On the structure of $\mathcal{IO}_n$

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Abstract

We study the structure of the semigroup $\mathcal{IO}_n$ of all order-preserving partial bijections on an n-element set. For this semigroup we describe maximal subsemigroups, maximal inverse subsemigroups, automorphisms and maximal nilpotent subsemigroups. We also calculate the maximal cardinality for the nilpotent subsemigroups in $\mathcal{IO}_n$ which happens to be given by the n-th Catalan number.

1 Introduction and setup

Let N denote the set $\{1, 2, \dots, n\}$. The semigroup $\mathcal{IO}_n$ of all, possibly partially defined or *partial* injections $a : N \to N$, which preserve the natural order on N (that is for every $x < y$ from the domain of a we have $x\,a < y\,a$), is a very interesting object. It is in fact the intersection of the full symmetric inverse semigroup $\mathcal{IS}_n$ of all partial injections from N to N with the semigroup of all order preserving transformations of the interval, the latter first being studied in [Ai1, Ai2, Ai3]. In [R] it is proved that $\mathcal{IO}_n$ contains exactly 1 element from every $\mathcal{H}$-class of $\mathcal{IS}_n$. In other words, $\mathcal{IO}_n$ is an $\mathcal{H}$–cross-section of $\mathcal{IS}_n$. In [CR] it is proved that for $n \neq 3$ all $\mathcal{H}$–cross-sections of $\mathcal{IS}_n$ can be obtained from $\mathcal{IO}_n$ by conjugation, i.e. has the form $\pi^{-1}\mathcal{IO}_n\pi$ for some permutation π from the symmetric group S_n. In [Gar] the subsemigroup $T \subset \mathcal{IO}_n$, generated by all nilpotent elements, was studied and its nilpotent rank was determined.

Some basic properties of $\mathcal{IO}_n$, in particular description of Green's relations, congruences and a presentation, were obtained in [F1, F2]. The main aim of this paper is to continue the study of $\mathcal{IO}_n$. Additionally to the basic properties listed above we describe ideals, systems of generators, maximal subsemigroups, maximal inverse subsemigroups and automorphisms of $\mathcal{IO}_n$. We also study the nilpotent subsemigroups in $\mathcal{IO}_n$, in fact, we classify all maximal nilpotent subsemigroups, give an isomorphism criterion for them and calculate the maximal cardinality for the nilpotent subsemigroups in $\mathcal{IO}_n$. The latter happens to be given by the n-th Catalan number.

We will try to keep the standard notation. For every partial transformation $a : N \to N$ by $\mathrm{dom}(a)$ and $\mathrm{im}(a)$ we denote the domain and the range of a respectively. If a is injective, the number $\mathrm{rank}(a) = |\,\mathrm{dom}(a)| = |\,\mathrm{im}(a)|$ is called the *rank* of a. Clearly, $\mathrm{rank}(ab) \leq \min(\mathrm{rank}(a), \mathrm{rank}(b))$. In this paper we will multiply transformations from the left to the right and use the corresponding notation for the left to right composition of transformations: $x\,ab = (x\,a)\,b$.

From the condition of order preservation it follows that every element $a \in \mathcal{IO}_n$ is uniquely determined by $\mathrm{dom}(a)$ and $\mathrm{im}(a)$ satisfying $|\,\mathrm{dom}(a)| = |\,\mathrm{im}(a)|$. Moreover, for every $A, B \subset N$ of the same cardinality there exists $a \in \mathcal{IO}_n$ such that $\mathrm{dom}(a) = A$ and $\mathrm{im}(a) = B$. Hence

$$|\mathcal{IO}_n| = \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

We will denote by $\pi_{A,B}$ the unique element $a \in \mathcal{IO}_n$ for which $A = \mathrm{dom}(a)$ and $B = \mathrm{im}(a)$. Since $\mathcal{IO}_n \subset \mathcal{IS}_n$ and the elements $e_A = \pi_{A,A}$, $A \subset N$, exhaust all idempotents in $\mathcal{IS}_n$, we have $E(\mathcal{IO}_n) = E(\mathcal{IS}_n)$.

$\mathcal{IO}_n$ is an inverse semigroup, moreover, $(\pi_{A,B})^{-1} = \pi_{B,A}$. Further, $\mathcal{IO}_n$ contains zero, which is the transformation 0 with $\mathrm{dom}(0) = \varnothing$. If $\mathrm{dom}(a) = \{i_1, \dots, i_k\}$ and $i_l\, a = j_l$, $l = 1, \dots, k$, it is convenient to use the following tableaux presentation of a:

$$a = \left(\begin{array}{cccc} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{array} \right).$$

One can also assume that $i_1 < i_2 < \dots < i_k$ and then $j_1 < j_2 < \dots < j_k$, however, this is not always convenient.

Sometimes, especially in Section 8, it will be convenient to use an analogue of the cyclic form for permutations. For partial injections from $\mathcal{IS}_n$ this is called the *chain decomposition* or the *chart decomposition* and we refer the reader to [L1, L2, GK1, GM2] for definitions. We explain this notion on one example. The element

$$a = \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 8 \\ 5 & 7 & 2 & 1 & 4 & 6 \end{array} \right) \in \mathcal{IS}_8$$

has the following graph of the action on N:

$$\begin{array}{ccc} 1 & \rightarrow & 5 \\ \uparrow & \swarrow & \\ 4 & & \end{array} \qquad 3 \to 2 \to 7 \qquad 8 \to 6,$$

and hence it is convenient to write it as $a = (1,5,4)[3,2,7][8,6]$. We will call $(1,5,4)$ a *cycle* and $[3,2,7]$ (as well as $[8,6]$) a *chain* of the element a. The chain decomposition of elements from $\mathcal{IO}_n$ can contain only cycles of length 1 or chains. Moreover, for every chain $[a_1, \dots, a_k]$ of an element $a \in \mathcal{IO}_n$ the condition of order preservation implies that either $a_1 > a_2 > \dots > a_k$ (we will call such chains *decreasing*) or $a_1 < a_2 < \dots < a_k$ (we will call such chains *increasing*).

We will also use the word *chain* for the sets with a fixed linear order and this will not lead us to the ambiguity as the meaning will always be clear from the context. The notion *anti-chain* will stand for the partially ordered sets on which the partial order coincides with the equality relation.

Let us now briefly describe the structure of the paper. In Section 2 we recall the structure of ideals and Green's relations on $\mathcal{IO}_n$. After this, in Section 3, we describe all irreducible systems of generators in $\mathcal{IO}_n$. In Section 4 we describe all maximal subsemigroups of $\mathcal{IO}_n$. It happens that there are exactly $2^n - 1$ such semigroups. Moreover, we also describe all maximal inverse subsemigroups of $\mathcal{IO}_n$ and it happens that there are exactly 2^{n-1} such semigroups. In Section 5 we prove that the semigroup $\mathcal{IO}_n$, $n > 1$, has only one non-trivial automorphism.

Sections 6-8 form the main part of the paper. In these sections we study the nilpotent subsemigroups of $\mathcal{IO}_n$. An element a of a semigroup S with the zero 0 is called *nilpotent* provided that $a^k = 0$ for some positive integer k. The element $a \in \mathcal{IO}_n$ is nilpotent if and only if its chain decomposition does not contain any cycles. A semigroup S with the zero 0 is called *nilpotent* provided that $S^k = 0$ for some positive integer k. The minimal k such that $S^k = 0$ is called the *nilpotency degree* of S and will be denoted by $n(S)$. It is clear that all elements of a nilpotent semigroup are nilpotent. The converse is not true in general, however, it is true for finite semigroups, see [Ar].

The study of nilpotent subsemigroups of a semigroup S containing the zero element 0 is a natural problem. Here one can study the general class off all nilpotent subsemigroups, including those, whose zeros do not coincide with 0 and hence can be arbitrary idempotent from $E(S)$. One can also study a more restrictive problem — only those nilpotent subsemigroups, which contain 0. In the present paper we will consider only those nilpotent subsemigroups of $\mathcal{IO}_n$, which contain the zero element 0.

In Section 6 we construct the general machinery for the study of nilpotent subsemigroups of (finite) transformation semigroups and get, as a corollary, a description of all maximal nilpotent subsemigroups of $\mathcal{IO}_n$ and a description of all nilpotent subsemigroups of $\mathcal{IO}_n$, which are maximal among nilpotent subsemigroups of nilpotency degree k, $1 \leq k \leq n$. In particular, $\mathcal{IO}_n$ contains exactly $n!$ maximal nilpotent subsemigroups, each of which has nilpotency degree n and naturally corresponds to some linear order on N.

In Section 7 we classify the maximal nilpotent subsemigroups in $\mathcal{IO}_n$ up to isomorphism and, finally, we study their cardinalities in Section 8. Here we show that cardinality of each maximal nilpotent subsemigroup of $\mathcal{IO}_n$ does not exceed the n-th Catalan number $t_n = \frac{1}{n+1}\binom{2n}{n}$, moreover, that this bound is exact, i.e. there exists a maximal nilpotent subsemigroup in $|\mathcal{IO}_n|$ of cardinality t_n. We complete the paper with a discussion on cardinalities of maximal nilpotent subsemigroups with a fixed nilpotency degree. In a special case we present a recursive formula for the computation of these cardinalities and compute closed formulas for two classes of maximal nilpotent subsemigroup with a fixed nilpotency degree. The general problem of computing cardinalities of the maximal nilpotent subsemigroups in $\mathcal{IO}_n$ seems to be very complicated.

2 Ideals and Green's relations

Green's relations in $\mathcal{IO}_n$ are described in [F1]. However, in this section we would like to present a more detailed description of the structure of left and right ideals in $\mathcal{IO}_n$. As a corollary we recover [F1, Proposition 2.3]. We start with the following observation.

Proposition 1. *Let $a \in \mathcal{IO}_n$. Then*

1. *the left principal ideal* $\mathcal{IO}_n \cdot a$ *equals* $\{b : \mathrm{im}(b) \subset \mathrm{im}(a)\}$,

2. *the right principal ideal* $a \cdot \mathcal{IO}_n$ *equals* $\{b : \mathrm{dom}(b) \subset \mathrm{dom}(a)\}$,

3. *the two-sided principal ideal* $\mathcal{IO}_n \cdot a \cdot \mathcal{IO}_n$ *equals* $\{b : \mathrm{rank}(b) \leq \mathrm{rank}(a)\}$.

Proof. We prove the last statement and the first two can be handled by analogous arguments. It is clear that $(a) \subset \{b : \mathrm{rank}(b) \leq \mathrm{rank}(a)\}$. Now let $\mathrm{rank}(b) \leq \mathrm{rank}(a)$ and $A = \mathrm{dom}(b)$, $B = \mathrm{im}(b)$. We choose arbitrary $C \subset \mathrm{dom}(a)$, $|C| = \mathrm{rank}(b)$, and let $D = a(C)$. Then $b = \pi_{A,B} = \pi_{A,C} \cdot a \cdot \pi_{D,B} \in (a)$, which gives the opposite inclusion. Hence $(a) = \{b : \mathrm{rank}(b) \leq \mathrm{rank}(a)\}$. □

For every k, $0 \leq k \leq n$, denote $I_k = \{b \in \mathcal{IO}_n : \mathrm{rank}(b) \leq k\}$.

Corollary 1. *([F2, Proposition 2.3]) All two-sided ideals of* $\mathcal{IO}_n$ *are principal and form the following chain:*

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{n-1} \subset I_n = \mathcal{IO}_n.$$

Proof. Let I be a two-sided ideal in $\mathcal{IO}_n$, $k = \max_{b \in I} \mathrm{rank}(b)$, and $a \in I$ be an element of rank k. Then Proposition 1 yields $I = (a) = I_k$. □

We denote by $\mathfrak{B}(N)$ the boolean of the set N, that is the set of all subsets of N, which is partially ordered by inclusions in the natural way.

