Combinatorics of nilpotents in $\mathcal{IS}_n$

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Abstract

We show how several famous combinatorial sequences appear in the context of nilpotent elements of the full symmetric inverse semigroup $\mathcal{IS}_n$. These sequences appear either as cardinalities of certain nilpotent subsemigroups or as the numbers of special nilpotent elements and include the Lah numbers, the Bell numbers, the Stirling numbers of the second kind, the binomial coefficients and the Catalan numbers.

1 Introduction

An element, a, of a semigroup, S, with the zero element 0 is called *nilpotent* provided that $a^k = 0$ for some positive integer k. If, for some positive integer k, $a_1 \cdot a_2 \cdots a_k = 0$ for arbitrary elements $a_1, \dots, a_k \in S$, then the semigroup S is also called *nilpotent* and the minimal possible k is called the *nilpotency degree* of the semigroup S. For a finite semigroup its nilpotency is equivalent to the nilpotency of all elements, [Ar, p. 179].

Up to (anti)-isomorphism, there exist exactly 4 semigroups of cardinality 2, and only one of them is nilpotent. However, the proportion of nilpotent semigroups among all semigroups with a fixed cardinality increases very rapidly. In [SYT] it is shown that, up to (anti)-isomorphism, 99% of almost $2 \cdot 10^9$ semigroups consisting of 8 elements are nilpotent. This shows that, for finite semigroups, nilpotency seems to be a very common property rather than an exceptional one. And hence the existence of a zero element should be even more frequent. Therefore the study of properties of nilpotent subsemigroups in the semigroups containing 0 is a natural and important problem. Many papers by different authors dedicated to this study have already appeared, see for example [Gar, GK2, GK3, GTS, GM1, GH, MS, Sh, Su] and references therein.

In the case of finite semigroups the study of nilpotent elements and subsemigroups naturally leads to interesting combinatorial problems. The present paper is dedicated to the solution of several problems of this kind, which appear during the study of the symmetric inverse semigroup $\mathcal{IS}_n$ consisting of all partial injections from the set $N = \{1, 2, \dots, n\}$ into itself, i.e. injections $a : A \to B$, where $A, B \subseteq N$. The zero element in $\mathcal{IS}_n$ is the totally undefined transformation 0, that is, the unique transformation whose domain is the empty set. It happens that many classical combinatorial number sequences, in particular, the Lah numbers, the Bell numbers, the Catalan numbers and the Stirling numbers of the second kind arise in relation to $\mathcal{IS}_n$ as the cardinalities of certain subsets (for example, subsemigroups) which contain only nilpotent elements.

The paper is organized as follows: in Section 2 we collect some elementary combinatorial results about $\mathcal{IS}_n$ including the formula for the number of nilpotent elements with a given defect. In fact, the latter is given by the signless Lah numbers. In Section 3 we describe the maximal nilpotent subsemigroups of $\mathcal{IS}_n$ and calculate the number of several classes of subsemigroups. In some cases the result is given by the Bell numbers, in some cases in terms of the Stirling numbers of the second kind, and in some cases just by the binomial coefficients. In Section 4 we find a formula for the cardinality of a maximal nilpotent subsemigroup of $\mathcal{IS}_n$ among the semigroups of nilpotency degree k. With every such subsemigroup we associate a certain polynomial with integer coefficients and the main, very amazing, result states that the desired cardinality is obtained if one substitutes x^i by the Bell number B_i in the standard expression for this polynomial. We finish the paper with showing in Section 5 how the Catalan numbers appear in the context of nilpotent elements in $\mathcal{IS}_n$. The semigroup $\mathcal{IS}_n$ contains the semigroup $\mathcal{IO}_n$ of all order-preserving injective maps, which is in fact an $\mathcal{H}$–cross-section of $\mathcal{IS}_n$. Then the Catalan numbers give the maximal order for nilpotent subsemigroups in $\mathcal{IO}_n$.

We will try to use the standard notation and the book [Ai] as the general reference. For $A \subseteq N$ we let $\overline{A} = N \setminus A$ denote the complement of A in N.

2 Preliminary combinatorics

From the definition of $\mathcal{IS}_n$ it follows immediately that every element $a \in \mathcal{IS}_n$ is uniquely determined by its domain $\mathrm{dom}(a)$, its range $\mathrm{im}(a)$ and a bijection from $\mathrm{dom}(a)$ to $\mathrm{im}(a)$. Hence (se also [Ho2, Exercise 5.11.3])

$$|\mathcal{IS}_n| = \sum_{k=0}^{n} \binom{n}{k}^2 k!.$$

The number $\mathrm{rank}(a) = |\,\mathrm{dom}(a)\,| = |\,\mathrm{im}(a)|$ is called the *rank* of a and the number $\mathrm{def}(a) = n - rank(a)$ is called the *defect* of a.

For elements from $\mathcal{IS}_n$ one can use their regular tableaux presentation

$$a = \left(\begin{array}{cccc} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{array} \right),$$

where $\mathrm{dom}(a) = \{i_1, \dots, i_k\}$ and $\mathrm{im}(a) = \{j_1, \dots, j_k\}$. However, sometimes it is more convenient to use the so-called *chain* (or *chart*) decomposition of a, which is analogous to the cyclic decomposition for usual permutations. This notion probably began in [Mu] (see also [Ho2, Exercise 5.11.7]) and we refer the reader to either these references or to [GK1, Li1, Mu] for rigorous definitions. However, this decomposition is easy to explain with the following example. The element

$$a = \left(\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 7 & 9 \\ 7 & 4 & 5 & 1 & 10 & 2 & 6 \end{array} \right) \in \mathcal{IS}_{10}$$

has the following graph of its action on $\{1, 2, \dots, 10\}$:

$$\begin{array}{ccc} 1 & \rightarrow & 7 \\ \uparrow & & \downarrow \\ 4 & \leftarrow & 2 \end{array} \qquad 3 \rightarrow 5 \rightarrow 10 \qquad 9 \rightarrow 6 \qquad 8,$$

and hence it is convenient to write it as $a = (1, 7, 2, 4)[3, 5, 10][9, 6][8]$. We call $(1, 7, 2, 4)$ a *cycle* and $[3, 5, 10]$ (as well as $[9, 6]$ and $[8]$) a *chain* of the element a. We remark that chains of length 1 (here length means the number of elements in the chain) correspond to those elements $x \in N$, which do not belong to $\mathrm{dom}(a) \cup \mathrm{im}(a)$. It is obvious that $\mathrm{def}(a)$ equals the number of chains in the chain decomposition of a.

