# On Kiselman's semigroup

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**Abstract**

We study the algebraic properties of the series $\mathrm{K}_n$ of semigroups, which is inspired by [Ki] and has origins in convexity theory. In particular, we describe Green's relations on $\mathrm{K}_n$, prove that there exists a faithful representation of $\mathrm{K}_n$ by $n \times n$ matrices with non-negative integer coefficients (and even explicitly construct such a representation), and prove that $\mathrm{K}_n$ does not admit a faithful representation by matrices of smaller size. We also describe the maximal nilpotent subsemigroups in $\mathrm{K}_n$, all isolated and completely isolated subsemigroups, all automorphisms and anti-automorphisms of $\mathrm{K}_n$. Finally, we explicitly construct all irreducible representations of $\mathrm{K}_n$ over any field and describe primitive idempotents in the semigroup algebra (which we prove is basic).

## 1 Introduction

Let $E$ be a real vector space and $\mathrm{Func}(E)$ be the set of all functions on $E$ with values in the extended real line $\mathbb{R} \cup \{-\infty, +\infty\}$. In convexity theory there appear three natural operators on $\mathrm{Func}(E)$, namely the operator $c$ of taking the convex hull of a function, the operator $l$ of taking the largest lower semicontinuous minorant of the function, and the operator $m$ defined via

$$m(f)(x) = \begin{cases} f(x), & \text{if } f \text{ is everywhere } > -\infty; \\ -\infty, & \text{otherwise.} \end{cases}$$

The operators $c, l, m$ generate a monoid, $G(E)$, with repsect to the usual composition. In [Ki] it was shown that this monoid consists of 18 elements and has the following presentation (as a monoid):

$$\begin{aligned} G(E) = \langle c, l, m : c^2 = c,\ l^2 = l,\ m^2 = m,& \\ clc = lcl = lc,\ cmc = mcm = mc,\ lml = mlm = ml\rangle.& \quad (1.1) \end{aligned}$$

Furthermore, the paper [Ki] also contains a detailed study of the algebraic structure of $G(E)$ and gives a faithful representation of $G(E)$ by $3\times 3$ matrices with non-negative integer coefficients.

There is a fairly straightforward way to generalize (1.1). Let $n$ be a positive integer. Denote by $\mathrm{K}_n$ the monoid defined via the following presentation:

$$\mathrm{K}_n = \langle a_1, \dots, a_n : \; a_i^2 = a_i,\, i = 1, \dots, n; \\ a_i a_j a_i = a_j a_i a_j = a_j a_i,\, 1 \leq i < j \leq n \rangle. \quad (1.2)$$

We will call $\mathrm{K}_n$ *Kiselman's semigroup* after the author of [Ki]. Obviously, we have $G(E) \cong \mathrm{K}_3$. The generalization (1.2) was proposed by O. Ganyushkin and the second author in 2002 (unpublished). In [Go] several results on the structure of $\mathrm{K}_n$ were announced. Unfortunately, the proofs have never appeared. So, we have decided to study $\mathrm{K}_n$ independently. In the present paper we prove all the results announced in [Go], in particular, we describe Green's relations on $\mathrm{K}_n$ (Section 7), prove that there exists a faithful representation of $\mathrm{K}_n$ by $n \times n$ matrices with non-negative integer coefficients (and even explicitly construct such a representation), and prove that $\mathrm{K}_n$ does not admit a faithful representation by matrices of smaller size (Subsection 11.1). We also obtain some additional results, in particular, we describe the maximal nilpotent subsemigroups in $\mathrm{K}_n$ (Section 8), all isolated and completely isolated subsemigroups (Section 9), all automorphisms of $\mathrm{K}_n$ and all anti-automorphisms of $\mathrm{K}_n$ (Section 6). We also explicitly construct all irreducible representations of $\mathrm{K}_n$ over any field and describe the primitive idempotents in the semigroup algebra (Subsection 11.2). We are convinced that $\mathrm{K}_n$ is a very beautiful combinatorial objects and might have a lot of further interesting combinatorial properties and applications.

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## 2 Finiteness of $\mathrm{K}_n$

We will denote by $e$ the unit element in $\mathrm{K}_n$. For a finite alphabet, $\mathcal{A}$, we denote by $\mathrm{W}(\mathcal{A})$ the set of all *finite* words over this alphabet, including the empty word (with respect to the usual operation of concatenation of words this is the same as the free monoid, generated by $\mathcal{A}$, which is sometimes denoted by $\mathcal{A}^*$). Let $\mathfrak{l} : \mathrm{W}(\mathcal{A}) \to \mathbb{N} \cup \{0\}$ denote the *length function*.

