$\mathcal{L}$– and $\mathcal{R}$–cross-sections in $\mathcal{IS}_n$

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Abstract

We classify cross-sections of the $\mathcal{L}$ and $\mathcal{R}$ Green's relations on the finite symmetric inverse semigroup $\mathcal{IS}_n$, determine which of them are isomorphic, and study their disposition with respect to the $\mathcal{H}$–cross-sections of $\mathcal{IS}_n$.

1 Introduction

Let ρ be an equivalence relation on a semigroup S. A subsemigroup $T \subset S$ is called a *cross-section* with respect to ρ provided that T contains exactly 1 element from every equivalence class. From an algebraic point of view, the most interesting cross-sections are those that arise from equivalence relations associated with the structure of the semigroup S in question. Foremost among these relations are Green's relations on S.

The cross-sections with respect to the $\mathcal{H}$– ($\mathcal{L}$–, $\mathcal{R}$–, $\mathcal{D}$–, $\mathcal{J}$–) Green's relations will be called *$\mathcal{H}$– ($\mathcal{L}$–, $\mathcal{R}$–, $\mathcal{D}$–, $\mathcal{J}$–) cross-sections* in the sequel. For the full inverse symmetric semigroup $\mathcal{IS}_n$ of all (possibly, partial) injections of the set $N = \{1, 2, \dots, n\}$ into itself the first example of an $\mathcal{H}$–cross-section was constructed in [R] and later in [CR] a complete description of all $\mathcal{H}$–cross-sections for $\mathcal{IS}_n$, was obtained (the characterization provided in [CR] is for $n \neq 3$, but clearly a complete description of all $\mathcal{H}$–cross-section of $\mathcal{IS}_3$ is an easy calculation exercise).

In the present paper we describe all $\mathcal{L}$– and $\mathcal{R}$–cross-sections of $\mathcal{IS}_n$. In contrast with $\mathcal{H}$–cross-sections, different $\mathcal{L}$– ($\mathcal{R}$–) cross-sections are not isomorphic in general. In the paper we give an isomorphism criterion for two different $\mathcal{L}$– ($\mathcal{R}$–) cross-sections and count both the number of different $\mathcal{L}$– ($\mathcal{R}$–) cross-sections of $\mathcal{IS}_n$ and the number of those cross-sections, which are pairwise non-isomorphic. In particular, the last number is equal to the number p_n of different partitions of the positive integer n into a sum of positive integers, if the order of summands is not important. We also count the number of $\mathcal{L}$– and $\mathcal{R}$–cross-sections, contained in a given $\mathcal{H}$–cross-section, and the number of $\mathcal{H}$–cross-sections, containing a given $\mathcal{L}$– ($\mathcal{R}$–) cross-section.

We would like to finish this introduction with the following two open problems, the solution of which would naturally complete the study, originated in [CR], and continued in the present paper.

Problem 1. *Describe all $\mathcal{D}$– (=$\mathcal{J}$–) cross-sections of the semigroup $\mathcal{IS}_n$ and give for them an isomorphism criterion.*

Problem 2. *Describe all $\mathcal{D}$– (=$\mathcal{J}$–) cross-sections of the semigroup $\mathcal{IS}_n$, which consist of idempotents, and give for them an isomorphism criterion.*

The paper is organized as follows. We collect all necessary preliminaries in Section 2. Section 3 is dedicated to the construction and classification of all $\mathcal{R}$– and $\mathcal{L}$–cross-sections in $\mathcal{IS}_n$. In Section 4 we describe the disposition of the $\mathcal{R}$– and $\mathcal{L}$–cross-sections with respect to the $\mathcal{H}$–cross-sections. Finally, in Section 5 we determine, which $\mathcal{R}$– ($\mathcal{L}$–) cross-sections are isomorphic.

After the paper was submitted that authors were informed that an analogous description of $\mathcal{R}$– and $\mathcal{L}$– cross-sections of $\mathcal{IS}_n$ has been recently obtained by by H.Yang and X.Yang, [YY], using quite different methods.

2 Preliminaries

For $a \in \mathcal{IS}_n$ we denote by $\mathrm{dom}(a)$ and $\mathrm{im}(a)$ the domain and the image of the element a respectively. The number $\mathrm{rank}(a) = |\,\mathrm{dom}(a)| = |\,\mathrm{im}(a)|$ is called the *rank* of a, and the number $\mathrm{def}(a) = n - \mathrm{rank}(a)$ is called the *defect* of a. The identity map $\mathrm{id}_N : N \to N$ is the unit element of $\mathcal{IS}_n$ and will be denoted by e.

For an element, $a \in \mathcal{IS}_n$, one can use the usual tableaux presentation

$$a = \begin{pmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{pmatrix},$$

if $\mathrm{dom}(a) = \{i_1, i_2, \dots, i_k\}$ and $a(i_r) = j_r$, $r = 1, 2, \dots, k$. However, it is often more convenient to use the so-called *chain decomposition* of a, which is an analogue of the cyclic decomposition for the usual permutations. We refer to [GM, L] for details, and explain this decomposition on the following example. The element

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 7 & 9 \\ 7 & 4 & 5 & 1 & 10 & 2 & 6 \end{pmatrix} \in \mathcal{IS}_{10}$$

has the following graph of the action on $\{1, 2, \dots, 10\}$:

$$\begin{array}{ccc} 1 & \to & 7 \\ \uparrow & & \downarrow \\ 4 & \leftarrow & 2 \end{array} \qquad 3 \to 5 \to 10 \qquad 9 \to 6 \qquad 8,$$

and hence it is convenient to write it as $a = (1, 7, 2, 4)[3, 5, 10][9, 6][8]$. We call $(1, 7, 2, 4)$ a *cycle* and $[3, 5, 10]$ (as well as $[9, 6]$ and $[8]$) a *chain* of the element a. By the *length* of a chain we will mean the number of elements in this chain. For example, the chain $[3, 5, 10]$ has length 3. We remark that chains of length 1 correspond to those elements $x \in N$, which do not belong to $\mathrm{dom}(a) \cup \mathrm{im}(a)$. It is obvious that $\mathrm{def}(a)$ equals the number of chains in the chain decomposition of a. If $1 \le p, q \le k$, then the chain $[i_1, i_2, \dots, i_p]$ is called a *beginning* and the chain $[i_q, i_{q+1}, \dots, i_k]$ is called an *ending* of the chain $[i_1, i_2, \dots, i_k]$. For example, $[3, 5]$ is a beginning and $[5, 10, 7]$ is an ending of the chain $[3, 5, 10, 7]$.