Proposition 1. *([Ho1, Lemma V.1.9]) The set $E(\mathcal{IS}_n)$ of idempotents in $\mathcal{IS}_n$ is a semigroup isomorphic to the semigroup $\mathfrak{B}_n = \{A : A \subseteq N\}$ with the intersection of sets as the corresponding binary operation. In particular, $|E(\mathcal{IS}_n)| = 2^n$.*

Proof. $a^2 = a$ if and only if $a(a(x)) = a(x)$ for each $x \in \mathrm{dom}(a)$, which means $a(x) = x$ for each $x \in \mathrm{dom}(a)$ since a is injective. At the same time for every $A = \{i_1, i_2, \dots, i_k\} \subset N$ the element

$$e_A = \left(\begin{array}{cccc} i_1 & i_2 & \dots & i_k \\ i_1 & i_2 & \dots & i_k \end{array} \right)$$

is an idempotent in $\mathcal{IS}_n$. It is obvious that $e_A \cdot e_B = e_{A \cap B}$. $\square$

Proposition 2. *The element $a \in \mathcal{IS}_n$ is nilpotent if and only if the chain decomposition of a contains only chains. The number of nilpotent elements in $\mathcal{IS}_n$ with the given defect k equals the signless Lah number $L'_{n,k} = \frac{n!}{k!}\binom{n-1}{k-1}$.*

Proof. The first statement is obvious. To prove the second statement we observe that to obtain the chain decomposition with defect k

$$[i_1, i_2, \dots, i_{m_1}][i_{m_1+1}, i_{m_1+2}, \dots, i_{m_2}] \dots [i_{m_{k-1}+1}, i_{m_{k-1}+2}, \dots, i_{m_k}]$$

from the permutation $i_1, i_2, \dots, i_n$ of $1, 2, \dots, n$, it is enough to choose the ends $i_{m_1}, i_{m_2}, \dots, i_{m_{k-1}}$ of the first $k-1$ chains (as $m_k = n$ automatically). This can be done in $\binom{n-1}{k-1}$ different ways. Going through all permutations we will get the chain decompositions for all nilpotent elements of defect k. Since the order of the chains is not important (because all chains in a chain decomposition commute), every nilpotent will be counted $k!$ times. Indeed, the chains in each chain decomposition can be permuted in $k!$ ways without changing the chain decomposition (as a partial transformation of N) but each of these permutations of the chains correspond to a different permutation of N, that is, $k!$ permutations of N give $k!$ repetitions of one chain decomposition. Hence the number of nilpotents of defect k equals $L'_{n,k} = \frac{n!}{k!}\binom{n-1}{k-1}$. $\square$

Let $a \in \mathcal{IS}_n$. Denote by l_i the number of cycles of length i and by m_i the number of chains of length i in the chain decomposition of a. The vector $(l_1, \dots, l_n; m_1, \dots, m_n)$ is called the *chain type* of a. Denote by $s(n)$ the number of different cycle types for elements in the symmetric group S_n (which is equal to the number of conjugacy classes in S_n) and put $s(0) = 1$.

Proposition 3. *The number of different chain types for the elements in the semigroup* $\mathcal{IS}_n$ *equals* $\sum_{k=0}^{n} s(k)s(n-k)$.

Proof. The term $s(k)s(n-k)$ equals the number of those chain types, where the sum of lengths of all chains is equal to k, $0 \le k \le n$. $\square$

Proposition 4. *The number of elements in* $\mathcal{IS}_n$ *of the chain type* $(l_1, \dots, l_n; m_1, \dots, m_n)$, *where* $\sum_{i=1}^{n} i(l_i + m_i) = n$, *equals*

$$\frac{n!}{\prod_{i=1}^{n}(l_i! \cdot i^{l_i} \cdot m_i!)}.$$

Proof. All cycle and chain terms in a chain decomposition of a commute. Hence, we can choose a chain decomposition, which we call *principal*, constructed in the following way: first we write down all cycles with respect to the increasing number of elements in these cycles; and after this we write down the chains in the same way. We remark that a principal chain decomposition for the element a is not uniquely defined, as the order of both, cycles and chains, of a fixed length can be chosen in an arbitrary way and, moreover, the order of elements $(a_1, \dots, a_k)$ in a given cycle is unique only up to a cyclic permutation. We say principal chain decompositions of a are *different* if they differ in one of these ways. In particular, if $(l_1, \dots, l_n, m_1, \dots, m_n)$ is the chain type of the element $a \in \mathcal{IS}_n$, then the number of different principal chain decompositions of a equals $t = \prod_{i=1}^{n}(l_i! \cdot i^{l_i} \cdot m_i!)$.

The construction above defines a natural bijection between the $n!$ permutations of $1, 2, \dots, n$ and the principal chain decompositions of the elements with a given chain type. Dividing this number with t we get the desired cardinality of the set of elements with a fixed chain type. $\square$

Corollary 1. *The number of nilpotent elements in* $\mathcal{IS}_n$ *of the nilpotency degree* k *equals*

$$\sum_{(x_1,x_2,\dots,x_k)\in\Lambda} \frac{n!}{x_1!x_2!\dots x_{k-1}!(x_k+1)!},$$

where $\Lambda = \{(x_1, x_2, \dots, x_k) \in (\mathbb{N} \cup \{0\})^k \,:\, 1 \cdot x_1 + 2 \cdot x_2 + \dots + k \cdot x_k = n - k\}$.

3 Maximal nilpotent subsemigroups

We start with the remark that for a semigroup, S, with the zero element 0 one can consider two different types of nilpotent subsemigroups. The subsemigroups of the first type are nilpotent subsemigroups of S containing 0, and thus, if T is such a subsemigroup, one has

$T^k = 0$ for some $k \geq 1$. The subsemigroups of the second type are subsemigroups of S which are nilpotent as semigroups, but whose zero element can differ from 0. For such a subsemigroup, T, one will have $T^k = \{e\}$ for some $k \geq 1$, where e is an idempotent in S. Moreover, we always assume that e is not invertible, i.e. does not coincide with the identity permutation. In what follows we will speak about the nilpotent subsemigroups of the second type if the opposite is not explicitly stated (as for example in the second part of Corollary 4, where we consider the nilpotent subsemigroups of $\mathcal{IS}_n$ of the first type).