Proposition 2. 1. *Every anti-chain* L *from* $\mathfrak{B}(N)$ *defines the right ideal*

$$I_L = \{a : \text{ there exists } A \in L \text{ such that } \mathrm{dom}(a) \subset A\}$$

of the semigroup $\mathcal{IO}_n$.

2. $L_1 \neq L_2$ *implies* $I_{L_1} \neq I_{L_2}$.

3. *For every right ideal* I *there exists an anti-chain* L *from* $\mathfrak{B}(N)$ *such that* $I = I_L$.

Proof. The first statement follows from $\mathrm{dom}(ab) \subset \mathrm{dom}(a)$.

If $L_1 \neq L_2$ then at least one of these anti-chains, L_1 say, contains an element, A say, such that for every $B \in L_2$ either A and B are not comparable or A properly contains B. This implies $e_A \in I_{L_1}$ and $e_A \notin I_{L_2}$ and hence $I_{L_1} \neq I_{L_2}$, which proves the second statement.

To prove the last statement we note that the maximal (with respect to inclusions) elements of the set $\{\mathrm{dom}(a) : a \in I\}$ form an anti-chain, L say. Then for every $a \in I$ there exists $A \in L$ such that $\mathrm{dom}(a) \subset A$. This yields $a \in I_L$ and $I \subset I_L$. From the other hand, for every $b \in I_L$ there exist $A \in L$ and $a \in I$ such that $\mathrm{dom}(b) \subset A$ and $\mathrm{dom}(a) = A$. Now from the second part of Proposition 1 it follows that $b \in a \cdot \mathcal{IO}_n \subset I$, which implies $I_L \subset I$. Therefore $I = I_L$ and the proof is complete. □

Corollary 2. *The map* $L \mapsto I_L$ *is a bijection between the set of all anti-chains in* $\mathfrak{B}(N)$ *and the set of all right ideals of the semigroup* $\mathcal{IO}_n$.

Using the anti-involution $a \mapsto a^{-1}$ we immediately get the following dual statements:

Proposition 3. *1. Every anti-chain L from $\mathfrak{B}(N)$ defines the left ideal*

$$_LI = \{a : \text{ there exists } A \in L \text{ such that } \mathrm{im}(a) \subset A\}$$

of the semigroup $\mathcal{IO}_n$.

2. $L_1 \neq L_2$ implies $_{L_1}I \neq {}_{L_2}I$.

3. For every left ideal I there exists an anti-chain L from $\mathfrak{B}(N)$ such that $I = {}_LI$.

Corollary 3. *The map $L \mapsto {}_LI$ is a bijection between the set of all anti-chains in $\mathfrak{B}(N)$ and the set of all left ideals of the semigroup $\mathcal{IO}_n$.*

With some small changes in the proof of Proposition 1 all arguments above remain valid for the semigroup $\mathcal{IS}_n$ as well. This gives the following.

Proposition 4. *Every left (right, two-sided, principal, left principal, right principal) ideal I of the semigroup $\mathcal{IO}_n$ has the form $I = J \cap \mathcal{IO}_n$ for some uniquely defined left (right, two-sided, principal, left principal, right principal respectively) ideal J of the semigroup $\mathcal{IS}_n$.*

From Proposition 4 we obtain the following description of the Green's relations in $\mathcal{IO}_n$.

Proposition 5. *([F1, Proposition 2.3]) Every Green relation $\mathcal{H}$, $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{D}$ on $\mathcal{IO}_n$ is the intersection of the corresponding Green relation on $\mathcal{IS}_n$ with $\mathcal{IO}_n \times \mathcal{IO}_n$. Namely, for $a, b \in \mathcal{IO}_n$ one has*

1. *$a\mathcal{L}b$ if and only if $\mathrm{im}(a) = \mathrm{im}(b)$,*
2. *$a\mathcal{R}b$ if and only if $\mathrm{dom}(a) = \mathrm{dom}(b)$,*
3. *$a\mathcal{H}b$ if and only if $a = b$,*
4. *$a\mathcal{D}b$ if and only if $\mathrm{rank}(a) = \mathrm{rank}(b)$.*

Moreover, $\mathcal{D} = \mathcal{J}$.

It is shown in [R] (see also [CR]) that $\mathcal{IO}_n$ contains exactly one element from each $\mathcal{H}$-class of the semigroup $\mathcal{IS}_n$. Then it is clear that every $\mathcal{R}$-, $\mathcal{L}$- and $\mathcal{D}$-class of the semigroup $\mathcal{IS}_n$ has a non-trivial intersection with $\mathcal{IO}_n$. In particular, egg-box diagrams for $\mathcal{IS}_n$ and $\mathcal{IO}_n$ have the same structure.

Corollary 4. *$\mathcal{IO}_n$ has exactly $n+1$ $\mathcal{D}$-classes D_0, $D_1, \dots, D_n$, where $D_k = \{a \in \mathcal{IO}_n : \mathrm{rank}(a) = k\}$. Moreover, $|D_k| = \binom{n}{k}^2$.*

3 Systems of generators

The main result of [F2] gives a presentation for $\mathcal{IO}_n$, which is related to the following system of generators: 1, $g_0 = [n, n-1, \dots, 1]$, $g_1 = (1)(2)\dots(n-2)[n-1, n]$, $g_2 = (1)(2)\dots(n-3)[n-2, n-1](n)$,..., $g_{n-1} = [1,2](3)(4)\dots(n)$. In this section we present a description of all irreducible systems of generators in $\mathcal{IO}_n$. It is obvious that $\mathcal{IO}_n$ contains the unique element of rank n. This is the identity e_N, which should be contained in every system of generators for $\mathcal{IO}_n$.

Lemma 1. *([F2, Lemma 2.7])* $\langle D_{n-1} \cup \{e_N\}\rangle = \mathcal{IO}_n$.

Lemma 2. *If S generates $\mathcal{IO}_n$ then $S \cap (D_{n-1} \cup \{e_N\})$ generates $\mathcal{IO}_n$ as well.*

Proof. The elements from $D_{n-1} \cup \{e_N\}$ can be generated only by elements from $S \cap (D_{n-1} \cup \{e_N\})$. Hence $\langle S \cap (D_{n-1} \cup \{e_N\})\rangle$ contains $D_{n-1} \cup \{e_N\}$. The rest follows from Lemma 1. □

To D_{n-1} we can associate a full oriented graph, K_N, with N being the set of vertices, in the following way: the vertex $i \in N$ is identified with the corresponding complement $\bar{i} \subset N$, and the element $\pi_{A,B}$ is considered as the arrow from $\overline{A}$ to $\overline{B}$. In this way we can interpret every subset $S \subset D_{n-1}$ as a subgraph, Γ_S, of K_N. We recall that an oriented graph, Γ, is called *strongly connected* provided that for every two vertices $a, b \in \Gamma$ there is an oriented path from a to b.

Theorem 1. *1. A set $S \subset \mathcal{IO}_n$ generates $\mathcal{IO}_n$ if and only if $e_N \in S$ and $\Gamma_{S\cap D_{n-1}}$ is strongly connected.*

2. A system S of generators of $\mathcal{IO}_n$ is irreducible if and only if $S = \{e_n\} \cup (S \cap D_{n-1})$ and $\Gamma_{S\cap D_{n-1}}$ is a minimal strongly connected oriented graph with n vertices.

3. Every irreducible system of generators in the semigroup $\mathcal{IO}_n$ contains at least $n+1$ elements. Moreover, there exist exactly $(n-1)!$ irreducible systems of generators containing exactly $n+1$ elements each.

Proof. By Lemma 2 a set $S \subset \mathcal{IO}_n$ is a system of generators if and only if $e_N \in S$ and $S \cap D_{n-1}$ generates D_{n-1}. For arbitrary $\pi_{A_1,B_1}, \pi_{A_2,B_2}, \dots, \pi_{A_k,B_k}$ from D_{n-1} their product $\pi_{A_1,B_1}\pi_{A_2,B_2}\dots\pi_{A_k,B_k}$ belongs to D_{n-1} if and only if $B_1 = A_2$, $B_2 = A_3$,..., $B_{k-1} = A_k$, or, in other words, when the arrows $\overline{A_1}\ \overline{B_1}$, $\overline{A_2}\ \overline{B_2}, \dots, \overline{A_k}\ \overline{B_k}$ form an oriented path from $\overline{A_1}$ to $\overline{B_k}$. Moreover, in this case $\pi_{A_1,B_1}\pi_{A_2,B_2}\dots\pi_{A_k,B_k} = \pi_{A_1,B_k}$. Hence $S \cap D_{n-1}$ generates D_{n-1} if and only if for every two vertices $a, b \in K_N$ the oriented graph $\Gamma_{S\cap D_{n-1}}$ contains an oriented path from a to b, or, in other words, if and only if $\Gamma_{S\cap D_{n-1}}$ is a strongly connected graph with n vertices. This proves the first statement.

The first part of the second statement follows immediately from Lemma 2. From the above proof it follows that for the irreducibility of S it is necessary and sufficient that, erasing arbitrary arrow from $\Gamma_{S\cap D_{n-1}}$, one breaks the property of $\Gamma_{S\cap D_{n-1}}$ to be strongly connected. The latter is equivalent to the requirement that $\Gamma_{S\cap D_{n-1}}$ is a minimal strongly connected oriented graph with n vertices. This proves the second statement.

To prove the last statement we remark that to ensure the fact that $\Gamma_{S\cap D_{n-1}}$ is strongly connected, there should exist, for each vertex, an arrow, starting from this vertex, and an arrow, terminating in this vertex. Hence, the total number of arrows is at least n and hence $S\cap D_{n-1}$ can not contain less than n elements. Therefore $|S|\geq n+1$. The equality $|S|=n+1$ is possible if and only if for each vertex there is exactly one arrow starting in it and exactly one arrow terminating in it. The latter is equivalent to the fact that $\Gamma_{S\cap D_{n-1}}$ is a union of disjoint oriented cycles. As $\Gamma_{S\cap D_{n-1}}$ is connected, we get that there exists the unique connected component and hence $\Gamma_{S\cap D_{n-1}}$ must be an oriented cycle of length n. Clearly, the number of such cycles is $(n-1)!$. □

4 Maximal (inverse) subsemigroups

Lemma 3. *Every maximal (maximal inverse) subsemigroups in $\mathcal{IO}_n$ contains the ideal I_{n-2}.*

Proof. Let S be a maximal (maximal inverse) subsemigroup of $\mathcal{IO}_n$. If $D_{n-1}\subset S$, then, according to Lemma 1, $I_{n-2}\subset I_{n-1}=\langle D_{n-1}\rangle\subset S$. If $D_{n-1}\not\subset S$, then $S\cup I_{n-2}$ is a proper (proper inverse) subsemigroup in $\mathcal{IO}_n$, and hence $S\cup I_{n-2}=S$ by maximality of S. This implies $I_{n-2}\subset S$. □

Let $N=N_1\cup N_2\cup\dots\cup N_k$ be a decomposition of N into k non-empty blocks and $\prec$ be a non-strict (i.e. reflexive) partial order on $\{1,2,\dots,k\}$. Denote

$$S(N_1,\dots,N_k,\prec)=\left\{a\in D_{n-1}\,:\,\overline{\mathrm{dom}(a)}\in N_i \text{ and } \overline{\mathrm{im}(a)}\in N_j\Rightarrow i\prec j\right\}.$$

If $\prec$ coincides with the equality relation $=$, we will write simply $S(N_1,\dots,N_k)$ instead of $S(N_1,\dots,N_k,=)$.

Lemma 4. *1. For every decomposition $N=N_1\cup N_2\cup\dots\cup N_k$ and every order $\prec$ as above the set $\{e_N\}\cup S(N_1,\dots,N_k,\prec)\cup I_{n-2}$ is a subsemigroup in $\mathcal{IO}_n$.*

2. The subsemigroups $\{e_N\}\cup S(N_1,\dots,N_k,\prec)\cup I_{n-2}$ is inverse if and only if the order $\prec$ coincides with the equality relation.

3. $\{e_N\}\cup S(N_1,\dots,N_k,\prec)\cup I_{n-2}\neq\mathcal{IO}_n$ if and only if $k>1$.