**Lemma 1.** *(i) Let $i \in \{1, \dots, n\}$ and $w \in \mathrm{W}(\{a_1, \dots, a_{i-1}\})$. Then we have $a_i w a_i = a_i w$ in $\mathrm{K}_n$.*

*(ii) Let $i \in \{1, \dots, n\}$ and $w \in \mathrm{W}(\{a_{i+1}, \dots, a_n\})$. Then we have $a_i w a_i = w a_i$ in $\mathrm{K}_n$.*

*Proof.* We prove (i). The statement (ii) is proved by similar arguments. We proceed by induction on $\mathfrak{l}(w)$. If $\mathfrak{l}(w) = 0$ or $\mathfrak{l}(w) = 1$, the statement follows directly from the presentation (1.2). Assume now that $\mathfrak{l}(w) > 1$ and write $w = w' a_j$ for some $j < i$. Then $w' \in \mathrm{W}(\{a_1, \dots, a_{i-1}\})$ and $\mathfrak{l}(w') = \mathfrak{l}(w) - 1$. We have

$$\begin{aligned} a_i w a_i = a_i w' a_j a_i = (a_i w') a_j a_i &= \text{(by the inductive assumption)} = \\ = (a_i w' a_i) a_j a_i = a_i w' a_i a_j a_i &= \text{(by (1.2))} = a_i w' a_i a_j = (a_i w' a_i) a_j = \\ = \text{(by the inductive assumption)} &= (a_i w') a_j = a_i w' a_j = a_i w. \end{aligned}$$

□

Define the function $L : \mathbb{N} \to \mathbb{N}$ as follows:

$$L(n) = \begin{cases} 2^{k+1} - 2, & n = 2k; \\ 3 \cdot 2^k - 2, & n = 2k + 1. \end{cases}$$

**Corollary 2.** *Let $\alpha \in \mathrm{K}_n$, $\alpha \neq e$, and let $w \in \mathrm{W}(\{a_1, \dots, a_n\})$ be a word of the shortest possible length such that $\alpha = w$ in $\mathrm{K}_n$. Then we have the following:*

*(i) For $i \leq \lceil \frac{n}{2} \rceil$ the letter $a_i$ occurs in $w$ at most $2^{i-1}$ times.*

*(ii) For $i \geq \lceil \frac{n+1}{2} \rceil$ the letter $a_i$ occurs in $w$ at most $2^{n-i}$ times.*

*(iii) $\mathfrak{l}(w) \leq L(n)$.*

*Proof.* We prove (i) by induction on $i$. If the letter $a_1$ occurs in $w$ more than once, the word $w$ can be reduced (shortened) using Lemma 1(ii). This gives us the basis of the induction. Let $1 < i \leq \lceil \frac{n}{2} \rceil$. From the inductive assumption we obtain that the total number of occurrences of the letters $a_1, \dots, a_{i-1}$ in $w$ does not exceed $2^{i-1} - 1$. Hence we can write $w = w_1 b_1 w_2 b_2 w_3 \dots w_{2^{i-1}-1} b_{2^{i-1}-1} w_{2^{i-1}}$, where $b_j \in \{a_1, \dots, a_{i-1}\}$ and $w_j \in \mathrm{W}(\{a_i, \dots, a_n\})$ for all appropriate $j$. If $a_i$ occurs in some $w_j$ more than once, the word $w_j$ and hence $w$ can be reduced using Lemma 1(ii). Hence the total number of occurrences of $a_i$ in $w$ does not exceed $2^{i-1}$. This proves (i). (ii) is proved by similar arguments. (iii) follows from (i) and (ii) since for all $n = 2k \in \mathbb{N}$ we have

$$L(n) = \sum_{i=1}^{k} 2^{i-1} + \sum_{i=k+1}^{n} 2^{n-i}$$

and for all $n = 2k + 1 \in \mathbb{N}$ we have

$$L(n) = \sum_{i=1}^{k+1} 2^{i-1} + \sum_{i=k+2}^{n} 2^{n-i}.$$

□

As an immediate corollary from the latter statement we have:

**Theorem 3.** *The semigroup* $\mathrm{K}_n$ *is finite, moreover*

$$|\mathrm{K}_n| \leq 1 + n^{L(n)}.$$

*Proof.* The semigroup $\mathrm{K}_n$ is generated by $n$ elements. By Corollary 2(iii), every element of $\mathrm{K}_n$, different from the unit element $e$, can be written as a product of at most $L(n)$ generators. Since all generators are idempotents, repeating the last generator, occurring in such a product, we conclude that every element of $\mathrm{K}_n$, different from the unit element $e$, can be written as a product of exactly $L(n)$ generators. The statement follows. □

**Question 4.** *Can one give an explicit formula for* $|\mathrm{K}_n|$*?*

**Remark 5.** In [Go] a slightly more general family of semigroups is considered: let $(I, <)$ be a partially ordered set. Define

$$\mathrm{K}_I = \langle a_i,\, i \in I :\; a_i^2 = a_i,\, i \in I;\; a_i a_j a_i = a_j a_i a_j = a_j a_i,\, i, j \in I,\, i < j \rangle.$$

[Go, Theorem 2] states that $\mathrm{K}_I$ is finite if and only if $I$ is finite and $<$ is linear. This is an immediate consequence of Theorem 3. Indeed, Theorem 3 gives us the sufficiency. The necessity follows from the trivial observation that for incomparable $i, j \in I$ the elements $(a_i a_j)^k \in \mathrm{K}_I$, $k \in \mathbb{N}$, are obviously different since there is no relation involving both $a_i$ and $a_j$.

