i x_j, \\ x_j y_j - q_j y_j x_j &= 1 + \sum_{i=1}^{j-1} (q_i - 1) y_i x_i, \end{aligned} \tag{3}$$

for $1 \le i < j \le n$.

Let $R = \mathbb{C}[t_1, \dots, t_n]$ be the polynomial ring in n variables, σ_i be the automorphisms of R defined by

$$\sigma_i : p(t_1, \dots, t_n) \to p(t_1, \dots, t_{i-1}, 1 + q_i t_i + \sum_{j=1}^{i-1} (q_j - 1) t_j, q_i t_{i+1}, \dots, q_i t_n),$$

and $M = (\mu_{ij})_{i,j=1}^n$, where $\mu_{ij} = \lambda_{ji}$ and $\mu_{ji} = q_i \lambda_{ij}$ for $i < j$. Then $A_n^{\overline{q},\Lambda}$ is isomorphic to a quotient of the algebra $\mathfrak{A}$ which is obtained from R, M, σ_i, t_i, $i \in \mathbb{N}_n$ by twisted generalized Weyl construction. It is easy to show that the maximal graded ideal of $\mathfrak{A}$ intersecting R trivially is generated by the elements $x_i x_j - q_i \lambda_{ij} x_j x_i$, $y_i y_j - \lambda_{ij} y_i y_j$, $1 \le i < j \le n$, hence $\mathfrak{A}(R, \sigma_1, \dots, \sigma_n, t_1, \dots, t_n, M) \simeq A_n^{\overline{q},\Lambda}$.

Assume that q_i, $\lambda_{ij} \in \mathbb{R} \setminus \{0\}$, $x_i^* = \varepsilon_i y_i$, $i,j = 1, \dots, n$. The involutions define $*$-structures in $\mathfrak{A}$ and the quantum Weyl algebra $A_n^{\overline{q},\Lambda}$.

Note that in the case $\lambda_{ji} = q_i \lambda_{ij} = \mu \in (0,1)$ relations (3) are known as twisted canonical commutation relations ([PW]). $*$-Representations of the algebra which correspond to the involution $x_i^* = y_i$, $i = 1, \dots, n$ were classified in [PW].

4. Q_{ij}**-CCR**. Let A_d be a $*$-algebra generated by elements a_i^*, a_i, $i = 1, \dots, d$, satisfying the following Q_{ij}- commutation relations:

$$a_i^* a_i - Q_{ii} a_i a_i^* = 1, \quad a_i^* a_j = Q_{ij} a_j a_i^*, \ i \neq j, \tag{4}$$

$$a_i a_j = Q_{ji} a_j a_i, \ i \neq j, \tag{5}$$

where $Q_{ii} \in (0,1)$, $|Q_{ij}| = 1$ if $i \neq j$, $Q_{ij} = \overline{Q_{ji}}$, $i, j = 1, \dots, d$. The $*$-algebra A_1 is a real form of the generalized Weyl algebra with $R = \mathbb{C}[T]$ and $t = t^* = T$, $\sigma(T) = Q_{11}^{-1}(T-1)$. For $d > 1$ set $R = A_{d-1} \oplus \mathbb{C}[T]$, where $T = T^*$ and $[T, a]$ for any $a \in A_{d-1}$. Let $\sigma(a_i) = Q_{id} a_i$, $i = 1, \dots d-1$, $\sigma(T) = Q_{dd}^{-1}(T-1)$ and $t = T$. Then $\mathfrak{A}_R^1 \simeq A_d$. Representations of Q_{ij}-CCR were studied in [Pr] using another method.

For other examples of twisted generalized Weyl constructions see also Remark 9.

3 $*$-Representations of twisted generalized Weyl constructions

3.1 Bounded representations of $\mathfrak{A}_R^{\varepsilon_1, \dots, \varepsilon_n}$

Let H be a separable Hilbert space. Throughout this section $L(H)$ denotes the set of all bounded operators on H. Let $\mathfrak{B}(\mathbb{R})$ be the class of Borel subsets of $\mathbb{R}$. For a selfadjoint operator A we will denote by $E_A(\cdot)$ the corresponding resolution of the identity.

Let M be any subset of $L(H)$. We denote by $\mathcal{M}'$ the commutant of $\mathcal{M}$, i.e. the set of those elements of $L(H)$ that commute with all the elements of $\mathcal{M}$. For a group G we will denote by G^* the set of its characters.

In this section we study bounded representations of $\mathfrak{A}_R^{\varepsilon_1, \dots, \varepsilon_n}$, i.e. $*$-homomorphisms $\pi : \mathfrak{A}_R^{\varepsilon_1, \dots, \varepsilon_n} \to L(H)$ up to unitary equivalence. We recall that representations of a $*$-algebra $\mathfrak{A}$ π in H and $\tilde{\pi}$ in $\tilde{H}$ are said to be unitarily equivalent if there exists a unitary operator $U : H \to \tilde{H}$ such that $U\pi(a) = \tilde{\pi}(a)U$ for any $a \in \mathfrak{A}$. Throughout the paper we will use the notation $\pi_1 \simeq \pi_2$ for unitarily equivalent representations π_1, π_2. We will assume also that $\mu_{ij} > 0$ for $i, j \in \mathbb{N}_n$.

Let π be a representation of $\mathfrak{A}_R^{\varepsilon_1, \dots, \varepsilon_n}$. We will denote the operators $\pi(x)$, $x \in \mathfrak{A}_R^{\varepsilon_1, \dots, \varepsilon_n}$ simply by x if no confusion can arise. Let $r = U_r|r|$ ($r \in R$), $X_i = U_i|X_i|$ be the polar decomposition of operators r and X_i respectively, where $|r| = (r^*r)^{1/2}$, $|X_i| = (X_i^* X_i)^{1/2}$, U_r and U_i are phases of the operators r and X_i. We recall, that the phase of an operator B is a partial isometry with the initial space $(\ker B)^\perp = (\ker B^*B)^\perp$ and the final space $(\ker B^*)^\perp = (\ker BB^*)^\perp$. Since $X_i^* X_i = \varepsilon_i t_i$ and $X_i X_i^* = \varepsilon_i \sigma_i(t_i)$, we have $\varepsilon_i t_i \geq 0$, $\varepsilon_i \sigma_i(t_i) \geq 0$, and $|X_i| = (\varepsilon_i t_i)^{1/2}$, $|X_i^*| = (\varepsilon_i \sigma_i(t_i))^{1/2}$, $i \in \mathbb{N}_n$.

Proposition 1. *For any bounded representation of the $*$-algebra $\mathfrak{A}_R^{\varepsilon_1, \dots, \varepsilon_n}$ the following relations hold*

$$U_i U_r = U_{\sigma_i(r)} U_i, \ U_i E_{|r|}(\Delta) = E_{|\sigma_i(r)|}(\Delta) U_i, \ \Delta \in \mathfrak{B}(\mathbb{R}), \ r \in R, \ i \in \mathbb{N}_n, \tag{6}$$

$$[U_i, U_j] P_{ij} = 0, \qquad [U_i, U_j^*] Q_{ij} = 0, \ i \neq j, \tag{7}$$

where P_{ij} and Q_{ij} are projections onto $(\ker t_it_j)^\perp$ and $(\ker \sigma_i(t_it_j))^\perp$ respectively. Moreover, U_i is centered, i.e.

$$\begin{array}{ll} [U_i^k(U_i^*)^k, U_i^l(U_i^*)^l] = 0, & [U_i^k(U_i^*)^k, (U_i^*)^l U_i^l] = 0, \\ [(U_i^*)^k U_i^k, (U_i^*)^l U_i^l] = 0 & \text{for any } k, l \in \mathbb{N}. \end{array} \tag{8}$$

Conversely, any family of operators $r = U_r|r|$, $X_i = U_i(\varepsilon_i t_i)^{1/2}$, $i \in \mathbb{N}_n$, determines a bounded representation of $\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n}$ if $|r|$, $\varepsilon_i t_i$, $\varepsilon_i\sigma_i(t_i)$, $i \in \mathbb{N}_n$ are bounded positive operators and U_r, U_i, $i \in \mathbb{N}_n$ are partial isometries satisfying (6)—(7) and the conditions $\ker U_r = \ker |r|$, $\ker U_i = \ker t_i$.

Proof. ¿From the relations in the algebra $\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n}$ it follows that $X_i|r|^2 = |\sigma_i(r)|^2 X_i$. By [SST, Theorem 2.1] we have $X_i E_{|r|}(\Delta) = E_{|\sigma_i(r)|}(\Delta) X_i$ for any $\Delta \in \mathfrak{B}(\mathbb{R})$. To obtain relations (6) we note that $|X_i| = (\varepsilon_i t_i)^{1/2}$, $\ker X_i = \ker U_i = \ker t_i$ and t_i commutes with any projection $E_{|r|}(\Delta)$ as a central element of the algebra R. By the definition of polar decomposition, $U_i^*U_i = E_{\varepsilon_i t_i}(\mathbb{R}\backslash\{0\})$ and $U_iU_i^* = E_{\varepsilon_i\sigma_i(t_i)}(\mathbb{R}\backslash\{0\})$. From this and relations (6) it follows that U_i is centered.

