VIRASORO-TYPE ALGEBRAS ASSOCIATED WITH HIGHER-RANK APERIODIC POINT SETS

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A method for the construction of infinite-dimensional Lie algebras of Virasoro-type is discussed, which uses aperiodic point sets as basic building blocks. The corresponding algebras have generators in a one-to-one correspondence with aperiodic point sets that are obtained via a projection formalism from higher dimensional lattices. They share structural similarities with the Virasoro algebra by construction, but exhibit different properties.

1. Introduction

The Virasoro algebra plays a crucial role in many areas of mathematical physics, and algebraic techniques for a deformation of this algebra are of interest, because they lead to perturbations of these theories. We present here a construction method for Virasoro-type algebras which is based on aperiodic point sets that can be obtained via a projection from a higher dimensional periodic lattice[1]. These aperiodic structures play an important role in the study of quasicrystals, that is alloys with noncrystallographic symmetries and long-range order[2], but the algebras presented here are not constructed for applications in this area but rather for potential applications in the theory of integrable systems as we discuss in the concluding remarks.

The main idea of the construction is to substitute the index set of the Virasoro algebra by a suitable aperiodic point set and to make modifications in the structure constants to ensure that the algebras remain Lie algebras. The construction leads to a family of infinite dimensional Lie algebras with properties that depend crucially on geometric properties of the associated aperiodic structures. The case of one-dimensional aperiodic point sets has been studied in a series of papers [3,4,5,6], and most recently, generalisations to two-dimensional aperiodic point sets have been achieved[7]. In these references, also the structure of highest weight modules and the irreducibility of Verma modules have been studied. Due to the fact that one-dimensional point sets are obtained from two-dimensional lattices by projection, each point is specified by two coordinates. Therefore the corresponding algebras also have structural resemblances with higher-rank Virasoro algebras[8,9,10].

It is the purpose of this contribution to review the construction principle with focus on recent developments and to direct the reader to the corresponding literature for further details. In particular, section 2 introduces the projection method for aperiodic point sets based on two explicit examples, and section 3 reviews the construction of the corresponding Virasoro-type algebras with emphasis on the most recent developments. In the concluding remarks, possible applications in the theory of integrable systems are pointed out.

2. Aperiodic point sets via the projection method

The construction principle is applicable to all aperiodic point sets that are model sets[1]. Here focus will be placed on two special cases, which correspond to a family of one- and a family of two-dimensional aperiodic point sets, respectively, because they will be used for the discussion of the Virasoro-type algebras in section 3.

2.1. *A one-dimensional example*

Let $\tau = \frac{1}{2}(1+\sqrt{5})$, and let $\mathbb{Z}[\tau] = \{a + \tau b | a, b \in \mathbb{Z}\}$ denote the ring of integers in the algebraic extension $\mathbb{Q}[\sqrt{5}]$ of the rational numbers by $\sqrt{5}$. Consider the Galois automorphism $' : \mathbb{Q}[\sqrt{5}] \to \mathbb{Q}[\sqrt{5}]$ defined via $(a+\sqrt{5}b)' = a - \sqrt{5}b$. Restricted to $\mathbb{Z}[\tau]$, this automorphism maps $a + \tau b$ to $a + \tau' b$ and links the two solutions of the equation $x^2 = x + 1$, namely the first solution τ and the second solution $\tau' = \frac{1}{2}(1-\sqrt{5})$. Examples of one-dimensional aperiodic point sets are then defined as follows.

Let Ω be a bounded interval in $\mathbb{R}$. Then

$$\Sigma(\Omega) := \{x \in \mathbb{Z}[\tau] | x' \in \Omega\} \tag{1}$$

is a one-dimensional aperiodic point set.

The interval Ω is called acceptance window because it controls how many points are admitted in the point set.

2.2. *A two-dimensional example*

In order to generalise this formalism to two dimensions, we consider a projection from the root lattice of A_4 onto two hyperplanes, each containing a copy of the root system of type H_2[11].