## 3 The canonical form for elements of $\mathrm{K}_n$

Let $\varphi : \mathrm{W}(\{a_1,\dots,a_n\}) \to \mathrm{K}_n$ denote the canonical epimorphism. For $w \in \mathrm{W}(\{a_1,\dots,a_n\})$ set $\overline{w} = \{x \in \mathrm{W}(\{a_1,\dots,a_n\}) : \varphi(x) = \varphi(w)\}$. If $w = a_{i_1}a_{i_2}\dots a_{i_k} \in \mathrm{W}(\{a_1,\dots,a_n\})$, then by a *subword* of $w$ we will mean an element of $\mathrm{W}(\{a_1,\dots,a_n\})$ of the form $a_{i_s}a_{i_{s+1}}a_{i_{s+2}}\dots a_{i_t}$ for some $1 \le s \le t \le k$. By a *quasi-subword* of $w$ we will mean an element of $\mathrm{W}(\{a_1,\dots,a_n\})$ of the form $a_{i_{l_1}}a_{i_{l_2}}a_{i_{l_3}}\dots a_{i_{l_t}}$ for some $1 \le l_1 < l_2 < l_3 < \dots < l_t \le k$ (including the empty quasi-subword). Each subword is, by definition, a quasi-subword.

The main result of this section is the following statement:

**Theorem 6.** *Let $w \in \mathrm{W}(\{a_1,\dots,a_n\})$.*

*(i) The set $\overline{w}$ contains a unique element of the minimal possible length.*

*(ii) $v \in \overline{w}$ has the minimal possible length if and only if the for each $i \in \{1,2,\dots,n\}$ the following condition is satisfied: if $a_iua_i$ is a subword of $v$ then $u$ contains some $a_j$ with $j > i$ and some $a_k$ with $k < i$.*

The words $v \in \mathrm{W}(\{a_1,\dots,a_n\})$, satisfying the condition of Theorem 6(ii), will be called *canonical*. If $w \in \mathrm{W}(\{a_1,\dots,a_n\})$ and $v \in \overline{w}$ is canonical, we will say that $v$ is the *canonical form* of $w$. By Theorem 6(i) the homomorphism $\varphi$ induces a bijection between the set of all canonical words in $\mathrm{W}(\{a_1,\dots,a_n\})$ and the elements of $\mathrm{K}_n$. In particular, it makes sense to speak about the *canonical form* of an element from $\mathrm{K}_n$.

**Remark 7.** The statement of Theorem 6(i) was announced in [Go, Theorem 1].

*Proof.* Define the binary relation $\to$ on $\mathrm{W}(\{a_1,\dots,a_n\})$ in the following way: for $w, v \in \mathrm{W}(\{a_1,\dots,a_n\})$ we set $w \to v$ if and only if there exists $i \in \{1,\dots,n\}$ such that $w = w_1a_iua_iw_2$ and either $v = w_1a_iuw_2$ and $u \in \mathrm{W}(\{a_1,\dots,a_{i-1}\})$, or $v = w_1ua_iw_2$ and $u \in \mathrm{W}(\{a_{i+1},\dots,a_n\})$. From Lemma 1 we obtain that $w \to v$ implies $v \in \overline{w}$. Obviously, $w \to v$ implies $\mathfrak{l}(v) = \mathfrak{l}(w) - 1$, in particular, any chain $v_1 \to v_2 \to v_3 \to \dots$ in $\mathrm{W}(\{a_1,\dots,a_n\})$ terminates in a finite number of steps. Denote by $\xrightarrow{*}$ the reflexive-transitive closure of $\to$.