A nilpotent subsemigroup, $T \subseteq \mathcal{IS}_n$, is called *maximal* if it is not contained in any other nilpotent subsemigroup $T_1 \subseteq \mathcal{IS}_n$, $T \neq T_1$. A nilpotent subsemigroup, $T \subseteq \mathcal{IS}_n$, is called *maximal for degree* k if the nilpotency degree of T equals k and T is not contained in any other nilpotent subsemigroup $T_1 \subseteq \mathcal{IS}_n$, $T \neq T_1$, of nilpotency degree k. The next statement is a natural generalization of the main result from [GK2].

Theorem 1. *For every idempotent $e \in E(\mathcal{IS}_n)$ such that $\mathrm{def}(e) = k \geq 1$, the semigroup $\mathcal{IS}_n$ contains exactly $k!$ maximal nilpotent subsemigroups containing e (and thus in which e is the zero element). If $\overline{\mathrm{dom}(e)} = \{a_1, \dots, a_k\}$, then every such subsemigroup corresponds to a permutation, $b_1, \dots, b_k$, of $a_1, \dots, a_k$ and has the following form:*

$$T_{b_1,\dots,b_k} = \{\pi \in \mathcal{IS}_n \,:\, \mathrm{dom}(e) \subseteq \mathrm{dom}(\pi); \pi(x) = x \textit{ for all } x \in \mathrm{dom}(e); \\ \pi(b_i) = b_j \textit{ implies } i < j \textit{ for all } b_i, b_j \in \mathrm{dom}(\pi) \setminus \mathrm{dom}(e)\}\,.$$

All these maximal nilpotent subsemigroups are isomorphic, their nilpotency degree equals k, and their cardinality equals the k-th Bell number B_k.

Proof. It is obvious that nilpotent elements can contain cycles only of length 1, which implies that if T is a maximal nilpotent subsemigroup of $\mathcal{IS}_n$ containing e, then $a(x) = x$ for every $a \in T$ and every $x \in \mathrm{dom}(e)$. In particular, a can be viewed as an element from $\mathcal{IS}(\overline{\mathrm{dom}(e)})$. Now the description of such subsemigroups follows from the general theory of nilpotent subsemigroups of transformation semigroups, see [GM2, Theorem 6] and [GM2, Corollary 8.1].

Let T be a maximal nilpotent subsemigroup of $\mathcal{IS}_n$ containing e. The chain decomposition of an element, $a \in T$, defines a partition of $\overline{\mathrm{dom}(e)}$. It is obvious that this correspondence between elements in T and partitions of $\overline{\mathrm{dom}(e)}$ is bijective and hence $|T| = B_k$. $\square$

Corollary 2. *The nilpotency degree of a nilpotent subsemigroup in $\mathcal{IS}_n$ does not exceed n and, for every k, $1 \leq k \leq n$, $\mathcal{IS}_n$ contains exactly $\binom{n}{k}k!$ maximal nilpotent subsemigroup whose nilpotency degree equals k.*

Proof. This follows from Theorem 1 and the fact that $\mathcal{IS}_n$ contains exactly $\binom{n}{k}$ idempotents of defect k. $\square$

As usual, we denote by $S_{n,m}$ the Stirling numbers of the second kind.

Corollary 3. *Let T be a maximal nilpotent subsemigroup in $\mathcal{IS}_n$ having nilpotency degree $k \geq 1$ and $a \in T$. Then $n-k \leq \mathrm{rank}(a) < n$ and, for every m, $n-k \leq m < n$, T contains exactly $S_{k,n-m}$ elements of rank m.*

Proof. Let e be the idempotent of T. According to Theorem 1, we have $T = T_{b_1,\dots,b_k}$, where $\{b_1,\dots,b_k\} = \overline{\mathrm{dom}(e)}$. We define the linear order $\prec$ on the set $\overline{\mathrm{dom}(e)}$ as follows: $b_1 \prec b_2 \prec \cdots \prec b_k$.

Since $a \in T$, we know $a(x) = x$ for all $x \in \mathrm{dom}(e)$, hence $\mathrm{rank}(a) = m \geq n-k$, and since $a^k = e$ for some $k \geq 1$ and e is not invertible, we get that a can not be surjective, so $\mathrm{rank}(a) < n$. As we have already ssen, a containes $n-k$ cycles of length 1, and, by Theorem 1, these are the only cycles contained in a. Now $\overline{\mathrm{im}\, a}$ contains $n-m$ elements, each of which forms a chain of length 1 (that is, lies in $\overline{\mathrm{dom}(e)}$) or is a start of a chain with length at least 2. Thus, the chain decomposition of a contains $n-m$ chains formed by elements of $\overline{\mathrm{dom}(e)}$, and a is completely determined by these chains. From the definition of $T_{b_1,\dots,b_k}$ we see that these chains are determined by unordered partitions of $\overline{\mathrm{dom}(e)}$ into $n-m$ non-empty blocks. The number of such partitions certainly equals $S_{k,n-m}$. Since the linear order $b_1 \prec b_2 \prec \cdots \prec b_k$ on $\overline{\mathrm{dom}(e)}$ is fixed, the elements in one block must increase with respect to this order and thus can form only one chain. This completes the proof. $\square$

Proposition 5. *Let T be a maximal nilpotent subsemigroup in $\mathcal{IS}_n$ of nilpotency degree $k \leq n$. Then for any m, $1 < m \leq k$, the cardinality $|T^m|$ of the subsemigroup $T^m = \{a_1 \dots a_m \,:\, a_1,\dots,a_m \in T\}$ equals B_{k-m+1}.*

Proof. Let $e \in T$ be an idempotent. Since e is an idempotent and T is nilpotent, it follows that for any $a \in T$ we have $ea = ae = e$ and $a^k = e$ for some k. It follows that for any $x \in \mathrm{dom}(e)$ one has $a(x) = x$, and for any $y \notin \mathrm{dom}(e)$ one has either $y \notin \mathrm{dom}(a)$ or $a(y) \neq y$. Hence, during the proof we will indicate only the action of $a \in T$ on $\overline{\mathrm{dom}(e)}$.

From Theorem 1 it follows that it is enough to consider the case, when $\overline{\mathrm{dom}(e)} = \{1,2,\dots,k\}$, and T corresponds to the trivial permutation of $1,2,\dots,k$.

