

# On conjugation in some transformation and Brauer-type semigroups

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## Abstract

We give, in terms of (generalized) cyclic types, a criterion for two elements of  $\mathcal{T}_n$ ,  $\mathcal{PT}_n$ ,  $\mathcal{IS}(\mathbb{N})$ ,  $\mathfrak{B}_n$ ,  $\mathcal{PB}_n$  and  $\mathfrak{C}_n$  to be conjugate with respect to the conjugation defined as the transitive closure of the  $xy \sim yx$  relation.

## 1 Introduction

The notion of conjugation for semigroups can be defined (or generalized from the corresponding notion for groups) in several ways. Perhaps the two most commonly used notions are the relations  $\sim_G$  and  $\sim_S$ , whose definitions below are taken from [La]. However, these relations are not the only notions of conjugations considered for semigroups, see for example [Ch, GT] for different notions.

Let  $S$  be a monoid and  $G$  its group of units. We will say that  $a$  and  $b$  from  $S$  are  $G$ -conjugate and write  $a \sim_G b$  if there exists  $g \in G$  such that  $a = g^{-1}bg$ . Obviously,  $\sim_G$  is an equivalence relation on  $S$ . If  $S$  is a group, that is  $S = G$ , then  $\sim_G$  coincides with the usual group conjugation.

We call the elements  $a, b \in S$  *primarily  $S$ -conjugate* if there exist  $x, y \in S$  such that  $a = xy$  and  $b = yx$ . This will be denoted by  $a \sim_{pS} b$ . The relation  $\sim_{pS}$  is reflexive and symmetric but not transitive in general. Denote by  $\sim_S$  the transitive closure of  $\sim_{pS}$ . If  $a \sim_S b$  we will say that the elements  $a$  and  $b$  are  $S$ -conjugate. The fact that  $\sim_{pS}$  is not transitive can be seen on the following example: one can show that the zero element of the symmetric inverse semigroup  $\mathcal{IS}_n$  is  $S$ -conjugate to any nilpotent element in this semigroup, however, such elements are rarely primarily  $S$ -conjugate, see [GK]. It is easy to see that in the case of a group the relation  $\sim_S$  coincides with the usual group conjugation and thus the relations  $\sim_G$  and  $\sim_S$  coincide. However, in the general case one only has the obvious inclusion  $\sim_G \subset \sim_S$ .

The notion of  $G$ -conjugacy for semigroups is usually much easier to understand than  $S$ -conjugacy. In many cases, for example for the classical transformation semigroups  $\mathcal{T}_n$ ,  $\mathcal{PT}_n$ , and  $\mathcal{IS}_n$  (that is the full transformation semigroup, the semigroup of all partial transformations, and the symmetric inverse semigroup on  $\{1, 2, \dots, n\}$ ), the description of  $G$ -conjugacy is fairly straightforward, see for example [Li, Chapter 13]. In contrast to this the description of  $S$ -conjugacy is usually a more subtle problem. In the present literature, as far as we know, the only semigroup, where the  $S$ -conjugacy classes have been described, is  $\mathcal{IS}_n$  (see [GK] or [GM, Theorem 9.1]). The aim of the present paper is to describe  $S$ -conjugacy classes in  $\mathcal{T}_n$ ,  $\mathcal{PT}_n$ ,  $\mathcal{IS}(\mathbb{N})$ , where  $\mathbb{N} = \{1, 2, \dots\}$ , and also in certain Brauer-type semigroups. In particular, in what follows by conjugacy we always mean  $S$ -conjugacy. It happens that in all of the cases we consider here one can define a proper generalization of the notion of the cyclic type of an element, after which the conjugacy criterion can be formulated in the same way as it is formulated for the symmetric group. Though all the answers we have obtained are rather similar, every class of semigroups mentioned above requires a thorough special study because of their quite different nature.

We also would like to pose the following **problem**: describe the conjugacy classes of  $\mathcal{T}(\mathbb{N})$  and  $\mathcal{PT}(\mathbb{N})$ .

The paper is organized as follows: in Section 2 we collect some preliminaries about  $S$ -conjugacy and in Section 3 we collect all necessary preliminaries about transformation semigroups. The conjugacy classes in  $\mathcal{T}_n$  and  $\mathcal{PT}_n$  are described in Section 4. In Section 5 we describe the conjugacy classes in  $\mathcal{IS}(\mathbb{N})$ . In Section 6 we collect all necessary preliminaries about Brauer-type semigroups. Finally, in Section 7 we describe the conjugacy classes in the semigroups  $\mathfrak{B}_n$ ,  $\mathcal{PB}_n$  and  $\mathfrak{C}_n$ .

## 2 Preliminaries about $S$ -conjugation

We start with some basic general facts about  $S$ -conjugacy.

**Proposition 1.** *Let  $S$  be a semigroup and let  $x, y \in S$ . Then*

- (1) *if  $x \sim_{pS} y$ , then  $x^i \sim_{pS} y^i$  for all  $i \in \mathbb{N}$ .*
- (2) *if  $e = x^i$  and  $f = y^j$  are idempotents and  $x \sim_{pS} y$ , then  $e \sim_{pS} f$ .*

*Proof.* Let  $x = ab$  and  $y = ba$  for some  $a, b \in S$ . Then  $x^i = (ab)^i = a((ba)^{i-1}b)$  and  $y^i = (ba)^i = ((ba)^{i-1}b)a$  implying  $x^i \sim_{pS} y^i$  and proving (1). To prove (2) we observe that  $e = x^i = x^{ij}$  and  $f = y^j = y^{ij}$  and that  $x^{ij} \sim_{pS} y^{ij}$  by (1).  $\square$

**Lemma 2.** *Let  $S$  be a semigroup and  $e, f, g$  be three idempotents from  $S$  such that  $e \sim_{pS} f$  and  $e \sim_{pS} g$ . Then  $f \sim_{pS} g$ .*

*Proof.* Let  $x, y, u, v \in S$  be such that  $e = xy, f = yx, e = uv, g = vu$ . Since  $e$  is an idempotent we also have  $e = e^2 = xyxy = xfy$  and analogously  $f = yex, e = ugv, g = veu$ . This implies  $g = vxfyu$  and  $f = yugvx$ . Therefore  $g = g^2 = (vxf)(yug)$  and  $f = f^2 = (yug)(vxf)$  and thus  $g \sim_{pS} f$ .  $\square$

**Corollary 3.** *Let  $S$  be a finite semigroup and  $x, y \in S$  be such that  $x \sim_S y$ . Let  $i, j \in \mathbb{N}$  be such that  $e = x^i$  and  $f = y^j$  are idempotents. Then  $e \sim_{pS} f$ .*

*Proof.* Let  $x = x_1, x_2, \dots, x_l = y$  be a sequence of elements from  $S$  such that  $x_i \sim_{pS} x_{i+1}$  for  $i = 1, \dots, l-1$ . Since  $S$  is finite, for every  $i = 2, \dots, l-1$  there exists  $m_i \in \mathbb{N}$  such that  $y_i = x_i^{m_i}$  is an idempotent. Let  $y_1 = e$  and  $y_l = f$ . From Proposition 1(2) we obtain  $y_i \sim_{pS} y_{i+1}$  for all  $i = 1, \dots, l-1$ . Applying Lemma 2 inductively, we get that  $e \sim_{pS} y_i$  for all  $i = 2, \dots, l$ . In particular,  $e \sim_{pS} f$ .  $\square$

We denote by  $\mathcal{D}, \mathcal{L}, \mathcal{R}$ , and  $\mathcal{H}$  the classical Green relations. For a semigroup,  $S$ , and an idempotent,  $e \in S$ , we denote by  $G(e)$  the maximal subgroup of  $S$ , corresponding to  $e$ . Recall, see for example [CP, § 1.7], that  $G(e)$  consists of all  $x \in S$  such that  $x e = e x = x$  and  $x^i = e$  for some  $i \in \mathbb{N}$ .

**Proposition 4.** *Let  $S$  be a finite semigroup and  $e, f$  be two idempotents from  $S$  such that  $e \sim_S f$ . Then*

- (1)  $e \sim_{pS} f$ .
- (2)  $e \mathcal{D} f$ , in particular, there exist  $x, y \in S$  such that  $x \mathcal{L} f, x \mathcal{R} e, y \mathcal{L} e, y \mathcal{R} f$ ,  $xy = e$  and  $yx = f$ .
- (3) The map  $\varphi : G(e) \rightarrow G(f)$ , defined via  $\varphi(a) = y a x$ , is an isomorphism with inverse  $\psi : G(f) \rightarrow G(e)$ ,  $\psi(b) = x b y$ .
- (4) For every  $a \in G(e)$  we have  $a \sim_{pS} \varphi(a)$ .

*Proof.* (1) is a special case of Corollary 3. Let  $x, y \in S$  be such that  $xy = e$  and  $yx = f$ . Then  $e = e^2 = xyxy = xfy$  and  $f = f^2 = yxyx = yex$ . In particular,  $e \mathcal{D} f$ , and the rest of (2) follows from [CP, Theorem 2.17]. Now (3) follows from [CP, Theorem 2.20]. Finally, let  $a \in G(e)$  and  $b = \varphi(a) = y a x$ . Then  $a x y = a e = a$ , which implies  $a \sim_{pS} b$ . This completes the proof.  $\square$

Finally, we have the following useful statement, which, together with Proposition 4(4), roughly describes the parts of  $S$ -conjugacy classes, which belong to the maximal subgroups of  $S$ :

**Proposition 5.** *Let  $S$  be a finite semigroup,  $e, f$  be two idempotents from  $S$ , and  $a \in G(e)$ ,  $b \in G(f)$ . Assume  $a \sim_S b$ . Then  $a \sim_{pS} b$ .*

*Proof.* Since  $a \in G(e)$ ,  $b \in G(f)$ , we have  $a \sim_S b$  implies  $e \sim_{pS} f$  by Corollary 3. From Proposition 4(2) we thus get  $a\mathcal{D}b$ .

Since  $a \sim_S b$ , there exist  $t_0, \dots, t_l, x_0, \dots, x_{l-1}$  and  $y_0, \dots, y_{l-1}$  in  $S$  such that  $a = t_0$ ,  $b = t_l$ , and  $t_i = x_i y_i$ ,  $t_{i+1} = y_i x_i$  for all  $i$ . Let  $N \in \mathbb{N}$  be such that  $e_i = t_i^N$  is an idempotent for every  $i = 0, \dots, l$ . Such  $N$  obviously exists. Then  $t_i \sim_{pS} t_{i+1}$  implies  $e_i \sim_{pS} e_{i+1}$  for all  $i = 0, \dots, l$  by Proposition 1(1). In particular, from Proposition 4(2) we have  $e_i \mathcal{D} e_j$  for all  $i, j \in \{0, \dots, l\}$ . Further,  $t_i^{N+1} \sim_{pS} t_{i+1}^{N+1}$  for all  $i = 0, \dots, l$  by Proposition 1(1). However,  $t_0^{N+1} = a$ ,  $t_l^{N+1} = b$ , and we have  $t_i^{N+1} \in G(e_i)$  for all  $i = 0, \dots, l$  (since  $t_i^{2N+1} = t_i^{N+1}$  and  $t_i^{N(N+1)} = t_i^N$ ). Substituting, if necessary,  $t_i$  with  $t_i^{N+1}$  we can assume that  $t_i \mathcal{D} t_j$  for all  $i, j \in \{0, \dots, l\}$  and that all  $t_i$  are group elements in the corresponding  $G(e_i)$ .