Proof. To prove the first statement it is enough to show that if the product ab of two elements $a,b\in S(N_1,\dots,N_k,\prec)$ is contained in D_{n-1}, then it is contained in the set $S(N_1,\dots,N_k,\prec)$. Let $ab\in D_{n-1}$. Then $\mathrm{im}(a)=\mathrm{dom}(b)$. Assume $\overline{\mathrm{dom}(a)}\in N_i$, $\overline{\mathrm{im}(a)}\in N_j$ and $\overline{\mathrm{im}(b)}\in N_l$. Then $\mathrm{dom}(ab)=\mathrm{dom}(a)$, $\mathrm{im}(ab)=\mathrm{im}(b)$ and from $i\prec j$ and $j\prec l$ it follows that $i\prec l$. Hence $ab\in S(N_1,\dots,N_k,\prec)$. The first statement is proved.

The sufficiency of the condition in the second statement is obvious. To prove the necessity we first remark that the sets $\{e_N\}$ and I_{n-2} are closed with respect to the operation of taking the inverse element in $\mathcal{IO}_n$. Therefore, to ensure that the semigroup

$\{e_N\} \cup S(N_1, \dots, N_k, \prec) \cup I_{n-2}$ is inverse it is sufficient and necessary to demand that the set $S(N_1, \dots, N_k, \prec)$ is closed under the operation of taking the inverse element in $\mathcal{IO}_n$ as well.

Let $a \in S(N_1, \dots, N_k, \prec)$, $\overline{\mathrm{dom}(a)} \in N_i$, $\overline{\mathrm{im}(a)} \in N_j$. Then $i \prec j$. But then $\mathrm{dom}(a^{-1}) = \mathrm{im}(a)$ and $\mathrm{im}(a^{-1}) = \mathrm{dom}(a)$ and from $a^{-1} \in S(N_1, \dots, N_k, \prec)$ it must follow $j \prec i$ and hence $i = j$. As a is arbitrary, $\prec$ coincides with the equality relation, proving the necessity in the second statement.

The last statement is obvious. $\square$

Theorem 2. *The subsemigroup $S \subset \mathcal{IO}_n$ is maximal in $\mathcal{IO}_n$ if and only if $S = I_{n-1}$ or $S = \{e_N\} \cup S(N_1, N_2, \prec) \cup I_{n-2}$, where $N_1 \cup N_2 = N$ is a decomposition of N into two non-empty blocks and $\prec$ is a linear order on $\{1, 2\}$. In particular, $\mathcal{IO}_n$ has exactly $2^n - 1$ maximal subsemigroups.*

Proof. It is clear that I_{n-1} is a maximal subsemigroup (even maximal inverse) and that each other maximal subsemigroup must contain e_N.

Let now S be a maximal subsemigroup such that $S \neq I_{n-1}$. Consider the oriented graph $\Gamma_{S \cap D_{n-1}}$. If $\Gamma_{S \cap D_{n-1}}$ contains arrows from $\overline{A}$ to $\overline{B}$ and from $\overline{B}$ to $\overline{C}$, then S contains the elements $\pi_{A,B}$ and $\pi_{B,C}$. Thus S contains the element $\pi_{A,B} \cdot \pi_{B,C} = \pi_{A,C}$ and hence $\Gamma_{S \cap D_{n-1}}$ contains the arrow from $\overline{A}$ to $\overline{C}$. This means that the oriented graph $\Gamma_{S \cap D_{n-1}}$ is transitive (i.e. the corresponding binary relation is transitive) and thus it defines an equivalence relation $\sim$ on N in the following way: $l \sim m$ if and only if $\Gamma_{S \cap D_{n-1}}$ contains arrows from l to m and from m to l. Let $N_1, \dots, N_k$ be the equivalence classes. We have $k > 1$ as otherwise $\Gamma_{S \cap D_{n-1}} = K_N$ and, according to Lemma 1, $S \supset I_{n-1}$. For arbitrary $i, j \in \{1, 2, \dots, k\}$ we set $i \prec j$ if and only if $\Gamma_{S \cap D_{n-1}}$ contains at least one arrow from N_i to N_j (transitivity of $\Gamma_{S \cap D_{n-1}}$ then guarantees that there is an arrow from every element in N_i to every element in N_j). The relation $\prec$ is a non-strict partial order on $\{1, 2, \dots, k\}$.

From Lemmas 3 and 4 we get that $S = \{e_N\} \cup S(N_1, \dots, N_k, \prec) \cup I_{n-2}$. It is well-known (see e.g. [CK]) that every partial order can be extended to a linear order. If $\prec$ is not linear, we extend it to some linear order $\prec_1$. Then we have proper inclusions:

$$S = \{e_N\} \cup S(N_1, \dots, N_k, \prec) \cup I_{n-2} \subset \{e_N\} \cup S(N_1, \dots, N_k, \prec_1) \cup I_{n-2} \subset \mathcal{IO}_n,$$

which contradicts the maximality of S.

Hence, the order $\prec$ has to be linear. If $k > 2$, we decompose the linearly ordered set $(\{1, \dots, k\}, \prec)$ into 2 non-empty intervals $\{1, \dots, k\} = J_1 \cup J_2$. Denote $N'_t = \cup_{i \in J_t} N_i$, $t = 1, 2$. Then for the decomposition $N = N'_1 \cup N'_2$ and the corresponding order $\prec_2$ on $\{1, 2\}$ we again have strict inclusions

$$S = \{e_N\} \cup S(N_1, \dots, N_k, \prec) \cup I_{n-2} \subset \{e_N\} \cup S(N'_1, N'_2, \prec_2) \cup I_{n-2} \subset \mathcal{IO}_n,$$

contradicting the choice of S.

Hence $k = 2$ and every maximal subsemigroup in $\mathcal{IO}_n$, different from I_{n-1}, has the form

$$S = \{e_N\} \cup S(N_1, N_2, \prec) \cup I_{n-2}, \tag{1}$$

where $\prec$ is a linear order on $\{1,2\}$. It is now left to show that all subsemigroups of this form are maximal. For this we take arbitrary subsemigroup S_1 in $\mathcal{IO}_n$, strictly containing S. Then S_1 contains elements from $D_{n-1}\setminus S$. But then the oriented graph $\Gamma_{S_1\cap D_{n-1}}$ is a proper transitive (in the same sense as above) extension of $\Gamma_{S\cap D_{n-1}}$ and hence coincides with K_N. Thus $S_1 \supset D_{n-1}$, which implies $S_1 = \mathcal{IO}_n$. The latter means that every subsemigroup of the form (1) is maximal.

To compute the number of subsemigroups of the form (1) we can assume $1 \prec 2$ without loss of generality. Then the semigroups of the form (1) bijectively correspond to proper subsets $N_1 \subset N$, which can be chosen in $2^n - 2$ different ways. Taking I_{n-1} into account, we get $2^n - 1$ maximal subsemigroups. This completes the proof. □

Theorem 3. *The subsemigroup $S \subset \mathcal{IO}_n$ is maximal inverse in $\mathcal{IO}_n$ if and only if $S = I_{n-1}$ or $S = \{e_N\} \cup S(N_1, N_2) \cup I_{n-2}$, where $N_1 \cup N_2 = N$ is a decomposition of N into two non-empty blocks. In particular, $\mathcal{IO}_n$ has exactly 2^{n-1} maximal inverse subsemigroups.*

Proof. Taking into account the second statement of Lemma 4, the proof is analogous to that of Theorem 2. We only remark that $S(N_1, N_2) = S(N_2, N_1)$. Hence the number of maximal inverse subsemigroups, which differ from I_{n-1}, equals $\frac{1}{2}(2^n - 2) = 2^{n-1} - 1$. Together with I_n we get 2^{n-1} subsemigroups. □

We remark that the last statement can also be derived from [Y, Proposition 3.4].

5 Automorphisms

Let $a = \begin{pmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{pmatrix}$, where $i_1 < \dots < i_k$ and $j_1 < \dots < j_k$, be arbitrary element from $\mathcal{IO}_n$. Then the element $a^* = \begin{pmatrix} n+1-i_1 & n+1-i_2 & \dots & n+1-i_k \\ n+1-j_1 & n+1-j_2 & \dots & n+1-j_k \end{pmatrix}$ belongs to $\mathcal{IO}_n$ as well.

Lemma 5. *The map $* : a \mapsto a^*$ is an automorphisms of $\mathcal{IO}_n$.*

Proof. Direct calculation. □

Theorem 4. $\mathrm{Aut}(IO_n) = \{\mathrm{id}, *\}$.

Proof. Since the ideals in $\mathcal{IO}_n$ form a chain, each ideal I_k has to be mapped to itself under every automorphism. This means that the automorphisms must also preserve $\mathcal{D}$-classes $D_k = I_k \setminus I_{k-1}$. In particular, all automorphisms preserve the ranks of elements.

Let $\varphi \in \mathrm{Aut}(IO_n)$. Then φ preserves idempotents and an easy induction shows that it also preserves the ranks of idempotents. Therefore for any $x \in N$ there exists a unique $y \in N$ such that $e_{\{x\}}\,\varphi = e_{\{y\}}$. Define a permutation μ of N such that $x\,\mu = y$ if $e_{\{x\}}\,\varphi = e_{\{y\}}$. Take $a \in (IO_n)$, then for $x, y \in N$ we have that $x\,a = y$ if and only if $e_{\{x\}}\,a = e_{\{y\}}$, or $(e_{\{x\}}\,\varphi)(a\,\varphi) = e_{\{y\}}\,\varphi$, that is $(x\,\mu)\;(a\,\varphi) = y\,\mu$.

If there exists a pair $i < j$ such that $i\,\mu < j\,\mu$, and if there exists a pair $k < l$ such that $l\,\mu < k\,\mu$, then for the element $a = \begin{pmatrix} i & j \\ k & l \end{pmatrix} \in \mathcal{IO}_n$ we get $a\,\varphi = \begin{pmatrix} i\,\mu & j\,\mu \\ k\,\mu & l\,\mu \end{pmatrix} \notin \mathcal{IO}_n$. Hence either $i\,\mu < j\,\mu$ holds for all $i < j$ or $j\,\mu < i\,\mu$ holds for all $i < j$. In the first case μ is the identity permutation and the corresponding automorphism is the identity map. In the second case $\mu = \begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{pmatrix}$ and the corresponding automorphism is $*$. This completes the proof. □

6 Nilpotent subsemigroups of transformation semigroups

Before we go to the study of nilpotent subsemigroups in $\mathcal{IO}_n$, we describe a general technique, which first appeared in [GK2] and later was used in [GK3, GK4, GM1, Sh1, Sh2] for the detailed study of nilpotent semigroups in different semigroups of transformations. However, in the general form this technique has never been written down in a regular way.

Let $\mathcal{PT}(M)$ denote the semigroup of all partial transformations of a (possibly infinite) set M and T denote arbitrary subsemigroup of $\mathcal{PT}(M)$ containing the totally undefined transformation 0. For every positive integer k we denote by $\mathrm{Nil}_k(T)$ the set of all nilpotent subsemigroups of T of nilpotency degree less or equal k, which contain 0. The set $\mathrm{Nil}_k(T)$ is partially ordered with respect to inclusions in a natural way. Denote by $\mathrm{Ord}_k(M)$ the ordered set of all strict partial orders on M in which the cardinalities of chains are bounded by k. If $k < m$, we have natural inclusions $\mathrm{Nil}_k(T) \hookrightarrow \mathrm{Nil}_m(T)$ and $\mathrm{Ord}_k(M) \hookrightarrow \mathrm{Ord}_m(M)$, which preserves the partial order. Hence we can consider the ordered (with respect to inclusions) sets

$$\mathrm{Nil}(T) = \bigcup_k \mathrm{Nil}_k(T) \quad \text{and} \quad \mathrm{Ord}(M) = \bigcup_k \mathrm{Ord}_k(M).$$

For every partial order $\rho \in \mathrm{Ord}(M)$ consider the set

$$\mathrm{Mon}(\rho, T) = \{a \in T \,:\, x \neq x\,a \text{ and } (x, x\,a) \in \rho \text{ for all } x \in \mathrm{dom}(a)\}, \tag{2}$$

and for every subsemigroup $S \in \mathrm{Nil}(T)$ the relation

$$\rho_S = \{(x,y) \,:\, \text{ there exists } a \in S, \text{ such that } x \in \mathrm{dom}(a) \text{ and } x\,a = y\} \subset M \times M. \tag{3}$$

Proposition 6. *1. For every positive integer k, the map $\rho \mapsto \mathrm{Mon}(\rho, T)$ is a homomorphism from the poset $\mathrm{Ord}_k(M)$ to the poset $\mathrm{Nil}_k(T)$.*

2. For every positive integer k, the map $S \mapsto \rho_S$ is a homomorphism from the poset $\mathrm{Nil}_k(T)$ to the poset $\mathrm{Ord}_k(M)$.