The relations connecting U_i and U_j follow from (2). Indeed,

$$X_iX_jt_i = \mu_{ji}X_jX_i\sigma_j^{-1}(t_i) \Leftrightarrow U_i(\varepsilon_i t_i)^{1/2}U_j(\varepsilon_j t_j)^{1/2}t_i = \mu_{ji}U_j(\varepsilon_j t_j)^{1/2}U_i(\varepsilon_i t_i)^{1/2}\sigma_j^{-1}(t_i)$$

¿From the relation $\varepsilon_i t_i U_j = U_j \varepsilon_i \sigma_j^{-1}(t_i)$ and the fact that $(\ker U_j)^\perp = (\ker t_j)^\perp$ is invariant with respect to $\sigma_j^{-1}(t_i)$ we can conclude that $\varepsilon_i\sigma_j^{-1}(t_i)|_{(\ker U_j)^\perp}$ is positive and $(\varepsilon_i t_i)^{1/2}U_j = U_j(\varepsilon_i\sigma_j^{-1}(t_i))^{1/2}$. This gives

$$\begin{gathered} U_iU_j(\varepsilon_i\varepsilon_j\sigma_j^{-1}(t_i)t_j)^{1/2}t_i = \mu_{ji}U_jU_i(\varepsilon_i\varepsilon_j\sigma_i^{-1}(t_j)t_i)^{1/2}\sigma_j^{-1}(t_i) \Leftrightarrow \\ U_iU_j(\varepsilon_i\varepsilon_j\sigma_j^{-1}(t_i)t_jt_i^2)^{1/2} = U_jU_i(\mu_{ji}\mu_{ij}\varepsilon_i\varepsilon_j\sigma_i^{-1}(t_j)t_i(\sigma_j^{-1}(t_i))^2)^{1/2} \Leftrightarrow \\ U_iU_j(\varepsilon_i\varepsilon_j\sigma_j^{-1}(t_i)t_jt_i^2)^{1/2} = U_jU_i(\varepsilon_i\varepsilon_j\sigma_j^{-1}(t_i)t_jt_i^2)^{1/2} \end{gathered}$$

By (1) we can conclude now that $[U_i, U_j]P_{ij} = 0$. Note that another relation of (2) will give us the same. Similar arguments applying to the equality $X_iY_j = \mu_{ij}Y_jX_i$ imply the relations connecting U_i and U_j^*.

The converse implication follows by standard arguments from spectral decomposition of the operators $|r|$. □

The question about classification of $*$-representation up to unitary equivalence can be very difficult in general for arbitrary $*$-algebra. If the $*$-algebra is of type I one has a more satisfactory theory (see [Dix2]). ¿From now on we will assume that R is an algebra of type I, i.e. for any representation π of R the W^*-algebra $\pi(R)'' = \{\pi(r), r \in R\}''$ is of type I (see [Dix2] for the precise definition). The algebra $\pi(R)''$ is said to be a W^*-algebra generated by $\pi(R)$. Moreover, we will restrict ourselves to the case countably generated $*$-algebras. Denote by $\hat{R}$ the set of equivalence classes of irreducible representations of R and by H_n the standard n-dimensional Hilbert space.

Theorem 1. *Let H be a separable Hilbert space, π a representation of R. Then there exist a standard Borel space Γ_π, mutually singular positive measures $(\mu_k)_{k\in K}$ on Γ_π, μ_k-measurable fields $\xi \to H_k(\xi)$ of Hilbert spaces, μ_k-measurable fields $\xi \to \pi_k(\xi)$ of non-trivial unitarily non-equivalent irreducible representations on $H_k(\xi)$ and an isomorphism of H onto $\oplus_{k\in K} n_k \int_{\Gamma_\pi}^{\oplus} H_k(\xi)d\mu_k(\xi)$, where the $n_k \in \mathbb{N}\cup\{\infty\}$ are mutually distinct, which transforms the representation π into*

$$\bigoplus_{k\in K} n_k \int_{\Gamma_\pi}^{\oplus} \pi_k(\xi)d\mu_k(\xi) \tag{9}$$

Moreover, the representations $\int_{\Gamma_\pi}^{\oplus} \pi_k(\xi)d\mu_k(\xi)$ are mutually disjoint and the set $(n_k)_{k\in K}$ is unique up to a permutation of the set of indices.

Proof. Let $\mathcal{A}$ be the closure of $\pi(R)$ in the operator norm. Since R is of type I and countably generated, we have that $\mathcal{A}$ is a separable C^*-algebra of type I. The theorem now follows from the general result about the same decomposition of any representation of $\mathcal{A}$ applied to the identity representation $\Psi(a) = a$ for any $a \in \mathcal{A}$ (see [Dix2, Theorem 8.6.6]). Here $\Gamma_\pi = \hat{\mathcal{A}}$ is the set of equivalence classes of irreducible representations of $\mathcal{A}$. We also note that measures μ_k on Γ_π are defined uniquely up to equivalence. $\square$

Using standard arguments one can show the uniqueness of the decomposition (9). Namely, if Γ_π^1, $(\mu_k^1)_{k\in K}$, $\xi \to H_k^1(\xi)$, $\xi \to \pi_k^1(\xi)$ have the same properties then there exists a μ_k-negligible set $N \in \Gamma_\pi$ and μ_k^1-negligible set $N_1 \in \Gamma_\pi^1$, $k \in K$, a Borel isomorphism ν of Γ_π/N onto Γ_π^1/N_1 transforming μ_k into a measure $\tilde{\mu}_k^1$ equivalent to μ_k^1 for any $k \in K$, an isomorphism $\xi \to V(\xi)$ of the field $\xi \to H_k(\xi)$, $\xi \in \Gamma_\pi/N$ onto the field $\xi_1 \to H_k^1(\xi_1)$, $\xi_1 \in \Gamma_\pi^1/N_1$ such that $V(\xi)$ transforms $\pi_k(\xi)$ into $\pi_k^1(\nu(\xi))$.

Define $\mu = \sum_{k\in K} \mu_k$ and $\pi(\xi) = \pi_k(\xi)$ for $\xi \in \Gamma_k$, Since π_k are mutually disjoint, there exists a set $M \subset \Gamma_\pi$, $\mu(M) = 0$ such that $\pi(\xi)$ and $\pi(\xi')$ are unitarily non-equivalent for any $\xi, \xi' \in \Gamma_\pi \setminus M$, $\xi \neq \xi'$. Let $m(\xi) = n_k$ for $\xi \in \Gamma_k$. The field $\Gamma_\pi \ni \xi \to m(\xi)$ is μ-measurable, hence $\xi \to \pi(\xi) \otimes I_\xi$ is a μ-measurable field of representations of R on Γ_π (here I_ξ is the identity operator in the space $K(\xi) = H_{m(\xi)}$) and $\pi \simeq \int_{\Gamma_\pi} \pi(\xi) \otimes I_\xi d\mu(\xi)$. This is the disintegration of π into primary components.

Remark 3. The set Γ_π depends on the representation π. If R is a C^*-algebra of type I then Γ_π can be identified with $\hat{R}$ with the Borel structure defined by the topology, which coincides with the Mackey structure (see [Dix2]). Also, in this case one has $\pi(\xi) \in \xi$ for any $\xi \in \hat{R}$ and the equivalence class of μ_k is uniquely determined by π_k.

Our basic assumption is the following: there exist a Borel set Γ and a one-to-one map $\varphi : \Gamma \to \hat{R}$ such that for any representation π of R there exist mutually singular standard measures μ_k, $k \in K$ on Γ and a $\sum_{k\in K} \mu_k$-measurable field $\xi \to \pi(\xi)$ of unitarily non-equivalent irreducible representations of R on Γ such that $\pi(\xi) \in \varphi(\xi)$ and $\pi \simeq \oplus_{k\in K} n_k \int_\Gamma \pi(\xi)d\mu_k(\xi)$ or equivalently $\pi \simeq \int_\Gamma \pi(\xi) \otimes I_\xi d\mu(\xi)$, where $\mu = \sum_{k\in K} \mu_k$ and I_ξ is the identity operator in $K(\xi)$ defined above. The set n_k is defined uniquely up to permutation of indices.

Clearly, $\pi(\sigma_i)$ is irreducible for any irreducible representation π of R and $i \in \mathbb{N}_n$. Moreover, any two representation π_1 and π_2 of R are unitarily equivalent if and only if $\pi_1(\sigma_i)$ and $\pi_2(\sigma_i)$ are unitarily equivalent. Thus, we can define the action of σ_i on $\hat{R}$ as follows: for any $\xi \in \hat{R}$ and any $\pi \in \xi$ set $\sigma_i(\xi)$ to be the equivalence class of the representation $\pi(\sigma_i)$. Since $\varphi : \Gamma \to \hat{R}$ is one-to-one, we can define $\sigma_i : \Gamma \to \Gamma$ to be $\sigma_i(\xi) = \varphi^{-1}(\sigma_i(\varphi(\xi)))$. In general $\sigma_i : \Gamma \to \Gamma$ is not necessarily Borel isomorphism.

If π is a representations of $\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n}$ then the restriction of π to R is a representation of R. Next Theorem is a realization of π in the space H of disintegration of $\pi(R)$ into primary components.

Theorem 2. *Let π be a representation of $\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n}$ in a Hilbert space $H = \int_\Gamma^\oplus H(\xi) \otimes K(\xi) d\mu(\xi)$ such that $\pi|_R = \int_\Gamma^\oplus \pi(\xi) \otimes I_\xi d\mu(\xi)$, where $\tilde{H}(\xi) = H(\xi) \otimes K(\xi)$, $\tilde{\pi}(\xi) = \pi(\xi) \otimes I_\xi$ and (Γ, μ) satisfy the basic assumption. Let $\Delta_i^1 = \{\xi \in \Gamma \mid \pi(\xi)(t_i) \neq 0\}$, $\Delta_i^2 = \{\xi \in \Gamma \mid \pi(\xi)(\sigma_i(t_i)) \neq 0\}$. Then there exist μ-negligible Borel sets $N_1, N_2 \subset \Gamma$ and Borel maps Φ_i, $i = 1, \dots, n$ such that Φ_i is an isomorphism of $\Delta_i^2 \setminus N_2$ onto $\Delta_i^1 \setminus N_1$ and*

$$(\pi(r)f)(\xi) = \tilde{\pi}(\xi)(r)f(\xi),$$
$$(\pi(X_i)f)(\xi) = \begin{cases} U_i(\Phi_i(\xi))\sqrt{\frac{d(\Phi_i(\mu))}{d\mu}(\Phi_i(\xi))}\sqrt{\varepsilon_i\tilde{\pi}(\Phi_i(\xi))(t_i)}f(\Phi_i(\xi)), & \xi \in \Delta_i^2 \setminus N_2 \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Here the measure $\chi_{\Delta_i^1}(\xi)d\Phi_i(\mu)(\xi)$ is absolutely continuous with respect to $d\mu(\xi)$, $\Delta_i^1 \setminus N_1 \ni \xi \to U_i(\xi)$ is a measurable field of unitary operators from $\tilde{H}(\xi)$ into $\tilde{H}(\Phi_i^{-1}(\xi))$ satisfying the relations

$$\begin{aligned} U_i(\xi)\tilde{\pi}(\xi)U_i^*(\xi) &= \tilde{\pi}(\Phi_i^{-1}(\xi))(\sigma_i), \\ U_j(\Phi_i^{-1}(\xi))U_i(\xi) &= U_i(\Phi_j^{-1}(\xi))U_j(\xi) \end{aligned} \tag{11}$$

μ-almost everywhere on $\Delta_i^1 \cup \Delta_j^1$. Moreover, $\sigma_i(\xi) = \Phi_i(\xi)$, $\xi \in \Delta_i^2$ μ-a.e.