We will multiply the elements in $\mathcal{IS}_n$ from the left to the right, that is, $(ab)(x) = b(a(x))$ for all $x \in \mathrm{dom}(ab)$. If $M \subset \mathcal{IS}_n$ is an arbitrary non-empty set, we denote by $\langle M \rangle$ the semigroup, generated by M.

It is well-known (see for example [GM]) that Green's relations on $\mathcal{IS}_n$ can be described as follows:

- $a\mathcal{R}b$ if and only if $\mathrm{dom}(a) = \mathrm{dom}(b)$;
- $a\mathcal{L}b$ if and only if $\mathrm{im}(a) = \mathrm{im}(b)$;
- $a\mathcal{H}b$ if and only if $\mathrm{dom}(a) = \mathrm{dom}(b)$ and $\mathrm{im}(a) = \mathrm{im}(b)$;
- $a\mathcal{D}b$ if and only if $a\mathcal{J}b$ if and only if $\mathrm{rank}(a) = \mathrm{rank}(b)$.

For $i = 0, 1, \dots, n$ we denote by D_i the set of all elements $x \in \mathcal{IS}_n$ satisfying $\mathrm{rank}(x) = i$. The decomposition $\mathcal{IS}_n = D_0 \cup \dots \cup D_n$ is the decomposition of $\mathcal{IS}_n$ into $\mathcal{D}$–classes.

From [R] it follows that the semigroup

$$\mathcal{IO}_n = \{a \in \mathcal{IS}_n \,:\, x < y \text{ implies } a(x) < a(y) \text{ for all } x, y \in \mathrm{dom}(a)\}$$

is an $\mathcal{H}$–cross-section in $\mathcal{IS}_n$. It is shown in [CR] that for $n \neq 3$ a subsemigroup, $H \subset \mathcal{IS}_n$, is an $\mathcal{H}$–cross-section of $\mathcal{IS}_n$ if and only if H is S_n-conjugated with $\mathcal{IO}_n$. This can be stated as follows.

Theorem 1. *For every linear order $\prec$ on the set N the semigroup*

$$H(\prec) = \{a \in \mathcal{IS}_n \,:\, x \prec y \textit{ implies } a(x) \prec a(y) \textit{ for all } x, y \in \mathrm{dom}(a)\}$$

is an $\mathcal{H}$–cross-section in $\mathcal{IS}_n$. Moreover, if $n \neq 3$ then every $\mathcal{H}$–cross-section in $\mathcal{IS}_n$ has the form $H(\prec)$ for some linear order $\prec$ on N.

3 Description of $\mathcal{L}$– and $\mathcal{R}$–cross-sections in $\mathcal{IS}_n$

Since for $a, b \in \mathcal{IS}_n$ the condition $a\mathcal{R}b$ is equivalent to the condition $\mathrm{dom}(a) = \mathrm{dom}(b)$, then the equalities $a = b$ and $\mathrm{dom}(a) = \mathrm{dom}(b)$ are equivalent for elements a, b from an arbitrary $\mathcal{R}$–cross-sections T of $\mathcal{IS}_n$. We will frequently use this fact in the paper.

Lemma 1. *Let T be an $\mathcal{R}$–cross-section of $\mathcal{IS}_n$. Then $c \notin \langle T \setminus \{c\} \rangle$ for every $c \in T \cap D_{n-1}$.*

Proof. We show that if $c \in T \cap D_{n-1}$, $a, b \in T$, and $c = ab$ then $c = a$ or $c = b$, from which the lemma follows. This is clear if $a = e$ or $b = e$, and hence we can assume that $a \neq e$ and $b \neq e$. Since $\mathrm{def}(c) = 1$, $\mathrm{def}(a) > 0$ and $\mathrm{def}(b) > 0$, the equality $c = ab$ implies $\mathrm{def}(a) = \mathrm{def}(b) = 1$. But this means that $\mathrm{dom}(a) = \mathrm{dom}(ab)$ and hence $a = ab$ implying $c = a$. Hence the equality $c = ab$ always means that at least one of the elements a and b is equal to c. □

The statement of Lemma 1 means that every system of generators of the $\mathcal{R}$–cross-section T must contain $T \cap D_{n-1}$. It is now natural to ask what this intersection can look like.

Lemma 2. *Let T be an $\mathcal{R}$–cross-sections of $\mathcal{IS}_n$ and $a \in T \cap D_{n-1}$. Then the chain decomposition of a contains exactly one chain, say of length k. Moreover, this decomposition also contains $n-k$ cycles of length 1.*

Proof. The first statement follows from the equality $\mathrm{def}(a) = 1$, and the second one from the equality $\mathrm{dom}(a^k) = \mathrm{dom}(a^{k+1})$. ☐

For $k = 1, \dots, n$ let $a_k = [1, 2, \dots, k](k+1)\dots(n)$.

Lemma 3. *$a_l \cdot a_k = a_{k-1} \cdot a_l$ for every l and k with $k \le l$.*

Proof. A direct calculation shows that for $k \equiv 0 (\mathrm{mod}\ 2)$ one has

$$a_l \cdot a_k = a_{k-1} \cdot a_l = [1, 3, 5, \dots, k-1][2, 4, 6, \dots, k, k+1, k+2, \dots, l](l+1)\dots(n),$$

and that for $k \equiv 1 (\mathrm{mod}\ 2)$ one has

$$a_l \cdot a_k = a_{k-1} \cdot a_l = [1, 3, 5, \dots, k, k+1, k+2, \dots, l][2, 4, 6, \dots, k-1](l+1)\dots(n).$$

☐

Corollary 1. *Every element a of the semigroup $\langle a_1, a_2, \dots, a_n\rangle$ can be written in the form $a = a_1^{\alpha_1} a_2^{\alpha_2} \dots a_n^{\alpha_n}$, where $\alpha_1, \alpha_2, \dots, \alpha_n \ge 0$.*

Corollary 2. *One has $a_k^{\alpha} = a_{k-1} \cdot a_k^{\alpha-1}$ for all $\alpha > 1$ and $k > 1$.*