Let  $i \in \{0, \dots, l-1\}$ . Then  $e_i = t_i^N = (x_i y_i)^N$  and  $e_{i+1} = t_{i+1}^N = (y_i x_i)^N$ . Thus we have

$$\begin{aligned} t_i &= t_i^{3N+1} = (x_i y_i)^{3N+1} = (x_i y_i)^N x_i (y_i x_i)^N y_i (x_i y_i)^N = e_i x_i e_{i+1} y_i e_i, \\ t_{i+1} &= t_{i+1}^{3N+1} = (y_i x_i)^{3N+1} = (y_i x_i)^N y_i (x_i y_i)^N x_i (y_i x_i)^N = e_{i+1} y_i e_i x_i e_{i+1}. \end{aligned}$$

Further, we have

$$t_i (x_i y_i)^{N-1} x_i = (x_i y_i)^N x_i (y_i x_i)^N y_i (x_i y_i)^N (x_i y_i)^{N-1} x_i = (x_i y_i)^N x_i (y_i x_i)^N,$$

implying  $e_i x_i e_{i+1} \mathcal{R} t_i$ . Analogously one shows that  $e_{i+1} y_i e_i \mathcal{R} t_{i+1}$ . Therefore, substituting, if necessary,  $x_i$  by  $e_i x_i e_{i+1}$  and  $y_i$  by  $e_{i+1} y_i e_i$ , we can assume that all  $t_i$ ,  $x_i$  and  $y_i$  belong to the same  $\mathcal{D}$ -class, moreover, that  $e_i x_i e_{i+1} = x_i$  and  $e_{i+1} y_i e_i = y_i$  for all  $i$  as well.

Let us now prove  $t_0 \sim_{pS} t_l$  by induction in  $l$ . For  $l = 1$  there is nothing to prove. To establish the induction step it is enough to prove the statement for  $l = 2$ . Since  $e_1 = (y_0 x_0)^N = (x_1 y_1)^N$ , we have

$$\begin{aligned} t_0 &= x_0 y_0 = (x_0 y_0)^{N+1} = x_0 (y_0 x_0)^N y_0 = x_0 (x_1 y_1)^N y_0 = \\ &= x_0 x_1 t_2^{N-1} y_1 y_0 = (x_0 x_1) (t_2^{N-1} y_1 y_0). \end{aligned}$$

On the other hand,

$$(t_2^{N-1} y_1 y_0) (x_0 x_1) = t_2^{N-1} y_1 y_0 x_0 x_1 = t_2^{N-1} y_1 x_1 y_1 x_1 = t_2^{N+1} = t_2,$$

which implies  $t_0 \sim_{pS} t_2$ . Induction completes the proof.  $\square$

### 3 Preliminaries about transformation semi-groups

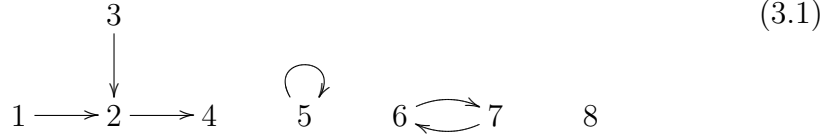
Let  $M$  be a set. The *full transformation semigroup*  $\mathcal{T}(M)$  on  $M$  is the set of all maps from  $M$  to  $M$  under the operation of composition of maps. The *symmetric inverse semigroup*  $\mathcal{IS}(M)$  is the set of all partial injective maps, i.e. injective maps  $f : M' \rightarrow M$ , where the *domain*  $\text{dom}(f) = M'$  of  $f$  is a subset of  $M$ , under the operation of composition of partial maps. Both  $\mathcal{T}(M)$  and  $\mathcal{IS}(M)$  contain the *symmetric group*  $\mathcal{S}(M)$  as the group of units. Moreover, they can be considered as (natural) extensions of  $\mathcal{S}(M)$  to the class of all semigroups and all inverse semigroups respectively. In particular, each semigroup is embeddable into  $\mathcal{T}(M)$  for some  $M$  and each inverse semigroup is embeddable into  $\mathcal{IS}(M)$  for some  $M$ . Combining the definitions of  $\mathcal{T}(M)$  and  $\mathcal{IS}(M)$  we get the semigroup  $\mathcal{PT}(M)$  of all partial but not necessarily injective maps from  $M$  to  $M$ . For  $\alpha \in \mathcal{PT}(M)$  we denote by  $\text{ran}(\alpha)$  the *range* of  $\alpha$ . For  $\alpha \in \mathcal{PT}(M)$  to indicate that  $x \notin \text{dom}(\alpha)$  we will sometimes write  $\alpha(x) = \emptyset$ .

Let  $S$  denote one of the semigroups  $\mathcal{IS}(M)$ ,  $\mathcal{T}(M)$  or  $\mathcal{PT}(M)$  and  $\alpha \in S$ . The *graph of the action of  $\alpha$*  is the directed graph  $\Gamma(\alpha)$ , whose set  $V(\alpha)$  of vertices coincides with  $M$ , and where two vertices,  $x, y \in V(\alpha)$ , are joined by a *directed edge*  $(x, y)$  if and only if  $\alpha(x) = y$ . The following statement is obvious.

**Proposition 6.** *Let  $\alpha, \beta \in S$ . Then  $\alpha \sim_{S(M)} \beta$  if and only if the graphs  $\Gamma(\alpha)$  and  $\Gamma(\beta)$  are isomorphic as directed graphs.*

Denote by  $\mathfrak{K}(\alpha)$  the set of all connected components, i.e. maximal connected directed subgraphs of  $\Gamma(\alpha)$ . We note that by a subgraph we will always mean a full subgraph (i.e. the one which inherits all possible edges). Hence, every  $K \in \mathfrak{K}(\alpha)$  is uniquely determined by its set of vertices, which is a subset of  $M$ . We will denote this subset also by  $K$  abusing notation (i.e. we identify  $K$  with its set of vertices). Denote by  $\alpha_K$  the element of  $S$  defined as follows:  $\text{dom}(\alpha_K) = (M \setminus K) \cup (\text{dom}(\alpha) \cap K)$ ,  $\alpha_K(x) = \alpha(x)$  if  $x \in \text{dom}(\alpha) \cap K$ , and  $\alpha_K(x) = x$  if  $x \notin K$ . The elements  $\alpha_K$  will be called *connected elements* and  $K$  will be called the *support* of  $\alpha_K$ . Clearly,  $\alpha_K$  is uniquely determined by  $K$  and  $\alpha$  and for any connected element, different from the identity element  $\text{id}$ , its support is uniquely determined. For  $\text{id}$  we will (slightly ambiguously) say that every one-point subgraph of  $\Gamma(\text{id})$  can be taken as its support. For example, consider the element  $\alpha \in \mathcal{PT}(\{1, \dots, 8\})$

with the following  $\Gamma(\alpha)$ :



This graph has four connected components and gives rise to three non-trivial connected elements (since one of the components is the identity). In particular, we have the following  $\Gamma(\alpha_{\{1,2,3,4\}})$ :



The following two statements are rather straightforward.

**Lemma 7.** *For all  $K, K' \in \mathfrak{K}(\alpha)$  we have  $\alpha_K \alpha_{K'} = \alpha_{K'} \alpha_K$ , moreover,*

$$\alpha = \prod_{K \in \mathfrak{K}(\alpha)} \alpha_K.$$

The decomposition, given by Lemma 7 is called the *connected decomposition* of  $\alpha$ .

**Proposition 8.** *Each element of  $S$  decomposes into a (possibly infinite) product of connected elements with pairwise disjoint supports. This product is unique up to permutations of factors and deletions of the unit element.*

It is natural to consider Proposition 8 as an analogue of the cyclic decomposition for elements of  $\mathcal{S}_n$ .

## 4 Conjugation in the full finite transformation semigroups $\mathcal{T}_n$ , $\mathcal{IS}_n$ and $\mathcal{PT}_n$

Assume that  $M$  is finite and  $n = |M|$ . For such  $M$  we denote the semigroups  $\mathcal{T}(M)$ ,  $\mathcal{IS}(M)$  and  $\mathcal{PT}(M)$  by  $\mathcal{T}_n$ ,  $\mathcal{IS}_n$  and  $\mathcal{PT}_n$  respectively. Let  $\alpha$  be an element of one of the semigroups  $\mathcal{T}_n$ ,  $\mathcal{IS}_n$  or  $\mathcal{PT}_n$ . Let  $k \in \mathbb{N}$ . We will call  $\alpha$  an *oriented cycle of length  $k$*  if there exists a set,  $K = \{x_1, x_2, \dots, x_k\} \subset M$  such that  $K \in \mathfrak{K}(\alpha)$  is the cycle  $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_k \rightarrow x_1$  and  $\alpha = \alpha_K$ . We denote the oriented cycle by  $(x_1, x_2, \dots, x_k)$ . We start with the following easy lemma.

**Lemma 9.** (1) Let  $\alpha \in \mathcal{T}_n$ . Then every connected component of  $\Gamma(\alpha)$  contains exactly one directed full subgraph, which is an oriented cycle.

(2) Let  $\alpha \in \mathcal{PT}_n$  or  $\alpha \in \mathcal{IS}_n$ . Then every connected component of  $\Gamma(\alpha)$  contains at most one directed full subgraph, which is an oriented cycle.

*Proof.* Assume first that  $\alpha$  is an element of either  $\mathcal{T}_n$  or  $\mathcal{PT}_n$  or  $\mathcal{IS}_n$ . Let us show that no connected component of  $\Gamma(\alpha)$  contains more than one oriented full subgraph, which is an oriented cycle. Indeed, suppose that this is not true and that some  $K \in \mathfrak{K}(\alpha)$  contains at least two different cycles, say  $a_1 \rightarrow a_2 \rightarrow \cdots \rightarrow a_s \rightarrow a_1$  and  $b_1 \rightarrow b_2 \rightarrow \cdots \rightarrow b_t \rightarrow b_1$ . Since every point is the source of exactly one arrow, these two cycles do not intersect. Both cycles belong to  $K$ , which is connected, hence there is a non-oriented finite path,  $a_i = x_1, x_2, \dots, x_{l-1}, x_l = b_j$ , from some  $a_i$  to some  $b_j$ . Without loss of generality we can assume that this path does not contain other points from our cycles. Since  $a_1$  belongs to an oriented cycle and every point is a source of at most one arrow, the arrow between  $a_1$  and  $x_2$  must terminate at  $a_1$ . The same argument implies that the arrow between  $x_2$  and  $x_3$  must terminate at  $x_2$  and so on. We will get that the arrow between  $x_{l-1}$  and  $x_l$  must terminate at  $x_{l-1}$ . This implies that there are at least two arrows starting at  $x_l = b_j$ , since the last point belongs to an oriented cycle. A contradiction. This proves the second statement.

Let now  $\alpha \in \mathcal{T}_n$  and  $K \in \mathfrak{K}(\alpha)$ . Since each point is the beginning of exactly one arrow,  $K$  has the same number of vertices and edges, and thus  $K$  is not a tree. Hence,  $K$  has at least one cycle, which is oriented by the same argument as above. This completes the proof.  $\square$

Let  $\alpha \in \mathcal{T}_n$  or  $\alpha \in \mathcal{IS}_n$  or  $\alpha \in \mathcal{PT}_n$ . For each  $i$ ,  $1 \leq i \leq n$ , let  $l_i$  denote the number of cycles of length  $i$  in  $\Gamma(\alpha)$ . The vector  $\text{ct}(\alpha) = (l_1, l_2, \dots, l_n)$  will be called the *cyclic type* of  $\alpha$ . Sometimes it will be convenient to compare the cyclic types of the elements from different semigroups and even for different  $n$ . In these cases all  $l_i$ , which are not defined, will be considered to be equal to 0.

Let  $\alpha \in \mathcal{IS}_n$  or  $\alpha \in \mathcal{PT}_n$  and  $k \in \mathbb{N}$ . We will call  $\alpha$  an *oriented chain of length  $k$*  if there exists a set,  $C = \{c_1, c_2, \dots, c_k\} \subset M$ , such that  $C \in \mathfrak{K}(\alpha)$  is the chain  $c_1 \rightarrow c_2 \rightarrow \cdots \rightarrow c_k$  (in particular,  $c_k \notin \text{dom}(\alpha)$ ) and  $\alpha = \alpha_K$ . Note that this chain may contain exactly 1 element, for example the element 8 for  $\alpha$  from the example (3.1) (in this example  $\alpha_{\{8\}}$  is an oriented chain of length 1). We will denote such  $\alpha$  by  $[c_1, c_2, \dots, c_k]$ . For such  $\alpha$  we have  $c_1 \notin \text{ran}(\alpha)$ ,  $\alpha(c_i) = c_{i+1}$ ,  $i = 1, \dots, k-1$ ,  $c_k \notin \text{dom}(\alpha)$ , and  $\alpha(a) = a$  for all  $a \notin C$ .