Proof. It is clear that the set $\mathrm{Mon}(\rho, T)$ is a semigroup for every partial order ρ. If $\rho \in \mathrm{Ord}_k(M)$ and $a_1 a_2 \dots a_k \neq 0$ for some $a_1, a_2, \dots, a_k \in \mathrm{Mon}(\rho, T)$, then the sequence x_1, $x_2 = x_1 a_1$, $x_3 = x_2 a_2, \dots$, $x_{k+1} = x_k a_k$ forms a chain with $k+1$ elements for every $x_1 \in \mathrm{dom}(a_1 a_2 \cdots a_k)$. This contradicts the choice of ρ. Hence $(\mathrm{Mon}(\rho, T))^k = 0$ for every $\rho \in \mathrm{Ord}_k(M)$ and $\mathrm{Mon}(\rho, T)$ is a nilpotent semigroup of nilpotency degree $n(\mathrm{Mon}(\rho, T)) \leq k$. That $\rho_1 \subset \rho_2$ implies $\mathrm{Mon}(\rho_1, T) \subset \mathrm{Mon}(\rho_2, T)$ is obvious. This proves the first statement.

If $x\,a = y$ and $y\,b = z$ then $x\,ab = z$. Hence the relation ρ_S is transitive. That ρ_S is anti-symmetric follows from the nilpotency of S. Indeed, if $x\,a = y$ and $y\,b = x$ then $x\,(ab)^k = x$ for all positive integer k and hence $(ab)^k \neq 0$ for all positive integer k as well.

Let now $S \in \mathrm{Nil}_k(T)$. Assume that ρ_S contains a chain $x_1, x_2, \dots, x_{k+1}$ of cardinality $k+1$. Then there exist elements $a_1, \dots, a_k$ in S such that $x_i\,a_i = x_{i+1}$, $i = 1, 2, \dots, k$. But the latter means that $x_1\,a_1 \dots a_k = x_{k+1}$ and thus $a_1 \dots a_k \neq 0$, which contradicts the choice of S. That $S_1 \subset S_2$ implies $\rho_{S_1} \subset \rho_{S_2}$ is obvious. This completes the proof. $\square$

Proposition 7. *Assume that $n(S) = k$. Then $\rho_S \in \mathrm{Ord}_k(M) \setminus \mathrm{Ord}_{k-1}(M)$.*

Proof. From the first part of the previous proposition it follows that it is enough to prove the existence of a chain of length k for ρ_S. But if $n(S) = k$, we have $S^{k-1} \neq 0$ and there exist $a_1, a_2 \dots, a_{k-1} \in S$ such that $a_1 a_2 \dots a_{k-1} \neq 0$. Take arbitrary $x_1 \in \mathrm{dom}(a_1 a_2 \dots a_{k-1})$ and we get that the elements x_1, $x_2 = x_1\,a_1, \dots$, $x_k = x_{k-1}\,a_{k-1}$ form the necessary chain. $\square$

Corollary 5. *Assume that M is finite. Then the nilpotency degree $n(S)$ of arbitrary nilpotent subsemigroup $S \subset \mathcal{PT}(M)$ with zero 0 does not exceed $|M|$.*

Proposition 8. *Let $S \in \mathrm{Nil}(T)$ and $\rho \in \mathrm{Ord}(M)$ be arbitrary. Then*

1. $\mathrm{Mon}(\rho_S, T) \supset S$, $\rho_{\mathrm{Mon}(\rho,T)} \subset \rho$;

2. $\mathrm{Mon}(\rho_{\mathrm{Mon}(\rho,T)}, T) = \mathrm{Mon}(\rho, T)$;

3. $\rho_{\mathrm{Mon}(\rho_S,T)} = \rho_S$.

Proof. The first statement is obvious.

As $\rho_{\mathrm{Mon}(\rho,T)} \subset \rho$, we can apply the first statement of Proposition 6 and get that $\mathrm{Mon}(\rho_{\mathrm{Mon}(\rho,T)}, T) \subset \mathrm{Mon}(\rho, T)$. Now let $a \in \mathrm{Mon}(\rho, T)$. Then, by (3), we have $(x, x\,a) \in \rho_{\mathrm{Mon}(\rho,T)}$ for all $x \in \mathrm{dom}(a)$. Further, $a \in \mathrm{Mon}(\rho_{\mathrm{Mon}(\rho,T)}, T)$ follows from (2), giving the opposite inclusion $\mathrm{Mon}(\rho, T) \subset \mathrm{Mon}(\rho_{\mathrm{Mon}(\rho,T)}, T)$ and completing the proof of the second statement.

Since $\mathrm{Mon}(\rho_S, T) \supset S$, we can apply the second statement of Proposition 6 and get $\rho_{\mathrm{Mon}(\rho_S,T)} \supset \rho_S$. At the same time for $\rho = \rho_S$ we get $\rho_{\mathrm{Mon}(\rho_S,T)} \subset \rho_S$ by the first statement, which completes the proof. $\square$

Recall that, according to [Co], a pair of maps $\varphi : P \to Q$ and $\psi : Q \to P$ defines a *Galois correspondence* between the posets P and Q if it satisfies the following conditions:

1. φ and ψ are antihomomorphisms of the partially ordered sets, that is $p_1 \leq p_2$ implies $p_1 \varphi \geq p_2 \varphi$ and $q_1 \leq q_2$ implies $q_1 \psi \geq q_2 \psi$;

2. $p \varphi\psi \geq p$ and $q \psi\varphi \geq q$ for all $p \in P$ and $q \in Q$;

3. $p \psi\varphi\psi = p \psi$ and $q \varphi\psi\varphi = q \varphi$ for all $p \in P$ and $q \in Q$.

Denote by $\mathrm{Ord}(M)^*$ the set $\mathrm{Ord}(M)$ with the order, which is opposite to the inclusion order.

Theorem 5. *The pair of maps $S \mapsto \rho_S$ and $\rho \mapsto \mathrm{Mon}(\rho, T)$ defines a Galois correspondence between the posets $\mathrm{Nil}(T)$ and $\mathrm{Ord}(M)^*$.*

Proof. It follows from Proposition 6 that the maps $S \mapsto \rho_S$ and $\rho \mapsto \mathrm{Mon}(\rho, T)$ are antihomomorphisms between the posets $\mathrm{Nil}(T)$ and $\mathrm{Ord}(M)^*$. The rest follows from Proposition 8. □

In the arguments that follow a very important role is played by the minimal ideal $I_1 = \{a \in \mathcal{PT}(M) : |\mathrm{dom}(a)| \leq 1\}$ of the semigroup $\mathcal{PT}(M)$. We remark that this ideal can be identified with the Brandt semigroup $B(E, M)$, where E is the identity group, in a natural way.

Proposition 9. *Let $I_1 \subset T \subset \mathcal{PT}(M)$. Then for maps $\varphi : \mathrm{Nil}(T) \to \mathrm{Ord}(M)$, $S \varphi = \rho_S$, and $\psi : \mathrm{Ord}(M) \to \mathrm{Nil}(T)$, $\rho \psi = \mathrm{Mon}(\rho, T)$, we have $\psi \cdot \varphi = id_{\mathrm{Ord}(M)}$. In particular, φ is surjective and ψ is injective.*

Proof. From the first statement of Proposition 8 we get $\rho_{\mathrm{Mon}(\rho,T)} \subset \rho$. Hence it is enough to prove only the opposite inclusion. Let $(x, y) \in \rho$. Then, for the element $a = [x, y] \in I_1$, it follows from (2) that $a \in \mathrm{Mon}(\rho, T)$ and further from (3) that $(x, y) \in \rho_{\mathrm{Mon}(\rho,T)}$. Therefore $\rho\, \psi\varphi = \rho_{\mathrm{Mon}(\rho,T)} = \rho$ for all $\rho \in \mathrm{Ord}(M)$ implying $\psi \cdot \varphi = id_{\mathrm{Ord}(M)}$. The second part of the statement is obvious. □

Lemma 6. *Assume that $T = I_1$. Then the map φ from Proposition 9 is injective.*

Proof. If φ is not injective, we can find two subsemigroups $S_1, S_2 \in \mathrm{Nil}(T)$ such that $\rho_{S_1} = \rho_{S_2}$. Without loss of generality we can assume $S_1 \setminus S_2 \neq \varnothing$. Let $a \in S_1 \setminus S_2$, $\mathrm{dom}(a) = \{x\}$, $x\, a = y$. Then $(x, y) \in \rho_{S_1}$, but $(x, y) \notin \rho_{S_2}$, which contradicts $\rho_{S_1} = \rho_{S_2}$. □

Corollary 6. *There exists a one-to-one correspondence between the nilpotent subsemigroups in the semigroup I_1 and those strict partial orders on M in which the cardinalities of chains are uniformly bounded by some positive integer.*

If T contains I_1, the map $\varphi : \mathrm{Nil}(T) \to \mathrm{Ord}(M)$, $S \mapsto \rho_S$, is only surjective in general and one can hardly hope for a nice description of all nilpotent subsemigroups of T. Nevertheless, the problem to describe all *maximal* nilpotent subsemigroups among all subsemigroups with a fixed nilpotency degree remains full in content. In other words this is the problem to describe the maximal elements in $\mathrm{Nil}_k(T)$ for every k. Since the maximal elements from $\mathrm{Nil}_k(T)$ are mapped by φ to the maximal elements of $\mathrm{Ord}_k(M)$ we start with description of the last ones.

By an *ordered partition of M in k blocks* we will mean the partition $M = M_1 \cup \cdots \cup M_k$ into k non-empty blocks in which the order of blocks is also taken into account. Each usual partition of M into k blocks gives, obviously, $k!$ ordered partitions. With every ordered partition $M = M_1 \cup \cdots \cup M_k$ we associate the set

$$\mathrm{ord}(M_1, \ldots, M_k) = \bigcup_{1 \le i < j \le k} M_i \times M_j \subset M \times M.$$

Lemma 7. *Fix a positive integer $k \le |M|$. Then for every ordered partition $M = M_1 \cup \cdots \cup M_k$ the set $\mathrm{ord}(M_1, \ldots, M_k)$ is a maximal element in $\mathrm{Ord}_k(M)$. Different ordered partitions of M correspond to different elements in $\mathrm{Ord}_k(M)$, and each maximal element in $\mathrm{Ord}_k(M)$ has the form $\mathrm{ord}(M_1, \ldots, M_k)$ for some ordered partition $M = M_1 \cup \cdots \cup M_k$.*

Proof. It is obvious that $\mathrm{ord}(M_1, \ldots, M_k)$ is a transitive and anti-symmetric relation on M and that different elements from arbitrary chain $x_1, \ldots, x_m$ of this order must belong to different blocks of the ordered decomposition. Hence $m \le k$ and $\mathrm{ord}(M_1, \ldots, M_k) \in \mathrm{Ord}_k(M)$.