Proof. The elements a_{k-1} and a_k act in the same way on elements from the set $N \setminus \{k-1, k\}$. If $\alpha > 1$ one has $k-1, k \notin \mathrm{dom}(a_k^{\alpha})$ and $k-1, k \notin \mathrm{dom}(a_{k-1} \cdot a_k^{\alpha-1})$. ☐

Corollary 3. *Every element a of the semigroup $\langle a_1, a_2, \dots, a_n\rangle$ can be written in the form $a = a_1^{\alpha_1} a_2^{\alpha_2} \dots a_n^{\alpha_n}$, where $\alpha_i \in \{0, 1\}$ for all $i = 1, 2, \dots, n$.*

Proof. Follows from Corollary 1, Corollary 2 and the equality $a_1^2 = a_1$. ☐

Lemma 4. *Let $1 \le i_1 < i_2 < \dots < i_k \le n$. Then*

$$a_{i_1} a_{i_2} \dots a_{i_k} = \begin{pmatrix} 1 & 2 & \dots & i_1 - 1 & i_1 + 1 & \dots & i_2 - 1 & i_2 + 1 & \dots & i_k - 1 & i_k + 1 & \dots & n \\ 1+k & 2+k & \dots & i_1 - 1 + k & i_1 + k & \dots & i_2 - 2 + k & i_2 - 1 + k & \dots & i_k & i_k + 1 & \dots & n \end{pmatrix}$$

(here in the upper line the elements $i_1, i_2, \dots, i_k$ are omitted and in the lower line the elements $1+k, 2+k, \dots, n$ are just written in a natural order).

Proof. Direct calculation. □

Corollary 4. *For* $1 \le i_1 < i_2 < \cdots < i_k \le n$ *one has that* $\mathrm{dom}(a_{i_1}a_{i_2}\dots a_{i_k}) = N \setminus \{i_1, i_2, \dots, i_k\}$ *and* $\mathrm{im}(a_{i_1}a_{i_2}\dots a_{i_k}) = N \setminus \{1, 2, \dots, k\}$.

Corollary 5. *The semigroup* $S_1 = \langle a_i, 1 \le i \le n \rangle$ *has the following presentation:* $\langle a_i, 1 \le i \le n : a_1^2 = a_1; a_k^2 = a_{k-1} \cdot a_k, k = 2, \dots, n; a_l \cdot a_k = a_{k-1} \cdot a_l, 1 \le k < l \le n \rangle$.

Proof. Set $S_2 = \langle x_i, 1 \le i \le n : x_1^2 = x_1; x_k^2 = x_{k-1} \cdot x_k, k = 2, \dots, n; x_l \cdot x_k = x_{k-1} \cdot x_l, 1 \le k < l \le n \rangle$. Then the natural map $\varphi : S_2 \to S_1$, $\varphi(x_i) = a_i$ is a homomorphism according to Lemma 3 and Corollary 2. But it follows from Corollary 4 that $|S_1| = 2^n - 1$ and by the same arguments as in the previous lemmas one computes $|S_2| = 2^n - 1$. This implies $S_1 \simeq S_2$. □

Corollary 6. *The semigroup* $K_n = \langle a_1, a_2, \dots, a_n \rangle \cup \{e\}$ *is an* $\mathcal{R}$*–cross-section of* $\mathcal{IS}_n$.

Proof. It follows from Corollaries 3 and 4 that for every subset $A \subset N$ the semigroup K_n contains exactly one element a such that $\mathrm{dom}(a) = A$. Hence K_n contains exactly one element from every $\mathcal{R}$–class of $\mathcal{IS}_n$. □

Lemma 5. *In the semigroup* K_n *the inequality* $x^{n-1} \ne x^n$ *has the unique solution* a_n.

Proof. It follows from Lemma 2 that the inequality $x^{n-1} \ne x^n$ is equivalent to the existence of a chain of length $\ge n$ in the chain decomposition of the element x. An element of K_n satisfies the last condition if and only if it has defect 1 and does not have any fixed point. From Corollary 3 and Lemma 4 it follows that a_n is the only element of K_n satisfying these conditions. □

Lemma 6. *Let* T *be an* $\mathcal{R}$*–cross-section of* $\mathcal{IS}_n$*. Assume that* T *contains the element* $a = a_k = [1, 2, \dots, k](k+1)\dots(n)$*. Let* $b \in T \cap D_{n-1}$ *be such that* $b(l) = l$ *for some* l, $1 \le l \le k$*. Then* $b(x) = x$ *for all* x, $l \le x \le k$.

Proof. Without loss of generality we can assume that l is the minimal number such that $b(l) = l$. Consider the set $\{x : l \le x \le k \text{ and } b(x) \ne x\}$ and assume that it is not empty. Let p be the maximal element in this set. We consider the following two cases:

Case 1. $p \notin \mathrm{dom}(b)$. Then $\mathrm{dom}(b) = N \setminus \{p\}$, and hence $\mathrm{dom}(ba^{k-p}) = \mathrm{dom}(a^{k-p+1}) = N \setminus \{p, p+1, \dots, k\}$. Thus $ba^{k-p} = a^{k-p+1}$. But $ba^{k-p}(l) = l+k-p \ne l+k-p+1 = a^{k-p+1}(l)$. Therefore this case is not possible.

Case 2. $p \in \mathrm{dom}(b)$. Set $b(p) = q \ne p$ and assume that the length of the chain in the chain decomposition of b equals m. Since $b(p) \ne p$, we get that p belongs to the chain of b, according to Lemma 2. This implies that for the element b^m, which is in fact an idempotent, we have $p, q \notin \mathrm{dom}(b^m)$. Set $A = \mathrm{dom}(a^k)$, $B = \mathrm{dom}(b^m)$, $C = A \cap B$ and $A_1 = A \setminus C$. All the elements of the set A_1 belong to the chain of the element b and hence $\mathrm{im}(b^m) \cap A_1 = \varnothing$. Moreover, all elements from $\{1, 2, \dots, l-1\}$ belong to the chain of the element b as well. Thus $\mathrm{im}(b^m) \subset \{l, l+1, \dots, k\} \cup C$. This implies that $\mathrm{im}(b^m \cdot a^{p-l}) \subset \{p, p+1, \dots, k\} \cup C \subset \mathrm{dom}(b)$. From this inclusion and the obvious statement that $\mathrm{im}(x) \subset \mathrm{dom}(y)$ implies $\mathrm{dom}(x) = \mathrm{dom}(xy)$ one immediately gets that $\mathrm{dom}(b^m \cdot a^{p-l}) = \mathrm{dom}(b^m \cdot a^{p-l} \cdot b)$ and hence $b^m \cdot a^{p-l} = b^m \cdot a^{p-l} \cdot b$. But

$$(b^m \cdot a^{p-l})(l) = a^{p-l}(b^m(l)) = a^{p-l}(l) = p \neq q = b(p) = (b^m \cdot a^{p-l} \cdot b)(l).$$

Hence this case is not possible either.