From [GM, Theorem 4.1] it follows that a connected element from  $\mathcal{IS}_n$  is either a chain or a cycle, and hence the unique decomposition of  $\alpha \in \mathcal{IS}_n$  stated in Proposition 8 is called the *chain decomposition* of  $\alpha$  (in [Li] the author calls it the *chart decomposition*). Let  $\alpha \in \mathcal{IS}_n$ . For each  $i$ ,  $1 \leq i \leq n$ , let  $m_i$  denote the number of chains of length  $i$  in the chain decomposition of  $\alpha$ . The vector  $\text{chat}(\alpha) = (m_1, m_2, \dots, m_n)$  will be called the *chain type* of  $\alpha$ . As a direct consequence from Proposition 6 we obtain that for  $\alpha, \beta \in \mathcal{IS}_n$  one has  $\alpha \sim_{\mathcal{S}_n} \beta$  if and only if  $\text{ct}(\alpha) = \text{ct}(\beta)$  and  $\text{chat}(\alpha) = \text{chat}(\beta)$  (see also [Li, Section 13]).

In [GK] it was shown that for  $\alpha, \beta \in \mathcal{IS}_n$  one has  $\alpha \sim_{\mathcal{IS}_n} \beta$  if and only if  $\text{ct}(\alpha) = \text{ct}(\beta)$ , while there are no restrictions on the chain types of  $\alpha$  and  $\beta$ . The following theorem provides analogous statements for the semigroups  $\mathcal{T}_n$  and  $\mathcal{PT}_n$ , moreover, all the results can be formulated in a unified way.

**Theorem 10.** *Let  $S$  denote one of the semigroups  $\mathcal{T}_n$ ,  $\mathcal{IS}_n$  or  $\mathcal{PT}_n$  and  $\alpha, \beta \in S$ . Then  $\alpha \sim_S \beta$  if and only if  $\text{ct}(\alpha) = \text{ct}(\beta)$ .*

*Proof.* For the  $\mathcal{IS}_n$  case we refer the reader to [GK], and hence we can let  $S$  be one of the semigroups  $\mathcal{T}_n$  or  $\mathcal{PT}_n$ . Let  $\alpha \in S$ . Since  $\text{ran}(\alpha) \supset \text{ran}(\alpha^2) \supset \dots$ , and at most  $n$  inclusions are strict, it follows that there is a positive integer,  $k$ , such that  $\text{ran}(\alpha^k) = \text{ran}(\alpha^{k+i})$  for all  $i \geq 1$ . The least  $k$  with this property will be called the *length* of  $\alpha$  and denoted by  $l(\alpha)$ . The set  $\text{stran}(\alpha) = \text{ran}(\alpha^k)$  will be called the *stable range* of  $\alpha$ . Note that, if  $\alpha = \prod_{K \in \mathfrak{R}(\alpha)} \alpha_K$  is the connected decomposition of  $\alpha$ , then  $l(\alpha) = \max_{K \in \mathfrak{R}(\alpha)} l(\alpha_K)$ . We also note that  $l(\alpha) = 1$  if and only if  $\text{ran}(\alpha) = \text{stran}(\alpha)$ . Further, for each  $\alpha$  the restriction of  $\alpha$  to  $\text{stran}(\alpha)$  is a permutation on  $\text{stran}(\alpha)$ . To proceed we need the following statement, which allows us to “splinter” all oriented chains of an element into oriented chains of length one staying within a given  $S$ -conjugacy class.

**Lemma 11.** *For each  $\alpha \in S$  there exists  $\alpha' \in S$  such that  $\alpha$  and  $\alpha'$  are  $S$ -conjugate,  $\Gamma(\alpha)$  and  $\Gamma(\alpha')$  have the same cycles, and  $l(\alpha') = 1$ .*

*Proof.* Without loss of generality we can assume  $l(\alpha) > 1$ . Fix  $K \in \mathfrak{R}(\alpha)$  such that  $l(\alpha_K) > 1$ . It follows from Lemma 9 that  $K$  contains either a unique cycle, or no cycles at all. If  $K$  contains exactly one cycle, say  $a_1 \rightarrow a_2 \rightarrow \dots \rightarrow a_s \rightarrow a_1$ , beside this cycle  $K$  can contain only some subtrees terminating at the points of the cycle. It follows easily from the definition of the stable range that  $A = \{a_1, a_2, \dots, a_s\} = K \cap \text{stran}(\alpha)$  and that the length of  $\alpha_K$  equals the length of the longest path from a point of  $K$  to the closest point of  $A$ . If  $K$  does not contain any cycle (which is possible only if  $S = \mathcal{PT}_n$ ), then it follows easily from the definition of the stable range that



$K \cap \text{stran}(\alpha) = \emptyset$  and that the length of  $\alpha_K$  equals the length of the longest path in  $K$ .

Let  $K \setminus \text{ran}(\alpha) = \{c_1, c_2, \dots, c_t\} = C$ . For every  $i \in \{1, \dots, t\}$  set  $b_i = \alpha(c_i)$  and  $d_i = \alpha(b_i)$  (we note that in the case of  $\mathcal{PT}_n$  some of  $b_i$ -s and  $d_i$ -s can be undefined, in particular, if some  $b_i$  is undefined then so is the corresponding  $d_i$ ). Set  $x_K = \alpha_K$ , and define  $y_K$  in the following way: set  $y_K(c_i) = d_i$  for every  $i$  (in the case when  $d_i$  is undefined we say that  $y_K(c_i)$  is undefined) and  $y_K(c) = c$  if  $c \notin C$ . Then  $y_K x_K = \alpha_K$ ,  $x_K y_K(c_i) = \alpha_K(d_i)$  for every  $i$ , and  $x_K y_K(c) = \alpha_K(c)$  for every  $c \notin C$ . Set  $\alpha_K^{(1)} = x_K y_K$ . From the construction of  $\alpha_K^{(1)}$  we immediately have that  $l(\alpha_K^{(1)}) = l(\alpha_K) - 1$ .

Set  $x = \prod_{K \in \mathfrak{K}(\alpha)} x_K$ ,  $y = \prod_{K \in \mathfrak{K}(\alpha)} y_K$ . Using Lemma 7 we easily get

$$\alpha = \prod_{K \in \mathfrak{K}(\alpha)} \alpha_K = \prod_{K \in \mathfrak{K}(\alpha)} (y_K x_K) = \prod_{K \in \mathfrak{K}(\alpha)} y_K \prod_{K \in \mathfrak{K}(\alpha)} x_K = yx.$$

For  $\alpha^{(1)} = \prod_{K \in \mathfrak{K}(\alpha)} \alpha_K^{(1)}$  we repeat the arguments above and get  $\alpha^{(1)} = xy$ . Hence,  $\alpha \sim_{pS} \alpha^{(1)}$ . Moreover, from the above remark on  $l(\alpha_K^{(1)})$  it follows that  $l(\alpha^{(1)}) = l(\alpha) - 1$ . Now inductive arguments on the length of  $\alpha$  show that  $\alpha$  is  $S$ -conjugate to some  $\alpha'$  satisfying the following conditions:

- $\alpha'(x) = \alpha(x)$  for all  $x \in \text{stran}(\alpha)$ ,
- for every  $K \in \mathfrak{K}(\alpha)$  containing a cycle and for all  $x \in K$  we have  $\alpha'(x) \in \text{stran}(\alpha)$ ,
- for every  $K \in \mathfrak{K}(\alpha)$  which does not contain any cycle and for all  $x \in K$  we have that  $\alpha'(x)$  is not defined.

□

Assume now that  $\text{ct}(\alpha) = \text{ct}(\beta)$ . It follows from Lemma 11 that there exist  $\alpha'$  and  $\beta'$  such that  $\text{ct}(\alpha) = \text{ct}(\alpha')$ ,  $\text{ct}(\beta) = \text{ct}(\beta')$  and  $l(\alpha') = l(\beta') = 1$ . Suppose first that  $\text{stran}(\alpha) \neq \emptyset$ . Fix any point  $a \in \text{stran}(\alpha) = \text{stran}(\alpha')$ . Set  $x = \alpha'$  and define  $y$  as follows:  $y(c) = c$  if  $c \in \text{stran}(\alpha')$  and  $y(c) = a$  otherwise. We immediately get  $yx = \alpha'$  and  $xy = \alpha''$ , where  $\alpha''(c) = \alpha'(c) = \alpha(c)$  if  $c \in \text{stran}(\alpha)$  and  $\alpha''(c) = \alpha(a)$  otherwise. Hence, we have that  $\alpha \sim_S \alpha''$ , moreover  $\Gamma(\alpha)$  and  $\Gamma(\alpha'')$  have the same cycles, and  $\alpha''(b) = \alpha(a)$  for every  $b \notin \text{stran}(\alpha'')$ . Analogously we construct the element  $\beta''$ . From the construction it follows that  $\Gamma(\alpha'')$  and  $\Gamma(\beta'')$  are isomorphic (for appropriate choices of  $a$  and the corresponding point in the construction of  $\beta''$ ), which, by Proposition 6, implies  $\alpha'' \sim_{S_n} \beta''$ . Finally, we obtain

$$\alpha \sim_S \alpha'' \sim_S \beta'' \sim_S \beta,$$

and, therefore,  $\alpha \sim_S \beta$ .

Suppose now  $\text{stran}(\alpha) = \emptyset$ . In this case we automatically have  $S = \mathcal{PT}_n$  and  $\alpha' = \beta' = 0$  ( $0 \in \mathcal{PT}_n$  is the unique element satisfying  $\text{dom}(0) = \emptyset$ ). Hence  $\alpha \sim_{\mathcal{PT}_n} \alpha' = \beta' \sim_{\mathcal{PT}_n} \beta$ . This proves sufficiency of our conditions.

To prove necessity it is enough to consider  $\alpha, \beta \in S$  such that  $\alpha \sim_{pS} \beta$ , that is  $\alpha = xy$  and  $\beta = yx$  for some  $x, y \in S$ . Using Proposition 1(1) we have  $\alpha^k \sim_{pS} \beta^k$  for all  $k \in \mathbb{N}$ . Choose  $k$  such that it is divisible by the lengths of all cycles in both  $\Gamma(\alpha)$  and  $\Gamma(\beta)$ , and  $k > \max\{\ell(\alpha), \ell(\beta)\}$ . Then  $\alpha^{k+1}$  and  $\beta^{k+1}$  are  $S$ -conjugate by Proposition 1(1). Moreover, they have length 1,  $\text{ct}(\alpha^{k+1}) = \text{ct}(\alpha)$ , and  $\text{ct}(\beta^{k+1}) = \text{ct}(\beta)$ . Hence, it remains to show that  $\text{ct}(\alpha^{k+1}) = \text{ct}(\beta^{k+1})$ .