If $\mathrm{ord}(M_1, \ldots, M_k)$ is not maximal, then there exists an order $\rho \in \mathrm{Ord}_k(M)$ which strictly contains $\mathrm{ord}(M_1, \ldots, M_k)$. This means that there exists $(x, y) \in \rho$ such that $x \in M_i$, $y \in M_j$ and $i \ge j$. First we assume $i = j$ and choose an element z_l in each block M_l, $l \ne i$. The sequence $z_1, z_2, \ldots, z_{i-1}, x, y, z_{i+1}, \ldots, z_k$ is then an increasing chain and has cardinality $k + 1$, which contradicts to $\rho \in \mathrm{Ord}_k(M)$. Next, if $i > j$, then ρ contains (y, x) as well, which contradicts to the fact that ρ is anti-symmetric. Altogether we get that $\mathrm{ord}(M_1, \ldots, M_k)$ is a maximal element in $\mathrm{Ord}_k(M)$. This proves the first part of the lemma.

That different ordered partitions give rise to different elements in $\mathrm{Ord}_k(M)$ is obvious.

Let now ρ be an arbitrary order from $\mathrm{Ord}_k(M)$. Denote by M_1 the set of all minimal elements with respect to ρ. Note that M_1 is not empty since ρ satisfies the decreasing chain condition. Further, for every increasing chain $x_1, \ldots, x_m$ of elements in $M \setminus M_1$ there exists $x_0 \in M_1$ such that $x_0, x_1, \ldots, x_m$ is an increasing chain in M. Hence the cardinality of every chain in $M \setminus M_1$ is bounded by $k - 1$. Now we denote by M_2 the set of all minimal elements in $M \setminus M_1$, consider $M \setminus (M_1 \cup M_2)$, and continue this procedure. In k steps we get the decomposition $M = M_1 \cup \cdots \cup M_k$. Obviously $\rho \subset \mathrm{ord}(M_1, \ldots, M_k)$ by construction. From the maximality of ρ we get $\rho = \mathrm{ord}(M_1, \ldots, M_k)$ completing the proof. □

For every subsemigroup $T \subset \mathcal{PT}(M)$ and an ordered partition $M = M_1 \cup \cdots \cup M_k$ we denote

$$T(M_1, \ldots, M_k) = \{a \in T \,:\, x \in M_i \text{ and } x\,a \in M_j \text{ imply } i < j \text{ for all } x \in \mathrm{dom}(a)\}.$$

Lemma 8. $T(M_1, \ldots, M_k) = \mathrm{Mon}(\mathrm{ord}(M_1, \ldots, M_k), T)$.

Proof. Clear from the definitions. □

Theorem 6. *Assume that* $I_1 \subset T \subset \mathcal{PT}(M)$. *Then for every ordered partition* $M = M_1 \cup \cdots \cup M_k$ *the semigroup* $T(M_1, \ldots, M_k)$ *is maximal in* $\mathrm{Nil}_k(T)$. *Different ordered partitions define different maximal elements in* $\mathrm{Nil}_k(T)$ *and every maximal element from* $\mathrm{Nil}_k(T)$ *has the form* $T(M_1, \ldots, M_k)$ *for some ordered partition* $M = M_1 \cup \cdots \cup M_k$.

Proof. According to Lemma 8, the set $T(M_1, \ldots, M_k)$ is an element in $\mathrm{Nil}_k(T)$. From Proposition 9 it follows that $\rho_{T(M_1,\ldots,M_k)} = \mathrm{ord}(M_1, \ldots, M_k)$. Let now $S \in \mathrm{Nil}_k(T)$ be such that $S \supset T(M_1, \ldots, M_k)$. Since $\mathrm{ord}(M_1, \ldots, M_k)$ is a maximal element in $\mathrm{Ord}_k(M)$ by Lemma 7, the second statement of Proposition 6 implies that $\rho_S = \mathrm{ord}(M_1, \ldots, M_k)$. Using the first statement of Proposition 8, previous equality, and Lemma 8, we get

$$S \subset \mathrm{Mon}(\rho_S, T) = \mathrm{Mon}(\mathrm{ord}(M_1, \ldots, M_k), T) = T(M_1, \ldots, M_k).$$

This implies the maximality of $T(M_1, \ldots, M_k)$.

The second part of the theorem is obvious and the third one follows from Proposition 6 and Lemmas 7 and 8. □

Corollary 7. *If* $|M| = \infty$ *and* $I_1 \subset T \subset \mathcal{PT}(M)$, *then there are no maximal nilpotent subsemigroups in* T *containing* 0.

Proof. If $|M| = \infty$, then for every ordered partition $M = M_1 \cup \cdots \cup M_k$ there exists at least one block containing more than 1 element (even infinitely many elements). If M_i is one of such blocks and $M_i = M_i' \cup M_i''$ is an ordered partition, we have that $T(M_1, \ldots, M_i, \ldots, M_l)$ is a proper subsemigroup of $T(M_1, \ldots, M_i', M_i'', \ldots, M_k)$. □

Corollary 8. *Let* $I_1 \subset T \subset \mathcal{PT}(M)$ *and* $|M| = n < \infty$. *Then*

1. *There exist exactly* $n!$ *maximal nilpotent subsemigroups of* T *containing* 0 *and the nilpotency degree of each of them equals* n.
2. *For every* k, $1 < k \leq n$, *there exists exactly* $\sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^n$ *maximal nilpotent subsemigroups of* T *of nilpotency degree* $\leq k$ *containing* 0. *Moreover, the nilpotency degree of each of these subsemigroups equals* k.

Proof. From Corollary 5 it follows that all nilpotent subsemigroups from T belong to $\mathrm{Nil}_n(T)$. By Theorem 6 and the arguments, analogous to that of Corollary 7, the maximal nilpotent subsemigroups bijectively correspond to the ordered decompositions of M into 1-element blocks, that is to the linear orders on M. Obviously there exists $n!$ different linear orders. As all maximal nilpotent subsemigroups belong to $\mathrm{Nil}_n(T) \setminus \mathrm{Nil}_{n-1}(T)$, their nilpotency degree is n. This proves the first statement.

The proof of the second statement differs from the proof of the first one only in the place where one counts the number of subsemigroups. To count the number of ordered partitions of M into k non-empty blocks we note that there is an obvious bijection between such partitions and surjections form M to $\{1, 2, \ldots, k\}$. The last can be easily computed using the inclusion-exclusion formula. □

Proposition 10. *Let* $I_1 \subset T \subset \mathcal{PT}(M)$ *and* $M = M'_1 \cup \cdots \cup M'_k = M''_1 \cup \cdots \cup M''_m$ *be two ordered partitions of* M. *The semigroup* $T_1 = T(M'_1, \ldots, M'_k)$ *is contained in* $T_2 = T(M''_1, \ldots, M''_m)$ *if and only if* $m \geq k$, *every block* M'_i *of the first partition is a union of several neighbor blocks of the second partition, and the linear order on the blocks of the first partition is induced from the linear order on the blocks of the second partition.*

Proof. The sufficiency is obvious, so we prove the necessity. Let $T_1 \subset T_2$. Then the inequality $m \geq k$ is obvious. First we prove that every block of the second partition belongs to some block of the first partition. Indeed, if there would exist $i < j$ and l such that $M''_l \cap M'_i \neq \varnothing$ and $M''_l \cap M'_j \neq \varnothing$, we would get

$$T_1 \setminus T_2 \supset \{[x, y] \in I_1 \, : \, x \in M''_l \cap M'_i \text{ and } y \in M''_l \cap M'_j\} \neq \varnothing,$$

which contradicts our assumptions.

Now we show that every block of the first partition is the union of some neighbor blocks of the second partition. Indeed, assume that there exist $i < l < j$ and r such that $M''_i \subset M'_r$, $M''_j \subset M'_r$ and $M''_l \not\subset M'_r$. Then $M''_l \subset M'_p$ for some $p \neq r$. If $r < p$, the set $T_1 \setminus T_2$ contains the non-empty set $\{[x, y] \in I_1 \, : \, x \in M''_j, y \in M''_l\}$, which is impossible. The case $p < r$ is analogs.

Finally, we show that $M''_i \subset M'_p$, $M''_j \subset M'_q$ and $i < j$ imply $p \leq q$. Indeed, if $p > q$ we have

$$T_1 \setminus T_2 \supset \{[x, y] \in I_1 \, : \, x \in M''_j, y \in M''_i\} \neq \varnothing,$$

which is impossible. Hence the linear order of the blocks of the first partition is induced from the linear order of the blocks of the second partition. □

Corollary 9. *Let* $I_1 \subset T \subset \mathcal{PT}(M)$ *and* $|M| = n < \infty$. *Then every nilpotent semigroup* $T(M_1, \ldots, M_k)$ *is contained in* $|M_1|!|M_2|! \ldots |M_k|!$ *different maximal nilpotent subsemigroups in* T.

We finish this section with application of the above theory to the semigroup $\mathcal{IO}_n$. As $I_1 \subset \mathcal{IO}_n \subset \mathcal{PT}(N)$, the statements of Theorem 6, Proposition 10 and Corollaries 8 and 9 hold for $\mathcal{IO}_n$. This can be formulated as follows:

Theorem 7. *For nilpotent subsemigroups from* $\mathcal{IO}_n$ *containing* 0 *one has the following:*

1. *There are exactly* $n!$ *maximal nilpotent subsemigroups in* $\mathcal{IO}_n$, *each of which corresponds to some linear order on* N *and has the nilpotency degree* n.
2. *There are exactly* $\displaystyle\sum_{i=0}^{k-1}(-1)^i \binom{k}{i} (k-i)^n$ *nilpotent subsemigroups of* $\mathcal{IO}_n$ *which are maximal among all subsemigroups of nilpotency degree* $\leq k$, $1 < k \leq n$. *Each of these semigroups has the form*

   $$S(M_1, \ldots, M_k) = \{a \in \mathcal{IO}_n \, : \, x \in M_i, x\,a \in M_j \textit{ imply } i < j \textit{ for all } x \in \mathrm{dom}(a)\},$$

   where $N = M_1 \cup \cdots \cup M_k$ *is an ordered partition of* N *into* k *non-empty blocks. The semigroup* $S(M_1, \ldots, M_k)$ *has nilpotency degree* k *and is contained in exactly* $|M_1|!|M_2|! \ldots |M_k|!$ *maximal nilpotent subsemigroups (of nilpotency degree* n*).*

7 Maximal nilpotent subsemigroups

Let T_1 and T_2 be two maximal nilpotent subsemigroups in $\mathcal{IO}_n$ corresponding to the linear orders $i_1 \prec i_2 \prec \cdots \prec i_n$ and $j_1 \prec' j_2 \prec' \cdots \prec' j_n$ respectively. We remark that nilpotent elements in $\mathcal{IO}_n$ do not contain cycles. To simplify our notations we are not going not write chains of length 1 in nilpotent element, understanding that all elements from N, which are not mentioned in the presentation of some nilpotent element $a \in \mathcal{IO}_n$, are not contained in $\mathrm{dom}(a)$.

Lemma 9. *Suppose there exists an isomorphisms $\varphi : T_1 \to T_2$. Then $[i_k, i_l]\,\varphi = [j_k, j_l]$ for every element $[i_k, i_l]$ of rank 1 from T_1.*

Proof. First we remark that $T_1^{n-1} = \{0, [i_1, i_n]\}$ and $T_2^{n-1} = \{0, [j_1, j_n]\}$. This implies $[i_1, i_n]\,\varphi = [j_1, j_n]$. From the decomposition $[i_1, i_n] = [i_1, i_2]\cdot\, [i_2, i_3] \cdots\, [i_{n-1}, i_n]$ we get $[j_1, j_n] = ([i_1, i_2]\,\varphi)\cdot\, ([i_2, i_3]\,\varphi) \cdot \cdots\, \cdot([i_{n-1}, i_n]\,\varphi)$. The last equality means that for all k, $1 \le k \le n$, the element j_k belongs to $\mathrm{dom}([i_k, i_{k+1}]\,\varphi)$ and $j_k\,([i_k, i_{k+1}]\,\varphi) = j_{k+1}$.