Proof. Here we retain the notations from Theorem 1. Denote by P_1^i, P_2^i, $i = 1, \dots, n$ the projections onto $(\ker \pi(t_i))^\perp$ and $(\ker \pi(\sigma_i(t_i)))^\perp$ respectively, $\mathcal{R}_\pi$ the von-Neumann algebra generated by $\pi(R)$ and $\mathcal{Z}$ the center of $\mathcal{R}_\pi$. Fix $i \in \mathbb{N}_n$. It is clear that P_1^i, $P_2^i \in \mathcal{Z} \subset \mathcal{R}_\pi$ and $P_j^i = \int_\Gamma^\oplus \chi_{\Delta_i^j}(\xi)d\mu(\xi)$ for $j = 1, 2$. Moreover, the subspace $P_j^i H$ is invariant with respect to $\pi(r)$, $r \in R$ which implies the operators $\{\pi_1^i(r) \equiv \pi(r)P_1^i \mid r \in R\}$ and $\{\pi_2^i(r) \equiv \pi(\sigma_i(r))P_1^i, r \in R\}$ define representations of R on $P_i^1 H$ and $P_i^2 H$ respectively and

$$\pi_1^i(r) = \int_{\Delta_i^1}^\oplus \tilde{\pi}(\xi)(r)d\mu(\xi), \ \pi_2^i(r) = \int_{\Delta_i^2}^\oplus \tilde{\pi}(\xi)(\sigma_i(r))d\mu(\xi).$$

Let U_i be the phase of $\pi(X_i)$. U_i is a partial isometry with the initial and final spaces $(\ker \pi(t_i))^\perp$ and $\ker \pi(\sigma_i(t_i))^\perp$ respectively and hence it is a unitary operator from $P_1^i H$ onto $P_2^i H$. Moreover, by proposition 1 we have $U_i E_{|\pi(r)|}(\Delta) P_1^i U_i^* = U_i E_{|\pi(r)|}(\Delta) U_i^* =$

$E_{|\pi(\sigma_i(r))|}(\Delta)U_iU_i^* = E_{|\pi(\sigma_i(r))|}(\Delta)P_2^i$ and $U_iU_{\pi(r)}P_1^iU_i^* = U_iU_{\pi(r)}U_i^* = U_{\pi(\sigma_i(r))}U_iU_i^* = U_{\pi(\sigma_i(r))}P_2^i$ which implies $U_i\pi_1^i(r)U_i^* = \pi_2^i(r)$, $r \in R$ and $U_i\pi_1^i(R)''U_i^* = \pi_2^i(R)''$ and $U_iZ_1^iU_i^* = Z_2^i$. Clearly, the center Z_j^i is the algebra of diagonalizable operators in $\int_{\Delta_i^j}^{\oplus} H(\xi)\otimes K(\xi)d\mu(\xi)$.

Now from [Dix1, Theorem 4, p.238] or [Ta, Theorem 8.23] we have that there exist μ-negligible Borel sets $N_1, N_2 \subset \Gamma$ and Borel isomorphisms Φ_i of $\Delta_i^2\backslash N_2$ onto $\Delta_i^1\backslash N_1$ such that the measures μ_1 and $\Phi_i(\mu_2)$ are equivalent (i.e. $\mu_1(\Delta) = 0$ if and only if $\mu_2(\Phi_i^{-1}(\Delta)) = 0$ for any Borel set $\Delta \subset \Delta_i^1$), where μ_1 and μ_2 are the restrictions of the measure μ onto Δ_i^1 and Δ_i^2 respectively. Further, there exists a μ-measurable field $\xi \to U_i(\xi)$, $\xi \in \Delta_i^1 \setminus N_1$ of unitary operators from $\tilde{H}(\xi)$ onto $\tilde{H}(\Phi_i^{-1}(\xi))$ such that $U_i(\xi)\tilde{\pi}(\xi)U_i^*(\xi) = \tilde{\pi}(\Phi_i^{-1}(\xi))(\sigma_i)$ and

$$U_i = \int_{\Delta_i^1}^{\oplus} U_i(\xi)\sqrt{\frac{d(\Phi_i(\mu))}{d\mu}(\xi)}d\mu(\xi).$$

Let $\pi_1^i(a)$ be a central element from $\pi_1^i(R)$. Then $\pi_2^i(a)$ is central in $\pi_2^i(R)$. Hence $\tilde{\pi}(\xi)(a) = \lambda_1(\xi)I_{\tilde{H}(\xi)}$, $\tilde{\pi}(\xi)(\sigma_i(a)) = \lambda_2(\xi)I_{\tilde{H}(\xi)}$, where $\lambda_j(\cdot)$, $j = 1,2$ are Borel complex functions. ¿From $(U_i\pi_1^i(a)U_i^* = \pi_2^i(a)$ we obtain $\lambda_1(\Phi_i(\xi)) = \lambda_2(\xi)$ μ-a.e. Δ_i^2. Moreover, it follows from the definition of the map $\sigma_i : \Gamma \to \Gamma$ that $\lambda_2(\xi) = \lambda_1(\sigma_i(\xi))$ μ-a.e. Thus $\lambda_1(\Phi_i(\xi)) = \lambda_1(\sigma_i(\xi))$ for allmost all $\xi \in \Delta_i^2 \setminus N_2$. Since $\pi_1^i(R)\cap\pi_1^i(R)'$ is dense in $\mathcal{Z}$ we get $f(\Phi_i(\xi)) = f(\sigma_i(\xi))$ for any Borel function and hence $\sigma_i(\xi) = \Phi_i(\xi)$ μ-a.e. on Δ_i^2. Finally, (11) follows from Proposition 1 ($[U_i, U_j]P_{ij} = 0$).

Conversely, one can easily check that any family of operators defined by (10) determines a representation of $\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n}$. □

It follows from the above theorem that $\sigma_i : \Gamma \to \Gamma$ is equal to a Borel function μ-almost everywhere if $\ker \pi(\sigma_i(t_i)) = \{0\}$.

¿From now on we will assume that σ_i, $i = 1, \dots, n$ are Borel. The mappings σ_i, $i \in \mathbb{N}_n$ determine an action of the group $\mathbb{Z}^n$ on Γ by $(i_1, \dots, i_n)\xi = \sigma_1^{i_1}(\sigma_2^{i_2}(\dots\sigma_n^{i_n}(\xi)\dots))$ and generate the *dynamical system* $(\Gamma, (\sigma_i)_{i=1}^n)$.

Let $\Omega_\xi = \{\sigma_1^{i_1}(\sigma_2^{i_2}(\dots(\sigma_n^{i_n}(\xi)\dots)) \mid (i_1, \dots, i_n) \in \mathbb{Z}^n\}$ denotes the orbit of $\xi \in \Gamma$. We will say that $(\Gamma, (\sigma_i)_{i=1}^n)$ is *simple* if there exists a Borel set $\tau \subset \Gamma$ intersecting any orbit of the dynamical system exactly in one point. This set is called a measurable section of the dynamical system.

Proposition 2. *Assume that $(\Gamma, (\sigma_i)_{i=1}^n)$ is simple. Then for any irreducible representation of $\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n}$ the corresponding measure μ is concentrated on a single orbit of the dynamical system.*

Proof. Let π be an irreducible representation of $\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n}$. Then μ is ergodic with respect to $(\sigma_i)_{i=1}^n$, i.e. given Borel set Δ such that $\sigma_i(\Delta) = \Delta$ for any $i = 1, \dots, n$, we have either $\mu(\Delta) = 0$ or $\mu(\Gamma \setminus \Delta) = 0$. In order to prove this, consider the projection $P = \int_\Gamma^{\oplus} \chi_\Delta(\xi)d\mu(\xi)$. By Theorem 2, P commutes with any operator from $\mathcal{R}_\pi$ and X_i, X_i^*, $i \in \mathbb{N}_n$. Thus, $P = \lambda I$ for some $\lambda \in \mathbb{R}$. Since P is a projection, we obtain that either $\lambda = 0$ or $\lambda = 1$ and the statement follows.

Now we show that the existence of measurable section is a sufficient condition for μ to be concentrated on an orbit. Indeed, suppose that this is not the case and denote by Δ the support of μ. Then there exists $(i_1, \dots, i_n) \in \mathbb{Z}^n$ such that $\mu(\sigma_1^{i_1}(\dots\sigma_n^{i_n}(\tau)\dots) \cap \Delta) \neq 0$. Let $\tilde{\mu}$ be the restriction of μ to the set $\sigma_1^{i_1}(\dots\sigma_n^{i_n}(\tau)\dots) \cap \Delta$ which will be denoted by $\tilde{\Delta}$. By the assumption, $\tilde{\mu}$ is not concentrated at one point x. Hence there exists a partition of $\tilde{\Delta}$ into two sets of positive measure: $\tilde{\Delta} = \tilde{\Delta}^1 \cup \tilde{\Delta}^2$, $\tilde{\Delta}^1 \cap \tilde{\Delta}^2 = \emptyset$, $\tilde{\mu}(\tilde{\Delta}^i) > 0$. The sets $\Omega(\tilde{\Delta}^i) = \{\sigma_1^{i_1}(\dots(\sigma_n^{i_n}(\xi)\dots) \mid \xi \in \tilde{\Delta}^i\}$, $i = 1, 2$ are invariant with respect to each σ_i. Moreover, $\Omega(\tilde{\Delta}^1) \cap \Omega(\tilde{\Delta}^2) = \emptyset$. This contradicts the ergodicity of the measure μ. Thus $\tilde{\mu}$ is concentrated at one point. This completes the proof. □

Remark 4. If the dynamical system does not have measurable section the problem of unitary classification of all irreducible representation of the $*$-algebra can be very difficult. In this case there might exist ergodic quasi-invariant measures which are not concentrated on a single orbit. Such measures generate a wide class of representations which are not of type I and the description of which is problematic (see, for example, [MN, OS1, V2]).