**Lemma 8.** *For all $u, v, w \in \mathrm{W}(\{a_1,\dots,a_n\})$, such that $u \neq v$, $w \to u$ and*

$w \to v$*, there exists* $x \in \mathrm{W}(\{a_1, \dots, a_n\})$ *such that*

$$\begin{tikzcd} & w \arrow[dl] \arrow[dr] & \\ u \arrow[dr, "*"] & & v \arrow[dl, "*"'] \\ & x & \end{tikzcd}$$

*Proof.* Both $u$ and $v$ are quasi-subwords of $w$ by the definition of $\to$. $u$ is obtained from $w$ by deleting some $a_i$, and $v$ is obtained from $w$ by deleting some $a_j$. If $i \neq j$, from Lemma 1 we obtain that we are allowed to delete the corresponding occurrence of $a_i$ in $v$ obtaining some $x$ such that $v \to x$. Moreover, again applying Lemma 1 we have that we are allowed to delete the corresponding occurrence of $a_j$ in $u$. Since these operations obviously commute we will get the same result $x$ and $u \to x$, as required.

Now assume that $i = j$. By the definition of $\to$, the deletion of $a_i$ involves two occurrences of $a_i$ in a word. If the corresponding two pairs of $a_i$'s in $w$ do not intersect, then the same argument as above works, implying that our deletion operations commute.

Without loss of generality, in the remaining cases we may assume $w = a_i \alpha a_i \beta a_i$, where $\alpha, \beta \in \mathrm{W}(\{a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n\})$. If $u = v$, we can obviously take $x = u = v$. Hence we are left to deal with the following cases:

1. $u = \alpha a_i \beta a_i$, $v = a_i \alpha \beta a_i$. Because of (1.2) this is possible if and only if $\alpha = e$, which gives us $u = v$. This case was considered above.

2. $u = \alpha a_i \beta a_i$, $v = a_i \alpha a_i \beta$. In this case we can take $x = \alpha a_i \beta$ and obviously have $u \to x$, $v \to x$.

3. $u = a_i \alpha a_i \beta$, $v = a_i \alpha \beta a_i$. Because of (1.2) this is possible if and only if $\beta = e$, which gives us $u = v$. This case was considered above.

The statement of the lemma follows. □

The statement (i) follows now from Lemma 8 and the Diamond Lemma (see e.g. [Ne]). The statement (ii) follows from the statement (i) and the definition of the relation $\to$. This completes the proof. □

From Corollary 2(i) we know that for any $w \in \mathrm{W}(\{a_1, \dots, a_n\})$ the length of the minimal representative in $\overline{w}$ does not exceed $L(n)$. Now we can show that this bound is sharp.

**Corollary 9.** *There exists* $w \in \mathrm{W}(\{a_1, \dots, a_n\})$ *such that the length of the minimal representative in* $\overline{w}$ *equals* $L(n)$.

*Proof.* Let $k = \lceil \frac{n}{2} \rceil$ and set $w_1 = a_1 a_n$, $w_2 = a_2 a_{n-1}$,..., $w_{k-1} = a_{k-1} a_{n-k+2}$,

$$w_k = \begin{cases} a_k a_{n-k+1}, & n \text{ is even;} \\ a_k, & n \text{ is odd.} \end{cases}$$

Define the words $v_i$, $i = 1, \dots, k$, recursively as follows: $v_1 = w_1$; if $v_i = w_{j_1} w_{j_2} \dots w_{j_s}$, then $v_{i+1} = w_{i+1} w_{j_1} w_{i+1} w_{j_2} w_{i+1} \dots w_{i+1} w_{j_s} w_{i+1}$. It follows immediately that $\mathfrak{l}(v_k) = L(n)$ and it is easy to see from the construction that $v_i$ is canonical for every $i$. The claim follows. □

# 4 Idempotents in $\mathrm{K}_n$

Let $w \in \mathrm{W}(\{a_1, \dots, a_n\})$. Define the *content* $\mathfrak{c}(w)$ of $w$ as the set of all those $i \in \{1, \dots, n\}$ such that the letter $a_i$ appears in $w$. In particular, $\mathfrak{c}(e) = \varnothing$ and $\mathfrak{c}(a_i) = \{i\}$ for all $i = 1, \dots, n$. From (1.2) it follows immediately that $\mathfrak{c}(v) = \mathfrak{c}(w)$ for every $v \in \overline{w}$, in particular, one can speak of the *content* of an element from $\mathrm{K}_n$. Furthermore, obviously $\mathfrak{c}(wv) = \mathfrak{c}(w) \cup \mathfrak{c}(v)$ for all $v, w \in \mathrm{W}(\{a_1, \dots, a_n\})$, which implies the following statement:

**Lemma 10.** *$\mathfrak{c}$ is an epimorphism from the semigroup $\mathrm{W}(\{a_1, \dots, a_n\})$ to the semigroup $(2^{\{1,2,\dots,n\}}, \cup)$. $\mathfrak{c}$ also induces an epimorphism from $\mathrm{K}_n$ to the semigroup $(2^{\{1,2,\dots,n\}}, \cup)$ (abusing notation we will denote this epimorphism also by $\mathfrak{c}$).*

Let $X \subset \{1, \dots, n\}$. If $X = \varnothing$, set $e_\varnothing = e$. If $X \neq \varnothing$, let $X = \{i_1, \dots, i_k\}$ such that $i_1 > i_2 > \dots > i_k$. Set $e_X = a_{i_1} a_{i_2} \cdots a_{i_k}$.