Assume that $\mathrm{rank}([i_k, i_{k+1}]\,\varphi) > 1$ for some k. Then there exists $l \neq k$ such that $j_l \in \mathrm{dom}([i_k, i_{k+1}]\,\varphi)$ and

$$([i_{l-1}, i_l]\,\varphi)([i_k, i_{k+1}]\,\varphi) \neq 0, \tag{4}$$

since $j_{l-1} \in \mathrm{dom}(([i_{l-1}, i_l]\,\varphi)([i_k, i_{k+1}]\,\varphi))$ by the arguments above. But $[i_{l-1}, i_l][i_k, i_{k+1}] = 0$ provided that $l \neq k$ and this contradicts the inequality (4). Thus $\mathrm{rank}([i_k, i_{k+1}]\,\varphi) = 1$ and $[i_k, i_{k+1}]\,\varphi = [j_k, j_{k+1}]$ for all k.

Now for arbitrary $k < l$ we get that

$$[i_k, i_l]\,\varphi = [i_k, i_{k+1}] \dots [i_{l-1}, i_l]\,\varphi = ([i_k, i_{k+1}]\,\varphi) \dots ([i_{l-1}, i_l]\,\varphi) = [j_k, j_{k+1}] \dots [j_{l-1}, j_l] = [j_k, j_l].$$

□

Set $s = \begin{pmatrix} i_1 & i_2 & \dots & i_n \\ j_1 & j_2 & \dots & j_n \end{pmatrix} \in S_n$.

Lemma 10. *Let $\varphi : T_1 \to T_2$ be an isomorphisms. Then $a\,\varphi = s^{-1}as$ for all $a \in T_1$. In particular, there exists only one isomorphism $\varphi : T_1 \to T_2$.*

Proof. Let $i_p \in \mathrm{dom}(a)$ and $i_p\, a = i_q$. If $p \neq 1$ and $q \neq n$, then $[i_1, i_p]a[i_q, i_n] = [i_1, i_n]$ and, using Lemma 9, we get $[j_1, j_p](a\,\varphi)[j_q, j_n] = [j_1, j_n]$, and hence $j_p \in \mathrm{dom}(a\,\varphi)$ and $j_p\,(a\,\varphi) = j_q$.

If $p = 1$, $q \neq n$, then $a[i_q, i_n] = [i_1, i_n]$ and hence $(a\,\varphi)[j_q, j_n] = [j_1, j_n]$. This implies $j_1 \in \mathrm{dom}(a\,\varphi)$ and $j_1\,(a\,\varphi) = j_q$. The case $p \neq 1$, $q = n$ can be treated in an analogous way.

Consider now those $a \in T_1$ for which $i_1 \in \mathrm{dom}(a)$ and $i_1\, a = i_n$. To prove that $j_1 \in \mathrm{dom}(a\,\varphi)$ and $j_1\,(a\,\varphi) = j_n$ it is enough to show that $\mathrm{rank}(a) = \mathrm{rank}(a\,\varphi)$. We use induction in $\mathrm{rank}(a)$. For elements of rank 1 the statement follows from Lemma 9. Let now rank$(a) = k$ and assume that we know that for all elements of smaller rang the statement is true. If rank$(a\,\varphi) < k$, we can use the inductive assumption to $a\,\varphi$ and the isomorphism $\varphi^{-1} : T_2 \to T_1$ and get rank$(a) < k$. Hence we have rank$(a\,\varphi) \geq k$.

If rank$(a\,\varphi) > k$, there exists at least k different elements m for which $[j_1, j_m](a\,\varphi) \neq 0$. But there exists only $k-1$ different m for which $[i_1, i_m]a \neq 0$. This contradiction implies rank$(a\,\varphi) = k =$ rank(a) and completes the proof. □

We recall that transformation semigroups (S_1, M_1) and (S_2, M_2) are called *similar* provided that there exists an isomorphism $\varphi : S_1 \to S_2$ and a bijection $f : M_1 \to M_2$ such that $(m\,s)\,f = (m\,f)\,(s\,\varphi)$ for all $s \in S_1$ and $m \in M_1$.

Corollary 10. *The maximal nilpotent subsemigroup T_1 and T_2 from $\mathcal{IO}_n$ are isomorphic if and only if they are similar as transformation semigroups on N.*

Proof. Necessity follows from Lemma 10 and sufficiency is trivial. □

With the linear order $i_1 \prec i_2 \prec \cdots \prec i_n$ on the set N we associate the vector $v_i = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$, where α_1 is the number of elements among i_2, i_3,..., i_{n-1}, smaller than i_1 with respect to the usual order on $\mathbb{N}$; and α_k, $k = 2, 3, \ldots, n-1$, is the number of elements among i_{k+1}, i_{k+2},..., i_n, smaller than i_k. We set

$$\overline{v_i} = (n-2-\alpha_1, n-2-\alpha_2, n-3-\alpha_3, \ldots, n-k-\alpha_k, \ldots, 2-\alpha_{n-2}, 1-\alpha_{n-1}).$$

Obviously $\overline{\overline{v_i}} = v_i$.

Theorem 8. *The maximal nilpotent semigroups T_1 and T_2 from $\mathcal{IO}_n$, associated with the linear orders $i_1 \prec i_2 \prec \cdots \prec i_n$ and $j_1 \prec' j_2 \prec' \cdots \prec' j_n$ respectively, are isomorphic if and only if $v_i = v_j$ or $v_i = \overline{v_j}$.*

Lemma 11. *Let T_1 and T_2 be isomorphic and $1 \leq k < l < p \leq n$. Then*

1. *$i_k > i_l > i_p$ implies $j_k > j_l > j_p$ or $j_k < j_l < j_p$;*
2. *$i_k > i_l < i_p$ implies $j_k > j_l < j_p$ or $j_k < j_l > j_p$;*
3. *$i_k < i_l > i_p$ implies $j_k < j_l > j_p$ or $j_k > j_l < j_p$;*
4. *$i_k < i_l < i_p$ implies $j_k < j_l < j_p$ or $j_k > j_l > j_p$.*

Proof. If $i_k > i_l > i_p$ then $\begin{pmatrix} i_k & i_l \\ i_l & i_p \end{pmatrix} \in T_1$ and $\begin{pmatrix} j_k & j_l \\ j_l & j_p \end{pmatrix} \in T_2$ by Lemma 10. Since the last transformation must preserve the order, we get $j_k > j_l > j_p$ or $j_k < j_l < j_p$. This proves the first statement and the last one is analogous.

If $i_k > i_l < i_p$ then $\begin{pmatrix} i_k & i_l \\ i_l & i_p \end{pmatrix} \notin T_1$ and hence $\begin{pmatrix} j_k & j_l \\ j_l & j_p \end{pmatrix} \notin T_2$. Therefore either $j_k > j_l < j_p$ or $j_k < j_l > j_p$ holds, which implies the second statement. The third statement can be treated by the same arguments. □

Proof of Theorem 8. We start with the necessity. Assume that there exist $k < l$, $k \neq 1$ or $l \neq n$, such that the inequalities in each pair i_k, i_l and j_k, j_l are directed in the same way (i.e. $i_k < i_l$ and $j_k < j_l$, or $i_k > i_l$ and $j_k > j_l$). If $k \neq 1$, Lemma 11 guarantees that the inequalities in the pairs i_{k-1}, i_k and j_{k-1}, j_k are directed in the same way as well. By analogous arguments, if $l \neq n$, then the inequalities in the pairs i_l, i_{l+1} and j_l, j_{l+1} are also directed in the same way. Using Lemma 11 several times we get that the inequalities in arbitrary two pairs i_m, i_{m+1} and j_m, j_{m+1} are directed in the same way. Again by Lemma 11 we get that the inequalities in arbitrary two pairs i_p, i_q and j_p, j_q; $p < q$, $p \neq 1$ or $q \neq n$, are directed in the same way. Hence, under the above assumption we get $v_i = v_j$. Obviously $v_i = \overline{v_j}$ in the opposite case.

Now we prove the sufficiency. With each permutation $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$ we can associate the vector $(\beta_1, \beta_2, \ldots, \beta_n)$ of inversions, where β_k is the number of i_l such that $i_l < i_k$ and $l > k$. This correspondence from S_n to the set of all vectors $(x_1, \ldots, x_n) \in \mathbb{Z}^n$, $0 \leq x_k \leq n - k$, $k = 1, 2, \ldots, n$, is bijective by [K]. The vector v_i can be obtained from $(\beta_1, \beta_2, \ldots, \beta_n)$ by deleting the last component $\beta_n = 0$ and substituting β_1 with $\beta_1 - 1$ if $i_1 > i_n$.

Now we have to determine for which $i_1 \prec i_2 \prec \cdots \prec i_n$ and $j_1 \prec' j_2 \prec' \cdots \prec' j_n$ one gets $v_i = v_j$. If the vectors of inversions for $i_1, \ldots, i_n$ and $j_1, \ldots, j_n$ would coincide, the permutations themselves and thus the corresponding linear order should coincide as well. Hence, we can assume that the vectors of inversions for $i_1, \ldots, i_n$ and $j_1, \ldots, j_n$ are different. In particular, we can assume $i_1 < i_n$ and $j_1 > j_n$.

For permutations $i_2, i_3, \ldots, i_n$ and $j_2, j_3, \ldots, j_n$ of $n-1$ elements the corresponding vectors of inversions coincide. Hence, if we write both permutations in the natural increasing orders

$$i_{p_1}, i_{p_2}, \ldots, i_{p_{n-1}}, \qquad \text{and} \qquad j_{p_1}, j_{p_2}, \ldots, j_{p_{n-1}}, \tag{5}$$

the corresponding permutation $p_1, \ldots, p_{n-1}$ of indices will be the same. Let us assume that $v_i = (k, \alpha_2, \ldots, \alpha_{n-2})$. If we add the elements i_1 and j_1 to (5) on the corresponding places, we get the sequences

$$i_{p_1}, i_{p_2}, \ldots, i_{p_k}, i_1, i_{p_{k+1}} \ldots, i_{p_{n-1}}, \qquad \text{and} \qquad j_{p_1}, j_{p_2}, \ldots, j_{p_k}, j_{p_{k+1}}, j_1, \ldots, j_{p_{n-1}},$$

each of which coincides with $1, 2, \ldots, n$. Hence $i_1 = j_{p_{k+1}}$, $j_1 = i_{p_{k+1}} = i_1 + 1$, and $i_m = j_m$ for $m \neq \{1, p_{k+1}\}$. The pair $i_n = j_n$ and $j_1 = i_1 + 1$ of equalities contradicts the assumption that $i_1 < i_n$ and $j_1 > j_n$. Hence $p_{k+1} = n$, $i_n = i_1 + 1$ and the linear order $j_1 \prec' j_2 \prec' \cdots \prec' j_n$ is obtained from $i_1 \prec i_2 \prec \cdots \prec i_n$ by the transposition of i_1 and i_n.

Now let $v_i = v_j$ and $a = \begin{pmatrix} i_{p_1} & i_{p_2} & \cdots & i_{p_m} \\ i_{q_1} & i_{q_2} & \cdots & i_{q_m} \end{pmatrix} \in T_1$, where $i_{p_1} < i_{p_2} < \cdots < i_{p_m}$ and $i_{q_1} < i_{q_2} < \cdots < i_{q_m}$. Since $n \notin \{p_1, \ldots, p_m\}$, $1 \notin \{q_1, \ldots, q_m\}$, and the partial orders $i_1 \prec i_2 \prec \cdots \prec i_n$ and $j_1 \prec' j_2 \prec' \cdots \prec' j_n$ differ only by the transposition of the minimal and the maximal elements (which in this case are neighbor positive integers), we get that $j_{p_1} < j_{p_2} < \cdots < j_{p_m}$ and $j_{q_1} < j_{q_2} < \cdots < j_{q_m}$ implying $\begin{pmatrix} j_{p_1} & j_{p_2} & \cdots & j_{p_m} \\ j_{q_1} & j_{q_2} & \cdots & j_{q_m} \end{pmatrix} \in T_2$. Hence the permutation $s = \begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$ defines an injection $\varphi : a \mapsto s^{-1}as$ from

T_1 to T_2. It is easy to check that φ is a homomorphism. By analogous arguments the map $b \mapsto sbs^{-1}$ is a monomorphism from T_2 to T_1. Hence $T_1 \simeq T_2$.