It follows that $\{x : l \leq x \leq k \text{ and } b(x) \neq x\} = \varnothing$ and the statement is proved. □

Lemma 7. *Let T be an $\mathcal{R}$–cross-section of $\mathcal{IS}_n$ and $a, b \in T \cap D_{n-1}$. Assume that the chains of a and b have at least one common element. Then one of these chains is a beginning of the other one.*

Proof. Without loss of generality we can assume that $a = [1, 2, \dots, k](k+1)\dots(n)$. Let $A = \{1, 2, \dots, k\}$, B be the set of all numbers from the chain of the element b and $C = A \cap B$. From Lemma 6 it follows that $C = \{1, 2, \dots, m\}$, where $m \leq k$. Moreover, if some $x \in N \setminus A$ takes the position in the chain of b before some element, say $y \in C$, then from $a(x) = x$ and Lemma 6 one gets that $a(y) = y$, which contradicts the choice of y. Hence $b = [i_1, i_2, \dots, i_m, j_1, \dots, j_l](f)\dots(g)$, where $i_1, i_2, \dots, i_m$ is a permutation of $1, 2, \dots, m$.

Let us assume that $(i_1, i_2, \dots, i_m) \neq (1, 2, \dots, m)$. Then there exist elements $u, v, p, q \in \{1, 2, \dots, m\}$ such that $i_p = u$, $i_q = v$, $u < v$, $p > q$. This implies $a^{v-u}(u) = v$, $b^{p-q}(v) = u$ and $(a^{v-u} \cdot b^{p-q})(u) = u$. For the element $c = a^{v-u} \cdot b^{p-q}$ there exists t such that c^t is an idempotent. Clearly, $u \in \mathrm{dom}(c^t)$ and $c^t(u) = u$. Moreover, $\mathrm{dom}(c^t) \subset \mathrm{dom}(c) \subset \mathrm{dom}(a)$. Now $c^{2t} = c^t$ implies that $\mathrm{dom}(c^t) = \mathrm{im}(c^t)$. Hence $\mathrm{dom}(c^t) = \mathrm{dom}(c^t \cdot a)$ and thus $c^t = c^t \cdot a$. But the last equality is impossible for $c^t(u) = u \neq u + 1 = (c^t \cdot a)(u)$.

Therefore $(i_1, i_2, \dots, i_m) = (1, 2, \dots, m)$ and $b = [1, 2, \dots, m, j_1, \dots, j_l](f)\dots(g)$. To complete the proof it is now enough to show that either $k = m$ or $l = 0$. Assume that this is not the case, that is, $k > m$ and $l > 0$. We remark that $\{m+1, \dots, k\} \cap \{j_1, \dots, j_l\} = \varnothing$, that a acts as the identity on all elements from $B \setminus C = \{j_1, \dots, j_l\}$, and that b acts as the identity on all elements from $A \setminus C = \{m+1, \dots, k\}$. This implies that $k, j_l \notin \mathrm{dom}(ab)$ and $k, j_l \notin \mathrm{dom}(ba)$. Since $\mathrm{def}(ab) \leq 2$ and $\mathrm{def}(ba) \leq 2$, we have that $\mathrm{dom}(ab) = \mathrm{dom}(ba) = N \setminus \{k, j_l\}$ and $ab = ba$. But $(ab)(m) = m+1 \neq j_1 = (ba)(m)$. This contradiction completes the proof of the lemma. □

Let now $N = M_1 \cup M_2 \cup \dots \cup M_k$ be an arbitrary decomposition of N into a disjoint union of non-empty blocks, where the order of blocks is not important. Assume that a linear order, $m_1^i, m_2^i, \dots, m_{|M_i|}^i$, is fixed on the elements of the block M_i for all $i = 1, 2, \dots, k$. The decomposition $N = M_1 \cup M_2 \cup \dots \cup M_k$ together with a fixed linear order on every block will be denoted by $\{\vec{M_1}, \vec{M_2}, \dots, \vec{M_k}\}$. The notation $\{\vec{M_1}, \vec{M_2}, \dots, \vec{M_k}\} \neq \{\vec{P_1}, \vec{P_2}, \dots, \vec{P_l}\}$ then means that either the decompositions $N = M_1 \cup M_2 \cup \dots \cup M_k$ and $N = P_1 \cup P_2 \cup \dots \cup P_k$ are different or there exists a block on which the fixed linear orders are different.

For every pair i, j, $1 \leq i \leq k$, $1 \leq j \leq |M_i|$, we denote by $a_{i,j}$ the element in D_{n-1}, containing the chain $[m_1^i, m_2^i, \dots, m_j^i,]$, which acts as the identity on the set $N \setminus \{m_1^i, m_2^i, \dots, m_j^i\}$. We denote by $R(\vec{M_1}, \vec{M_2}, \dots, \vec{M_k})$ the semigroup $\langle a_{i,j} : 1 \leq i \leq k, 1 \leq j \leq |M_i| \rangle \cup \{e\}$.

Theorem 2. *For an arbitrary decomposition $N = M_1 \cup M_2 \cup \dots \cup M_k$ and arbitrary linear orders on the elements of every block of this decomposition the semigroup $R(\vec{M_1}, \vec{M_2}, \dots, \vec{M_k})$ is an $\mathcal{R}$–cross-section of $\mathcal{IS}_n$. If $\{\vec{M_1}, \vec{M_2}, \dots, \vec{M_k}\} \neq \{\vec{P_1}, \vec{P_2}, \dots, \vec{P_l}\}$ then one has that $R(\vec{M_1}, \vec{M_2}, \dots, \vec{M_k}) \neq R(\vec{P_1}, \vec{P_2}, \dots, \vec{P_l})$. Moreover, every $\mathcal{R}$–cross-section of $\mathcal{IS}_n$ has the form $R(\vec{M_1}, \vec{M_2}, \dots, \vec{M_k})$ for some decomposition $N = M_1 \cup M_2 \cup \dots \cup M_k$ and some linear orders on the elements of every block.*