**Proposition 11.** *Each $e_X$ is an idempotent in $\mathrm{K}_n$ and every idempotent in $\mathrm{K}_n$ has the form $e_X$ for some $X \subset \{1, \dots, n\}$. In particular, the semigroup $\mathrm{K}_n$ contains $2^n$ idempotents.*

*Proof.* As the word $a_{i_1} a_{i_2} \cdots a_{i_k}$ is canonical we have $e_X \neq e_Y$ if $X \neq Y$. That $e_X e_X = e_X$ follows immediately from Lemma 1(i). Hence we have only to show that any idempotent in $\mathrm{K}_n$ has the form $e_X$ for some $X \subset \{1, \dots, n\}$. Let $x \in \mathrm{K}_n$ be an idempotent. Then $x^k = x$ for all $k \in \mathbb{N}$ and the necessary statement follows from the following lemma:

**Lemma 12.** *Let $w \in \mathrm{W}(\{a_1, \dots, a_n\})$. Then $w^k = e_{\mathfrak{c}(w)}$ for all $k \geq |\mathfrak{c}(w)|$.*

*Proof.* Set $N = |\mathfrak{c}(w)|$. Let $X \subset \{1, \dots, n\}$. From Lemma 1(i) and the definition of $e_X$ it follows that $e_X a_i = e_X$ for every $i \in X$. Hence it is enough to show that $w^N = e_{\mathfrak{c}(w)}$. For $i \in \{1, \dots, n\}$ denote by $\partial_i : \mathrm{W}(\{a_1, \dots, a_n\}) \to$

$W(\{a_1,\dots,a_{i-1},a_{i+1},\dots,a_n\})$ the operation of deleting all occurrences of the letter $a_i$ in a word. Let $\mathfrak{c}(w)=\{i_1,\dots,i_N\}$ and $i_1>i_2>\dots>i_N$. Using Lemma 1(i) we inductively compute:

$$\begin{aligned}w^N=\underbrace{www w\dots w}_{N\text{ times}}=w\partial_{i_1}(w)\partial_{i_1}(w)\partial_{i_1}(w)\dots\partial_{i_1}(w)=\\ =w\partial_{i_1}(w)\partial_{i_2}\partial_{i_1}(w)\partial_{i_2}\partial_{i_1}(w)\dots\partial_{i_2}\partial_{i_1}(w)=\dots=\\ =w\partial_{i_1}(w)\partial_{i_2}\partial_{i_1}(w)\partial_{i_2}\partial_{i_3}\partial_{i_1}(w)\dots(\partial_{i_{N-1}}\dots\partial_{i_2}\partial_{i_1})(w).\end{aligned}\tag{4.1}$$

For $j=1,\dots,N-1$ set $w_j=\partial_{i_j}\dots\partial_{i_2}\partial_{i_1}(w)$. Again, from the computation (4.1) and Lemma 1(ii) we inductively derive:

$$\begin{aligned}w^N=ww_1w_2\dots w_{N-1}=\partial_{i_N}(w)\partial_{i_N}(w_1)\partial_{i_N}(w_2)\dots\partial_{i_N}(w_{N-2})w_{N-1}=\\ =\partial_{i_{N-1}}\partial_{i_N}(w)\partial_{i_{N-1}}\partial_{i_N}(w_1)\dots\partial_{i_{N-1}}\partial_{i_N}(w_{N-3})\partial_{i_N}(w_{N-2})w_{N-1}=\dots=\\ =(\partial_{i_2}\dots\partial_{i_{N-1}}\partial_{i_N})(w)(\partial_{i_3}\dots\partial_{i_{N-1}}\partial_{i_N})(w_1)\dots\partial_{i_N}(w_{N-2})w_{N-1}.\end{aligned}\tag{4.2}$$

Now it is left to observe that

$$\mathfrak{c}((\partial_{i_2}\dots\partial_{i_{N-1}}\partial_{i_N})(w))=\{i_1\},\mathfrak{c}((\partial_{i_3}\dots\partial_{i_{N-1}}\partial_{i_N})(w_1))=\{i_2\},\dots,\\ \mathfrak{c}(w_{N-1})=\{i_N\}.$$

Hence the product in the formula (4.2) results in the product $a_{i_1}a_{i_2}\dots a_{i_N}$, which is equal to $e_{\mathfrak{c}(w)}$. Therefore $w^N=e_{\mathfrak{c}(w)}$ and the statement is proved. □

The statement of Proposition 11 follows immediately from Lemma 12. □

**Remark 13.** It is easy to see that different idempotents in $\mathrm{K}_n$ do not commute. Furthermore, the set of all idempotents in $\mathrm{K}_n$ is not a subsemigroup of $\mathrm{K}_n$, as it follows from the next statement.