For the rest of the paper we will assume that the dynamical system defined above is simple. The investigation of irreducible representations can be restricted to that of irreducible representations with $\operatorname{supp}\mu \subset \gamma$ for a fixed orbit $\gamma \subset \Gamma$. For $i \in \mathbb{N}_n$ we will call an element $\lambda \in \gamma$ a forward (a backward) i-break if $\pi(\lambda)(t_i) = 0$ ($\pi(\lambda)(\sigma_i(t_i)) = 0$), where $\pi(\lambda)$ is an irreducible representation in $H(\lambda)$. For $\lambda \in \gamma$ denote by P_λ the set of all $\xi \in \gamma$ with the following properties: $\varepsilon_i \pi(\xi)(t_i) \geq 0$, $\varepsilon_i \pi(\xi)(\sigma_i(t_i)) \geq 0$, there exists $(i_1, \dots, i_k) \in \mathbb{Z}^n$ such that $\xi = \sigma_{i_k}^{\delta_k}(\dots(\sigma_{i_1}^{\delta_1}(\lambda)\dots))$, $\delta_l = \pm 1$, $1 \leq l < k$ and each $\sigma_{i_{l+1}}^{-\delta_{l+1}}(\dots\sigma_{i_k}^{-\delta_k}(\xi)\dots)$, $1 \leq l \leq k$ is not forward i_l-break if $\delta_l = 1$ and is not backward i_l-break if $\delta_l = -1$.

Let g_i denote the canonical generator $\underbrace{(0, \dots, 0, 1, 0, \dots, 0)}_{i}$ of $\mathbb{Z}^n$. We define $\tilde{P}_\lambda$ to be the set of all elements $g \in \mathbb{Z}^n$ such that there exists a decomposition $g = g_{i_s}^{\delta_s} \dots g_{i_1}^{\delta_1}$, where $g_{i_k}^{\delta_k} \dots g_{i_1}^{\delta_1} \lambda \in P_\lambda$ for any $k \leq s$. Below we present some constructions of irreducible representations of $\mathfrak{A}_R^{\varepsilon_1, \dots, \varepsilon_n}$.

For a representation π with $\operatorname{supp}\pi = P_\lambda$ consider the stabilizer K^λ of $\lambda \in \gamma$, i.e. $K^\lambda = \{g \in \mathbb{Z}^n \mid g\lambda = \lambda\}$, and put $\tilde{K}^\lambda = K^\lambda \cap \tilde{P}_\lambda$. Note that if $\lambda_1, \lambda_2 \in P_\lambda$ then $\tilde{K}_{\lambda_1} = \tilde{K}_{\lambda_2}$.

Consider the following two models of representations of $\mathfrak{A}_R^{\varepsilon_1, \dots, \varepsilon_n}$.

($\mathfrak{M}_1$): Assume that K^λ is trivial and $P_\lambda \neq \emptyset$. Let $H = \oplus_{g \in \tilde{P}_\lambda} H(g\lambda)$ where $H(g\lambda) = H(\lambda)$ for any $g \in \tilde{P}_\lambda$. We define

$$(\pi(r)f)(g\lambda) = \pi(\lambda)(\sigma_1^{i_1} \dots \sigma_n^{i_n}(r))f(g\lambda)$$

for $g = (i_1, \dots, i_n) \in \mathbb{Z}^n$, and partial isometries U_i, $i = 1, \dots, n$ as follows:

$$(U_i f)(g\lambda) = \begin{cases} 0, & \pi(g\lambda)(\sigma_i(t_i)) = 0, \\ f(g_i g\lambda), & \text{otherwise.} \end{cases}$$

($\mathfrak{M}_2$): Assume that K^λ is non-trivial, $P_\lambda \neq \emptyset$. Put $H = \oplus_{g \in \tilde{P}_\lambda} \tilde{H}(g\lambda)$, where $\tilde{H}(g\lambda) = H(g\lambda) \otimes K(g\lambda)$, $H(g\lambda) = H(\lambda)$ and $K(g\lambda) = K(\lambda)$, $g \in \tilde{P}_\lambda$ is a separable Hilbert spaces.

We define

$$(\pi(r)(f\otimes h))(g\lambda)=\pi(\lambda)(\sigma_1^{i_1}\dots\sigma_n^{i_n}(r))f(g\lambda)\otimes h(g\lambda)$$

for $g=(i_1,\dots,i_n)\in\mathbb{Z}^n$. The operators U_i, $i\in\mathbb{N}_n$ are defined by imposing the following conditions:

$$U_i\tilde{H}(g\lambda)\subset\tilde{H}(g_i^{-1}g\lambda),\ U_i^*\tilde{H}(g\lambda)\subset\tilde{H}(g_ig\lambda),\quad g\in\tilde{P}_\lambda,$$
$$(U_{i_k}^{\delta_k}\dots U_{i_1}^{\delta_1}(f\otimes h))(g\lambda)=0,\quad g=g_{i_k}^{-\delta_k}\dots g_{i_1}^{-\delta_1}$$

provided that there exists $1\le s\le k$ such that for $g^s=g_{i_{s-1}}^{-\delta_{s-1}}\dots g_{i_1}^{-\delta_1}$ we have that either $\pi(g^s\lambda)(t_{i_s})=0$ and $\delta_s=1$, or $\pi(g_{i_s}g^s\lambda)(t_{i_s})=0$ and $\delta_s=-1$, (by $U_i^{\delta_i}$, where $\delta_i=-1$ we mean U_i^*), otherwise

$$(U_{i_k}^{\delta_k}\dots U_{i_1}^{\delta_1}(f\otimes h))(g\lambda)=\begin{cases}f(\lambda)\otimes h(\lambda), & g\notin\tilde{K}^\lambda\\ W(g_{i_k}^{-\delta_k}\dots g_{i_1}^{-\delta_1})f(\lambda)\otimes S(g_{i_k}^{-\delta_k}\dots g_{i_1}^{-\delta_1})h(\lambda), & g\in\tilde{K}^\lambda\end{cases}$$

Here $W(g_{i_k}^{-\delta_k}\dots g_{i_1}^{-\delta_1})$ is a unitary operator acting in $H(\lambda)$ and such that

$$W(g_{i_k}^{-\delta_k}\dots g_{i_1}^{-\delta_1})\pi(\lambda)(r)W^{-1}(g_{i_k}^{-\delta_k}\dots g_{i_1}^{-\delta_1})=\pi(\lambda)(\sigma_{i_k}^{\delta_k}\dots\sigma_{i_1}^{\delta_1}(r)),\tag{12}$$

$S(\cdot)$ is a unitary irreducible projective representation of $\tilde{K}^\lambda$ on the space $K(\lambda)$ with multiplier $c(k_1,k_2)$ defined by $c(k_1,k_2)I=W(k_1)W(k_2)W^{-1}(k_1k_2)$, i.e.

$$S(k_1k_2)=c(k_1,k_2)S(k_1)S(k_2)\text{ for any }k_1,k_2\in\tilde{K}^\lambda.$$

(see, for example, [Kir2] for the definition).

The existence of such operators $W(k)$, $k\in\tilde{K}_\lambda$ follows from the fact that $\pi(\lambda)$ and $\pi(\lambda)(\sigma_{i_k}^{\delta_k}\dots\sigma_{i_1}^{\delta_1})$ are unitarily equivalent irreducible representations for $k=g_{i_k}^{-\delta_k}\dots g_{i_1}^{-\delta_1}\in\tilde{K}^\lambda$. One can show also that $W(k_1)W(k_2)W^{-1}(k_1k_2)$, $k_1,k_2\in\tilde{K}_\lambda$ commutes with $\pi(\lambda)(r)$, $r\in R$ and hence is a multiple of the identity operator.

Remark 5. If R is commutative then any irreducible representation of R is one-dimensional. Hence dim $H(g\lambda)=1$. Moreover, since any $W(k)=\lambda(k)\in\mathbb{C}$ for some $|\lambda(k)|=1$, we have $[S(k_1),S(k_2)]=0$ and hence any irreducible representation $s(\cdot)$ is one-dimensional. This gives us dim $\tilde{H}(g\lambda)=1$.

Theorem 3. *Any irreducible representation of $\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n}$ is unitarily equivalent to one described in the models $\mathfrak{M}_i$, $i=1,2$.*

Proof. Let π be an irreducible representation of $\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n}$ in a Hilbert space H. By Theorem 2 and Proposition 2, there exits an orbit γ of the dynamical system generated by σ_i, $i\in\mathbb{N}_n$ such that

$$H=\bigoplus_{\lambda\in\operatorname{supp}\pi\subseteq\gamma}\tilde{H}(\lambda),$$

where $\tilde{H}(\lambda)=H(\lambda)\otimes K(\lambda)$,

$$(\pi(r)(f\otimes h))(\lambda)=\pi(\lambda)(r)f(\lambda)\otimes h(\lambda),$$

and

$$U_i\tilde{H}(\lambda) \subset \tilde{H}(g_i^{-1}\lambda),\ U_i^*\tilde{H}(\lambda) \subset \tilde{H}(g_i\lambda). \tag{13}$$

Let $\mathcal{Z}$ be the center of $\pi(R)''$. Denote by $\mathcal{A}_0$ the W^*-algebra generated by $\pi(R)$ and polynomials in U_i, U_i^* that commute with any operator of $\mathcal{Z}$. Then

$$\mathcal{A}_0 = \bigoplus_{\lambda \in \operatorname{supp}\pi} \mathcal{A}_{0,\lambda}$$

where $\mathcal{A}_{0,\lambda}$ is the subalgebra of operators acting in $\tilde{H}(\lambda)$.