Proof. Every block M_i, $i = 1, 2, \dots, k$, is invariant with respect to the action of the semigroup $R(\vec{M_1}, \vec{M_2}, \dots, \vec{M_k})$. Moreover, the only generators, which act identically on $N \setminus M_i$ are the elements $a_{i,j}$, $1 \leq j \leq |M_i|$. Hence $R(\vec{M_1}, \vec{M_2}, \dots, \vec{M_k}) = R_1 \times \dots \times R_k$, where $R_i = \langle a_{i,j} : 1 \leq j \leq |M_i| \rangle \cup \{e\}$, $i = 1, \dots, k$. The map $a_j \mapsto a_{i,j}$, $1 \leq j \leq |M_i|$, induces a natural isomorphism of the semigroup $K_{|M_i|}$ from Corollary 6 and the semigroup R_i. This isomorphism is coordinated with the actions of $K_{|M_i|}$ and R_i on $\{1, 2, \dots, |M_i|\}$ and M_i respectively. This allows one to identify $R_1 \times \dots \times R_k$ with $K_{|M_1|} \times \dots \times K_{|M_k|}$ as the semigroups of partial transformations of the set N. As $K_{|M_i|}$ is an $\mathcal{R}$-cross-section of $\mathcal{IS}_{|M_i|}$, we get that $|K_{|M_i|}| = 2^{|M_i|}$ and

$$|R(\vec{M_1}, \vec{M_2}, \dots, \vec{M_k})| = |K_{|M_1|}| \times \dots \times |K_{|M_k|}| = 2^{|M_1| + \dots + |M_k|} = 2^n.$$

If $f = (f_1, \dots, f_k)$ and $g = (g_1, \dots, g_k)$ are different elements from $R(\vec{M_1}, \vec{M_2}, \dots, \vec{M_k})$, then $f_i \neq g_i$ for some i. Hence $\mathrm{dom}(f_i) \neq \mathrm{dom}(g_i)$, according to Corollary 6. But $\mathrm{dom}(f_i) = \mathrm{dom}(f) \cap M_i$ and $\mathrm{dom}(g_i) = \mathrm{dom}(g) \cap M_i$. This implies $\mathrm{dom}(f) \neq \mathrm{dom}(g)$. Therefore different elements from $R(\vec{M_1}, \vec{M_2}, \dots, \vec{M_k})$ have different domains and thus belong to different $\mathcal{R}$–classes. Since the number of different $\mathcal{R}$–classes in $\mathcal{IS}_n$ is exactly $2^n = |R(\vec{M_1}, \vec{M_2}, \dots, \vec{M_k})|$, we get that the semigroup $R(\vec{M_1}, \vec{M_2}, \dots, \vec{M_k})$ is an $\mathcal{R}$–cross-section of $\mathcal{IS}_n$. This completes the proof of the first part of our theorem.

To prove the second statement we consider for every element $x \in N$ its *orbit* $orb(x) = \{a(x) : a \in R(\vec{M_1}, \vec{M_2}, \dots, \vec{M_k}) \text{ and } x \in \mathrm{dom}(a)\}$. Then the blocks $M_1, \dots, M_k$ are maximal orbits with respect to inclusions. Hence the equality $R(\vec{M_1}, \vec{M_2}, \dots, \vec{M_k}) = R(\vec{P_1}, \vec{P_2}, \dots, \vec{P_l})$ implies that the decompositions $N = M_1 \cup \dots \cup M_k$ and $N = P_1 \cup \dots \cup P_l$ coincide (as unordered decompositions). Further, the semigroup R_i is exactly the set of all those elements from $R(\vec{M_1}, \vec{M_2}, \dots, \vec{M_k})$, which act identically on $N \setminus M_i$. Since $R_i = K_{|M_i|}$, we can apply Lemma 5 and get that the inequality $x^{|M_i|-1} \neq x^{|M_i|}$ has the unique solution $a = [m^i_1, \dots, m^i_{|M_i|}]$ in R_i. This uniquely defines the linear order on the elements of M_i and proves the second statement of the theorem.

Let now R be an $\mathcal{R}$–cross-section of $\mathcal{IS}_n$. By Lemma 7 the chains of two arbitrarily chosen elements from $R \cap D_{n-1}$ either are disjoint or one of these chains is a beginning of the other one. Then the chains of the elements from $R \cap D_{n-1}$, which are not proper beginnings of any other element from $R \cap D_{n-1}$ define a disjoint family $M_1, \dots, M_k$ of subsets of N. Moreover, each such chain defines a natural linear order on the corresponding M_i in the following way: the chain $[x_1, \dots, x_m]$ defines $\{x_1, \dots, x_m\}$ with the order $x_1, \dots, x_m$. For every $x \in N$ there exists $a \in R \cap D_{n-1}$ such that $\mathrm{dom}(a) = N \setminus \{x\}$. This means that x belongs to the chain of a and hence x belongs to some M_i. Hence R defines a decomposition $N = M_1 \cup \dots \cup M_k$ of N. From the construction of this decomposition it follows immediately that $R \cap D_{n-1} = R(\vec{M}_1, \vec{M}_2, \dots, \vec{M}_k) \cap D_{n-1}$ and hence

$$R \supset \langle R(\vec{M}_1, \vec{M}_2, \dots, \vec{M}_k) \cap D_{n-1} \rangle \cup \{e\} = R(\vec{M}_1, \vec{M}_2, \dots, \vec{M}_k).$$

Since all $\mathcal{R}$–cross-sections of $\mathcal{IS}_n$ have the same cardinality 2^n we finally get the equality $R = R(\vec{M}_1, \vec{M}_2, \dots, \vec{M}_k)$ completing the proof. □

The involution $a \mapsto a^{-1}$ interchanges $\mathcal{R}$– and $\mathcal{L}$–classes in every inverse semigroup. Clearly, this involution also maps $\mathcal{L}$–cross-sections to $\mathcal{R}$–cross-sections and vice-versa. Hence, dualizing Theorem 2, one immediately gets the description of the $\mathcal{L}$–cross-sections in $\mathcal{IS}_n$. To formulate this theorem it is convenient to introduce the following notation.