Lemma 1. *For $b \in T$ we have $b \in T^m$ if and only if the inequality $b(x) \geq x+m$ holds for all $x \in \mathrm{dom}(b) \cap \overline{\mathrm{dom}(e)}$.*

Proof. The necessity follows from the fact that $c(x) \geq x+1$ for all $c \in T$ and $x \in \mathrm{dom}(c) \cap \overline{\mathrm{dom}(e)}$.

Assume now that $b = \begin{pmatrix} i_1 & i_2 & \dots & i_l \\ j_1 & j_2 & \dots & j_l \end{pmatrix} \in T$ satisfies the condition of the lemma for some m. Then all the elements

$$b_1 = \begin{pmatrix} i_1 & \dots & i_l \\ i_1+1 & \dots & i_l+1 \end{pmatrix}, b_2 = \begin{pmatrix} i_1+1 & \dots & i_l+1 \\ i_1+2 & \dots & i_l+2 \end{pmatrix}, \dots,$$
$$b_{m-1} = \begin{pmatrix} i_1+m-2 & \dots & i_l+m-2 \\ i_1+m-1 & \dots & i_l+m-1 \end{pmatrix}, b_m = \begin{pmatrix} i_1+m-1 & \dots & i_l+m-1 \\ j_1 & \dots & j_l \end{pmatrix}$$

belong to T and $b_1 b_2 \dots b_m = b$. Hence $b \in T^m$. $\square$

Consider now the maximal nilpotent subsemigroup $R \subset \mathcal{IS}_n$ of nilpotency degree $k-m+1$, which contains the idempotent $e_{\{1,2,\dots,k-m+1\}}$ and corresponds to the trivial permutation of $1,2,\dots,k-m+1$. By Theorem 1, we have that $|R| = B_{k-m+1}$. At the same time, Lemma 1 guarantees that the maps

$$\varphi : R \to T^m, \quad \begin{pmatrix} i_1 & i_2 & \dots & i_l \\ j_1 & j_2 & \dots & j_l \end{pmatrix} \mapsto \begin{pmatrix} i_1 & i_2 & \dots & i_l \\ j_1+m-1 & j_2+m-1 & \dots & j_l+m-1 \end{pmatrix}$$

and

$$\psi : T^m \to R, \quad \begin{pmatrix} p_1 & p_2 & \dots & p_r \\ q_1 & q_2 & \dots & q_r \end{pmatrix} \mapsto \begin{pmatrix} p_1 & p_2 & \dots & p_r \\ q_1-m+1 & q_2-m+1 & \dots & q_l-m+1 \end{pmatrix}$$

are both injective. Hence $|T^m| = |R| = B_{k-m+1}$. □

Remark 1. The semigroups T^m and R are not isomorphic if $k \geq 4$ and $1 < m < k-1$, since they have different nilpotency degrees. Indeed, the nilpotency degree of T equals $\lceil k/m \rceil$ and the one of R equals $k-m+1$. For $m = k$ and $m = k-1$ the semigroups T^m and R are obviously isomorphic.

Theorem 2. *[GTS, Theorem 3] The group* $\mathrm{Aut}(T)$ *of automorphisms of the maximal nilpotent subsemigroup* $T \subseteq \mathcal{IS}_n$*, having nilpotency degree* $k > 2$*, is the elementary abelian* 2*-group of cardinality* $|\mathrm{Aut}(T)| = 2^{B_{k-2}}$.

For some k, $1 \leq k \leq n$, let $M_1, \dots, M_k$ be an ordered collection of k arbitrary non-empty disjoint subsets of N (i.e. $M_1 \ll M_2 \ll \dots \ll M_k$ for some linear order $\ll$), and $M = M_1 \cup \dots \cup M_k$. Set

$$T(M_1,\dots,M_k) = \big\{\pi \in \mathcal{IS}_n \,:\, \overline{M} \subseteq \mathrm{dom}(\pi); \pi(x) = x \text{ for all } x \in \overline{M}; \\ x \in M_i \text{ and } \pi(x) \in M_j \text{ implies } i < j \text{ for all } x \in \mathrm{dom}(\pi) \cap M\big\}.$$

We remark that for $k = 1$ we get $T_{M_1} = \{e_{\overline{M_1}}\}$.

Theorem 3. *[GK3, Theorem 1],[GK4, Theorem 1] The semigroup* $T(M_1,\dots,M_k)$ *is a maximal nilpotent subsemigroup of* $\mathcal{IS}_n$ *for degree* k*. If the ordered collections* $M_1,\dots,M_k$ *and* $M'_1,\dots,M'_k$ *are different then* $T(M_1,\dots,M_k) \neq T(M'_1,\dots,M'_k)$*. Moreover, every maximal nilpotent subsemigroup* T *of* $\mathcal{IS}_n$ *for degree* k *has the form* $T = T(M_1,\dots,M_k)$ *for some ordered collection* $M_1,\dots,M_k$ *of non-empty disjoint subsets from* N.

Corollary 4. *Let* n_k *denote the number of maximal nilpotent subsemigroups in* $\mathcal{IS}_n$ *for degree* k*. Then*

$$n_k = \sum_{m=k}^{n} \sum_{i=0}^{k-1} (-1)^i \binom{n}{m} \binom{k}{i} (k-i)^m = \sum_{m=k}^{n} \binom{n}{m} \cdot S_{m,k} \cdot k!. \tag{1}$$

In particular, $\mathcal{IS}_n$ *contains* $\sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^n = S_{n,k} \cdot k!$ *maximal nilpotent subsemigroups for degree* k*, whose zero element is* 0.

Proof. To prove the equality (1) we first choose an m-element subset, $M = M_1 \cup \cdots \cup M_k$. Then we can consider surjections $\varphi : M \to \{1, 2, \dots, k\}$ and identify M_i with $\varphi^{-1}(i)$, which, summing over all m, gives the double summation in (1). Note that $T(M_1, \dots, M_k)$ is defined in terms of an ordered partition of M, not an unordered one. Hence, for every unordered partition of M into k blocks, we can order these blocks in $k!$ ways and this, after summation, gives us the right hand side of (1).