**Proposition 14.** *Let $X,Y\subset\{1,2,\dots,n\}$. Then the following conditions are equivalent:*

*(a) $e_Xe_Y$ is an idempotent.*

*(b) $e_Xe_Y=e_{X\cup Y}$.*

*(c) For every $i\in X\setminus Y$ and every $j\in Y\setminus X$ we have $i>j$.*

*Proof.* The implication (b)⇒(a) is obvious. By Lemma 10 we have $\mathfrak{c}(e_Xe_Y)=X\cup Y$. At the same time $e_{X\cup Y}$ is the only idempotent of $\mathrm{K}_n$ with content $X\cup Y$. The implication (a)⇒(b) follows.

If $|X| = 0$, the implication (c)⇒(b) is trivial. Hence we may assume $|X| > 0$. We prove the implication (c)⇒(b) by induction on $|Y|$. If $|Y| = 0$, we have $e_Y = e$ and the claim is obvious. Let $|Y| > 0$ and $y$ be the minimal element of $Y$. Let $x$ be the minimal element of $X$. If $x = y$, we have

$$e_X e_Y = e_{X\setminus\{x\}} a_y e_{Y\setminus\{y\}} a_y = e_{X\setminus\{x\}} e_{Y\setminus\{y\}} a_y$$

by Lemma 1(ii). The sets $X \setminus \{x\}$ and $Y \setminus \{y\}$ still satisfy (c) and hence by induction we get

$$e_{X\setminus\{y\}} e_{Y\setminus\{y\}} a_y = e_{(X\cup Y)\setminus\{y\}} a_y = e_{X\cup Y}.$$

If $x \neq y$, then $x > y$ by (c). Hence the sets $X$ and $Y \setminus \{y\}$ satisfy (c) and hence by induction we get

$$e_X e_Y = e_X e_{Y\setminus\{y\}} a_y = e_{(X\cup Y)\setminus\{y\}} a_y = e_{X\cup Y}.$$

This proves the implication (c)⇒(b).

Finally, assume that (c) is not satisfied. Let $i \in X \setminus Y$ and $j \in Y \setminus X$ be such that $i < j$. Then the letter $a_i$ occurs in $e_X e_Y$ to the left of the letter $a_j$. Moreover, both $a_i$ and $a_j$ occur only once. Hence, applying Lemma 1 we will not be able to switch the occurrences of these letters. This and Proposition 11 imply that $e_X e_Y$ is not an idempotent. This proves the implication (a)⇒(c) and completes the proof. □

**Corollary 15.** *All maximal subgroups of* $\mathrm{K}_n$ *are trivial (that is consist of one element).*

*Proof.* Let $f \in \mathrm{K}_n$ be an idempotent and $x \in \mathrm{K}_n$ be an element, which belongs to the maximal subgroup of $\mathrm{K}_n$, corresponding to $f$. Then $x^k = f$ for some $k \in \mathbb{N}$ and $fx = x^{k+1} = x$. Now Lemma 12 implies $x = f$, completing the proof. □

**Remark 16.** The idempotent $e_{\{1,\dots,n\}}$ is the zero element of $\mathrm{K}_n$. This follows from Lemma 1.

Recall the following *natural* order on the idempotents: $f_1 \leq f_2$ if and only if $f_1 f_2 = f_2 f_1 = f_1$. We have:

**Proposition 17.** *Let* $f_1, f_2 \in \mathrm{K}_n$ *be idempotents. Then* $f_1 \leq f_2$ *if and only if* $\mathfrak{c}(f_2) \subset \mathfrak{c}(f_1)$.

*Proof.* If $\mathfrak{c}(f_2) \subset \mathfrak{c}(f_1)$ then $f_1 f_2 = f_2 f_1 = f_1$ follows from Remark 16. Assume that $f_1 f_2 = f_2 f_1 = f_1$. Then, by Lemma 10, we have $\mathfrak{c}(f_1 f_2) = \mathfrak{c}(f_1) \cup \mathfrak{c}(f_2) = \mathfrak{c}(f_1)$. Hence $\mathfrak{c}(f_2) \subset \mathfrak{c}(f_1)$. The statement is proved. □

# 5 Kiselman's linear representation of $\mathrm{K}_n$

For $i = 1, \dots, n$ denote by $A_i$ the following $(0,1)$-matrix of size $n \times n$:

$$A_i = \begin{pmatrix} 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

where the $i$-th row is zero and the $i$-th column equals $(1, \dots, 1, 0, \dots, 0)^t$ (the first $i-1$ elements are equal to 1). The following proposition is inspired by [Ki, Theorem 3.3].