For a subset, $M_i \subset N$, with the fixed linear order $m^i_1, m^i_2, \dots, m^i_{|M_i|}$ on its elements we denote by $b_{i,j}$ the element from D_{n-1}, which has the chain $[m^i_j, m^i_{j+1}, \dots, m^i_{|M_i|}]$ and acts as the identity on the set $N \setminus \{m^i_j, m^i_{j+1}, \dots, m^i_{|M_i|}\}$. For a decomposition, $N = M_1 \cup \dots \cup M_k$, and fixed linear orders on all blocks we define the following semigroup:

$$L(\vec{M}_1, \vec{M}_2, \dots, \vec{M}_k) = \langle b_{i,j} \ : \ 1 \le i \le k, 1 \le j \le |M_i| \rangle \cup \{e\}.$$

We remark that the involution $a \mapsto a^{-1}$ sends $R(\vec{M}_1, \vec{M}_2, \dots, \vec{M}_k)$ to $L(\overleftarrow{M}_1, \overleftarrow{M}_2, \dots, \overleftarrow{M}_k)$, where $\overleftarrow{M}_i$ denotes the order on M_i, which is opposite to the order $\vec{M}_i$.

Theorem 3. *For an arbitrary decomposition $N = M_1 \cup M_2 \cup \dots \cup M_k$ and arbitrary linear orders on the elements of every block of this decomposition the semigroup $L(\vec{M}_1, \vec{M}_2, \dots, \vec{M}_k)$ is an $\mathcal{L}$–cross-section of $\mathcal{IS}_n$. If $\{\vec{M}_1, \vec{M}_2, \dots, \vec{M}_k\} \ne \{\vec{P}_1, \vec{P}_2, \dots, \vec{P}_l\}$ then one has that $L(\vec{M}_1, \vec{M}_2, \dots, \vec{M}_k) \ne L(\vec{P}_1, \vec{P}_2, \dots, \vec{P}_l)$. Moreover, every $\mathcal{L}$–cross-section of $\mathcal{IS}_n$ has the form $L(\vec{M}_1, \vec{M}_2, \dots, \vec{M}_k)$ for some decomposition $N = M_1 \cup M_2 \cup \dots \cup M_k$ and some linear orders on the elements of every block.*

Corollary 7. *The semigroup $\mathcal{IS}_n$ contains exactly $\displaystyle\sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1}$ different $\mathcal{R}$– ($\mathcal{L}$–) cross-section.*

Proof. It is enough to prove the statement for $\mathcal{R}$–cross-section. Partition the set of all $\mathcal{R}$–cross-sections $R(\vec{M}_1, \vec{M}_2, \dots, \vec{M}_k)$ into n disjoint classes with respect to the number k of the blocks in the corresponding decomposition $N = M_1 \cup M_2 \cup \dots \cup M_k$ of N. For a fixed k consider an arbitrary permutation, $i_1, \dots, i_n$ of $1, 2, \dots, n$, and a $(k-1)$–element subset, $\{j_1, \dots, j_{k-1}\}$, of $\{1, 2, \dots, n-1\}$. We can assume that $j_1 < j_2 < \dots < j_{k-1}$. This defines a decomposition of N into k blocks $M_1 = \{i_1, \dots, i_{j_1}\}$, $M_2 = \{i_{j_1+1}, \dots, i_{j_2}\}, \dots,$

$M_k = \{i_{j_{k-1}+1}, \dots, i_n\}$ together with linear orders on these blocks. Since the order of the blocks is not important for $R(\vec{M}_1, \vec{M}_2, \dots, \vec{M}_k)$, we get that each $\mathcal{R}$–cross-section is counted $k!$ times. Therefore we have exactly $n!\binom{n-1}{k-1}\frac{1}{k!}$ different $\mathcal{R}$–cross-sections for a fixed k. Summing this over all k we get the necessary statement. □

Corollary 8. *The only subsemigroup of $\mathcal{IS}_n$, which is an $\mathcal{R}$– and an $\mathcal{L}$–cross-section at the same time is the semigroup $E(\mathcal{IS}_n)$ of all idempotents of $\mathcal{IS}_n$.*

Proof. Let $R = R(\vec{M}_1 \vec{M}_2, \dots, \vec{M}_n)$ be an $\mathcal{R}$–cross-section. It follows from Lemma 4 that for a fixed linear order, $m^i_1, \dots, m^i_{|M_i|}$, on the elements of the block M_i, the conditions $a \in R$ and $|\operatorname{dom}(a) \cap M_i| = 1$ imply $\operatorname{im}(a) \cap M_i = m^i_{|M_i|}$. Hence, the necessary condition for R to be an $\mathcal{L}$–cross-section as well is that all blocks M_i contain not more than 1 element. From the other hand one can easily check that

$$R(\vec{\{1\}}\vec{\{2\}}, \dots, \vec{\{n\}}) = L(\vec{\{1\}}\vec{\{2\}}, \dots, \vec{\{n\}}) = E(\mathcal{IS}_n).$$

This completes the proof. □

The referee has remarked that the statement of Corollary 8 also follows from the easy observation that if $|M_i| > 2$ then $a_{i,1} \neq a_{i,2} \in M_i$ are $\mathcal{L}$–related, since they have the same image.

Corollary 9. *The semigroup $E(R)$, consisting of all idempotents of the $\mathcal{R}$–cross-section $R = R(\vec{M}_1, \vec{M}_2, \dots, \vec{M}_k)$, is a lattice with respect to the natural partial order on $\mathcal{IS}_n$. This lattice decomposes into a direct product of k chains of lengths $|M_1|$, $|M_2|$,..., $|M_k|$ respectively.*

Proof. That $E(R)$ is a semigroup follows from the fact that $\mathcal{IS}_n$ is inverse.

For a set, M, a linear order, $m_1, \dots, m_{|M|}$, on the elements of M, and a number r, $0 \leq r \leq |M|$, we denote by M^r the set $\{m_{|M|-r+1}, m_{|M|-r+2}, \dots, m_{|M|}\}$. In particular, $M^0 = \varnothing$. From Lemma 4 and Corollary 4 it follows that the element $a \in R$ is an idempotent if and only if $\operatorname{dom}(a) = M_1^{r_1} \cup \dots \cup M_k^{r_k}$ for some vector $(r_1, \dots, r_k)$, where $0 \leq r_i \leq |M_i|$ for all $1 \leq i \leq k$. It is easy to check that the map $a \mapsto (r_1, \dots, r_k)$ is an isomorphism of the poset $E(R)$ onto the poset $P = \{(r_1, \dots, r_k) : 0 \leq r_i \leq |M_i| \text{ for all } 1 \leq i \leq k\}$ with the partial order, defined as follows:

$$(r'_1, \dots, r'_k) \leq (r''_1, \dots, r''_k) \text{ if and only if } r'_i \leq r''_i \text{ for all } i = 1, \dots, k.$$

It is obvious that P is a lattice and that P is a direct product of k chains of lengths $|M_1|$, $|M_2|$,..., $|M_k|$ respectively. □

4 Relation between $\mathcal{R}$– ($\mathcal{L}$–) cross-sections and $\mathcal{H}$–cross-sections

To proceed we need a more detailed result about the structure of $\mathcal{H}$–cross-sections in $\mathcal{IS}_n$.