If a nilpotent subsemigroup, T, of $\mathcal{IS}_n$ contains 0, then $M = \overline{\mathrm{dom}(0)} = N$ and hence the number of such maximal nilpotent subsemigroups for degree k is obtained in termes of partitions of N. The latter corresponds to the $m = n$ case above. □

Theorem 4. *[GK4, Lemma 10] Let $k > 2$. The maximal nilpotent subsemigroups $T_1 = T(M_1, \dots, M_k)$ and $T_2 = (M_1', \dots, M_k')$ are isomorphic if and only if $|M_i| = |M_i'|$ for all i, $1 \leq i \leq k$.*

Corollary 5. *The semigroup $\mathcal{IS}_n$ contains $\binom{n}{k}$ pairwise non-isomorphic maximal nilpotent subsemigroups for a fixed degree k, $2 < k \leq n$.*

Proof. According to Theorem 4, the isomorphism classes of the maximal nilpotent subsemigroups for a fixed degree k, $2 < k \leq n$, bijectively correspond to the solutions of the inequality $m_1 + m_2 + \cdots + m_k \leq n$, where all m_i's are positive integers. The map $(m_1, \dots, m_k) \mapsto \{m_1, m_1 + m_2, \dots, m_1 + m_2 + \cdots + m_k\}$ then gives a bijection between the set of solutions to this inequality and the set of all k-element subsets in N. □

We remark that the number of pairwise non-isomorphic maximal nilpotent subsemigroups containing 0 for a fixed degree k, $2 < k \leq n$, equals $\binom{n-1}{k-1}$ ([GK4, Corollary 3]).

The nilpotency degree 2 corresponds to the nilpotent semigroups with zero multiplication. The inner structure of such semigroups is quite poor and two such semigroups are isomorphic if and only if they have the same cardinalities. Hence the statement of Theorem 4 does not hold for maximal nilpotent subsemigroups of $\mathcal{IS}_n$ having nilpotency degree 2. Although it is quite easy to see that

$$|T(M_1, M_2)| = \sum_{i=0}^{\min(m_1, m_2)} \binom{m_1}{i} \binom{m_2}{i} i!, \tag{2}$$

where $m_i = |M_i|$, $i = 1, 2$, it is not very easy to determine how many of these semigroups will be pairwise non-isomorphic. However, for a fixed zero element this can be done.

Proposition 6. *[GK2, Theorem 5] $\mathcal{IS}_n$ contains exactly $\left[\frac{k+1}{2}\right]$ pairwise non-isomorphic maximal nilpotent subsemigroups for degree 2, whose zero element is an idempotent of defect k.*

4 Cardinalities of the maximal nilpotent subsemigroups

From Theorem 4 and the equality (2) it follows that the cardinality of the maximal nilpotent subsemigroup $T(M_1, \dots, M_k)$ for degree k depends only on cardinalities $m_i = |M_i|$ of the blocks M_i, $i = 1, \dots, k$. The nilpotent semigroups of nilpotency degree 1 satisfy $T = T^1 = e$, where e is an idempotent, and hence contain only one element. This corresponds to the fact that $|T(M_1)| = 1$ in the case $k = 1$. Let now $|M_i| = m_i$, $i = 1, \dots, k$. The vector $(m_1, \dots, m_k)$ will be called the *type* of the semigroup $T(M_1, \dots, M_k)$. If $T(M_1, \dots, M_k)$ has type $(m_1, \dots, m_k)$ we denote $t(m_1, \dots, m_k) = |T(M_1, \dots, M_k)|$. In particular, $t(m) = 1$ for all $m \geq 1$.

Lemma 2.

$$\begin{aligned} t(m_1, \dots, m_{i-1} + 1, m_{i+1}, \dots, m_k) &= \\ &= t(m_1, \dots, m_{i-1}, 1, m_{i+1}, \dots, m_k) - m_{i-1} \cdot t(m_1, \dots, m_{i-1}, m_{i+1}, \dots, m_k). \end{aligned}$$

Proof. We decompose the semigroup $T = T(M_1, \dots, M_{i-1}, \{x\}, M_{i+1}, \dots, M_k)$ into two sets: $T_1 = \{\pi \in T : x \notin \pi(M_{i-1})\}$ and $T_2 = \{\pi \in T : x \in \pi(M_{i-1})\}$. The set T_1 coincides with $T(M_1, \dots, M_{i-1} \cup \{x\}, M_{i+1}, \dots, M_k)$. Moreover, if $M_{i-1} = \{b_1, \dots, b_{m_{i-1}}\}$, the set T_2 further decomposes into the disjoint union $T_2 = T_2^1 \cup \dots \cup T_2^{m_{i-1}}$, where $T_2^j = \{\pi \in T_2 : \pi(b_j) = x\}$, $1 \leq j \leq m_{i-1}$. For a fixed j, with every $\pi \in T_2^j$ we can associate the element π' of the semigroup $T_3 = T(M_1, \dots, M_{i-1}, M_{i+1}, \dots, M_k)$ in the following way: $\pi'(x) = \pi(x)$ for $x \neq b_j$ and $\pi'(b_j) = \pi(x)$. Then the map $\pi \mapsto \pi'$ from T_2^j to T_3 is bijective. Hence, for every j we get $|T_2^j| = |T_3| = t(m_1, \dots, m_{i-1}, m_{i+1}, \dots, m_k)$. This gives

$$\begin{aligned} t(m_1, \dots, m_{i-1}, 1, m_{i+1}, \dots, m_k) &= |T| = |T_1| + |T_2| = \\ &= t(m_1, \dots, m_{i-1} + 1, m_{i+1}, \dots, m_k) + m_{i-1} \cdot t(m_1, \dots, m_{i-1}, m_{i+1}, \dots, m_k). \end{aligned}$$

and completes the proof. □

For a polynomial, $f(x) = a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0$, with integer coefficients we set $f(B) = a_m B_m + a_{m-1} B_{m-1} + \dots + a_1 B_1 + a_0$, where B_k is the k-th Bell number. Set $[x]_k = x(x-1)(x-2)\dots(x-k+1)$. The following lemma is obvious.

Lemma 3. *Let $f(x), g(x) \in \mathbb{Z}[x]$ and $r \in \mathbb{Z}$. Then $(f+g)(B) = f(B) + g(B)$ and $(rf)(B) = rf(B)$.*

The following quite amazing result was first proved in [GP], however, we would like to reprove it here because of the very poor availability of [GP] to the general audience.