Denote  $\alpha^{k+1}$  by  $\alpha_1$  and  $\beta^{k+1}$  by  $\beta_1$ . From  $\alpha_1 \sim_{pS} \beta_1$  it follows that there exist  $x, y$  such that  $\alpha_1 = xy$ ,  $\beta_1 = yx$ . This implies  $\alpha_1^2 = x\beta_1y$ , and therefore  $|\text{ran}(\alpha_1)| = |\text{ran}(\alpha_1^2)| \leq |\text{ran}(\beta_1)|$ . Analogously we obtain the opposite inequality:  $|\text{ran}(\beta_1)| \leq |\text{ran}(\alpha_1)|$ , that is  $|\text{ran}(\alpha_1)| = |\text{ran}(\beta_1)|$ . Let  $\gamma \in \mathcal{S}_n$  be such that it maps  $\text{ran}(\beta_1)$  onto  $\text{ran}(\alpha_1)$ . Set  $\beta_2 = \gamma\beta_1\gamma^{-1}$  and  $x_1 = x\gamma^{-1}$ ,  $y_1 = \gamma y$ . In this notation  $\beta_2 = y_1x_1$ ,  $\alpha_1 = x_1y_1$ , in particular,  $\beta_2 \sim_{pS} \alpha_1$ . Since  $\text{stran}(\beta_2) = \text{stran}(\alpha_1)$ , from the definition of the stable range it follows that both  $x_1$  and  $y_1$  must map  $\text{stran}(\beta_2)$  to  $\text{stran}(\beta_2)$ . Hence we can consider the restrictions  $\bar{\beta}_2, \bar{\alpha}_1, \bar{y}_1$  and  $\bar{x}_1$  of  $\beta_2, \alpha_1, y_1$ , and  $x_1$  respectively to  $\text{stran}(\beta_2)$ . They all belong to  $\mathcal{S}(\text{stran}(\beta_2))$  and  $\bar{\beta}_2 = \bar{y}_1\bar{x}_1$ ,  $\bar{\alpha}_1 = \bar{x}_1\bar{y}_1$ . Therefore,  $\bar{\beta}_2$  and  $\bar{\alpha}_1$  are  $\mathcal{S}(\text{stran}(\beta_2))$ -conjugate. From the description of conjugacy classes in the symmetric group  $\mathcal{S}(\text{stran}(\beta_2))$  we have  $\text{ct}(\bar{\beta}_2) = \text{ct}(\bar{\alpha}_1)$ . To complete the proof we have only to note that  $\text{ct}(\bar{\beta}_2) = \text{ct}(\beta_1)$  and  $\text{ct}(\bar{\alpha}_1) = \text{ct}(\alpha_1)$  by construction.  $\square$

In particular, we have the following easy corollary, which reveals a kind of “stable behavior” of the notion of  $S$ -conjugacy for  $\mathcal{IS}_n, \mathcal{T}_n$  or  $\mathcal{PT}_n$ :

**Corollary 12.** *Let  $S = \mathcal{IS}_n, \mathcal{T}_n$  or  $\mathcal{PT}_n$  and  $x \in S$ . Let  $i \in \mathbb{N}$  be such that  $e = x^i$  is an idempotent. Then  $x \sim_S xe$ .*

*Proof.* It is easy to see that  $x$  and  $xe$  have the same cyclic types and the statement follows from Theorem 10.  $\square$

## 5 Conjugation in the symmetric inverse semi-group on a countable set

In this section  $M$  will denote a countable set. Let  $\alpha \in \mathcal{IS}(M)$ . It turns out that apart from the finite cycles and chains the graph  $\Gamma(\alpha)$  can contain three other types of connected components:

- infinite “bijective” connected components, which we will call *infinite cycles*; such components have the form  $K = \cdots \rightarrow x_{-1} \rightarrow x_0 \rightarrow x_1 \rightarrow \cdots$ , where  $\alpha(x_i) = x_{i+1}$  for all  $i \in \mathbb{Z}$ , and we will denote such component by  $(\dots, x_{-1}, x_0, x_1, \dots)$ ;
- infinite non-surjective connected components, which we will call *injective chains*; such components have the form  $K = x_1 \rightarrow x_2 \rightarrow \cdots$ , where  $\alpha(x_i) = x_{i+1}$  for all  $i \in \mathbb{N}$  and  $x_1 \notin \text{ran}(\alpha)$ , and we will denote such component by  $[x_1, x_2, \dots]$ ;
- infinite surjective, but not “bijective”, connected components, which we will call *surjective chains*, such components have the form  $K = \cdots \rightarrow x_2 \rightarrow x_1$ , where  $\alpha(x_i) = x_{i-1}$  for all  $i \in \{2, 3, \dots\}$  and  $x_1 \notin \text{dom}(\alpha)$ , and we will denote such component by  $[\dots, x_2, x_1]$ .

Using the arguments, analogous to those of [GM, Theorem 4.1], it is easy to see that these five types of connected components exhaust all possible types of connected components in  $\Gamma(\alpha)$  if  $\alpha \in \mathcal{IS}(M)$ . In particular, the notion of a chain decomposition immediately extends to the elements of  $\mathcal{IS}(M)$  with  $M$  countable.

Let  $\alpha \in \mathcal{IS}(M)$ . For each positive integer  $i$  let  $l_i \in \mathbb{N} \cup \{0, \omega\}$  denote the number of cycles of length  $i$  in the chain decomposition of  $\alpha$ , and let  $l_\omega \in \mathbb{N} \cup \{0, \omega\}$  denote the number of infinite cycles in the chain decomposition of  $\alpha$ . The vector  $\text{ct}(\alpha) = (l_\omega, l_1, l_2, \dots)$  is called the *cyclic type* of  $\alpha$ .

For each positive integer  $i$  let  $m_i \in \mathbb{N} \cup \{0, \omega\}$  denote the number of chains of length  $i$  in the chain decomposition of  $\alpha$ , and let  $m_\omega^s \in \mathbb{N} \cup \{0, \omega\}$ ,  $m_\omega^i \in \mathbb{N} \cup \{0, \omega\}$  denote the numbers of infinite surjective and injective chains in the chain decomposition of  $\alpha$  respectively. The vector  $\text{chat}(\alpha) = (m_\omega^s, m_\omega^i, m_1, m_2, \dots)$  is called the *chain type* of  $\alpha$ .

**Proposition 13.** *Let  $\alpha, \beta \in \mathcal{IS}(M)$  be such that  $\alpha \sim_{\mathcal{IS}(M)} \beta$ . Then  $\text{ct}(\alpha) = \text{ct}(\beta)$ .*

*Proof.* It is of course enough to consider the case when  $\alpha \sim_{p\mathcal{IS}(M)} \beta$ . Then  $\alpha = xy$ ,  $\beta = yx$  for some  $x, y \in \mathcal{IS}(M)$ . Fix any positive integer  $k$ . Denote by  $l_k(\alpha)$  and  $l_k(\beta)$  the numbers of cycles of length  $k$  in the cyclic types of  $\alpha$  and  $\beta$  respectively. Let us show that  $l_k(\alpha) \leq l_k(\beta)$  by constructing an injective map  $\pi$  from the cycles of length  $k$  in  $\alpha$  to the cycles of length  $k$  in  $\beta$ .

Take a cycle,  $K = (a_1, a_2, \dots, a_k) \in \Gamma(\alpha)$ . Set  $b_i = y(a_i)$ ,  $1 \leq i \leq k$ . Since  $\alpha(a_i) = xy(a_i) = x(b_i)$ , it follows that  $x(b_i) = a_{i+1}$ ,  $1 \leq i \leq k-1$ , and  $x(b_k) = a_1$ . Therefore,  $\beta(b_i) = yx(b_i) = y(a_{i+1}) = b_{i+1}$ ,  $1 \leq i \leq k-1$ , and  $\beta(b_k) =$

$yx(b_k) = y(a_1) = b_1$ . Hence,  $\Gamma(\beta)$  has the cycle  $\pi(K) = (b_1, b_2, \dots, b_k)$ . By construction,  $\pi$  is injective and thus  $l_k(\alpha) \leq l_k(\beta)$ . Switching  $\alpha$  and  $\beta$  we obtain  $l_k(\beta) \leq l_k(\alpha)$ . Hence,  $l_k(\alpha) = l_k(\beta)$  for all  $k \geq 1$ . That  $l_\omega(\alpha) = l_\omega(\beta)$  is proved similarly. Therefore,  $\text{ct}(\alpha) = \text{ct}(\beta)$  and the proof is complete.  $\square$

Let  $\alpha \in \mathcal{IS}(M)$ . Denote by  $\mathbf{M}_i(\alpha)$  the set of all (maximal) chains in  $\Gamma(\alpha)$  of length  $i$ ,  $i \geq 1$ , and by  $\mathbf{M}_\omega^s(\alpha)$  and  $\mathbf{M}_\omega^i(\alpha)$  the sets of all (maximal) surjective and injective infinite chains respectively. Set

$$\mathbf{M}^{fin}(\alpha) = \cup_{i \geq 1} \mathbf{M}_i(\alpha), \quad \text{and} \quad \mathbf{M}(\alpha) = \mathbf{M}_\omega^s(\alpha) \cup \mathbf{M}_\omega^i(\alpha) \cup \mathbf{M}^{fin}(\alpha).$$

We will say that the length of an infinite chain equals  $\omega$ .

**Lemma 14.** *For all  $\alpha, \beta \in \mathcal{IS}(M)$  we have:  $\alpha \sim_{p\mathcal{IS}(M)} \beta$  if and only if  $\text{ct}(\alpha) = \text{ct}(\beta)$  and there is a partial bijection,  $\pi$ , from  $\mathbf{M}(\alpha)$  to  $\mathbf{M}(\beta)$  such that the following conditions are satisfied:*

- (i)  $\mathbf{M}(\alpha) \setminus \text{dom}(\pi) \subset \mathbf{M}_1(\alpha)$ ;
- (ii)  $\mathbf{M}(\beta) \setminus \text{ran}(\pi) \subset \mathbf{M}_1(\beta)$ ;
- (iii)  $\pi(\mathbf{M}_1(\alpha) \cap \text{dom}(\pi)) \subset \mathbf{M}_1(\beta) \cup \mathbf{M}_2(\beta)$ ;
- (iv)  $\pi(\mathbf{M}_i(\alpha)) \subset \mathbf{M}_{i-1}(\beta) \cup \mathbf{M}_i(\beta) \cup \mathbf{M}_{i+1}(\beta)$  for all  $i \geq 2$ ;
- (v)  $\pi(\mathbf{M}_\omega^s(\alpha)) = \mathbf{M}_\omega^s(\beta)$ ;
- (vi)  $\pi(\mathbf{M}_\omega^i(\alpha)) = \mathbf{M}_\omega^i(\beta)$ .

*Proof. Sufficiency.* Let  $\alpha, \beta \in \mathcal{IS}(M)$  be such that  $\text{ct}(\alpha) = \text{ct}(\beta)$  and assume that there exists  $\pi$  satisfying (i)-(vi). Let us show that  $\alpha \sim_{p\mathcal{IS}(M)} \beta$ . For this we are going to construct  $x, y \in \mathcal{IS}(M)$  such that  $\alpha = xy$ ,  $\beta = yx$ .

**Construction of  $x$  and  $y$ .**

1. If  $A = [a_1, a_2, \dots, a_t] \in \text{dom}(\pi)$ ,  $t > 1$ , is a finite chain such that  $B = [b_1, b_2, \dots, b_{t-1}] = \pi(A) \in M_{t-1}(\beta)$ , then we set

$$x(b_1) = a_2, \dots, x(b_{t-1}) = a_t, y(a_1) = b_1, \dots, y(a_{t-1}) = b_{t-1}, y(a_t) = \emptyset.$$

2. If  $A = [a_1, a_2, \dots, a_t] \in \text{dom}(\pi)$ ,  $t > 1$ , is a finite chain such that  $B = [b_1, b_2, \dots, b_t] = \pi(A) \in M_t(\beta)$ , then we set

$$x(b_1) = a_2, \dots, x(b_{t-1}) = a_t, x(b_t) = \emptyset, y(a_1) = b_1, \dots, y(a_t) = b_t.$$

3. If  $A = [a_1, a_2, \dots, a_t] \in \text{dom}(\pi)$  is a finite chain such that we have  $B = [b_1, b_2, \dots, b_{t+1}] = \pi(A) \in M_{t+1}(\beta)$ , then we set

$$x(b_1) = a_1, \dots, x(b_t) = a_t, x(b_{t+1}) = \emptyset, y(a_1) = b_2, \dots, y(a_t) = b_{t+1}.$$

4. If  $A = [\dots, a_2, a_1]$  is a surjective infinite chain and  $B = \pi(A) = [\dots, b_2, b_1]$ , then we set

$$x(b_1) = \emptyset, x(b_2) = a_1, x(b_3) = a_2, \dots, y(a_1) = b_1, y(a_2) = b_2, \dots$$

5. If  $A = [a_1, a_2, \dots]$  is an injective infinite chain and  $B = \pi(A) = [b_1, b_2, \dots]$ , then we set

$$x(b_1) = a_2, x(b_2) = a_3, x(b_3) = a_4, \dots, y(a_1) = b_1, y(a_2) = b_2, \dots$$

6. Fix any bijection,  $\gamma$ , from the union  $N$  of the supports of all cycles in  $\Gamma(\alpha)$  to the union  $N'$  of the supports of all cycles in  $\Gamma(\beta)$ , which preserves cycles (this is possible since  $\text{ct}(\alpha) = \text{ct}(\beta)$ ). For  $a \in N$  set  $y(a) = \gamma(a)$  and for  $a \in N'$  set  $x(a) = \alpha(\gamma^{-1}(a))$ .