We claim that $\mathcal{A}_{0,\lambda} = L(\tilde{H}(\lambda))$, $\lambda \in \operatorname{supp}\pi$. Suppose, that the statement is false. Then $\mathcal{A}_{0,\lambda} = \mathcal{A}^1_{0,\lambda} \oplus \mathcal{A}^2_{0,\lambda}$ which implies $\mathcal{A} = \mathcal{A}^1 \oplus \mathcal{A}^2$, where $\mathcal{A}$ is the W^*-algebra generated by $\pi(\mathfrak{A}_R^{\varepsilon_1,\cdots,\varepsilon_n})$ and $\mathcal{A}^i = \mathcal{A}\mathcal{A}^i_{0,\lambda}$, $i = 1, 2$. Indeed, given $X = X_1X_2\dots X_n \in \mathcal{A}$, where $X_i \in \{r \in R, U_i, U_i^*, i = 1, \dots, n\}$, we have either $X\tilde{H}(\lambda)$ is orthogonal to $\tilde{H}(\lambda)$ or $X\tilde{H}(\lambda) \subseteq \tilde{H}(\lambda)$. If $X\tilde{H}(\lambda) \subseteq \tilde{H}(\lambda)$ and $X\tilde{H}(\lambda) \neq 0$ then one can easily see that $XP_{\tilde{H}(\lambda)}$ commutes with any operator from $\mathcal{Z}$, where $P_{\tilde{H}(\lambda)}$ is the projection onto $\tilde{H}(\lambda)$. Hence $X\mathcal{A}^1_{0,\lambda} \subseteq \mathcal{A}^1_{0,\lambda}$ and $X\mathcal{A}^1_{0,\lambda} \perp Y\mathcal{A}^2_{0,\lambda}$ for any $X, Y \in \mathcal{A}$ which forces immediately $\mathcal{A}^1 \perp \mathcal{A}^2$. The decomposition $\mathcal{A} = \mathcal{A}^1 \oplus \mathcal{A}^2$ is impossible due to irreducibility.

Fix now $\lambda \in \operatorname{supp}\pi$. It is clear that the standard gradation of $\mathfrak{A}_R^{\varepsilon_1,\cdots,\varepsilon_n}$ induces the gradation on the algebra $\mathcal{A}$. From (13) and the equalities $\ker U_i = \ker t_i$, $\ker U_i^* = \ker \sigma_i(t_i)$ it follows that $U_{i_k}^{\delta_k}\dots U_{i_1}^{\delta_1}\tilde{H}(\lambda) = 0$ if there exists $1 \le s \le k$ such that either $\pi(g^s\lambda)(t_{i_s}) = 0$ and $\delta_s = 1$ or $\pi(g_{i_s}g^s\lambda)(t_{i_s}) = 0$ and $\delta_s = -1$, where $g^s = g_{i_{s-1}}^{-\delta_{s-1}}\dots g_{i_1}^{-\delta_1}$ (for $\delta_l = -1$ we mean by $U_{i_l}^{\delta_l}$ the operator $U_{i_l}^*$). In the other case, $U^*U|_{\tilde{H}(\lambda)} = (U_{i_k}^{\delta_k}\dots U_{i_1}^{\delta_1})^*U_{i_k}^{\delta_k}\dots U_{i_1}^{\delta_1}|_{\tilde{H}(\lambda)} = I$ which implies $U^*U \in \mathcal{Z}$.

Let S_1, S_2 be two products of some operators U_i, U_i^*, $i = 1, \dots, n$ such that $\deg S_1 = \deg S_2$. We claim that $S_1|_{\tilde{H}(\lambda)} = S_2|_{\tilde{H}(\lambda)}$ if both $S_1|_{\tilde{H}(\lambda)}$ and $S_2|_{\tilde{H}(\lambda)}$ are non-zero. This follows from the fact that $U_i^{\delta_i}U_j^{\delta_j}f \neq U_j^{\delta_j}U_i^{\delta_i}f$, $f \in \tilde{H}(\xi)$, $\xi \in \operatorname{supp}\pi$ implies either $U_i^{\delta_i}U_j^{\delta_j}f = 0$ or $U_j^{\delta_j}U_i^{\delta_i}f = 0$. Indeed, suppose $U_iU_jf \neq U_jU_if$ then

$$P_{ij}|_{\tilde{H}(\xi)} = U_j^*U_jU_i^*U_i|_{\tilde{H}(\xi)} = U_i^*U_iU_j^*U_j|_{\tilde{H}(\xi)} = 0$$

which yield that either $U_if = 0$ or $U_jf = 0$. The same conclusion can be drawn for other cases. Thus for any $g \in \tilde{P}_\lambda$ we can define an operator $U(g^{-1})P_{H(\lambda)}$ to be the non-zero operator of the form $U_{i_s}^{-\delta_s}\dots U_{i_1}^{-\delta_1}P_{H(\lambda)}$ for some decomposition $g = g_{i_1}^{\delta_1}\dots g_{i_s}^{\delta_s}$. It is obvious that the subspace $\oplus_{g\in\tilde{P}_\lambda}U(g^{-1})\tilde{H}(\lambda)$ is invariant with respect to $\pi(\mathfrak{A}_R^{\varepsilon_1,\cdots,\varepsilon_n})$, hence it coincides with H. Clearly, operators $U(g^{-1})$ are unitary operators from $\tilde{H}(\lambda)$ to $U(g^{-1})\tilde{H}(\lambda)$. Moreover, $U_iU(g^{-1})\tilde{H}(\lambda) = 0$ if $\pi(g\lambda)(t_i) = 0$ and $U_iU(g^{-1})\tilde{H}(\lambda) = U(g_ig^{-1})\tilde{H}(\lambda)$ if $g_i^{-1}g \in \tilde{P}_\lambda$. Analogously we have that $U_i^*U(g^{-1})\tilde{H}(\lambda) = 0$ if $\pi(g_ig\lambda)(t_i) = 0$ and $U_i^*U(g^{-1})\tilde{H}(\lambda) = U((g_ig)^{-1})\tilde{H}(\lambda)$ if $g_ig \in \tilde{P}_\lambda$.

Assume now that K^λ is trivial. Then $U\tilde{H}(\lambda) := U_{i_k}^{\delta_k}\dots U_{i_1}^{\delta_1}\tilde{H}(\lambda) \subseteq \tilde{H}(\lambda)$ if and only if either $U|_{\tilde{H}(\lambda)} = 0$, or $g_{i_k}^{\delta_k}\dots g_{i_1}^{\delta_1} = e$ and in the last case $U|_{\tilde{H}(\lambda)} = U(e)|_{\tilde{H}(\lambda)} = I$. From this one can conclude that $\mathcal{A}_0 = (\pi(R))''$, $\mathcal{A}_{0,\lambda} = (\pi(\lambda)(R))''$, and $\tilde{H}(\lambda) = H(\lambda)$. Since

$$U^*(g^{-1})\pi(r)U(g^{-1})|_{H(\lambda)} = \pi(\lambda)(\sigma_{i_s}^{\delta_s} \dots \sigma_{i_1}^{\delta_1})(r),$$

one can easily get that the representation π is unitarily equivalent to one of the representations from $\mathfrak{M}_1$.

If the dynamical system is not free then $\mathcal{A}_0 \neq (\pi(R))''$ in general. Fix $\lambda \in \gamma$. By the same arguments as given above it follows that the representation of our algebra is irreducible if and only if the family of operators $\mathcal{A}_{0,\lambda}$ is irreducible or, equivalently, the family $\{\pi(r), U_{i_k}^{\delta_k} \dots U_{i_1}^{\delta_1} \mid r \in R, g_{i_k}^{\delta_k} \dots g_{i_1}^{\delta_1} \in K^\lambda\}$ is irreducible as operators on $\tilde{H}(\lambda)$. As before, for $g_{i_1}^{\delta_1} \dots g_{i_k}^{\delta_k} \in K^\lambda$ we have that either the restriction of the operator $U_{i_k}^{-\delta_k} \dots U_{i_1}^{-\delta_1}$ onto $\tilde{H}(\lambda)$ is zero or it is unitary as an operator from $\tilde{H}(\lambda)$ to $\tilde{H}(\lambda)$. Thus for any $k \in \tilde{K}^\lambda$ we can define the operators $U(k^{-1})P_{\tilde{H}(\lambda)}$ as it was done above. It is clear that $U(k^{-1})P_{\tilde{H}(\lambda)}$ transforms any irreducible representation into a unitarily equivalent one. Let $W(k)$ be a unitary operator that transforms $\pi(\lambda)$ into $\pi(\lambda)(\sigma_l^{-i_l} \dots \sigma_1^{-i_1})$ for $k = (i_1, \dots, i_l) \in \tilde{K}^\lambda$, i.e.

$$W(k)\pi(\lambda)(r)W^{-1}(k) = \pi(\lambda)(\sigma_l^{-i_l} \dots \sigma_1^{-i_1})(r), \qquad r \in R. \tag{14}$$

Then $U(k^{-1})(W^{-1}(k) \otimes I_{K(\lambda)})$ commutes with any operator $\pi(\lambda)(r) \otimes I_{K(\lambda)}$, $r \in R$ which implies $U(k^{-1})(W^{-1}(k) \otimes I_\lambda) = I_{H(\lambda)} \otimes S(k)$, where $S(k)$ is an operator on $K(\lambda)$. Since $U(k_1)$ and $U(k_2)$ commute for any $k_1, k_2 \in \tilde{K}^\lambda$,

$$[W(k_1) \otimes S(k_1), W(k_2) \otimes S(k_2)] = 0.$$

¿From this it follows easily that there exists $s(k_1, k_2) \in \mathbb{C}$ such that $W(k_2)W(k_1) = s(k_2, k_1)W(k_1)W(k_2)$ and $S(k_2)S(k_1) = s(k_1, k_2)^{-1}S(k_1)S(k_2)$. Moreover, the corresponding representation of our algebra is irreducible if and only if the family $\{S(k), k \in \tilde{K}^\lambda\}$ is irreducible.

In the same way we can see that there exist $c(k_1, k_2)$, $k_1, k_2 \in \tilde{K}^\lambda$ such that $c(k_1, k_2) = W(k_1)W(k_2)W^{-1}(k_1k_2)$ and $S(k_1k_2) = c(k_1, k_2)S(k_1)S(k_2)$ for any k_1, $k_2 \in \tilde{K}$. In particular, $c(k_2, k_1)c(k_1, k_2)^{-1} = s(k_1, k_2)$. The representation π is unitarily equivalent to a representation from the model $\mathfrak{M}_2$. □

Corollary 1. *Let π be an irreducible representation of the $*$-algebra $\mathfrak{A}_R^{\varepsilon_1, \dots, \varepsilon_n}$, $\lambda \in \operatorname{supp} \pi$. Then $\operatorname{supp} \pi = P_\lambda$.*

Remark 6. Theorem 3 shows that there is one-to-one correspondence between irreducible representations π of the $*$-algebra $\mathfrak{A}_R^{\varepsilon_1, \dots, \varepsilon_n}$ and irreducible projective unitary representations of $\tilde{K}^\lambda$, $\lambda \in \operatorname{supp} \pi$.