If $j_k = n + 1 - i_n$ for all $k = 1, 2, \ldots, n$, then we have $v_i = \overline{v_j}$. Hence, since $v_i = \overline{v_j}$, we get that $j_1 \prec' j_2 \prec' \cdots \prec' j_n$ is obtained from $i_1 \prec i_2 \prec \cdots \prec i_n$ by the transformation $j_k = n{+}1{-}i_k$, $k = 1, 2, \ldots, n$, and, if i_1 and i_n are neighbor positive integers, possibly by the transposition of the minimal and the maximal elements following the first transformation.

Hence, for $v_i = \overline{v_j}$ the equalities $i_{p_1} < \cdots < i_{p_m}$ and $i_{q_1} < \cdots < i_{q_m}$, where $n \notin \{p_1, \ldots, p_m\}$ and $1 \notin \{q_1, \ldots, q_m\}$, yield $j_{p_1} > \cdots > j_{p_m}$ and $j_{q_1} > \cdots > j_{q_m}$. Hence $\begin{pmatrix} i_{p_1} & i_{p_2} & \cdots & i_{p_m} \\ i_{q_1} & i_{q_2} & \cdots & i_{q_m} \end{pmatrix} \in T_1$ anyway implies $\begin{pmatrix} j_{p_1} & j_{p_2} & \cdots & j_{p_m} \\ j_{q_1} & j_{q_2} & \cdots & j_{q_m} \end{pmatrix} \in T_2$. An isomorphism from T_1 to T_2 follows in the same way as above. This completes the proof. □

Corollary 11. *1. All maximal nilpotent subsemigroups in $\mathcal{IO}_2$ are isomorphic.*

2. Let $n > 2$. Then $\mathcal{IO}_n$ contains exactly $\frac{1}{2}(n! - (n-1)!)$ pairwise non-isomorphic maximal nilpotent subsemigroups.

Proof. The first statement is obvious, so we prove the second one. Decompose the linear orders on N into equivalence classes with respect to the isomorphism between the corresponding maximal nilpotent subsemigroups. From the proof of Theorem 8 it follows that those linear orders in which the minimal and the maximal elements are neighbor positive integers form equivalence classes with 4 elements and all other orders form equivalence classes with 2 elements. The number of linear orders of the first type is $2 \cdot (n-1) \cdot (n-2)! = 2 \cdot (n-1)!$. Hence the number of pairwise non-isomorphic maximal nilpotent subsemigroup in $\mathcal{IO}_n$ equals

$$\frac{2 \cdot (n-1)!}{4} + \frac{n! - 2 \cdot (n-1)!}{2} = \frac{n! - (n-1)!}{2}.$$

□

8 On the cardinality of nilpotent subsemigroups

Proposition 11. *The maximal nilpotent subsemigroup T associated with the natural linear order has the maximal cardinality among all nilpotent subsemigroups in $\mathcal{IO}_n$.*

Proof. Since every nilpotent subsemigroup is contained in a maximal nilpotent subsemigroup, it is enough to show that T has the maximal cardinality among all maximal nilpotent subsemigroups in $\mathcal{IO}_n$.

Let T_1 be arbitrary maximal nilpotent subsemigroup of $\mathcal{IO}_n$, $a \in T_1$ and $u = [a_1, \ldots, a_k]$ be a chain in a. The intervals $[a_1, a_k]$ and $[a_k, a_1]$ of $\mathbb{N}$ for increasing and decreasing u respectively will be called *the interval* of the chain u and denoted I_u.

Now we prove that the intervals I_u and I_v do not intersect provided that the element $a \in T_1$ has an increasing chain $u = [a_1, a_2, \ldots, a_k]$ and a decreasing chain, $v = [b_1, b_2, \ldots, b_m]$. Indeed, assume $I_u \cap I_v \neq \varnothing$. Then either $a_1 \in I_v$, or $a_k \in I_v$, or $I_v \subset I_u$ and $b_1 \in I_u$. In the first case there exists j such that $b_j < a_1 < b_{j-1}$. Since a preserves the order,

$a_2 = a_1 a < b_{j-1} a = b_j$ and hence $a_2 < b_j < a_1$, which contradicts the increasing of u. In the second case there exists j such that $b_j < a_k < b_{j-1}$. Then the inequality $a_{k-1} < a_k < b_{j-1}$ and the fact that a preserves the order yield $a_k = a_{k-1} a < b_{j-1} a = b_j$, which contradicts to $b_j < a_k$. The impossibility of the third case can be proved analogously.

Therefore, for a fixed $a \in T_1$, the set N decomposes into a disjoint union $N = I^1 \cup I^2 \cup \dots \cup I^m$ of intervals in $\mathbb{N}$ such that every chain of a is defined on one of these intervals only. Moreover, all chains defined on a given interval I^l either increase or decrease. Thus, substituting every decreasing chain $v = [b_1, \dots, b_m]$ of a by the corresponding increasing chain $\overline{v} = [b_m, \dots, b_1]$ we get the element $\overline{a} \in T$. It is now clear that the map $a \mapsto \overline{a}$ is injective and hence $|T_1| \leq |T|$. □

Theorem 9. *The maximal cardinality of a nilpotent subsemigroup in $\mathcal{IO}_n$ equals the n-th Catalan number $t_n = \frac{1}{n+1}\binom{2n}{n}$.*

Proof. According to Proposition 11, this maximal cardinality is the cardinality of T associated with the natural order on N. Each element $a \in T$ defines a decomposition of N into chains and all these chains will be increasing since $a \in T$.

Assume that we are given a decomposition $N = M_1 \cup \dots \cup M_k$ of N into k blocks, where $M_i = \{a_1^j, a_2^j, \dots, a_{m_i}^j\}$ and $m_1 + m_2 + \dots + m_k = n$. We will assume that the elements in each block are ordered: $a_1^j < a_2^j < \dots < a_{m_j}^j$. Then the element

$$a = [a_1^1, \dots, a_{m_1}^1] \dots [a_1^k, \dots, a_{m_k}^k]$$

is a nilpotent element in $\mathcal{IS}_n$ having k increasing chains. However, it is possible that a does not belong to T. Clearly, $a \in T$ if and only if a preserves the natural order on N, if and only if $a \in \mathcal{IO}_n$. This means that $a \notin T$ if and only if there exist $x, y \in N$ such that $x < y$ and $x\,a > y\,a$. The latter means that two elements $y, y\,a$ of one chain are contained between the neighbor elements $x, x\,a$ of another chain.

Therefore the decomposition $N = M_1 \cup \dots \cup M_k$ defines an element from T if and only if no pair of elements of one block lies between the neighbor elements of another block (here everything is taken with respect to the usual order on N). Further, $|T|$ equals the number p_n of the decompositions satisfying this condition.

For arbitrary positive integers n, m, $1 \leq m \leq n$, we denote by $T_{n,m}$ the set of all decompositions of N into blocks (as described above) in which the last m elements $n-m+1$, $n-m+2, \dots, n-1$, n belong to different blocks. Denote $t_{n,m} = |T_{n,m}|$ and put $t_{n,0} = t_{n,1}$. Obviously $p_n = t_{n,1}$ and $t_{n,n} = 1$.

For every decomposition $\rho \in T_{n+1,m}$ we denote by k_ρ the maximal element from the set $\{1, 2, \dots, n-m+1\}$, which belongs to the same block as $n+1$, if such element exists. Otherwise (i.e in the case when $\{n+1\}$ forms a separate block) we say that k_ρ is undefined. Moreover,

$$|\{\rho \in T_{n+1,m} \,:\, k_\rho = i\}| = t_{n,n+1-i}.$$

Now, if we decompose $T_{n+1,m}$ into subsets with respect to the value of k_ρ (this one is either undefined or an element from $\{1, 2, \dots, n-m+1\}$), we get the equality $t_{n+1,m} = t_{n,m-1}+$

$t_{n,m}+ t_{n,m+1}+ \ldots +t_{n,n}$. This implies $t_{n+1,m} = t_{n,m-1} + t_{n+1,m+1}$ for all $0 < m < n+1$. Set $a_{n,k} = t_{n,n-k}$. For the numbers $a_{n,k}$ we have

$$0 \leq k \leq n, a_{n,n} = a_{n,n-1}, a_{n,0} = 1, \text{ and } a_{n+1,k} = a_{n,k} + a_{n+1,k-1} \text{ for } 0 < k < n+1. \quad (6)$$

From relations (6) it follows that $a_{n,k}$ can be interpreted as the number of paths from the point (n, k), $n \geq k$, in $\mathbb{Z}^2$ to the point $(0, 0)$ if it is allowed to go along the vectors $(-1, 0)$ or $(-1, -1)$ on every step and it is not allowed to go above the diagonal $\{(x, x)\}$. It is well known, see for example [Gr, Section 1.6], that the number $a_{n,n}$ of such paths from the point (n, n) equals the n-th Catalan number $t_n = \frac{1}{n+1}\binom{2n}{n}$. Hence

$$p_n = t_{n,1} = t_{n,0} = a_{n,n} = \frac{1}{n+1}\binom{2n}{n}.$$

This completes the proof. □

The problem about the cardinalities of all nilpotent subsemigroups in $\mathcal{IO}_n$ seems to be very complicated. The main reason for this is the appearance of two interacting orders on N, namely the natural one, and the second one coming from $\mathrm{ord}(M_1, \ldots, M_k)$. Even in the case, when all blocks of the second order are 1-element, that is, in the case when the semigroup in question is maximal nilpotent, Theorem 8 gives only an upper bound for the cardinality.

As a comparison we remind that, for the semigroup $\mathcal{IS}_n$, where the description of the maximal nilpotent subsemigroups of the fixed nilpotency degree is the same as in the case of $\mathcal{IO}_n$ (see [GK3]), the cardinality of the maximal nilpotent subsemigroup $T_{\mathcal{IS}_n}(M_1, \ldots, M_k)$ of nilpotency degree k, which corresponds to the ordered decomposition $N = M_1 \cup \cdots \cup M_k$, can be calculated (see [GM2, Section 15]). In particular, one has that this cardinality depends only on the cardinalities $m_i = |M_i|$, $i = 1, \ldots, k$, and does not depend on the order of m_i's. This is no longer true for $\mathcal{IO}_n$ even in the "best" case, when the decomposition into blocks is coordinated with the natural order on N, that is, when the blocks are intervals $M_1 = \{1, 2, \ldots, m_1\}$, $M_2 = \{m_1 + 1, m_1 + 2, \ldots, m_2\}$ and so on. For instance, $|T(\{1, 2\}, \{3\}, \{4\})| = 9$ and $|T(\{1\}, \{2, 3\}, \{4\})| = 10$.

However, several partial results in this direction still can be obtained. Now we are going to discuss some of them. Until the end of the paper we will consider only those ordered decompositions $N = M_1 \cup \cdots \cup M_k$ of N for which $M_1 = \{1, 2, \ldots, m_1\}$, $M_2 = \{m_1 + 1, m_1 + 2, \ldots, m_2\}$ and so on. We denote $m_i = |M_i|$, $i = 1, 2, \ldots, k$, and set $F(m_1, \ldots, m_k) = |T(M_1, \ldots, M_k)|$.