7. Finally, for each  $a \in M$ , which forms the support of a chain,  $[a] \notin \text{dom}(\pi)$ , we set  $y(a) = \emptyset$ , and for each  $a \in M$ , which forms the support of a chain,  $[a] \notin \text{ran}(\pi)$ , we set  $x(a) = \emptyset$ .

It is straightforward to verify that  $x, y \in \mathcal{IS}(M)$  and that  $\alpha = xy$ ,  $\beta = yx$ . Thus  $\alpha \sim_{p\mathcal{IS}(M)} \beta$ , which completes the proof of the sufficiency.

**Necessity.** Take  $\alpha, \beta \in \mathcal{IS}(M)$  such that  $\alpha \sim_{p\mathcal{IS}(M)} \beta$ . By Proposition 13, we have  $\text{ct}(\alpha) = \text{ct}(\beta)$ . Consider  $x, y \in \mathcal{IS}(M)$  such that  $\alpha = xy$ ,  $\beta = yx$ . Fix  $i \geq 2$ . Consider a chain,  $A = [a_1, a_2, \dots, a_i] \in M_i(\alpha)$ . For every  $j$ ,  $1 \leq j \leq i-1$ , one gets that  $a_j \in \text{dom}(\alpha)$  implies that  $y(a_j) \in \text{dom}(x)$ , and hence  $a_j \in \text{dom}(y)$ . Denote  $b_j = y(a_j)$ . In this notation  $a_{j+1} = xy(a_j) = x(b_j)$ . Consider four possible cases.

*Case 1.*  $a_1 \in \text{ran}(x)$  and  $a_i \in \text{dom}(y)$ . Denote  $b_0 = x^{-1}(a_1)$  and  $b_i = y(a_i)$ . Since  $a_1 \notin \text{ran}(\alpha)$  and  $a_i \notin \text{dom}(\alpha)$ , it follows that  $b_0 \notin \text{ran}(y)$  and  $b_i \notin \text{dom}(x)$ . Thus we have that  $\beta = yx$  has the maximal chain  $B = [b_0, b_1, \dots, b_i] \in M_{i+1}(\beta)$ . Let  $B = \pi(A)$ .

*Case 2.*  $a_1 \in \text{ran}(x)$  and  $a_i \notin \text{dom}(y)$ . Denote  $b_0 = x^{-1}(a_1)$ . Similarly to the Case 1 we obtain  $b_0 \notin \text{ran}(y)$ . Then  $\beta = yx$  has the maximal chain  $B = [b_0, b_1, \dots, b_{i-1}] \in M_i(\beta)$ . Let  $B = \pi(A)$ .

*Case 3.*  $a_1 \notin \text{ran}(x)$  and  $a_i \in \text{dom}(y)$ . Denote  $b_i = y(a_i)$ . Similarly to the previous cases we get that the element  $\beta = yx$  has the maximal chain  $B = [b_1, \dots, b_i] \in M_i(\beta)$ . Let  $B = \pi(A)$ .

*Case 4.*  $a_1 \notin \text{ran}(x)$  and  $a_i \notin \text{dom}(y)$ . Similarly to the previous cases we obtain that the element  $\beta = yx$  has the maximal chain  $B = [b_1, \dots, b_{i-1}] \in M_{i-1}(\beta)$ . Let  $B = \pi(A)$ .

In the cases when  $A$  is an infinite (surjective or injective) chain the construction of  $\pi(A)$  is analogous.

Let  $A = [a_1] \in M_1(\alpha)$ . In case when  $a_1 \in \text{dom}(y) \cap \text{ran}(x)$  we set  $b_1 = x^{-1}(a_1)$  and  $b_2 = y(a_1)$ . We have  $b_1 \notin \text{ran}(y)$ ,  $\beta(b_1) = b_2$  and  $b_2 \notin \text{dom}(x)$ . Thus  $\beta$  has the maximal chain  $B = [b_1, b_2]$ . Set  $B = \pi(A)$ . Otherwise we set  $\pi(A) = \emptyset$ .

From the construction it follows immediately that the map  $\pi$  satisfies (i)-(vi). This completes the proof.  $\square$

**Theorem 15.** *Let  $\alpha, \beta \in \mathcal{IS}(M)$ . Then the following conditions are equivalent:*

- (a)  $\alpha \sim_{\mathcal{IS}(M)} \beta$ .
- (b)  $\text{ct}(\alpha) = \text{ct}(\beta)$ ,  $m_\omega^s(\alpha) = m_\omega^s(\beta)$ ,  $m_\omega^i(\alpha) = m_\omega^i(\beta)$ , and there exist  $k \in \mathbb{N}$  and a partial bijection,  $\pi : \mathbf{M}^{fin}(\alpha) \rightarrow \mathbf{M}^{fin}(\beta)$ , such that the following conditions are satisfied:
  - (i)  $\mathbf{M}_i(\alpha) \subset \text{dom}(\pi)$  for all  $i \geq k + 1$ ;
  - (ii)  $\mathbf{M}_i(\beta) \subset \text{ran}(\pi)$  for all  $i \geq k + 1$ ;
  - (iii)  $\pi(\mathbf{M}_i(\alpha) \cap \text{dom}(\pi)) \subset \mathbf{M}_1(\beta) \cup \dots \cup \mathbf{M}_{i+k}(\beta)$  for all  $1 \leq i \leq k$ ;
  - (iv)  $\pi(\mathbf{M}_i(\alpha)) \subset \mathbf{M}_{i-k}(\beta) \cup \dots \cup \mathbf{M}_{i+k}(\beta)$  for all  $i \geq k + 1$ .
- (c)  $\text{ct}(\alpha) = \text{ct}(\beta)$ ,  $m_\omega^s(\alpha) = m_\omega^s(\beta)$ ,  $m_\omega^i(\alpha) = m_\omega^i(\beta)$ , and there exist  $k \in \mathbb{N}$  and an infinite matrix  $(m_{i,j})$ , where  $i \geq -k + 1$  and  $-k \leq j \leq k$  with entries from  $\mathbb{N} \cup \{0, \omega\}$ , such that

$$m_t(\alpha) = \sum_{j=-k}^k m_{t,j}; \quad m_t(\beta) = \sum_{j=-k}^k m_{t-j,j} \quad (5.1)$$

for each  $t \geq 1$  (we assume that  $i + \omega = \omega + \omega = \omega$  for all  $i \in \mathbb{N}$ ).

*Proof.* (a) $\Rightarrow$ (b). Assume that  $\alpha \sim_{\mathcal{IS}(M)} \beta$ . Then, by definition, there exist  $k \in \mathbb{N}$  and a sequence,  $\alpha = \alpha_0, \alpha_1, \dots, \alpha_k = \beta \in \mathcal{IS}(M)$ , such that  $\alpha_0 \sim_{p\mathcal{IS}(M)} \alpha_1, \dots, \alpha_{k-1} \sim_{p\mathcal{IS}(M)} \alpha_k$ . For  $i = 1, \dots, k - 1$  let  $\pi_i : \mathbf{M}(\alpha_i) \rightarrow \mathbf{M}(\alpha_{i+1})$  be a partial bijection, given by Lemma 14. Set  $\pi = \pi_{k-1} \circ \dots \circ \pi_0 : \mathbf{M}(\alpha) \rightarrow \mathbf{M}(\beta)$ . The equality  $\text{ct}(\alpha) = \text{ct}(\beta)$  is implicit in Lemma 14,  $m_\omega^s(\alpha) = m_\omega^s(\beta)$  follows from Lemma 14(v),  $m_\omega^i(\alpha) = m_\omega^i(\beta)$  follows from Lemma 14(vi). In particular,  $\pi$  restricts to a partial bijection

from  $\mathbf{M}^{fin}(\alpha)$  to  $\mathbf{M}^{fin}(\beta)$ . The conditions (bi)–(biv) of our theorem follow from Lemma 14(i)–(iv). This proves the implication (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (c). Assume that we are given  $\pi : \mathbf{M}^{fin}(\alpha) \rightarrow \mathbf{M}^{fin}(\beta)$ , satisfying (bi)–(biv) for some  $k \in \mathbb{N}$ . For  $i \geq -k + 1$  and  $-k \leq j \leq k$  define

$$m_{i,j} = \begin{cases} 0, & i < 0; \\ 0, & i < -j; \\ 0, & i = j = 0; \\ |\mathbf{M}_j(\beta) \setminus \pi(\mathbf{M}^{fin}(\alpha))|, & i = 0, 1 \leq j \leq k; \\ |\mathbf{M}_i(\alpha) \setminus \text{dom}(\pi)|, & i = -j > 0; \\ |\{c \in \mathbf{M}_i(\alpha) \mid \pi(c) \in \mathbf{M}_{i+j}(\beta)\}|, & \text{otherwise.} \end{cases}$$

Then the equalities from (5.1) follow directly from the definition and the fact that  $\pi$  is a partial bijection. This proves the implication (b) $\Rightarrow$ (c). Moreover, one easily shows the reverse implication (c) $\Rightarrow$ (b).

(c) $\Rightarrow$ (a). We prove this statement by induction in  $k$ . If  $k = 1$  then, since we know that (c) $\Rightarrow$ (b), we have  $\alpha \sim_{p\mathcal{IS}(M)} \beta$  by Lemma 14 and hence  $\alpha \sim_{\mathcal{IS}(M)} \beta$ . Assume that  $m \geq 2$  and our statement is true for all  $k < m$ . Let us prove the statement for  $k = m$ . Let  $\alpha, \beta \in \mathcal{IS}(M)$  be such that (c) is satisfied with  $k = m$ . We are going to construct an element,  $\gamma \in \mathcal{IS}(M)$ , and two matrices,  $(m'_{i,j})$  and  $(m''_{i,j})$  such that for  $\alpha$ ,  $\gamma$ , and  $(m'_{i,j})$  the condition (c) is satisfied with  $k = m - 1$ , and for  $\beta$ ,  $\gamma$ , and  $(m''_{i,j})$  the condition (c) is satisfied with  $k = 1$ .

**Construction of  $\gamma$ :** Since all our conditions are given in terms of cyclic and chain types, it will be enough to specify these types for  $\gamma$ . So, we take some  $\gamma$  for which  $\text{ct}(\alpha) = \text{ct}(\gamma)$ ,  $m_\omega^s(\alpha) = m_\omega^s(\gamma)$ ,  $m_\omega^i(\alpha) = m_\omega^i(\gamma)$ , and for which the lengths of finite chains are prescribed as follows: For  $i \geq -m + 2$  and  $-m + 1 \leq j \leq m - 1$  define

$$m'_{i,j} = \begin{cases} m_{i,j}, & -m + 2 \leq j \leq m - 2; \\ m_{i,-m} + m_{i,-m+1}, & j = -m + 1; \\ m_{i,m} + m_{i,m-1}, & j = m - 1. \end{cases}$$

Then  $m_i(\alpha) = \sum_{j=-m+1}^{m-1} m'_{i,j}$ . For  $i \geq 1$  set

$$m_i(\gamma) = \sum_{j=-m+1}^{m-1} m'_{i-j,j}.$$

In this way we completely prescribe all cycles and chains in  $\gamma$ . Now we take any  $\gamma$  which satisfies this prescription. It obviously exists.