Theorem 5. *The cardinality $t(m_1, \dots, m_k)$ of the maximal nilpotent subsemigroup $T = T(M_1, \dots, M_k)$ for degree k and of type $(m_1, \dots, m_k)$ in $\mathcal{IS}_n$ equals $f_{m_1, \dots, m_k}(B)$, where the polynomial $f_{m_1, \dots, m_k}(x)$ is defined as follows: $f_{m_1, \dots, m_k}(x) = [x]_{m_1}[x]_{m_2} \dots [x]_{m_k}$.*

Proof. We prove the statement using induction in the sum m' of all $m_i \neq 1$. If $m' = 0$, then $m_i = 1$ for all i and the semigroup T has type $(1, 1, \dots, 1)$. According to Theorem 1, T is a maximal nilpotent subsemigroup and $|T| = B_k = f_{1,1,\dots,1}(B)$ for $f_{1,1,\dots,1}(x) = [x]_1[x]_1 \dots [x]_1 = x^k$. Thus the statement is true in the case $m' = 0$.

Since m' can not be equal to 1, let us now assume that $m' \geq 2$ and choose i such that $m_i \geq 2$. By replacing $m_{i-1}+1$ in Lemma 2 by $(m_i - 1) + 1$ and using the inductive assumption we get

$$t(m_1, \dots, m_i, m_{i+1}, \dots, m_k) = t(m_1, \dots, m_i - 1, 1, m_{i+1}, \dots, m_k) -$$
$$- (m_i - 1) \cdot t(m_1, \dots, m_i - 1, m_{i+1}, \dots, m_k) = f_{m_1, \dots, m_i - 1, 1, m_{i+1}, \dots, m_k}(B) -$$
$$- (m_i - 1) \cdot f_{m_1, \dots, m_i - 1, m_{i+1}, \dots, m_k}(B).$$

The equality $[x]_m = (x - (m-1))[x]_{m-1} = [x]_{m-1}[x]_1 - (m-1)[x]_{m-1}$ implies

$$f_{m_1, \dots, m_i, m_{i+1}, \dots, m_k}(x) = f_{m_1, \dots, m_i - 1, 1, m_{i+1}, \dots, m_k}(x) - (m_i - 1) \cdot f_{m_1, \dots, m_i - 1, m_{i+1}, \dots, m_k}(x).$$

Now the equality

$$t(m_1, \dots, m_i, m_{i+1}, \dots, m_k) = f_{m_1, \dots, m_i, m_{i+1}, \dots, m_k}(B).$$

follows from Lemma 3. □

Corollary 6. *Let $i_1, \dots, i_k$ be a permutation of $1, \dots, k$. Then*

$$t(m_{i_1}, m_{i_2}, \dots, m_{i_k}) = t(m_1, m_2, \dots, m_k).$$

Proof. This follows from Theorem 5 and the equality

$$[x]_{m_{i_1}}[x]_{m_{i_2}} \dots [x]_{m_{i_k}} = [x]_{m_1}[x]_{m_2} \dots [x]_{m_k}$$

□

5 $\mathcal{H}$–cross-sections, order-preserving nilpotents and Catalan numbers

Let ρ be an equivalence relation on the semigroup S. A subsemigroup, T, of S is said to be a *cross-section* with respect to ρ provided that T contains exactly 1 element from every equivalence class. Certainly, the most important are cross-sections with respect to the equivalence relations, which are somehow related to the semigroup structure on S. And here the primary candidates are the Green relations. Among the Green relations the smallest one is the relation $\mathcal{H}$. We recall that the elements $a, b \in S$ are contained in the same $\mathcal{H}$-class if and only if they generate the same right and the same left principal ideals, that is, if $\{a\} \cup aS = \{b\} \cup bS$ and $Sa \cup \{a\} = Sb \cup \{b\}$. In the semigroup $\mathcal{IS}_n$ the study of $\mathcal{H}$–cross-sections naturally leads to the study of the subsemigroup $\mathcal{IO}_n$ of those elements, which preserve the natural order on N, that is,

$$\mathcal{IO}_n = \{\pi \in \mathcal{IS}_n \,:\, x, y \in \mathrm{dom}(\pi) \text{ and } x < y \text{ imply } \pi(x) < \pi(y)\}.$$

We remark that the zero element 0 belongs to $\mathcal{IO}_n$.

Theorem 6. *[Re, Section 2],[CR, Theorem 3.2] $\mathcal{IO}_n$ is an $\mathcal{H}$–cross-section of $\mathcal{IS}_n$. Moreover, for $n \neq 3$ a subsemigroup, T, of $\mathcal{IS}_n$ is an $\mathcal{H}$–cross-section of $\mathcal{IS}_n$ if and only if there exists $s \in S_n$ such that $T = s^{-1} \cdot \mathcal{IO}_n \cdot s$.*

From the definition of $\mathcal{IO}_n$ it follows immediately that every element in $\mathcal{IO}_n$ is uniquely determined by its domain and its range, and that for arbitrary subsets $A, B \subseteq N$ such that $|A| = |B|$ there exists an element, $\pi \in \mathcal{IO}_n$, such that $\mathrm{dom}(\pi) = A$ and $\mathrm{im}(\pi) = B$. Hence

$$|\mathcal{IO}_n| = \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

According to Corollary 2, the nilpotency degree of arbitrary nilpotent subsemigroups in $\mathcal{IO}_n$ does not exceed n.

Theorem 7. *[GM2, Theorem 7] $\mathcal{IO}_n$ contains exactly $n!$ maximal nilpotent subsemigroups whose zero element is 0. Each of these subsemigroups has nilpotency degree n, corresponds to a permutation, $i_1, \dots, i_n$, of $1, 2, \dots, n$, and has the following form:*

$$\tilde{T}_{i_1,\dots,i_n} = \{\pi \in \mathcal{IO}_n \,:\, \pi(i_k) = i_m \textit{ implies } m > k \textit{ for all } i_k \in \mathrm{dom}(\pi)\}.$$

In particular, one can easily see that $\tilde{T}_{i_1,\dots,i_n} = T_{i_1,\dots,i_n} \cap \mathcal{IO}_n$. Hence, using Theorems 1 and 7, one gets the following:

Corollary 7. *The map $T \mapsto T \cap \mathcal{IO}_n$ from the set of all maximal nilpotent subsemigroups of $\mathcal{IS}_n$ containing 0 to the set of all maximal nilpotent subsemigroup of $\mathcal{IO}_n$ containing 0 is a bijection.*