Lemma 8. *Let $H(\prec_1)$ and $H(\prec_2)$ be two $\mathcal{H}$–cross-sections of the semigroup $\mathcal{IS}_n$. The semigroups $H(\prec_1)$ and $H(\prec_2)$ coincide if and only if the linear orders $\prec_1$ and $\prec_2$ are either equal or opposite.*

Proof. The sufficiency of this condition is obvious. To prove the necessity we assume that the linear orders $\prec_1$ and $\prec_2$ are neither equal nor opposite. Then there exist the elements $x, y, z \in N$ such that $x \prec_1 y \prec_1 z$ and either $x \prec_2 y$ and $z \prec_2 y$ or $y \prec_2 x$ and $y \prec_2 z$. In both cases the element $\begin{pmatrix} x & y \\ y & z \end{pmatrix}$ belongs to $H(\prec_1)$ and does not belong to $H(\prec_2)$. Hence $H(\prec_1) \neq H(\prec_2)$. □

Lemma 9. *Assume that the $\mathcal{R}$–cross-section $R = R(\vec{M}_1, \vec{M}_2, \dots, \vec{M}_k)$ is contained in the $\mathcal{H}$–cross-section $H = H(\prec)$. Then the following statements hold:*

1. *The elements from every block M_i form an interval of the linearly ordered set $(N, \prec)$.*
2. *If the linear order on M_i (coming from R) has the form $m_1^i, m_2^i, \dots, m_{|M_i|}^i$ then either $m_1^i \prec m_2^i \prec \cdots \prec m_{|M_i|}^i$ or $m_{|M_i|}^i \prec m_{|M_i|-1}^i \prec \cdots \prec m_1^i$.*

Proof. Let the element m_p^j from the block M_j be placed between the elements m_q^i and $m_{|M_i|}^i$ from the block M_i. Assume that $m_q^i \prec m_p^j \prec m_{|M_i|}^i$ (the case $m_{|M_i|}^i \prec m_p^j \prec m_q^i$ is analogous). It follows from Theorem 2 and Lemma 4 that the semigroup R contains the elements $\begin{pmatrix} m_q^i & m_p^j \\ m_{|M_i|}^i & m_{|M_j|}^j \end{pmatrix}$ and $\begin{pmatrix} m_p^j & m_{|M_i|}^i \\ m_{|M_j|}^j & m_{|M_i|}^i \end{pmatrix}$. Since $R \subset H(\prec)$, we have $m_{|M_i|}^i \prec m_{|M_j|}^j$ for the first element and $m_{|M_j|}^j \prec m_{|M_i|}^i$ for the second one. This contradiction proves the first statement of the lemma.

From Lemma 4 it follows that for arbitrary p and q, $1 \leq p < q \leq |M_i|$, the semigroup R contains the element $\begin{pmatrix} m_p^i & m_q^j \\ m_{|M_i|-1}^i & m_{|M_i|}^i \end{pmatrix}$. Hence the inequality $m_{|M_i|-1}^i \prec m_{|M_i|}^i$ implies $m_p^i \prec m_q^i$ for all $p < q$. Analogously, the inequality $m_{|M_i|}^i \prec m_{|M_i|-1}^i$ implies $m_q^i \prec m_p^i$ for all $p < q$. This completes the proof. □

Proposition 1. *Assume that $n > 3$ and that the decomposition $N = M_1 \cup \cdots \cup M_k$ contains exactly m blocks containing more than 1 element. Then for arbitrary linear orders on the blocks $M_1, \dots, M_k$, the $\mathcal{R}$–cross-section $R = R(\vec{M}_1, \vec{M}_2, \dots, \vec{M}_k)$ is contained in exactly $k! \cdot 2^{m-1}$ different $\mathcal{H}$–cross-sections.*

Proof. Let us fix some ordering, $M_{l_1}, M_{l_2}, \dots, M_{l_k}$, of the blocks $M_1, M_2, \dots, M_k$ and write down the elements in the following way: we start with elements of M_{l_1}, then elements of M_{l_2} and so on. Moreover, the elements of every block are written either with respect to the linear order fixed on this block (this order comes from R) or with respect to the opposite order. In this way we get some linear order on N, which we denote by $\prec$. From Theorem 2 and Lemma 4 we get that $R \subset H(\prec)$. Moreover, from Theorem 1 and Lemma 9 it follows that for $n > 3$ every $\mathcal{H}$–cross-section, containing R, has the form $H(\prec)$, where the order $\prec$ is constructed exactly by the procedure, described above. Changing the order of blocks or the orders of writing down the elements for the blocks, containing more than 1 element, we get $k! \cdot 2^m$ different linear orders.

It is obvious, that if some order $\prec$ can be constructed by this procedure, then the opposite order can be constructed by this procedure as well. Taking into account Lemma 8, we get $(k! \cdot 2^m)/2 = k! \cdot 2^{m-1}$ different $\mathcal{H}$–cross-sections, containing R. This completes the proof. □

We remark that, using the involution $a \mapsto a^{-1}$, one gets the same statement for the $\mathcal{L}$–cross-sections as well.

Proposition 2. *Let $n > 3$. Then every $\mathcal{H}$-cross-section of the semigroup $\mathcal{IS}_n$ contains exactly*

$$1 + \sum_{l=1}^{[n/2]} \left(2^l \cdot \sum_{m=0}^{n-2l} \binom{m+l}{l} \binom{n-m-l-1}{l-1} \right)$$

different $\mathcal{R}$– ($\mathcal{L}$–) cross-section.