**Proposition 18.** *The assignment $a_i \mapsto A_{n-i+1}$ extends uniquely to a homomorphism, $\psi_n : \mathrm{K}_n \to \mathrm{Mat}_{n\times n}(\mathbb{Z})$. Moreover, we have $\psi_n(e_{\{1,\dots,n\}}) = 0$.*

*Proof.* Because of (1.2) it is enough to check that $A_i^2 = A_i$ for all $i = 1, \dots, n$; and $A_iA_jA_i = A_jA_iA_j = A_jA_i$ for all $i, j$ such that $1 \leq i < j \leq n$. This is a straightforward calculation. That $\psi_n(e_{\{1,\dots,n\}}) = 0$ is also a straightforward calculation. □

**Remark 19.** In [Ki, Theorem 3.3] it is proved that $\psi_3$ is faithful. Unfortunately, already $\psi_4$ is not faithful. For example, both, $a_3a_4a_2a_1a_3a_2$ and $a_3a_2a_4a_3a_1a_2$, are different canonical words and hence represent different elements from $\mathrm{K}_4$. However, one easily computes that $\psi_4(a_3a_4a_2a_1a_3a_2) = \psi_4(a_3a_2a_4a_3a_1a_2)$.

# 6 (Anti)automorphisms of $\mathrm{K}_n$

**Proposition 20.** *(a) The only automorphism of $\mathrm{K}_n$ is the identity.*

*(b) The map $a_i \mapsto a_{n-i+1}$ extends uniquely to an antiautomorphism of $\mathrm{K}_n$. This is the only antiautomorphism of $\mathrm{K}_n$.*

*Proof.* Let $\sigma : \mathrm{K}_n \to \mathrm{K}_n$ be an automorphism. Obviously $\sigma(e) = e$. The map $\mathfrak{c} \circ \sigma : \mathrm{K}_n \to 2^{\{1,\dots,n\}}$ must be an epimorphism since $\mathfrak{c}$ is an epimorphism by Lemma 10. For every $i \in \{1, \dots, n\}$ the set $2^{\{1,\dots,n\}} \setminus \{\varnothing, \{i\}\}$ is closed under $\cup$, and $\mathfrak{c}^{-1}(\{i\}) = a_i$. This implies that $\sigma$ must induce a permutation on the generators $a_1, \dots, a_n$. Let us prove that $\sigma(a_i) = a_i$ by induction on

$n$. For $n = 1$ the statement is obvious. By (1.2), the letter $a_n$ may be characterized as the only letter $a_i$ among $a_1, \dots, a_n$ such that there does not exist any $a_j$, $j \neq i$, with the property $a_j a_i = a_i a_j a_i = a_j a_i a_j$. Hence $\sigma(a_n) = a_n$. In particular, $\sigma$ induces a permutation of the remaining letters $a_1, \dots, a_{n-1}$, that is an automorphism of $\mathrm{K}_{n-1}$. By the inductive assumption, this automorphism is trivial. Hence $\sigma$ is also trivial. This proves (a).

That $a_i \mapsto a_{n-i+1}$ extends uniquely to an antiautomorphism of $\mathrm{K}_n$ follows from the fact that it preserves the defining relations (1.2). That this antiautomorphism is unique is proved analogously to (a). This completes the proof. □

We will denote the unique antiautomorphism of $\mathrm{K}_n$ by $\tau$.

**Question 21.** *Is it possible to classify endomorphisms of* $\mathrm{K}_n$*?*

# 7 Green's relations on $\mathrm{K}_n$

**Theorem 22.** *Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$, $\mathcal{H}$, and $\mathcal{J}$ for $\mathrm{K}_n$ are trivial (that is all equivalence classes of these equivalence relations consist of one element each).*

To prove this theorem we will need the following notion: let $A = (a_{i,j})$ be an $n \times n$ matrix with coefficients from some ring. Define the *height* $\mathfrak{h}(A)$ of $A$ as follows:

$$\mathfrak{h}(A) = \sum_{i=1}^{n} |\{j \in \{1, \dots, n\} : a_{i,j} \neq 0\}| \cdot 2^i.$$

For $x \in \mathrm{K}_n$ we define the *height* $\mathfrak{h}(x)$ of $x$ as $\mathfrak{h}(\psi_n(x))$.