Proposition 12. *The numbers $F(m_1, \ldots, m_k)$ satisfy the following recursion:*

1. *$F(1, 1, \ldots, 1) = t_k$ (here 1 appears exactly k times as the argument of F).*

2.

$$F(m_1,\ldots,m_{i-1},m_i+1,m_{i+1},\ldots,m_k)= \\ = F(m_1,\ldots,m_{i-1},m_i,1,m_{i+1},\ldots,m_k)- \\ - F(m_1,\ldots,m_{i-1},m_i)\cdot\sum_{j=1}^{m_i}F(j,m_{i+1},\ldots,m_k).$$

Proof. The first statement is Theorem 9, so we prove the second one. We consider the ordered decomposition $M_1\cup\cdots\cup M_{i-1}\cup M_i\cup\{x\}\cup M_{i+1}\cup\cdots\cup M_k$ and the corresponding subsemigroup $T=T(M_1,\ldots,M_i,\{x\},M_{i+1},\ldots,M_k)$. Define $T^{(1)}$ as the set of all those elements a from T such that $y\,a\neq x$ for all $y\in M_i$. Set $T^{(2)}=T\setminus T^{(1)}$. The condition on elements in $T^{(1)}$ allows one to consider them as elements from $T(M_1,\ldots,M_{i-1},M_i\cup\{x\},M_{i+1},\ldots,M_k)$ and it is easy to see that this correspondence is bijective. Hence $|T^{(1)}|=|T(M_1,\ldots,M_{i-1},M_i\cup\{x\},M_{i+1},\ldots,M_k)|=F(m_1,\ldots,m_{i-1},m_i+1,m_{i+1},\ldots,m_k)$.

Let $M_i=\{l,l+1,\ldots,l+m_i-1\}$. For $j=1,\ldots m_i$ define $T^{(2,j)}$ as the set of all elements a from $T^{(2)}$ such that $l+j-1\,a=x$. Clearly $T^{(2)}$ is the disjoint union of $T^{(2,j)}$, $j=1,\ldots m_i$. Now let us calculate $|T^{(2,j)}|$. If $a\in T^{(2,j)}$ and $y<l$ (that is $y\in M_1\cup\cdots\cup M_{i-1}$), we have $y\,a<l\,a\leq l+j-1\,a=x$ by the definition of $\mathcal{IO}_n$. Hence the element $a'\in\mathcal{IS}(M_1\cup\cdots\cup M_i)$, defined as follows: $\mathrm{dom}(a')=\mathrm{dom}(a)\cap M_1\cup\cdots\cup M_{i-1}$ and $y\,a'=y\,a$, $y\in\mathrm{dom}(a')$, belongs to $\mathcal{IO}(M_1\cup\cdots\cup M_i)$ and, moreover, can be considered as an element from $T(M_1\cup\cdots\cup M_i)$.

Further, with each element $a\in T^{(2,j)}$ we can associate another element, $a''\in T(\{l+j-1,\ldots,m_i\},M_{i+1},\ldots,M_k)$, in the following way:

$$\begin{gathered}\mathrm{dom}(a'')\cap(\{l+j,\ldots,m_i\}\cup M_{i+1}\cup\cdots\cup M_k)= \\ =\mathrm{dom}(a)\cap(\{l+j,\ldots,m_i\}\cup M_{i+1}\cup\cdots\cup M_k)\,; \\ l+j-1\in\mathrm{dom}(a'')\text{ if and only if }x\in\mathrm{dom}(a); \\ y\,a''=y\,a,\quad y\in\mathrm{dom}(a'')\cap(M_{i+1}\cup\cdots\cup M_k)\,; \\ \{l+j-1,\ldots,m_i\}\,a''=\{l+j,\ldots,m_i,x\}\,a.\end{gathered}$$

The last condition uniquely defines the action of a'' on $\{l+j-1,\ldots,m_i\}$ since $|\,\mathrm{dom}(a'')\cap\{l+j-1,\ldots,m_i\}|=|\,\mathrm{dom}(a)\cap\{l+j,\ldots,m_i,x\}|$ and thus we get that $a''\in\mathcal{IO}(\{l+j-1,\ldots,m_i\},M_{i+1},\ldots,M_k)$.

It is easy to see that the map $a\mapsto(a',a'')$ from $T^{(2,j)}$ into $T(M_1\cup\cdots\cup M_i)\times T(\{l+j-1,\ldots,m_i\},M_{i+1},\ldots,M_k)$ is bijective. Hence $|T^{(2,j)}|=F(m_1,\ldots,m_{i-1},m_i)\cdot F(j,m_{i+1},\ldots,m_k)$. This completes the proof. □

Using reduction, given by Proposition 12, one can express $F(m_1,\ldots,m_k)$ as a polynomial in t_i, $i=1,\ldots,n$. However, it is possible that the final expression will depend on the reduction process. We formulate our belief that the answer does not depend on the reduction in the following conjecture, which we have checked for many small values of m_i:

Conjecture. The expression of $F(m_1,\ldots,m_k)$ in terms of $F(1,\ldots,1)=t_i$, $i=1,\ldots,n$, which can be obtained using Proposition 12, does not depend on the choice of the reduction procedure even as a non-commutative polynomial.

Problem. Give a closed formula for $F(m_1, \ldots, m_k)$.

We finish our paper with two general formulas for $F(m_1, \ldots, m_k)$ in the case of special $(m_1, \ldots, m_k)$. The first case is when $m_2 = m_3 = \cdots = m_k = 1$ and $m_1 = p$. In this case we denote $F(m_1, \ldots, m_k)$ by $G(p, k-1)$ for simplicity.

Proposition 13. *For $p \in \mathbb{N}$ and $k \in \mathbb{Z}_+$ we have*

$$G(p,k) = \sum_{i \geq 0} (-1)^i \binom{p-i}{i} t_{p+k-i}.$$

Lemma 12. *For $i \in \mathbb{Z}_+$ and $j \in \mathbb{N}$ the numbers $b_{i,j} = (-1)^i \binom{j-i}{i}$ satisfy the recursion $b_{0,1} = 1$, $b_{0,j} + b_{1,j} + \cdots + b_{i,j} = b_{i,j-1}$ (every undefined number is supposed to be zero).*

Proof. The first part is obvious and for the second part we can use induction in i. By these arguments we get

$$\sum_{s=0}^{i} b_{s,j} = b_{i,j} + \sum_{s=0}^{i-1} b_{s,j} = b_{i,j} + b_{i-1,j-2},$$

which reduces the problem to the identity $b_{i,j} + b_{i-1,j-2} = b_{i,j-1}$. The latter follows by a straightforward verification. □

Proof of Proposition 13. We start with the remark that in our case it is obvious that the expression for $G(p,k)$ does not depend on the reduction, because the reduction is unique. However, the formula we are going to prove does not necessarily coincide with the result of our reduction. We prove it by showing that it satisfies the recursion, given by Proposition 12. From Proposition 12 we get the following recursion for $G(p,k)$:

1. $G(1,k) = t_{k+1}$;

2. $G(p+1,k) = G(p,k+1) - G(p,0) \left(\sum_{i=1}^{p} G(i,k) \right)$.

Taking into account $G(p,0) = 1$ the last expression reads $G(p+1,k) + G(p,k) + \cdots + G(1,k) = G(p,k+1)$. Set $G(p,k) = \sum_{i \geq 0} a_{p,i,k} t_{n+k-i}$. Then for the numbers $a_{p,i,k}$, $p \in \mathbb{N}$, $i,k \in \mathbb{Z}_+$, we derive the following recursion: $a_{1,0,k} = 1$; $a_{p+1,i,k} + a_{p,i+1,k} + \cdots + a_{1,i+p,k} = a_{p,i,k+1}$. But from Lemma 12 it follows immediately that the numbers $b_{i,p} = a_{p,i,k}$ satisfy this recursion. This completes the proof. □

For $k = 0$ we have $G(p,k) = 1$ and hence we obtain the following combinatorial identity for Catalan numbers.

Corollary 12. *For all $p \in \mathbb{N}$ we have*

$$\sum_{i \geq 0} (-1)^i \binom{p-i}{i} t_{p-i} = 1.$$

The last case we are going to treat here is the case when $|m_i| \leq 2$ for all i. In this case among all decompositions τ of N into a union of intervals (with respect to the usual order on N) satisfying the following condition: the intersection of every block of τ with each M_i contains at most one element, there exists the unique decompositions, ρ say, containing the minimal number of blocks. For a decomposition τ of N into a union of intervals we write $\rho \vdash \tau$ provided that each block of ρ is contained is some block of τ. Clearly, the number of those τ for which $\rho \vdash \tau$ equals $2^{|\rho|-1}$, where $|\rho|$ denotes the number of blocks in ρ. Let τ be a decomposition of N into a union of intervals. Denote by $f(\tau)$ the following number:

$$f(\tau) = \prod_{B-\text{ block in } \tau} t_{|B|},$$

where the product is taken over all blocks of τ. For $A \subset N$ we write $A \ltimes \tau$ if A is contained in a block of τ.

Proposition 14. *Assume that $|m_i| \leq 2$ for all i and let ρ be as above. Then*

$$F(m_1, \ldots, m_k) = \sum_{\rho \vdash \tau} (-1)^{|\tau|+1} f(\tau), \tag{7}$$

where the sum is taken over the set of all decomposition τ of N into a union of intervals satisfying $\rho \vdash \tau$.

Proof. It is easy to see that under the condition $|m_i| \leq 2$ for all i the expression for $F(m_1, \ldots, m_k)$, obtained by reduction from Proposition 12, does not depend on the reduction process. Still the expression (7) will not coincide with this one in general. If all $|m_i| = 1$ then the validity of the formula (7) is obvious. So, we can use induction in the number of 2-element blocks and assume that $|m_i| = 2$. Then the recursion formula from Proposition 12 gives the following:

$$\begin{aligned} &F(m_1, \ldots, m_{i-1}, m_i, m_{i+1}, \ldots, m_k) = \\ &\qquad = F(m_1, \ldots, m_{i-1}, 1, 1, m_{i+1}, \ldots, m_k) - F(m_1, \ldots, m_{i-1}, 1) \cdot F(1, m_{i+1}, \ldots, m_k). \end{aligned}$$

Let $M_i = \{l, l+1\}$, A be the block of ρ containing l and B be the block of ρ containing $l+1$. The ordered decomposition ρ', which corresponds to $M_1 \cup \cdots \cup M_{i-1} \cup \{l\} \cup \{l+1\} \cup M_{i+1} \cup \cdots \cup M_k$ is obtained from ρ by gluing A and B. Hence, by induction,

$$F(m_1, \ldots, m_{i-1}, 1, 1, m_{i+1}, \ldots, m_k) = \sum_{\rho' \vdash \tau'} (-1)^{|\tau|+1} f(\tau) = \sum_{\rho \vdash \tau, A \cup B \ltimes \tau} (-1)^{|\tau|+1} f(\tau).$$

Denote by ρ_1 and ρ_2 the restrictions of ρ to $\{1, 2, \ldots, l\}$ and $\{l+1, l+2, \ldots, n\}$ respectively. Every τ, such that A and B are contained in different blocks, induces a pair of decompositions: a decomposition τ' of $\{1, 2, \ldots, l\}$ into intervals and a decomposition τ''

of $\{l+1, l+2, \ldots, n\}$ into intervals. Moreover, $\rho_1 \vdash \tau'$, $\rho_2 \vdash \tau''$. Then, by induction,

$$F(m_1, \ldots, m_{i-1}, 1) \cdot F(1, m_{i+1}, \ldots, m_k) = \\ = \left(\sum_{\rho_1 \vdash \tau'} (-1)^{|\tau'|+1} f(\tau') \right) \left(\sum_{\rho_2 \vdash \tau''} (-1)^{|\tau''|+1} f(\tau'') \right) = \\ = \sum_{\rho \vdash \tau, A \cup B \not\prec \tau} (-1)^{|\tau|+1} f(\tau).$$

this completes the proof. □

Corollary 13. *Assume that* $|M_l| = 2$ *and* $|M_i| = 1$ *for* $i \neq l$. *Then* $F(1, \ldots, 1, 2, 1, \ldots, 1) = t_n - t_l t_{n-l}$.

We remark once more that the expressions, obtained in Propositions 13 and 14 do not coincide with the result of the reduction provided by Proposition 12 in general. For example, one sees that we often used the identity $t_1 = 1$ and hence have possibly lost a lot of "trivial" factors in the formulae.

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