It is clear that $\tilde{K}^\lambda$ is a subgroup of $\mathbb{Z}^n$. Hence $\tilde{K}^\lambda \simeq \mathbb{Z}^l$ for some $l \leq n$. Let $k_1, \dots, k_l$ be the generators of $\tilde{K}^\lambda$ and fix unitary operators $W(k_1), \dots, W(k_l)$ satisfying (14). Defining $W(k) = W(k_l)^{\delta_l} \dots W(k_1)^{\delta_1}$ for $k = k_l^{\delta_l} \dots k_1^{\delta_1}$ we get $S(k) = S(k_l)^{\delta_l} \dots S(k_1)^{\delta_1}$, where $S(k_i)$ are unitary operators satisfying the relations

$$S(k_i)S(k_j) = s(k_j, k_i)^{-1}S(k_j)S(k_i) \tag{15}$$

with $s(k_j, k_i)$ described in the proof of the last theorem. The numbers $s(k_i, k_j)$, $i, j = 1, \dots, l$ are uniquely defined by the representation. Moreover, any representation can be obtained in this way. Note that any family of unitary operators $S(k_i)$ satisfying (15) defines a representation of C^*-algebra known as the non-commutative tori A_Θ, where $\Theta = (\theta_{ij})$, $e^{2\pi\theta_{ij}} = s(k_j, k_i)$ (see [R]). In particular, if $l = 2$, A_Θ is a rotational algebra. The problem of unitary classification of all representations of such C^*-algebras might be very difficult. It is known, for example, that a 2-tori A_Θ is of type I if and only if $\theta_{12} = -\theta_{21} \in \mathbb{Q}$. For details we refer the reader to [R].

Next theorem provides a description of irreducible $*$-representations of twisted GWA. Suppose that R is commutative. Then any irreducible representation R is one-dimensional. For $\pi(x) \in \varphi(x) \in \hat{R}$ and $r \in R$ we will denote by $r(x)$ the operator $\pi(x)(r)$.

Theorem 4. *Any irreducible representations of the real form $\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n}$ of twisted GWA is unitarily equivalent to one of the following:*
1. Let orbit γ and $\lambda \in \gamma$ be such that $\tilde{K}^\lambda$ is trivial, $P_\lambda \neq \emptyset$. Then $H = l_2(P_\lambda)$, $re_x = r(x)e_x$, $X_i e_x = \sqrt{\varepsilon_i t_i(x)} u_i(x) e_{\sigma_i^{-1}(x)}$ where $x \in P_\lambda$, $\sigma_i(r)(x) = r(\sigma_i(x))$, $i = 1, \dots, n$ and

$$u_i(x) = \begin{cases} 1, \sigma_i^{-1}(x) \in P_\lambda, \\ 0, \sigma_i^{-1}(x) \notin P_\lambda, \end{cases}$$

2. Let orbit γ and $\lambda \in \gamma$ be such that $\tilde{K}^\lambda$ is non-trivial, $P_\lambda \neq \emptyset$ and let $s(\cdot) \in (\tilde{K}^\lambda)^*$. Then $H = l_2(P_\lambda)$,

$$re_x = r(x)e_x, \ X_i e_x = \sqrt{\varepsilon_i t_i(x)} U_i e_x$$

where $x \in P_\lambda$, $\sigma_i(r)(x) = r(\sigma_i(x))$ and $U_{i_k}^{\delta_k} \dots U_{i_1}^{\delta_1} e_\lambda = 0$ if there exists $1 \leq s \leq k$ such that either $\pi(g^s\lambda)(t_i) = 0$ and $\delta_s = 1$ or $\pi(g_i g^s \lambda)(t_i) = 0$ and $\delta_s = -1$ (here $g^s = g_{i_{s-1}}^{-\delta_{s-1}} \dots g_{i_1}^{-\delta_1}$). Otherwise $U_{i_k}^{\delta_k} \dots U_{i_1}^{\delta_1} e_\lambda = e_{g_{i_k}^{-\delta_k} \dots g_{i_1}^{-\delta_1} \lambda}$ if $g_{i_k}^{-\delta_k} \dots g_{i_1}^{-\delta_1} \notin \tilde{K}^\lambda$ or $U_{i_k}^{\delta_k} \dots U_{i_1}^{\delta_1} e_\lambda = s(g_{i_k}^{-\delta_k} \dots g_{i_1}^{-\delta_1}) e_\lambda$, if $g_{i_k}^{-\delta_k} \dots g_{i_1}^{-\delta_1} \in \tilde{K}^\lambda$.

Remark 7. It is clear that the formulae above defines an action of all operators U_i on any vector $e_x \in l_2(P_\lambda)$.

Proof. It follows from remark 5 that dim $\tilde{H}(\lambda) = 1$. For representations from the model $\mathfrak{M}_2$ we have $W(g) = w(g) \in \mathbb{C}$, $|w(g)| = 1$. If we take $w(g) = 1$, $g \in \tilde{K}_\lambda$ we get $S(g) = S_(g_l)^{\delta_l} \dots S(g_1)^{\delta_1} \in \mathbb{C}$ for $g = g_l^{\delta_l} \dots g_1^{\delta_1}$. Setting $s(g) = W(g) \otimes S(g)$ we obtain $s(\cdot) \in (\tilde{K}^\lambda)^*$. The rest follows from Theorem 3. □

Remark 8. It was shown in [MT] that the supports of finite-dimensional modules of twisted GWA might have more interesting geometrical structure than in the case of classical GWA. We constructed an example which provides some analogue between the structure of supports of finite-dimensional modules over classical simple Lie algebras and supports of finite-dimensional modules of some twisted GWA. The same analogue can be obtained for the $*$-algebras $\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n}$ and some real forms of simple Lie algebras.

Remark 9. Using the above results one can obtain a complete classification of irreducible representations of the $*$-algebras from examples 1–4. This technique was also applied in [STP] to study collections $(u, v, j_1, \dots, j_n)$ of selfadjoint unitary operators satisfying the following commutation relations:

$$j_i j_k = (-1)^{g(i,k)} j_k j_i,$$
$$u j_k = (-1)^{h(k)} j_k u, \ v j_k = (-1)^{w(k)} j_k v,$$

here $g(i,k) = g(k,i) \in \{0,1\}$, $g(i,i) = 0$, $h(k)$, $w(k) \in \{0,1\}$ for any $i, k = 1, \dots, n$. The corresponding $*$-algebra can be treated as a $*$-algebra obtained by twisted generalized Weyl construction and has applications, for example, to a study of operator Banach algebras containing a dense $*$-subalgebra, and construction of invertibility symbols for operators in algebra. For details we refer the reader to [STP].

3.2 Unbounded representations of $\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n}$

Often $*$-algebras do not have any bounded representations or their structure is not interesting. In general we have to deal with representations by unbounded operators. The problem of describing all such representations might be very difficult. Usually one studies some classes of "well-behaved" representations. In the case of Lie algebras there were considered the so-called integrable representations which can be extended to unitary representations of the corresponding Lie group. For arbitrary $*$-algebras there is no canonical way how to select good unbounded representations. The problem of defining integrability for some $*$-algebras and operator relations was studied by many authors (see [S1, OS1, OT, S2, V2, Wor] and the bibliography therein). In this section we will try to define unbounded representations for the $*$-algebra $\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n}$. Throughout this section we will use some notions and facts from the theory of unbounded operator algebras that can be found in [S1].

Let $\mathfrak{A}$ be a $*$-algebra with a unit element. A $*$-representation π of $\mathfrak{A}$ in a Hilbert space $H(\pi)$ is a homomorphism from $\mathfrak{A}$ into the family of closable operators defined on a dense domain $D(\pi)$ that is invariant with respect to $\pi(a)$, $a \in \mathfrak{A}$ and $\pi(a^*) \subseteq (\pi(a))^*$ for any $a \in \mathfrak{A}$.

For a closable operator A in a Hilbert space H we denote by $\mathcal{D}(A)$ the domain of A and $(A)'_s$ the set $\{c \in L(H) \mid cA \subseteq Ac\}$ which is called the strong commutator of A. The strong commutator of a representation π of a $*$-algebra $\mathfrak{A}$ is defined by $\pi(\mathfrak{A})'_s = \cap_{a\in\mathfrak{A}} (\pi(a))'_s$. We will say that the set $D \subseteq H$ is a core for a closed operator A if $\overline{A|_D} = A$. Here $\overline{A}$ denotes the closure of the operator A.

Let π be a representation of $\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n}$. For simplicity we shall write r instead of $\overline{\pi(r)}$ for $r \in R$ if it does not lead to any confusion. Here r is always supposed to be closed. Consider the polar decompositions of the operators $r = U_r|r|$, where $|r| = (r^* r^{1/2}$. Let Z denote the center of $\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n}$.

We will say that closed operators $r \in R$, X_i, X_i^* $i = 1, \dots, n$ acting on a Hilbert space H define a representation of $\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n}$ if

1. the operators $r \in R$ generate a closed representation of R on a dense domain $D \subseteq H$,

2. the operators $t \in Z$ are normal and $E_t(\Delta) \in (r)'_s$ for any $\Delta \in \mathfrak{B}(\mathbb{C})$, $r \in R$,

3. D is a core for any $r \in R$,

4. $X_i^* X_i = \varepsilon_i t_i$, $X_i X_i^* = \varepsilon_i \sigma_i(t_i)$ for all $i = 1, \dots, n$,

5. relations (6) $-$ (7) hold on H, where U_i is the phase of the operator X_i.

Proposition 3. *Assume that conditions* $1-5$ *hold. Then the operators* $r \in R$, X_i, X_i^* $i \in \mathbb{N}_n$ *define a closed representation* π *of* $\mathfrak{A}_R^{\varepsilon_1, \dots, \varepsilon_n}$ *on* D. *Moreover,* D *is a core for* $r \in R$, X_i, X_i^*, $i = 1, \dots, n$.