Lemma 4. *For every nilpotent subsemigroup $T \subseteq \mathcal{IO}_n$ there exists a maximal nilpotent subsemigroup $T' \subseteq \mathcal{IO}_n$ containing 0, such that $|T| \leq |T'|$.*

Proof. Let e be the idempotent of T. The set N forms a linearly ordered poset under the natural order on N and the elements of $\overline{\mathrm{dom}(e)}$ may appear anywhere in this poset. Thus, under this order, $\mathrm{dom}(e)$ is partitioned into subsets $I_1, \dots, I_p$ of N which in effect are intervals of N and are separated by at least one element of $\mathrm{dom}(e)$. For $1 \leq q \leq p$ we denote $F_q = \{\pi \in \mathcal{IO}_n \,:\, \mathrm{dom}(\pi) \cup \mathrm{im}(\pi) \subseteq I_q\}$. Clearly, F_q is isomorphic to $\mathcal{IO}_{|I_q|}$ in a natural way.

Since every $\pi \in T$ is order-preserving, and the intervals $I_1, \dots, I_p$ are separated by elements of $\mathrm{dom}(e)$ which are fixed by π, we know that each of $I_1, \dots, I_p$ is invariant under π. Hence the natural projection $\varphi_q : \pi \mapsto \pi|_{I_q}$ is a homomorphism from T to F_q. Moreover, since all $\pi \in T$ act trivially on $\mathrm{dom}(e)$, the map

$$T \to F_1 \times \cdots \times F_p, \qquad \pi \mapsto (\varphi_1(\pi), \dots, \varphi_p(\pi))$$

is a monomorphism. Further, every projection $\varphi_q(T)$ is a nilpotent subsemigroup in F_q and hence $\varphi_q(T)$ is contained in some maximal nilpotent subsemigroup $T_q \subseteq F_q$. According to Theorem 7, T_q corresponds to a permutation, τ_q of the elements from I_q. Consider the permutation $i_1, \dots, i_n$ of the elements from N, in which all elements from $\mathrm{dom}(e)$ remain on their places and such that on each interval I_q this permutation coincides with τ_q. The embedding $T \hookrightarrow F_1 \times \cdots \times F_p$ induces an embedding $T \hookrightarrow T_1 \times \cdots \times T_p$ and this direct product can be embedded in $T' = \tilde{T}_{i_1,\dots,i_n}$ resulting $|T| \leq |T'|$ and completing the proof. $\square$

Lemma 5. *[GM2, Proposition 11] The semigroup $\tilde{T}_{1,\dots,n}$ has the maximal cardinality among all maximal nilpotent subsemigroups in $\mathcal{IO}_n$.*

Theorem 8. *[GM2, Theorem 9] The maximal cardinality for nilpotent subsemigroups of $\mathcal{IO}_n$ equals the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.*

In [GM2] this statement is proved using the recursion for the cardinalities of certain subsets in $\tilde{T}_{1,\dots,n}$. In the present paper we give a more direct combinatorial proof, constructing a bijection between the elements from $\tilde{T}_{1,\dots,n}$ and the set Bin_n of all rooted binary trees with n vertices. It is well known, see for example [Do] or [St], that $|Bin_n| = C_n$.

Proof. According to Lemmas 4 and 5, this maximal cardinality is $|\tilde{T}_{1,\dots,n}|$. Let us now construct a bijection between $\tilde{T}_{1,\dots,n}$ and the set Bin_n. For this we first choose a special enumeration of the vertices for every binary tree $\Gamma \in Bin_n$, using the following algorithm. To number the vertices of Γ we make a list of binary trees, where initially the list contains only Γ. Now let us assume the first k vertices of Γ are already numbered. We take the first tree Γ_1 from the present list $(\Gamma_1, \dots, \Gamma_m)$ and assign the number $k+1$ to its root. After this we add the left subtree $\Gamma_{1,l}$ of Γ_1 (if it is non-empty) to the end of the list and substitute Γ_1 with the right subtree $\Gamma_{1,r}$ of Γ_1 (if it is non-empty). If the tree $\Gamma_{1,r}$ is empty, we delete Γ_1. In this way we get a new list $(\Gamma_{1,r}, \Gamma_2, \dots, \Gamma_m, \Gamma_{1,l})$, where the trees $\Gamma_{1,r}$ $\Gamma_{1,l}$ are omitted in the case when they are empty. After this we go to assigning the number $k+2$. For example, at the satrt, 0 vertices are numbered and Γ is the only tree in the list, so 1 is assigned to the root of Γ, and the list (Γ) is substituted by $(\Gamma_{1,r}, \Gamma_{1,l})$, where the trees $\Gamma_{1,r}$ $\Gamma_{1,l}$ are omitted in the case when they are empty.

From now on we will use *vertex x* instead of *vertex with number x*. Moreover, we will consider a binary tree as an oriented graph, in which from every vertex we have a "left" and a "right" arrow to the root of the respective left or right subtree correspondingly, provided that this subtree is not empty.

To every $\Gamma \in Bin_n$ we can associate a partial map, $\pi_\Gamma : N \to N$, where $\pi_\Gamma(x) = y$ if and only if Γ contains a "left" arrow from x to y.

For every $\pi \in \tilde{T}_{1,\dots,n}$ we construct the oriented graph Γ_π in the following way: the set of vertices is N; there is a "left" arrow from x to y provided that $\pi(x) = y$; and there is a "right" arrow from x to $x+1$ provided that $x+1 \notin \mathrm{im}(\pi)$.

The statement of the theorem now follows from the following two lemmas and the fact that the sets $\tilde{T}_{1,\dots,n}$ and Bin_n are finite.

Lemma 6. *The map $\varphi : \Gamma \mapsto \pi_\Gamma$ from Bin_n to $\tilde{T}_{1,\dots,n}$ is injective.*

Proof. Obviously, $\pi_\Gamma \in \mathcal{IS}_n$. From the algorithm of the vertex numbering it follows immediately that $\pi_\Gamma(x) > x$ and thus $\pi_\Gamma \in T_{1,2,\dots,n}$. If $x < y$, then, during the numeration of the vertices in Γ, the left subtree of the subtree with the root x appears in the list of trees earlier than the left subtree of the subtree with the root y. Hence $\pi_\Gamma(x) < \pi_\Gamma(y)$ and hence $\pi_\Gamma \in \mathcal{IO}_n$. Therefore $\pi_\Gamma \in \mathcal{IO}_n \cap T_{1,2,\dots,n} = \tilde{T}_{1,\dots,n}$.