Proof. Clearly, it is enough to prove the statement for, say $\mathcal{R}$–cross-sections. For every $n > 3$ each $\mathcal{H}$–cross-section from $\mathcal{IS}_n$ has the form $H(\prec)$, according to Theorem 1. By Lemma 9, to define an $\mathcal{R}$–cross-sections, $R = R(\vec{M}_1, \vec{M}_2, \dots, \vec{M}_k)$, which is contained in $H(\prec)$, one has to define a decomposition of the poset $(N, \prec)$ into a disjoint union of intervals, and then on every interval, consisting of more than 1 element, to fix a linear order, which either is inherited from $\prec$ or is the opposite one. According to Theorem 2, all $\mathcal{R}$–cross-section, obtained in this way, will be different.

Let us now calculate in how many different ways one can decompose $(N, \prec)$ into intervals, among which there will be exactly l intervals, containing more than 1 element, and exactly m intervals, containing 1 element (clearly, $0 \leq l \leq [n/2]$ and for $l > 0$ one has $0 \leq m \leq n - 2l$). Since the linear order $\prec$ induces a linear order on the collection of disjoint intervals, we first choose in the sequence of $l + m$ intervals l places for those intervals, which contain more than 1 element. This can be done in $\binom{l+m}{l}$ different ways. Fix now these places for intervals containing more than 1 element and take away one element from every interval. Then in all intervals, which originally contained more than 1 element, we will have $t_1, t_2, \dots, t_l$ elements respectively and we reduce our problem to the problem of decomposition of the number $n-l-m$ into a sum of l non-zero positive integers: $n - l - m = t_1 + t_2 + \cdots + t_l$, where the order of summands is important. It is well-known, see for example [Gr], that the number of such decompositions equals $\binom{n-m-l-1}{l-1}$.

If we fix the number of intervals and the places for intervals, containing more than 1 element, then the constructed correspondence between the decompositions of $(N, \prec)$ into intervals and the solutions to $n - l - m = t_1 + t_2 + \cdots + t_l$ is bijective. Moreover, every decomposition of this kind gives exactly 2^l different $\mathcal{R}$–cross-section, contained in $H(\prec)$. Hence, summing over all possible values of l and m we get exactly

$$1 + \sum_{l=1}^{[n/2]} \left(2^l \cdot \sum_{m=0}^{n-2l} \binom{m+l}{l} \binom{n-m-l-1}{l-1} \right)$$

different $\mathcal{R}$–cross-section, contained in $H(\prec)$. □

5 Classification of $\mathcal{R}$– ($\mathcal{L}$–) cross-sections up to isomorphism

Let $R = R(\vec{M}_1, \vec{M}_2, \dots, \vec{M}_k)$ be an $\mathcal{R}$–cross-section of $\mathcal{IS}_n$. The vector $(u_1, \dots, u_n)$, where $u_m = |\{i : |M_i| = m\}|$, $1 \le m \le n$, will be called the *type* of R. Analogously one defines the type of an $\mathcal{L}$–cross-section.

Theorem 4. *Two $\mathcal{R}$– ($\mathcal{L}$–) cross-section in $\mathcal{IS}_n$ are isomorphic if and only if they have the same type.*

Proof. Clearly, it is enough to prove the statement for, say $\mathcal{R}$–cross-sections. Let $R_1 = R(\vec{M}_1, \vec{M}_2, \dots, \vec{M}_k)$ and $R_2 = R(\vec{P}_1, \vec{P}_2, \dots, \vec{P}_l)$ be two arbitrary $\mathcal{R}$–cross-sections of types $(u_1, \dots, u_n)$ and $(v_1, \dots, v_n)$ respectively.

Assume first that $R_1 \simeq R_2$ and let $p = \max_{u_m \neq 0} m$, $q = \max_{v_m \neq 0} m$. As it was done in the proof of Theorem 2, we can identify R_1 with $K_{|M_1|} \times \cdots \times K_{|M_k|}$ and R_2 with $K_{|P_1|} \times \cdots \times K_{|P_l|}$. Using this identification and Lemmas 4 and 5 we get that in the semigroup R_1 the inequality $x^m \neq x^{m-1}$ does not have solutions for all $m > p$ and has exactly $2^{u_p} - 1$ solutions for $m = p$. Since the same is true for R_2 as well, we get $p = q$ and $u_p = u_q$.

The fact that the types of the $\mathcal{R}$–cross-sections R_1 and R_2 coincide can now be easily proved by induction. Indeed, assume that $u_m = v_m$ is already proved for all $m > t$. The inequality $x^{t-1} \neq x^t$ should have in both R_1 and R_2 the same number of solutions. Using the inductive assumption and Lemma 5, we derive $u_t = v_t$.

Now let us assume that the types of R_1 and R_2 are the same, that is, $(u_1, \dots, u_n) = (v_1, \dots, v_n)$. Then we have, in particular, $k = l$. Let us order the blocks of the decomposition $N = M_1 \cup \cdots \cup M_k$ with respect to the growth of their cardinalities (in general this is not uniquely defined). Now write down the elements of N in the following way: start with the elements from the block M_1, taking into account the linear order on M_1, coming from R_1, proceed with M_2 and so on. We get a permutation, $y_1, y_2, \dots, y_n$, of the elements $1, 2, \dots, n$. Analogously, from the decomposition $N = P_1 \cup \cdots \cup P_k$ we can construct a permutation, say $z_1, \dots, z_n$. Set $\pi = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ z_1 & z_2 & \dots & z_n \end{pmatrix}$. Direct calculation shows that the map $x \mapsto \pi^{-1}x\pi$ is an isomorphism from R_1 to R_2. $\square$

Corollary 10. *Two $\mathcal{R}$– ($\mathcal{L}$–) cross-sections T_1 and T_2 are isomorphic if and only if they are conjugated with respect to the natural action of the group S_n, that is, if and only if there exists $\pi \in S_n$ such that $T_2 = \pi^{-1}T_1\pi$.*

Proof. The sufficiency is obvious and the necessity follows from the proof of Theorem 4. $\square$

Denote by p_n the number of decompositions of the positive integer n into the sum of positive integers, where the order of summands is not important.

Corollary 11. *The number of pairwise non-isomorphic $\mathcal{R}$– ($\mathcal{L}$–) cross-sections in the semigroup $\mathcal{IS}_n$ equals p_n.*

Proof. This follows from Theorem 4 and easy counting of the number of different types for $\mathcal{R}$– ($\mathcal{L}$–) cross-sections. $\square$

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