Proof. We first prove that D is invariant with respect to U_i, U_i^*, $i \in \mathbb{N}_n$. Indeed, since $U_i E_{|r|}(\Delta) = E_{|\sigma_i(r)|}(\Delta) U_i$ for any $\Delta \in \mathfrak{B}(\mathbb{R})$, $(U_i|r|\varphi, \psi) = (U_i\varphi, |\sigma_i(r)|\psi)$ for any $\varphi \in \mathcal{D}(|r|)$, $\psi \in \mathcal{D}(|\sigma_i(r)|)$, which gives $U_i\varphi \in \mathcal{D}(|\sigma_i(r)|)$ and $U_i|r|\varphi = |\sigma_i(r)|U_i\varphi$ for $\varphi \in \mathcal{D}(|r|)$. From this and relations (6)–(7) it follows that $U_i r\varphi = \sigma_i(r)U_i\varphi$ for any $\varphi \in \mathcal{D}(r)$. By condition 1, the family $\{r \mid r \in R\}$ defines a closed representation π of R on D and hence $D = \cap_{r\in R}\mathcal{D}(\overline{\pi(r)}) = \cap_{r \in R}\mathcal{D}(r)$, the last equality is true due to condition 3. From this we have $U_i D \subseteq D$. The same holds for U_i^*, $i \in \mathbb{N}_n$.

Since $X_i^* X_i = \varepsilon_i t_i$, $\varepsilon_i t_i \geq 0$. By condition 2, $E_{\varepsilon_i t_i}(\Delta)\mathcal{D}(r) \subseteq \mathcal{D}(r)$ and $E_{\varepsilon_i t_i}(\Delta) r\varphi = rE_{\varepsilon_i t_i}(\Delta)\varphi$ for any $\varphi \in \mathcal{D}(r)$, $\Delta \in \mathfrak{B}(\mathbb{R})$. In the same manner we get $(\varepsilon_i t_i)^{1/2} D \subseteq D$, and hence $X_i D \subseteq D$. Analogously we obtain that D is invariant with respect to X_i^*. From this we can conclude that $r \in R$, X_i, X_i^*, $i \in \mathbb{N}_n$ determine a representation π of $\mathfrak{A}_R^{\varepsilon_1, \dots, \varepsilon_n}$ on the domain D. Since D is a core for t_i, $\sigma_i(t_i)$, D is a core for $(\varepsilon_i t_i)^{1/2}$, $(\varepsilon_i \sigma_i(t_i))^{1/2}$ and hence for X_i, X_i^*. Finally π is closed since $\pi|_R$ is closed and $D = \cap_{r\in R} D(r)$ is invariant with respect to X_i, X_i^*, $i = 1, \dots, n$. $\square$

In the rest of the paper we will consider only representations satisfying 1–5. Denote by $\mathcal{R}_\pi$ the von Neumann algebra generated by $E_{|r|}(\Delta)$ and U_r, where $r \in R$ and $\Delta \in \mathfrak{B}(\mathbb{R})$. We will need the following auxiliary lemma.

Lemma 1. *Let* π *be a representation of* $\mathfrak{A}_R^{\varepsilon_1, \dots, \varepsilon_n}$ *defined above. Then*

$$\mathcal{R}'_\pi = \pi(R)'_s \cap (\pi(R)'_s)^*.$$

Proof. By the definition, the restriction of π to R is closed and D is a core for all operators r. Hence $\pi(R)'_s \cap (\pi(R)'_s)^* = \{c \in L(H) \mid cr \subseteq rc, c^*r \subseteq rc^*, r \in R\}$ (see [S1, Proposition 7.2.10]). We denote this set by $\pi(R)'_{ss}$.

Let $c \in \pi(R)'_{ss}$. Then $c\mathcal{D}(r) \subseteq \mathcal{D}(r)$ and $c^*\mathcal{D}(r) \subseteq \mathcal{D}(r)$ for every $r \in R$.

Given $\varphi \in \mathcal{D}(r^*r)$, $\psi \in \mathcal{D}(r)$, we obtain

$$(cr^*r\varphi, \psi) = (r^*r\varphi, c^*\psi) = (r\varphi, rc^*\psi) = (r\varphi, c^*r\psi) = (cr\varphi, r\psi) = (rc\varphi, r\psi),$$

which implies $rc\varphi \in \mathcal{D}(r^*)$ and $cr^*r\varphi = r^*rc\varphi$, $\varphi \in \mathcal{D}(r^*r)$. From this $[c, E_{|r|^2}(\Delta)] = 0$ and $[c, E_{|r|}(\Delta)] = 0$ for any $\Delta \in \mathfrak{B}(\mathbb{R})$. Using spectral properties of selfadjoint operators one obtains $c\mathcal{D}(|r|) \subseteq \mathcal{D}(|r|)$ and $c|r|\varphi = |r|c\varphi$ for any $\varphi \in \mathcal{D}(|r|)$.

Given $\varphi \in \mathcal{D}(r)$ it follows that

$$(cU_r|r|\varphi, \psi) = (U_r|r|c\varphi, \psi) = (|r|c\varphi, U_r^*\psi) = (c|r|\varphi, U_r^*\psi) = (|r|\varphi, c^*U_r^*\psi).$$

On the other hand, $(cU_r|r|\varphi, \psi) = (|r|\varphi, U_r^*c^*\psi)$. Therefore, $cU_r\varphi = U_rc\varphi$ for any φ from the image $\mathfrak{R}(|r|)$ of the operator $|r|$. This implies $cU_rP = U_rcP$ where P is the projection onto the space $(\ker |r|)^\perp$. Since $P = E_{|r|}(\mathbb{R} \setminus \{0\}) = U_r^*U_r$, $U_rP = U_r$ and c, P commute. This gives $cU_r = U_rc$ and consequently $c \in \mathcal{R}'_\pi$.

Let $c \in \mathcal{R}'_\pi$. One has $c|r| \subseteq |r|c$ and $c^*|r| \subseteq |r|c^*$ which implies $cr = cU_r|r| = U_rc|r| \subseteq U_r|r|c = rc$. By the same arguments we obtain $c^*r \subseteq rc^*$. This easily forces $\mathcal{R}'_\pi = \pi(R)'_{ss}$. □

A representation π of an algebra $\mathfrak{A}$ on a Hilbert space H is said to be irreducible if there is no non-trivial closed subspace K of H such that every operator $\pi(a)$, $a \in \mathfrak{A}$ can be written as a direct sum $\pi(a) = a_1 \oplus a_2$ where a_1 and a_2 are linear operators on K and $H \cap K^\perp$ respectively. A non-trivial closed subspace $K \subseteq H$ possessing the above properties is called reducing for π.

Lemma 2. *Let π be a $*$-representation of $\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n}$ by unbounded operators $r \in R$, X_i, X_i^*, $i = 1,\dots,n$ defined on a domain D of a Hilbert space H. For each closed linear subspace K of H the following conditions are equivalent:*

1. *K is reducing for π;*
2. *The projection P_K onto K belongs to the set*

$$T_\pi = \mathcal{R}'_\pi \bigcap \cap_{i=1}^n (U_i)'_s.$$

Proof. By [S1, lemma 8.3.3] K is reducing for π if and only if $P_K \in \pi(\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n})'_s$. To obtain the statement it is sufficient to prove that $\{c \in T_\pi \mid c = c^*\} = \{c \in \pi(\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n})'_s \mid c = c^*\}$.

Let $c = c^* \in \pi(\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n})'_s$. It follows that $cD \subseteq D$ and $c \in \pi(R)'_s \cap (\pi(R)'_s)^* = \mathcal{R}'_\pi$. Hence the operators c and t_i (respectively c and $(\varepsilon_i t_i)^{1/2}$) strongly commute, i.e. their spectral projections commute. Applying the same argument as in Lemma 1 we get $cU_i = U_ic$ and finally, $c \in T_\pi$.

Let $c = c^* \in T_\pi$. Thus $c \in \pi(R)'_s$ and hence $cD \subseteq D$. Since $U_iD \subseteq D$, $U_i^*D \subseteq D$ and $(\varepsilon t_i)^{1/2}D \subseteq D$ (see the proof of proposition 3), we get $cX_iD = cU_i(\varepsilon t_i)^{1/2}D = U_ic(\varepsilon t_i)^{1/2}D = U_i(\varepsilon t_i)^{1/2}cD = X_icD$. This implies $c \in \pi(\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n})'_s$. □

Corollary 2. *A representation π of $\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n}$ is irreducible if and only if the family*

$$\{x, U_i, U_i^* \mid x \in \mathcal{R}_\pi, i = 1,\dots,n\}$$

is irreducible.

Now we will make the following assumption. Let $\hat{R}$ be the set of equivalence classes of irreducible representations of R. As before, the set $\{\sigma_i\}_{i\in\mathbb{N}_n}$ define the maps σ_i on $\hat{R}$. We will assume that there exist a Borel set Γ and an injective map $\varphi : \Gamma \to \hat{R}$ such that $\varphi(\Gamma)$ is invariant with respect to σ_i, $i = 1, \dots, n$ and any representation of $\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n}$ restricted to R can be decomposed into a direct integral of primary representations of R which are multiples of irreducible representations from $\varphi(\Gamma)$. Namely, given a separable Hilbert space H and a $*$-representation π of $\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n}$ with a domain $D(\pi)$, there exist a standard measure μ on Γ , μ-measurable fields $\xi \to H(\xi)$, $\xi \to K(\xi)$ of Hilbert spaces on Γ, a μ-measurable field $\xi \to \pi'(\xi)$ of non-trivial irreducible representations on $D(\pi'(\xi)) \subseteq H(\xi)$ such that $\pi'(\xi) \in \varphi(\xi)$ and an isomorphism U of H onto $\int_\Gamma^\oplus H(\xi) \otimes K(\xi) d\mu(\xi)$ that transforms the representation π into π' such that

$$D(\pi') = \int_\Gamma^\oplus D(\pi'(\xi)) \otimes K(\xi) d\mu(\xi)$$

and

$$\pi'(r) = \int_\Gamma^\oplus \pi'(\xi)(r) \otimes I_\xi d\mu(\xi),$$

where I_ξ is the identity operator on $K(\xi)$.