From the definition of  $m_i(\gamma)$  we immediately have that  $\alpha$ ,  $\gamma$ , and  $(m'_{i,j})$  satisfy the condition (c) with  $k = m - 1$ . For  $i \geq 0$  and  $j = -1, 0, 1$  define

$$m''_{i,j} = \begin{cases} \sum_{j=-m+1}^{m-1} m_{i-j,j}, & j = 0, \\ m_{i+m-1,-m}, & j = -1, \\ m_{i-m+1,m}, & j = 1. \end{cases}$$

Then for  $i \geq 1$  we have

$$m_i(\beta) = \sum_{j=-m}^m m_{i-j,j} = m''_{i+1,-1} + m''_{i,0} + m''_{i-1,1}$$

and

$$\begin{aligned} m_i(\gamma) &= \sum_{j=-m+1}^{m-1} m'_{i-j,j} = m_{i+m-1,-m} + \sum_{j=-m+1}^{m-1} m_{i-j,j} + m_{i-m+1,m} = \\ &= m''_{i,-1} + m''_{i,0} + m''_{i,1}. \end{aligned}$$

Hence  $\beta$ ,  $\gamma$ , and  $(m''_{i,j})$  satisfy (c) with  $k = 1$ . Now from the inductive assumption we have  $\alpha \sim_{\mathcal{IS}(M)} \gamma$  and  $\gamma \sim_{\mathcal{IS}(M)} \beta$  and hence  $\alpha \sim_{\mathcal{IS}(M)} \beta$  by transitivity. This completes the implication (c) $\Rightarrow$ (a) and the proof of the theorem.  $\square$

**Corollary 16.** *Let  $\alpha \in \mathcal{IS}(M)$ . Then*

- (1)  *$\alpha$  is conjugate to some element from  $\mathcal{S}(M)$  if and only if the cardinality of the union of the supports of all cycles of  $\alpha$  is  $\omega$  and the lengths of all chains of  $\alpha$  are finite and uniformly bounded.*
- (2)  *$\alpha$  is conjugate to the zero element of  $\mathcal{IS}(M)$  if and only if  $\alpha$  does not have any cycles and the lengths of all chains of  $\alpha$  are finite and uniformly bounded.*

## 6 Preliminaries about the Brauer-type semi-groups

Let  $n \in \mathbb{N}$ ,  $M = \{1, 2, \dots, n\}$  and  $M' = \{1', 2', \dots, n'\}$ . We consider  $' : M \rightarrow M'$  as a fixed bijection and will denote the inverse bijection by the same symbol, that is  $(a')' = a$ .



Denote by  $\mathfrak{B}_n = \mathfrak{B}(M)$  the set of all possible partitions of  $M \cup M'$  into two-element subsets; by  $\mathcal{P}\mathfrak{B}_n = \mathcal{P}\mathfrak{B}(M)$  the set of all possible partitions of  $M \cup M'$  into one- and two-element subsets; and by  $\mathfrak{C}_n = \mathfrak{C}(M)$  the set of all possible partitions of  $M \cup M'$  into arbitrary subsets. It follows from the definition that we have obvious inclusions  $\mathfrak{B}_n \subset \mathcal{P}\mathfrak{B}_n \subset \mathfrak{C}_n$ . If  $\alpha \in \mathfrak{C}_n$  and  $a, b \in M \cup M'$  we will write  $a \equiv_\alpha b$  if and only if  $a$  and  $b$  belong to the same subset of the partition  $\alpha$ .

Let  $\alpha = X_1 \cup \dots \cup X_k$  and  $\beta = Y_1 \cup \dots \cup Y_l$  be two elements from  $\mathfrak{C}_n$ . We define a new element  $\gamma = \alpha\beta \in \mathfrak{C}_n$  in the following way:

- for  $a, b \in M$  we have  $a \equiv_\gamma b$  if and only if  $a \equiv_\alpha b$  or there is a sequence,  $c'_1, \dots, c'_{2s}$ ,  $s \geq 1$ , of elements in  $M'$ , such that  $a \equiv_\alpha c'_1$ ,  $c_1 \equiv_\beta c_2$ ,  $c'_2 \equiv_\alpha c'_3, \dots, c_{2s-1} \equiv_\beta c_{2s}$ , and  $c'_{2s} \equiv_\alpha b$ ;
- for  $a', b' \in M'$  we have  $a' \equiv_\gamma b'$  if and only if  $a' \equiv_\beta b'$  or there is a sequence,  $c_1, \dots, c_{2s}$ ,  $s \geq 1$ , of elements in  $M$ , such that  $a' \equiv_\beta c_1$ ,  $c'_1 \equiv_\alpha c'_2$ ,  $c_2 \equiv_\beta c_3, \dots, c'_{2s-1} \equiv_\alpha c'_{2s}$ , and  $c_{2s} \equiv_\beta b'$ ;
- for  $a \in M$  and  $b' \in M'$  we have  $a \equiv_\gamma b'$  if and only if there is a sequence,  $c'_1, \dots, c'_{2s-1}$ ,  $s \geq 1$ , of elements in  $M'$ , such that  $a \equiv_\alpha c'_1$ ,  $c_1 \equiv_\beta c_2$ ,  $c'_2 \equiv_\alpha c'_3, \dots, c'_{2s-2} \equiv_\alpha c'_{2s-1}$ , and  $c_{2s-1} \equiv_\beta b'$ .

One can think about the elements from  $\mathfrak{C}_n$  as a certain "microchips" with  $n$  pins on the left hand side (corresponding to  $M$ ) and  $n$  pins on the right hand side (corresponding to  $M'$ ). Having  $\alpha \in \mathfrak{C}_n$  we connect two pins in the corresponding chip if and only if they belong to the same set of the partition  $\alpha$ . The operation described above can then be viewed as a "composition" of such chips: having  $\alpha, \beta \in \mathfrak{C}_n$  we identify (connect) the right pins of  $\alpha$  with the corresponding left pins of  $\beta$ , which uniquely defines a connection of the remaining pins (which are the left pins of  $\alpha$  and the right pins of  $\beta$ ). Note that during the operation we can obtain some "dead circles" formed by some identified pins from  $\alpha$  and  $\beta$  (for example the two lowest identified pins on Figure 1). These circles should be disregarded. From such realization it follows immediately that the defined above composition of elements from  $\mathfrak{C}_n$  is associative. The obtained semigroup is called the semigroup of all partitions, see [Mar, Xi, Ma2].

It is easy to see that both  $\mathfrak{B}_n$  and  $\mathcal{P}\mathfrak{B}_n$  are subsemigroups of  $\mathfrak{C}_n$ . The semigroup  $\mathfrak{B}_n$  is called the *Brauer* semigroup (or Brauer monoid) and was first constructed in [Br]. The semigroup  $\mathcal{P}\mathfrak{B}_n$  is the *partial Brauer* semigroup, defined in [Ma1].  $\mathcal{P}\mathfrak{B}_n$  is also known as the *rook Brauer* monoid. The corresponding finite-dimensional associative algebras and their deformations (Brauer algebras, rook Brauer algebras) play important role in representation

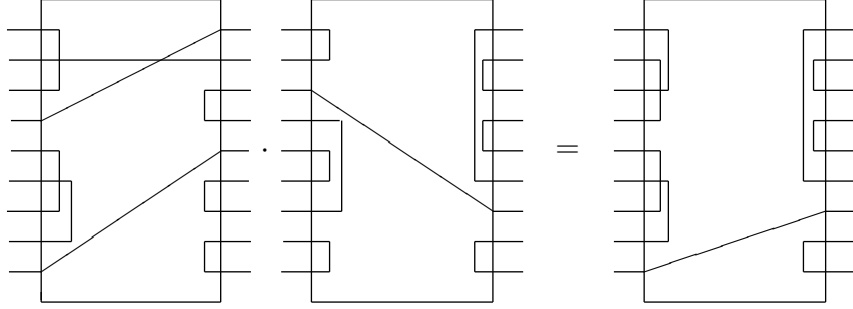


Figure 1: Chips and their multiplication.

theory and have been extensively studied. An example of multiplication of two elements in  $\mathfrak{B}_n$  is given on Figure 1.

It is easy to see that there are natural monomorphisms  $\mathcal{S}_n \hookrightarrow \mathfrak{B}_n$  and  $\mathcal{IS}_n \hookrightarrow \mathcal{PB}_n$ , defined via  $\mathcal{S}_n \ni \sigma \mapsto \{\sigma(1), 1'\} \cup \dots \cup \{\sigma(n), n'\}$ .

Let  $\alpha \in \mathfrak{C}_n$ . A subset,  $X \subset M$ , will be called  $\alpha$ -invariant provided that for any  $a \in X \cup X'$  and any  $b \in M \cup M'$  the condition  $a \equiv_\alpha b$  implies  $b \in X \cup X'$ . If  $X$  is invariant with respect to  $\alpha$ , then we define the element  $\alpha|_X \in \mathfrak{C}_n$  in the following way:

- for all  $a, b \in X \cup X'$  we have  $a \equiv_{\alpha|_X} b$  if and only if  $a \equiv_\alpha b$ ;
- for all  $a \in M \setminus X$  and  $b \in M \cup M'$  we have  $a \equiv_{\alpha|_X} b$  if and only if  $b = a'$ .

The element  $\alpha|_X$  is called the *restriction* of  $\alpha$  to  $X$ . We also define the element  $\alpha|^{X'} \in \mathfrak{C}(X)$  as follows:

- for all  $a, b \in X \cup X'$  we have  $a \equiv_{\alpha|^{X'}} b$  if and only if  $a \equiv_\alpha b$ .

Note that if  $X$  is  $\alpha$ -invariant then  $M \setminus X$  is  $\alpha$ -invariant as well.

Let  $\alpha \in \mathfrak{C}_n$ . A subset,  $X \in \alpha$ , will be called a *line* provided that  $X \cap M \neq \emptyset$  and  $X \cap M' \neq \emptyset$ . The number of lines in  $\alpha$  will be called the *rank* of  $\alpha$  and denoted by  $\text{rank}(\alpha)$ . For  $\alpha \in \mathfrak{C}_n$  we define the *stable rank*  $\text{strank}(\alpha)$  of  $\alpha$  as  $\text{rank}(\alpha^k)$ , where  $k$  is such that  $\alpha^{2k} = \alpha^k$ . Obviously  $\text{strank}(\alpha) \leq \text{rank}(\alpha)$ .

Let  $\pi \in \mathfrak{C}_n$  be an idempotent and  $M = N_1 \cup \dots \cup N_l$  be a decomposition of  $M$  into a disjoint union of the minimal  $\pi$ -invariant subsets (for example, if  $\pi$  is the leftmost element on Figure 1, then the corresponding decomposition is  $\{1, 3, 4\} \cup \{2\} \cup \{5, 6, 7, 8, 9\}$  if one counts pins from above). We call  $N_i$ ,  $i = 1, \dots, l$ , *non-degenerate* provided that there exist  $a, b \in N_i$  such

that  $a \equiv_{\pi} b'$ , and *degenerate* in the opposite case. Note that the number of non-degenerate  $N_i$ 's is precisely  $\text{strank}(\pi) = \text{rank}(\pi)$ . The union of all non-degenerate  $N_i$ 's will be called the *stable support* of  $\pi$ . For  $\alpha \in \mathfrak{C}_n$  the *stable support* of  $\alpha$  is defined as the stable support of  $\alpha^k$  such that  $\alpha^{2k} = \alpha^k$ . In [Ma1] the maximal subgroups of  $\mathfrak{B}_n$  and  $\mathcal{P}\mathfrak{B}_n$  are determined ([Ma1, Theorem 1 and Theorem 2]). We now extend this result by describing the maximal subgroups of  $\mathfrak{C}_n$  (and giving a unified proof for all cases).