For the vertex x of the tree Γ we set $l(x) = 1$ provided that the subtree with the root x has a non-empty left subtree and $l(x) = 0$ otherwise. We define the "right" function $r(x)$ analogously.

Assume now that the trees Γ and Γ' are different. Choose the minimal k for which $(l(k), r(k)) \neq (l'(k), r'(k))$. Assume that $l(k) \neq l'(k)$. Without loss of generality we can assume that $l(k) = 1$ and $l'(k) = 0$. Then $k \in \mathrm{dom}(\pi_\Gamma)$ and $k \notin \mathrm{dom}(\pi_{\Gamma'})$, which implies $\pi_\Gamma \neq \pi_{\Gamma'}$.

Now assume that $r(k) \neq r'(k)$. Without loss of generality we can assume that $r(k) = 1$ and $r'(k) = 0$. Then the tree Γ contains a "right" arrow from k to $k+1$ and $k+1 \notin \mathrm{im}(\pi_\Gamma)$, since it is not possible that a vertex is terminal for a "right" and a "left" arrow at the same time.

In the list of subtrees, which is formed during our process of numbering the vertices in a binary tree, all subtrees, except, possibly, the very first one, are left subtrees of the corresponding subtrees. Hence in the case $r'(k) = 0$ the number $k+1$ is assigned in Γ' to the root of some left subtree and $k+1 \in \mathrm{im}(\pi_{\Gamma'})$. Hence $\pi_\Gamma \neq \pi_{\Gamma'}$ is this case either.

This proves that the map φ is injection and completes the proof of the lemma. $\square$

Lemma 7. *The map $\psi : \pi \mapsto \Gamma_\pi$ from $\tilde{T}_{1,\dots,n}$ to Bin_n is injective.*

Proof. From the definition of Γ_π it follows immediately that it is not possible that at the same time a vertex is terminal for both a "right" and a "left" arrow. Moreover, if $x \neq 1$, then there is at least one arrow from a vertex with a smaller number, terminating in x. Hence, there is an oriented path from 1 to arbitrary vertex and thus $\Gamma_\pi \in Bin_n$. From the definition of φ and ψ it now follows that $\pi_{\Gamma_\pi} = \pi$ and hence $\psi \cdot \varphi$ (composition from the left to the right) is the identity on $\tilde{T}_{1,\dots,n}$. Therefore ψ is injective. $\square$

$\square$

Remark 2. Catalan numbers as cardinalities of some transformation semigroups appeared in [Hi]. In [Hi, Theorem 3.1] it is shown that C_n is equal to the cardinality of the semigroup $\mathcal{C}_n$ of all order-preserving, decreasing, and everywhere defined transformations on N (that is maps $f : N \to N$ satisfying $f(i) \leq f(j)$ for all $i, j \in N$, $i \leq j$, and $f(i) \leq i$ for all $i \in N$). Although the semigroups $\tilde{T}_{1,\dots,n}$ and $\mathcal{C}_n$ are not isomorphic (the first one is nilpotent and the second one contains 2^{n-1} idempotents), one can construct a bijection between these two sets in the following way (this also gives an alternative proof of Theorem 8). The permutation $\sigma = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ n & n-1 & \dots & 2 & 1 \end{pmatrix}$ defines an isomorphism $a \mapsto \sigma^{-1} a \sigma$ of $\mathcal{C}_n$ onto the semigroup $\hat{\mathcal{C}}_n$ of all order-preserving increasing transformations of N. With $a = \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix} \in \hat{\mathcal{C}}_n$ we associate a partially defined map $\psi(a) : N \to N$, such that $x \notin \mathrm{dom}(\psi(a))$ if and only if $a(x) = x$ or $a(x) = a(x-1)$, and $\psi(a)(x) = a(x)$

otherwise. By a (quite long) direct computation one can check that $\psi : \hat{\mathcal{C}}_n \to \tilde{T}_{1,\dots,n}$ is a bijection.

Proposition 7. *The maximal semigroups among subsemigroups from $\mathcal{IO}_n$ containing* 0 *and having zero multiplication (i.e. of nilpotency degree* 2*) have cardinality* $\binom{n}{k}$*, where* $0 < k < n$*. The number of maximal subsemigroups with the zero multiplication of cardinality* $\binom{n}{k}$ *equals* $2\binom{n}{k}$ *if* $n \neq 2k$ *and* $\binom{n}{k}$ *if* $n = 2k$*.*

Proof. From [GM2, Theorem 7] it follows that every maximal subsemigroup with zero multiplication in $\mathcal{IO}_n$ has the form

$$T_M = \left\{\pi \in \mathcal{IO}_n \,:\, \mathrm{dom}(\pi) \subseteq M, \mathrm{im}(\pi) \subseteq \overline{M}\right\},$$

where $M \subseteq N$ and $M \neq \varnothing, N$. Every element $\pi \in \mathcal{IO}_n$ is uniquely determined by $\mathrm{dom}(\pi)$ and $\mathrm{im}(\pi)$, and hence there is a natural bijection between the elements from T_M and pairs (A, B) of subsets such that $A \subseteq M$ and $B \subseteq \overline{M}$, for which $|A| = |B|$. Let now $|M| = k$. Then for every k-element subset $K \subseteq N$ we have $|M \setminus K| = |K \cap \overline{M}|$ and for every pair $A \subseteq M$, $B \subseteq \overline{M}$ such that $|A| = |B|$, the set $(M \setminus A) \cup B$ contains exactly k elements. Therefore we get $|T_M| = \binom{n}{k}$ in the case $|M| = k$.

To prove the rest it is enough to observe that $|T_M| = |T_{\overline{M}}|$. □

Acknowledgements

An essential part of the paper was written during the visit of the first author to Uppsala University, which was supported by The Royal Swedish Academy of Sciences. The paper was completed during the visit of the first author to Uppsala University, which was supported by The Swedish Institute. The financial support of The Royal Swedish Academy of Sciences of The Swedish Institute and the hospitality of Uppsala University are gratefully acknowledged. For the second author the research was partially supported by The Swedish Research Council.

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