Let $\xi := \exp(\frac{2\pi i}{5})$. Then the root system $\Delta_2 \subset \mathbb{C}$ of type H_2 contains ten roots, which are given in terms of the simple roots $\alpha_1 := \xi^0$ and $\alpha_2 := \xi^2$ as follows:

$$\Delta_2 = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \tau\alpha_2), \pm(\tau\alpha_1 + \alpha_2), \pm(\tau\alpha_1 + \tau\alpha_2)\}\,.$$

Denote by $\mathbb{Z}[\tau]\Delta_2$ the $\mathbb{Z}[\tau]$–lattice with basis $\{\alpha_1, \alpha_2\}$. The assignment $\xi^* = \xi^2$ uniquely extends to a $\mathbb{Z}[\tau]$–semilinear bijection * on $\mathbb{Z}[\tau]\Delta_2$. This map is usually called the *star map*[11] . Moreover, according to this reference, this map uniquely extends to a $\mathbb{Z}[\tau]$–semilinear bijection on $\mathbb{Q}[\tau]\Delta_2$, which we will also denote by *.

It is immediate that the star-map defined above acts like the original Galois automorphism $' : \mathbb{Q}[\sqrt{5}] \to \mathbb{Q}[\sqrt{5}]$ when restricted to $\mathbb{Z}[\tau]$. Indeed, $\tau^* =(-\xi^2 - \xi^3)^* = (-\xi^4 - \xi^6) =\tau'$. Obviously, the star map is a permutation on $\mathbb{Z}[\tau]\Delta_2$ (or $\mathbb{Q}[\tau]\Delta_2$) of order 4.

Let $\Omega \subset \mathbb{C}$ be a bounded set. Then

$$\Sigma(\Omega) = \{x \in \mathbb{Z}[\tau]\Delta_2 | x^* \in \Omega\} \tag{2}$$

is a planar aperiodic point set.

For applications to the algebraic setting, we consider in particular the case $\Omega \equiv \Omega_P^T := T\xi^0 + \Omega_P$, where Ω_P denotes the regular unital pentagon and Ω_P^T is a translation of Ω_P by $T\xi^0$, $T > 0$.

3. Virasoro-type algebras

In this section we introduce Lie algebras with generators in a one-to-one correspondence with the aperiodic point sets in (1) and (2).

3.1. *The one-dimensional case*

Witt-type algebras related to the point sets in (1) are given by the following Lemma[3]:

Lemma 3.1. *Let $\Sigma(\Omega)$ be a one-dimensional quasicrystal with acceptance window $[a, b]$, $(a, b]$, $[a, b)$, or (a, b), where $a \neq b \in \mathbb{R}$ and $0 \leq ab < \infty$. Then the algebra $Q(\Omega)$, defined as a linear span of*

$$B(Q(\Omega)) = \{L_n \mid n \in \Sigma(\Omega)\} \tag{3}$$

with commutation relations

$$[L_n, L_m] = \begin{matrix} (m-n)L_{n+m} & \text{if} \;\; n+m \in \Sigma(\Omega) \\ 0 & \text{if} \;\; n+m \notin \Sigma(\Omega) \end{matrix}\,, \tag{4}$$

is a Lie algebra.

Note that the Jacobi identity holds only provided that $ab \geq 0$ which implies a restriction on the aperiodic point sets that may be used as building blocks in the construction. Observe furthermore that in contrast to the Witt algebra (which is simple), many commutators in $Q(\Omega)$ vanish because $m, n \in \Sigma(\Omega)$ does not imply $m+n \in \Sigma(\Omega)$ in general. As a consequence, the algebras are locally finite, that is the closure of any finite set of generators from $B(Q(\Omega))$ under the Lie operation gives a finite dimensional subalgebra of $Q(\Omega)$.

Structure and existence of a central extension for these algebras depend on the choice of Ω and on the choice of the structure constants. It has been shown[5] that there exists a unique central extension for the algebras in Lemma 3.1 for $\Omega = [0,1]$ if the structure constants are given in terms of the second component of points $m = m_1 + \tau m_2 \in \Sigma(\Omega)$, that is if one chooses the structure constants $(m_2 - n_2)$ instead of $(m - n)$. The corresponding algebra has been called the *Aperiodic Virasoro algebra* $AV(\Omega)$. Highest weight representations for this algebra have been discussed in this reference and a conjecture for an analog to the Kac-determinant formula[12] has been suggested. The latter has later been proved[6].

3.2. *The two-dimensional case*

In this section we discuss a generalisation of the results in section 3.1 to the aperiodic point sets in (2)[7]. In order to introduce the algebras, we need the following terminology.

Definition 3.1. An acceptance window, Ω, is called *admissible* if for each triple n, m, $k \in \Sigma(\Omega)$ one has the condition

$$n^* + m^* + k^* \in \Omega \text{ and } n^* + m^* \in \Omega \Rightarrow m^* + k^* \in \Omega \text{ and } n^* + k^* \in \Omega\,. \quad (5)$$

The concept of admissibility is necessary to ensure that the aperiodic point sets are such that the corresponding algebras fulfill the Jacobi identity.