**Proposition 17.** *Let  $S$  be  $\mathfrak{B}_n$ ,  $\mathcal{P}\mathfrak{B}_n$  or  $\mathfrak{C}_n$  and  $\pi \in S$  be an idempotent. Then the maximal subgroup  $G(\pi)$  of  $S$ , corresponding to  $\pi$ , is isomorphic to  $\mathcal{S}_{\text{strank}(\pi)}$ .*

*Proof.* Recall that the maximal subgroup  $G(\pi)$ , corresponding to  $\pi$ , consists of all  $\alpha \in \mathfrak{C}_n$  such that  $\alpha^k = \pi$  for some  $k$  and  $\pi\alpha = \alpha\pi = \alpha$ . Let  $X$  be the stable support of  $\pi$ . Let  $\alpha \in S$  be such that  $\pi\alpha = \alpha\pi = \alpha$ . Then for any  $a, b \in (M \setminus X) \cup (M' \setminus X')$  we obviously have  $a \equiv_{\alpha} b$  if and only if  $a \equiv_{\pi} b$ . Let  $N_1, \dots, N_l$  be a complete list of non-degenerate minimal  $\pi$ -invariant subsets of  $M$ .

Let  $i \in \{1, \dots, l\}$ . Let us show that  $\pi|^{N_i} = N_i^{(1)} \cup N_i^{(2)} \cup \dots$  where  $N_i^{(j)} \cap M = \emptyset$  or  $N_i^{(j)} \cap M' = \emptyset$  for every  $j > 1$ , moreover,  $N_i^{(1)} \cap M \neq \emptyset$  and  $N_i^{(1)} \cap M' \neq \emptyset$ . Indeed, the existence of  $N_i^{(1)}$  with the necessary properties follows immediately from the non-degeneracy of  $N_i$ . The uniqueness of  $N_i^{(1)}$  follows easily from the minimality of  $N_i$  and the fact that  $\pi$  is an idempotent.

For every  $i = 1, \dots, l$  set  $K_i = N_i^{(1)} \cap M$  and  $P_i = N_i^{(1)} \cap M'$ . Since  $\alpha \in G(\pi)$  is equivalent to  $\alpha^k = \pi$  for some  $k$  and  $\pi\alpha = \alpha\pi = \alpha$ , it follows that  $\alpha \in G(\pi)$  if and only if  $\alpha$  satisfies the following conditions:

- for any minimal degenerate  $\pi$ -invariant subset  $N$  we have  $\pi|^{N} = \alpha|^{N}$ ;
- $\alpha$  contains all  $N_i^{(j)}$  for  $i = 1, \dots, l$  and  $j > 1$ .
- there is a permutation,  $\sigma \in \mathcal{S}_l$ , such that  $\alpha$  contains the set  $K_i \cup P'_{\sigma(i)}$ .

We see that  $\alpha$  depends only on  $\sigma \in \mathcal{S}_l$  and it is easy to see that the corresponding map is a group isomorphism.  $\square$

**Remark 18.** From the proof of Proposition 17 we even have that the group  $G(\pi)$  acts on the set  $\{1, 2, \dots, l\}$ , the elements of which index  $N_i$ 's, in a natural way, and that this action is similar to the standard action of  $\mathcal{S}_l$  on  $\{1, 2, \dots, l\}$ . In particular, we can speak about the cyclic types of the elements of  $G(\pi)$  in an unambiguous way.

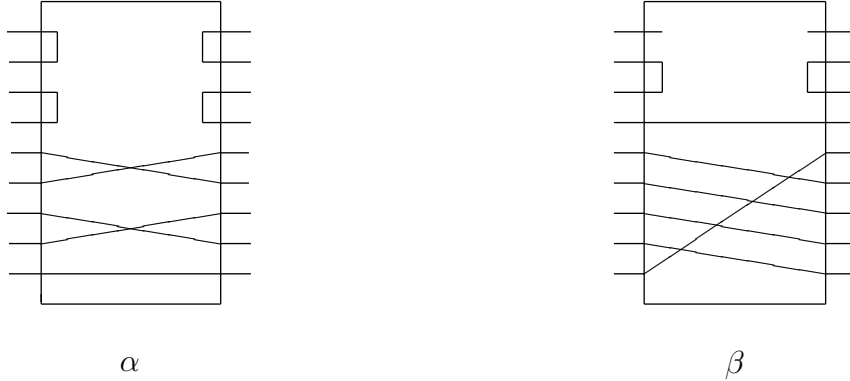


Figure 2: Two canonical elements in  $\mathfrak{C}_9$ .

## 7 Conjugacy in the semigroups $\mathfrak{B}_n$ , $\mathcal{PB}_n$ and $\mathfrak{C}_n$

In this section we describe the conjugacy classes in  $\mathfrak{B}_n$ ,  $\mathcal{PB}_n$  and  $\mathfrak{C}_n$ .

Call  $\alpha \in \mathfrak{C}_n$  *canonical* of *index*  $\text{ind}(\alpha) = k$ ,  $0 \leq k \leq n$ , if the following conditions are satisfied:

- $X = \{1, 2, \dots, k\}$  is  $\alpha$ -invariant;
- $\alpha|_X \in \mathcal{S}_n$ ;
- $\alpha$  contains  $\{k+1, k+2\}$ ,  $\{k+3, k+4\}, \dots$ , and  $\{(k+1)', (k+2)'\}$ ,  $\{(k+3)', (k+4)'\}, \dots$ .
- if  $k$  and  $n$  have different parities, then  $\alpha$  contains  $\{n\}$  and  $\{n'\}$ .

If  $\alpha$  is canonical then, by definition, its index coincides with the number of subsets of  $\alpha$ , which have form  $\{a, b'\}$ , where  $a \in M$  and  $b' \in M'$ . In particular, if  $\alpha$  is canonical then  $\text{strank}(\alpha) = \text{rank}(\alpha) = \text{ind}(\alpha)$ . Following Remark 18 we define the *cyclic type* of  $\alpha$  as  $\text{ct}(\alpha) = \text{ct}(\alpha|_{\{1, \dots, \text{ind}(\alpha)\}})$ . Note that the cyclic types of elements with different indices are automatically different. Note also that any canonical element, whose index has the same parity as  $n$ , belongs to  $\mathfrak{B}_n$ , and in the opposite case it belongs to  $\mathcal{PB}_n \setminus \mathfrak{B}_n$ . Two examples of canonical elements from  $\mathfrak{C}_9$  are given in Figure 2 (the pins of these elements are numbered  $1, 2, \dots, n$  from bottom to top). The element  $\alpha$  has index 5 and the cyclic type  $(1, 2, 0, 0, 0)$ , the element  $\beta$  has index 6 and the cyclic type  $(1, 0, 0, 0, 1, 0)$ .

Our main result about the canonical elements will be the following:

**Proposition 19.** *Let  $S$  denote one of the semigroups  $\mathfrak{B}_n$ ,  $\mathcal{P}\mathfrak{B}_n$ , or  $\mathfrak{C}_n$ . Then each element from  $S$  is conjugate to a canonical element from  $S$ .*

Further we show the following:

**Proposition 20.** *Let  $S$  denote one of the semigroups  $\mathfrak{B}_n$ ,  $\mathcal{P}\mathfrak{B}_n$ , or  $\mathfrak{C}_n$ , and  $\alpha, \beta$  be two canonical elements in  $S$ . Then  $\alpha \sim_S \beta$  if and only if  $\text{ct}(\alpha) = \text{ct}(\beta)$ .*

Let  $S$  denote one of the semigroups  $\mathfrak{B}_n$ ,  $\mathcal{P}\mathfrak{B}_n$ , or  $\mathfrak{C}_n$ . Using Proposition 19, for every  $\alpha \in S$  we can define the *cyclic type*  $\text{ct}(\alpha)$  as  $\text{ct}(\beta)$ , where  $\beta$  is canonical and  $\alpha \sim_S \beta$ . By Proposition 20 the cyclic type of  $\alpha$  is well-defined. Our main result can now be formulated in the following way:

**Theorem 21.** *Let  $S$  denote one of the semigroups  $\mathfrak{B}_n$ ,  $\mathcal{P}\mathfrak{B}_n$ , or  $\mathfrak{C}_n$  and  $\alpha, \beta \in S$ . Then  $\alpha \sim_S \beta$  if and only if  $\text{ct}(\alpha) = \text{ct}(\beta)$ .*

*Proof.* This is an immediate corollary of Proposition 19 and Proposition 20 and the fact that  $\sim_S$  is an equivalence relation.  $\square$

Now we prove Proposition 19 and Proposition 20 starting from Proposition 20. We will need the following lemma:

**Lemma 22.** (1) *A canonical element,  $\xi \in S$ , is an idempotent if and only if  $\xi$  contains  $\{i, i'\}$  for all  $i = 1, \dots, \text{ind}(\xi)$ .*

(2) *Let  $\xi, \zeta \in S$  be canonical idempotents. Then  $\xi \sim_S \zeta$  if and only if  $\xi = \zeta$ .*

*Proof.* The first statement and the "if" part of the second statement are obvious. Hence we are left to prove the "only if" part of the second statement. By Proposition 4(1),  $\xi \sim_S \zeta$  implies  $\xi \sim_{pS} \zeta$ . Let  $k = \text{rank}(\xi)$ ,  $l = \text{rank}(\zeta)$ ,  $\xi = xy$  and  $\zeta = yx$  for some  $x, y \in S$ . As  $\xi$  is canonical, there should exist  $a'_1, \dots, a'_k \in M'$  such that  $i \equiv_x a'_i$  for all  $i = 1, \dots, k$ . Moreover, there should also exist  $b_1, \dots, b_k \in M$  such that  $b_i \equiv_y i'$  for all  $i = 1, \dots, k$ . As  $\zeta = yx$ , we obtain  $b_i \equiv_\zeta a'_i$  for all  $i = 1, \dots, k$ , in particular,  $l \geq k$ . Because of the symmetry we also obtain  $k \geq l$ , that is  $k = l$ . This implies  $\xi = \zeta$ .  $\square$

**Lemma 23.** *Let  $\xi \in S$  be canonical and  $\zeta \in S$  be arbitrary. Then  $\xi \sim_S \zeta$  implies  $\text{rank}(\xi) \leq \text{rank}(\zeta)$  and  $\text{strank}(\xi) = \text{strank}(\zeta)$ .*

*Proof.* Let  $i \in \mathbb{N}$  be such that both  $x = \xi^i$  and  $y = \zeta^i$  are idempotents. Then Corollary 3 implies  $x \sim_{pS} y$ . Let  $x = ab$  and  $y = ba$  for some  $a, b \in S$ . Then  $x = ab = abab = ayb$  and  $y = ba = baba = bxa$ . This implies  $\text{rank}(x) = \text{rank}(y)$ . Since  $\xi$  is canonical we have  $\text{rank}(\xi) = \text{strank}(\xi) = \text{rank}(x)$ . From  $y = \zeta^i$  we have  $\text{rank}(y) = \text{strank}(\zeta) \leq \text{rank}(\zeta)$ . The statement follows.  $\square$