Lemma 3. *Under the above assumptions we have*

$$\overline{\pi'(r)} = \int_\Gamma^\oplus \overline{\pi'(\xi)(r)} \otimes I_\xi d\mu(\xi).$$

The operator U transforms the center $\mathcal{Z}$ of $\mathcal{R}_\pi$ into the algebra of diagonalizable operators with respect to the decomposition. Moreover, if

$$\mathcal{R}_{\pi'} := U^{-1}\mathcal{R}_\pi U = \int_\Gamma^\oplus \mathcal{R}_{\pi'}(\xi) \otimes I_\xi d\mu(\xi)$$

then $\mathcal{R}_{\pi'}(\xi) = \mathcal{R}_{\pi'(\xi)}$ almost everywhere with respect to μ.

Proof. It follows from general properties of decomposable operators (see, for example, [S1, section 12]). □

In a natural way we can define maps $\sigma_i : \Gamma \to \Gamma$, $i = 1, \dots, n$. We will assume that these maps are Borel. Next theorem is an unbounded analogue of Theorem 2.

Theorem 5. *Let $\mathcal{R}_{\pi'} = \int_\Gamma^\oplus \mathcal{R}_{\pi'}(\xi) \otimes I_\xi d\mu(\xi)$ be a direct integral of von-Neumann algebras on $\int_\Gamma^\oplus \tilde{H}(\xi) d\mu(\xi)$, $\tilde{H}(\xi) = H(\xi) \otimes K(\xi)$ from lemma 3 and U_i be the phase of the operator $\overline{\pi'(X_i)}$ ($i = 1, \dots, n$). Let $\Delta_i^1 = \{\xi \in \Gamma \mid \pi'(\xi)(t_i) \neq 0\}$, $\Delta_i^2 = \{\xi \in \Gamma \mid \pi'(\xi)(\sigma_i(t_i)) \neq 0\}$. Then*

$$\begin{aligned} &(xf)(\xi) = x(\xi) f(\xi),\ x \in \mathcal{R}_{\pi'}, \\ &(U_i f)(\xi) = \begin{cases} U_i(\sigma_i(\xi)) \sqrt{\frac{d\sigma_i(\mu)}{d\mu}(\sigma_i(\xi))} f(\sigma_i(\xi)), & \xi \in \Delta_i^2, \\ 0, & \textit{otherwise} \end{cases} \end{aligned} \tag{16}$$

almost everywhere with respect to μ. *Here the measure* $\chi_{\Delta_i^1}(\xi)d\sigma_i(\mu)(\xi)$ *is absolutely continuous with respect to* $d\mu(\xi)$, $\xi \to U_i(\xi)$ *is a measurable field of operators on* $\tilde{H}(\xi)$ *into* $\tilde{H}(\sigma_i^{-1}(\xi))$ *which are unitary* μ*-almost everywhere on* Δ_i^1 *and*

$$U_i(\xi)x(\xi)U_i^*(\xi) = \sigma_i(x)(\sigma_i^{-1}(\xi)),$$
$$U_j(\sigma_i^{-1}(\xi))U_i(\xi) = U_i(\sigma_j^{-1}(\xi))U_j(\xi) \tag{17}$$

almost everywhere on $\Delta_i^1 \cup \Delta_j^1$ *with respect to* μ. *Here* $\sigma_i(x)$ *is defined by* $\sigma_i(x) = U_i x U_i^*$, $x \in \mathcal{R}_\pi$.

Proof. Is similar to that of Theorem 2. □

To describe the structure of unbounded representations of $\mathfrak{A}_R^{\varepsilon_1,\dots,\varepsilon_n}$, we use the following two models which coordinate with the ones given in the previous section and have some corrections connected with unboundedness of representations.

Let $\lambda \in \hat{R}$ be an equivalence class of a closed irreducible representation $\pi(\lambda)$ of R defined on a dense domain $D(\lambda) \subseteq H(\lambda)$ such that $\overline{\pi(\lambda)(r)}$, $r \in R$, and $D(\lambda)$ satisfy the conditions $1, 2, 3$. We retain the notations K^λ, $\tilde{K}^\lambda$, P_λ from the previous section.

($\mathfrak{M}_1^{un}$): Assume that K^λ is trivial and $P_\lambda \neq \emptyset$. Put $H = \oplus_{g\in\tilde{P}_\lambda} H(g\lambda)$, where $H(g\lambda) = H(\lambda)$ for any $g \in \tilde{P}_\lambda$, and $D = \oplus_{g\in\tilde{P}_\lambda} D(g\lambda)$, where $D(g\lambda) = D(\lambda)$. We define

$$(rf)(g\lambda) = \overline{\pi(\lambda)(\sigma_1^{i_1}\dots\sigma_n^{i_n}(r))}f(g\lambda), \quad f \in D$$

for $g = (i_1,\dots,i_n) \in \mathbb{Z}^n$, $r = \overline{r|_D}$ and partial isometries U_i, $i = 1,\dots,n$ as follows:

$$(U_i f)(g\lambda) = \begin{cases} 0, & \text{if } \pi(g\lambda)(\sigma_i(t_i)) = 0, \\ f(g_i g\lambda), & \text{otherwise,} \end{cases}$$

The operators X_i are defined by $X_i = U_i(\varepsilon_i t_i)^{1/2}$.

($\mathfrak{M}_2^{un}$): Assume that K^λ is non-trivial, $P_\lambda \neq \emptyset$. Put $H = \oplus_{g\in\tilde{P}_\lambda}\tilde{H}(g\lambda)$, where $\tilde{H}(g\lambda) = H(g\lambda)\otimes K(g\lambda)$ and $H(g\lambda)$, $K(g\lambda)$ are separable Hilbert spaces such that $K(g\lambda) = K(\lambda)$, $H(g\lambda) = H(\lambda)$ for any $g \in \tilde{K}^\lambda$. We define $D = \oplus_{g\in\tilde{P}_\lambda} D(g\lambda)\otimes I_{g\lambda}$, where $D(g\lambda) = D(\lambda)$ and $I_{g\lambda}$ is the identity operator in $K(g\lambda)$. The closed operators r are defined by requiring

$$(r(f\otimes h))(g\lambda) = \overline{\pi(\lambda)(\sigma_1^{i_1}\dots\sigma_n^{i_n}(r))}f(g\lambda)\otimes h(g\lambda), \quad f\otimes h \in D$$

for $g = (i_1,\dots,i_n) \in \mathbb{Z}^n$, and $r = \overline{r|_D}$.

Partial isometries U_i, $i = 1,\dots,n$ are defined by imposing the following conditions:

$$U_i\tilde{H}(g\lambda) \subset \tilde{H}(g_i^{-1}g\lambda), U_i^*\tilde{H}(g\lambda) \subset \tilde{H}(g_i g\lambda)$$

and

$$U_{i_k}^{\delta_k}\dots U_{i_1}^{\delta_1}|_{H(\lambda)\otimes K(\lambda)} = 0$$

if there exists $1 \leq s \leq k$ such that either $\pi(g^s\lambda)(t_i) = 0$ and $\delta_s = 1$, or $\pi(g_i g^s \lambda)(t_i) = 0$ and $\delta_s = -1$, where $g^s = g_{i_{s-1}}^{-\delta_{s-1}} \dots g_{i_1}^{-\delta_1}$ (by $U_i^{\delta_i}$ where $\delta_i = -1$ we mean U_i^*), otherwise

$$(U_{i_k}^{\delta_k} \dots U_{i_1}^{\delta_1}(f \otimes h))(g\lambda) = \begin{cases} f(\lambda) \otimes h(\lambda), & g \notin \tilde{K}^\lambda \\ W(g_{i_k}^{-\delta_k} \dots g_{i_1}^{-\delta_1}) f(\lambda) \otimes S(g_{i_k}^{-\delta_k} \dots g_{i_1}^{-\delta_1}) h(\lambda), & g \in \tilde{K}^\lambda \end{cases}$$

Here $W(g_{i_k}^{-\delta_k} \dots g_{i_1}^{-\delta_1})$ is a unitary operator such that $W(g)D(\lambda) \subseteq D(\lambda)$ and

$$W(g)\pi(\lambda)(r)W^{-1}(g)\varphi = \pi(\lambda)(\sigma_{i_k}^{\delta_k} \dots \sigma_{i_1}^{\delta_1}(r))\varphi, \ r \in R, \varphi \in D(\lambda) \tag{18}$$

$S(\cdot)$ is a unitary irreducible projective representation of $\tilde{K}^\lambda$ on the space $K(\lambda)$ with a multiple $c(k_1, k_2) = W(k_1)W(k_2)W^{-1}(k_1 k_2)$, i.e. $S(k_1 k_2) = c(k_1, k_2)S(k_1)S(k_2)$ for any $k_1, k_2 \in \tilde{K}^\lambda$.

Theorem 6. *Assume that there exists a Borel set $N \subset \Gamma$ such that N is invariant with respect to σ_i, $i = 1, \dots, n$, the dynamical system $(N, (\sigma_i)_{i=1}^n)$ is simple and for any representation of $\mathfrak{A}_R^{\varepsilon_1, \dots, \varepsilon_n}$ the corresponding measure μ is based essentially on N (i.e. $\operatorname{supp} \mu \subset N$). Then any irreducible representation is unitarily equivalent to one given in the models $\mathfrak{M}_i^{un}$, $i = 1, 2$.*

Proof. Let π be an irreducible representation of $\mathfrak{A}_R^{\varepsilon_1, \dots, \varepsilon_n}$. Then by corollary 2, the family $\{x, U_i, U_i^* \mid x \in \mathcal{R}_\pi, i = 1, \dots, n\}$ is irreducible. As in the proof of Theorem 3 it follows that any irreducible family $\{x, U_i, U_i^* \mid x \in \mathcal{R}_\pi, i = 1, \dots, n\}$ is unitarily equivalent to one given in the models $\mathfrak{M}_i^{un}$ with $\mathcal{R}_\pi$ and $\mathcal{R}_\pi(\lambda)$ instead of $\pi(R)$ and $\pi(\lambda)(R)$ respectively. The statement of the theorem now follows from the fact that $\mathcal{R}_\pi(\lambda) = \mathcal{R}_{\pi(\lambda)}$ for any irreducible $\pi(\lambda)$. $\square$

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Mechanics and Mathematics department
Kyiv Taras Shevchenko University
64, Volodymyrska st.
252033 Kyiv
Ukraine
e-mail: mazorchu@uni-alg.kiev.ua

Institute of Mathematics
3, Tereshchenkivska st.
252603 Kyiv
Ukraine
e-mail: turowska@imath.kiev.ua