In particular, one obtains the following planar generalisation of Lemma 3.1[7].

Lemma 3.2. *Let $\mathbb{F}$ be any number field such that $\mathbb{F} \supset \mathbb{Q}[\tau]$ and let T be such that Ω_P^T is admissible. Let furthermore $\varphi : \Sigma(\Omega_P^T) \to \mathbb{F}$ be a $\mathbb{Z}$–linear map. Then for each such T the $\mathbb{F}$–span of $\{L_n | n \in \Sigma(\Omega_P^T)\}$ with the Lie* *bracket defined by*

$$[L_n, L_m] = \begin{cases} \varphi(m-n)L_{n+m} & \text{if } \; n+m \in \Sigma(\Omega_P^T) \\ 0 & \text{if } \; n+m \notin \Sigma(\Omega_P^T) \end{cases} \\ = \chi_{\Omega_P^T}(n^* + m^*)\varphi(m-n)L_{n+m}\,. \tag{6}$$

is a Lie algebra.

As in the one-dimensional case, the structural properties of these algebras depend on the geometry of the acceptance window Ω_P^T[7]. Moreover, as has been shown in this reference, these Lie algebras allow for a central extension. For a natural triangular decomposition the corresponding Verma modules have been studied.

In particular, if we denote by $\mathcal{V}(\Omega_P^T)^+$ the subalgebras generated by elements L_n, where $\Sigma(\Omega_P^T) \ni n = n_1\xi^0 + n_2\xi^2$ and $n_2' < 0$, and by $\mathcal{V}(\Omega_P^T)^-$ the subalgebras generated by elements L_n, where $\Sigma(\Omega_P^T) \ni n = n_1\xi^0 + n_2\xi^2$ and $n_2' > 0$, and call $\mathfrak{h}_T$ the subalgebra, generated by L_n, where $\Sigma(\Omega_P^T) \ni n = n_1\xi^0$, and by all central terms $\hat{c}_{v_\lambda}$ with $v_\lambda \in \mathbb{Z}[\tau] \cap [2T-\tau, T+1]$, then there is a triangular decomposition given by[7]

$$\mathcal{V}(\Omega_P^T) = \mathcal{V}(\Omega_P^T)^- \oplus \mathfrak{h}_T \oplus \mathcal{V}(\Omega_P^T)^+\,.$$

Let furthermore $\lambda \in \mathfrak{h}_T^*$ and consider $\mathbb{F}_\lambda = \mathbb{F}$ as an $\mathfrak{h}_T$-module with

$$\begin{aligned} hz &= \lambda(h)z \qquad \forall h \in \mathfrak{h}_T\,, z \in \mathbb{F}\,, \\ \mathcal{V}(\Omega_P^T)^+\mathbb{F}_\lambda &= 0 \end{aligned} \tag{7}$$

Then $\mathbb{F}_\lambda$ can be viewed as a $\mathfrak{b} = \mathfrak{h}_T \oplus \mathcal{V}(\Omega_P^T)^+$-module. Consider the induced module $M(\lambda) = U(\mathcal{V}(\Omega_P^T)) \otimes_{U(\mathfrak{b})} \mathbb{F}_\lambda$ and denote the unique simple quotient of $M(\lambda)$ by $L(\lambda)$. Then one has[7]:

Theorem 3.1.

(1) $M(\lambda)$ is always reducible.
(2) The module $L(\lambda)$ is one dimensional if and only if λ is zero on $\mathfrak{h}_T^{ess} := \langle \hat{c}_{v_\alpha}, L_{v'_\alpha} | v_\alpha \in \mathbb{Z}[\tau] \cap [2T-\tau, T+1]\rangle$. Otherwise, $L(\lambda)$ is infinite-dimensional.

4. Concluding remarks

The Virasoro algebra plays an important role in the framework of Calogero-Sutherland models[13,14,15]. In particular, the structure of its highest weight representations and especially its singular vectors allows one to construct explicit solutions[16,17]. It has been shown that a similar approach, based on the Aperiodic Virasoro algebra[18], that is for the Virasoro-type algebras related to one-dimensional aperiodic point sets, is possible. We expect that, along the same lines, the algebras related to two-dimensional aperiodic point sets that have been pointed out here should have applications in the framework of Calogero-Sutherland models.

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