*Proof of Proposition 20.* That  $\text{ct}(\alpha) = \text{ct}(\beta)$  implies  $\alpha \sim_S \beta$  follows from the definition of canonical elements and the description of the conjugacy classes in the symmetric group. Now let  $\alpha, \beta$  be two canonical elements in  $S$  such that  $\alpha \sim_S \beta$ . From Lemma 23 we have  $\text{rank}(\alpha) = \text{rank}(\beta) = k$ . From this and the definition of the canonical element we immediately get that if  $e = \alpha^i$  and  $f = \beta^j$  are idempotents, then  $e = f$  and  $\alpha, \beta \in G(e)$ . Hence Proposition 5 implies  $\alpha \sim_{pS} \beta$ . Now we almost repeat the arguments from Lemma 22. Assume  $\alpha = xy$  and  $\beta = yx$  for some  $x, y \in S$ . As  $\alpha$  is canonical, there exist  $a'_1, \dots, a'_k \in M'$  such that  $i \equiv_x a'_i$  for all  $i = 1, \dots, k$ . Moreover, there also exist  $b_1, \dots, b_k \in M$  such that  $b_i \equiv_y i'$  for all  $i = 1, \dots, k$ . As  $\beta = yx$  and is canonical, we obtain  $b_i \equiv_\beta a'_i$  for all  $i = 1, \dots, k$ , which also implies  $\{a_1, \dots, a_k\} = \{b_1, \dots, b_k\} = \{1, \dots, k\}$ . This shows that  $\alpha|_{\{1, \dots, k\}}$  and  $\beta|_{\{1, \dots, k\}}$  must be conjugate as elements from  $\mathcal{S}_k$ , in particular,  $\text{ct}(\alpha) = \text{ct}(\alpha|_{\{1, \dots, k\}}) = \text{ct}(\beta|_{\{1, \dots, k\}}) = \text{ct}(\beta)$ . This completes the proof.  $\square$

*Proof of Proposition 19.* For  $X \subset M$  we denote by  $\alpha_X$  the idempotent in  $\mathfrak{C}_n$  defined as follows:

$$\alpha_X = X \bigcup X' \bigcup (\cup_{i \notin X} \{i, i'\})$$

Let us first consider the cases  $S = \mathfrak{B}_n$  and  $S = \mathcal{P}\mathfrak{B}_n$ . Let  $\alpha \in S$ . We first claim that any  $\alpha \in S$  is conjugate to some element  $\beta$  such that there exists  $k \in \{0, \dots, n\}$  such that  $\{1, \dots, k\}$  is  $\beta$ -invariant and the following conditions are satisfied:

- (a)  $\beta|_{\{1, \dots, k\}} \in \mathcal{S}_k$ ;
- (b)  $\beta|_{\{k+1, \dots, n\}}$  does not contain lines.

Observe that if there are no  $X \subset M$ , contained in  $\alpha$ , then  $\alpha \in \mathcal{S}_n$  and the statement is obvious for all  $n$  as  $\alpha$  satisfies both (a) and (b). In the remaining cases, we proceed by induction on  $n$ . Assume that  $X \subset M$  is contained in  $\alpha$ . Then  $\alpha_X \alpha = \alpha$  and hence  $\alpha \alpha_X \sim_{pS} \alpha$ . Moreover,  $\alpha \alpha_X$  contains both  $X$  and  $X'$ . Let  $l = |X|$ ,  $X = \{x_1\}$  if  $l = 1$ , and  $X = \{x_1, x_2\}$  if  $l = 2$ . Conjugating, if necessary, with the (invertible) element, which switches  $i_j$  with  $n - l + j$  and stabilizes all other points, we obtain  $\alpha \alpha_X \sim_S \alpha_1$ , where  $\{n, \dots, n - l + 1\}$  is  $\alpha_1$ -invariant and  $\alpha_1|_{\{n, \dots, n - l + 1\}} = \{n, \dots, n - l + 1\} \cup \{n', \dots, (n - l + 1)'\}$ . Now we can apply induction to  $\alpha_1|_{\{1, \dots, n - l\}}$  and the statement follows. The procedure we have just described will be called the *cleaning procedure*.

Now take some  $\beta \in S$  such that  $\beta \sim_S \alpha$  and both (a) and (b) are satisfied. Set  $k = \text{rank}(\beta)$  and let  $\gamma$  be any canonical element of rank  $k$ . Define

$$\begin{aligned} x &= (\cup_{X \in \gamma, X \subset M'} X) \bigcup (\cup_{X \in \beta, X \subset M} X) \bigcup (\cup_{i=1}^k \{i, i'\}) ; \\ y &= (\cup_{X \in \gamma, X \subset M'} X) \bigcup (\cup_{X \in \beta, X \subset M} X) \bigcup (\cup_{i=1}^k \{i, i'\}) . \end{aligned}$$

We have  $\beta = x\beta$  and thus  $\beta x \sim_S \beta$ , moreover,  $\beta xy = \beta x$  and thus  $y\beta x \sim_S \beta x \sim_S \beta$ . By construction we also have that  $y\beta x$  is canonical. This completes the proof in the cases  $S = \mathfrak{B}_n, \mathcal{P}\mathfrak{B}_n$ .

Now let us consider the case  $S = \mathfrak{C}_n$ . Assume that  $X' \subset M'$  is contained in  $\alpha$ , Then  $\alpha\alpha_X = \alpha$  and hence  $\alpha_X\alpha \sim_{pS} \alpha$ . Following, in this setup, the definition of the cleaning procedure, one defines the *right cleaning procedure* (this is necessary since for  $\mathfrak{C}_n$  it might happen that  $X \notin \alpha$  for any  $X \subset M$ , whereas there exists  $X' \subset M'$  such that  $X' \in \alpha$ ). Using the cleaning procedure, the right cleaning procedure and the arguments from the previous paragraph we get that any  $\alpha \in \mathfrak{C}_n$  is conjugate to some  $\beta \in \mathfrak{C}_n$  such that the following conditions are satisfied:

- any  $X \in \beta|_{\{1, \dots, k\}}$  is a line;
- $\beta|_{\{k+1, \dots, n\}}$  coincides with  $\gamma|_{\{k+1, \dots, n\}}$  for some canonical element  $\gamma$  of rank  $k$ .

It is now left to show that any element  $\beta \in \mathfrak{C}_n$ , consisting entirely of lines, is conjugate to a canonical element. If  $n - \text{rank}(\beta) = 0$  then  $\beta \in \mathfrak{S}_n$  and hence  $\beta$  is canonical. Therefore in this case the statement is true for all  $n$ . Suppose now that  $n - \text{rank}(\beta) > 0$ . We proceed by induction on  $n$ . The case  $n = 1$  is trivial since in this case  $\mathfrak{C}_n = \mathcal{P}\mathfrak{B}_n$ . Assume that  $n > 1$ . Then there exists  $X \in \beta$  such that  $|X \cap M| > 1$  and let  $a, b \in X \cap M$  be two different elements. Consider the element

$$x = \{a, b, b'\} \bigcup \{a'\} \bigcup (\cup_{i \notin \{a, b\}} \{i, i'\}). \quad (7.1)$$

Then  $x\beta = \beta$  and hence  $\beta \sim_S \beta x$ , moreover,  $\{a'\} \in \beta x$ . We call this the *deleting procedure*. Applying now the right cleaning procedure to  $\beta x$  we reduce  $n$  (since  $\{a'\} \in \beta x$ ) and thus we can apply the inductive assumption. The statement follows.  $\square$

It happens that one can also define  $\text{ct}(\alpha)$  intrinsically, that is only in terms of the semigroup, generated by  $\alpha$ , and without any reference to the canonical elements.

**Proposition 24.** *Let  $S$  denote one of the semigroups  $\mathfrak{B}_n, \mathcal{P}\mathfrak{B}_n$ , or  $\mathfrak{C}_n$  and  $\alpha \in S$ . Let  $i \in \mathbb{N}$  be such that  $\alpha^i = \pi$  is an idempotent. Then  $\text{ct}(\alpha)$  coincides with the cyclic type of  $\pi\alpha\pi \in G(\pi) \cong \mathfrak{S}_{\text{strank}(\alpha)}$ .*

*Proof.* Let  $\alpha \in S$  and  $i \in \mathbb{N}$  be such that  $\pi = \alpha^i$  is an idempotent. Set  $\text{ct}'(\alpha) = \text{ct}(\pi\alpha\pi)$ . Let  $\tau$  denote either the element  $\alpha_X$  from the (right) cleaning procedure or the element  $x$  from (7.1). To prove our statement it

is enough to show that  $\text{ct}'(\alpha) = \text{ct}'(\alpha')$ , where  $\alpha' = \tau\alpha\tau$  (this is the element obtained from  $\alpha$  after one step of the (right) cleaning or deleting procedure). To do this we will need the following:

**Lemma 25.** *The map  $F : S \rightarrow S$  defined via  $F(y) = \tau y \tau$  has the following properties:*

- (i) *For each  $k \geq 1$  we have  $F(\alpha^k) = (\alpha')^k$ .*
- (ii) *The element  $F(\alpha^i) = (\alpha')^i$  is an idempotent, which we denote  $\pi'$ .*
- (iii)  *$\pi'\pi\pi' = \pi'$  and  $\pi\pi'\pi = \pi$ , in particular,  $\pi \sim_{pS} \pi'$ .*
- (iv)  *$F$  induces an isomorphism from  $G(\pi)$  to  $G(\pi')$ , which further gives rise to the similarity of the actions of  $G(\pi)$  and  $G(\pi')$  on the set  $\{1, 2, \dots, \text{strank}(\alpha)\}$ .*

*Proof.* First we note that  $\tau\alpha = \alpha$  if  $\tau$  is taken from the cleaning and deleting procedure, and that  $\alpha\tau = \alpha$  if  $\tau$  is taken from the right cleaning procedure. We consider the first case, the arguments in the second case are analogous.

Since  $\tau\alpha = \alpha$  and  $\tau^2 = \tau$ , we have

$$F(\alpha^k) = \tau\alpha^k\tau = \tau\alpha\tau\alpha \cdots \tau\alpha\tau = (\alpha')^k,$$

which proves (i). From (i) we have  $(\alpha')^{2i} = F(\alpha^{2i}) = F(\alpha^i) = (\alpha')^i$ , which implies (ii). Note that  $\pi' = F(\alpha^i) = \tau\alpha^i\tau = \tau\pi\tau$  by (ii), and  $\tau\pi = \tau\alpha^i = \alpha^i = \pi$ . Thus we have

$$\begin{aligned} \pi'\pi\pi' &= \tau\pi\tau\pi\tau\pi\tau = \tau\pi\pi\pi\tau = \tau\pi\tau = \pi', \\ \pi\pi'\pi &= \pi\tau\pi\tau\pi = \pi\pi\pi = \pi, \end{aligned}$$

which proves (iii).

Take now any element  $z \in G(\pi)$ . Then  $z = \pi z \pi$  and, using  $\tau\pi = \pi$ , we have

$$F(z) = \tau z \tau = \tau\pi z \pi \tau = \tau\pi\pi z \pi\pi\tau = \tau\pi\tau\pi z \pi\tau\pi\tau = \pi'\pi z \pi\pi'.$$

Now the first part of (iv) follows from Proposition 4(3). Obviously, the elements  $\alpha$  and  $\alpha'$  act in the same way on the indexing set of the minimal non-degenerate  $\pi$ - (resp.  $\pi'$ -) invariant subsets (see Proposition 17 and Remark 18). This means that  $F$  gives rise to the similarity of the actions of  $G(\pi)$  and  $G(\pi')$  on  $\{1, 2, \dots, \text{strank}(\alpha)\}$ .  $\square$

Lemma 25(iv) implies that for any  $\alpha \in S$  we have  $\text{ct}'(\alpha) = \text{ct}'(\beta)$ , where  $\beta$  is canonical and conjugate to  $\alpha$ . Since we obviously have  $\text{ct}'(\beta) = \text{ct}(\beta)$  it follows that  $\text{ct}'(\alpha) = \text{ct}(\alpha)$  by the definition of  $\text{ct}$ . This completes the proof.  $\square$



In particular, we have an analogue of Corollary 12:

**Corollary 26.** *Let  $\alpha \in \mathfrak{C}_n$  and  $i \in \mathbb{N}$  be such that  $\pi = \alpha^i$  is an idempotent. Then  $\alpha \sim_S \pi\alpha\pi$ .*

*Proof.* Follows from Proposition 24 and Theorem 21. □

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