We will need the following property of the height:

**Lemma 23.** *Let $\alpha \in \mathrm{K}_n$ and $i \in \{1, \dots, n\}$ be such that $a_i\alpha \neq \alpha$. Then $\mathfrak{h}(a_i\alpha) < \mathfrak{h}(\alpha)$. In particular, if $\alpha, \beta \in \mathrm{K}_n$ are such that $\alpha\beta \neq \beta$, then $\mathfrak{h}(\alpha\beta) < \mathfrak{h}(\beta)$.*

*Proof.* By the definition of $\mathfrak{h}$ we have to show that $\mathfrak{h}(\psi_n(a_i)\psi(\alpha)) < \mathfrak{h}(\psi_n(\alpha))$. Set $j = n - i + 1$. Because of the definition of $\psi_n(a_i) = A_j$, the matrix $\psi_n(a_i)\psi_n(\alpha)$ is obtained from the matrix $\psi_n(\alpha)$ by the following sequence of elementary operations: the $j$-th row of $\psi_n(\alpha)$ is added to all rows with numbers $1, 2, \dots, j-1$, and then the $j$-th row of the resulting matrix is multiplied with 0. Let $m$ be the number of non-zero entries in the $j$-th row of $\psi_n(\alpha)$. This contributes $m2^j$ to $\mathfrak{h}(\alpha)$. Since $\psi_n(\alpha)$ has only non-negative coefficients, adding the $j$-th row of $\psi_n(\alpha)$ to the rows with numbers

$1, 2, \dots, j-1$ we can create at most $m$ new non-zero elements in all these rows. These new elements will contribute at most $m(2^{j-1}+2^{j-2}+\cdots+2^1) < m2^j$ to $\mathfrak{h}(a_i\alpha)$. Hence $\mathfrak{h}(a_i\alpha) < \mathfrak{h}(\alpha)$ and the first statement of the lemma is proved. The second statement follows immediately from the first one. □

Now we are ready to prove Theorem 22:

*Proof of Theorem 22.* Let us prove the statement for the $\mathcal{L}$ relation. Assume that $a, b \in \mathrm{K}_n$ are such that $a \neq b$ and $a\mathcal{L}b$. This means that there exists $x, y \in \mathrm{K}_n$ such that $xa = b$ and $yb = a$. Hence from Lemma 23 we obtain $\mathfrak{h}(b) = \mathfrak{h}(xa) < \mathfrak{h}(a)$ and $\mathfrak{h}(a) = \mathfrak{h}(yb) < \mathfrak{h}(b)$. This implies $\mathfrak{h}(a) < \mathfrak{h}(a)$, a contradiction. Therefore, every $\mathcal{L}$-class consists of exactly one element and thus $\mathcal{L}$ is trivial.

Since the relation $\mathcal{L}$ is trivial, applying $\tau$ we obtain that the relation $\mathcal{R}$ is trivial as well. From the definition of $\mathcal{H}$ and $\mathcal{D}$ it then follows that both $\mathcal{H}$ and $\mathcal{D}$ are trivial. Since $\mathrm{K}_n$ is finite, we have $\mathcal{D} = \mathcal{J}$, completing the proof. □

**Remark 24.** The statement of Theorem 22 was announced in [Go, Theorem 3].

# 8 Maximal nilpotent subsemigroups of $\mathrm{K}_n$

Recall that a semigroup, $S$, with the zero element 0 is called *nilpotent* provided that there exists $k \in \mathbb{N}$ such that $S^k = \{0\}$. The minimal possible $k$ with this property is called the *nilpotency class* of $S$. For every $X \subset \{1, \dots, n\}$ denote by $\mathrm{Nil}(X)$ the set $\{w \in \mathrm{K}_n | \mathfrak{c}(w) = X\}$.

**Theorem 25.** *(i) For each $X \subset \{1, \dots, n\}$ the set $\mathrm{Nil}(X)$ is a maximal nilpotent subsemigroup of $\mathrm{K}_n$ (with the zero element $e_X$). $\mathrm{Nil}(X)$ has nilpotency class $|X|$ if $|X| > 0$, and nilpotency class 1 if $|X| = 0$.*

*(ii) Every maximal nilpotent subsemigroup of $\mathrm{K}_n$ has the form $\mathrm{Nil}(X)$ for some $X \subset \{1, \dots, n\}$.*

*(iii) We have the following decomposition into a disjoint union of maximal nilpotent subsemigroups: $\mathrm{K}_n = \cup_{X \subset \{1,\dots,n\}} \mathrm{Nil}(X)$.*

*Proof.* That $\mathrm{Nil}(X)$ is a subsemigroup of $\mathrm{K}_n$ follows from Lemma 10. That $e_X$ is the zero element of $\mathrm{Nil}(X)$ and the only idempotent of $\mathrm{Nil}(X)$ follows from Lemma 12. Hence $\mathrm{Nil}(X)$ is a nilpotent semigroup by [Ar, Fact2.30, page 179]. If $w \in \mathrm{K}_n \setminus \mathrm{Nil}(X)$, then $w^{|\mathfrak{c}(w)|}$ is an idempotent, different from $e_X$. This means that the semigroup, generated by $\mathrm{Nil}(X)$ and such $w$, can not be nilpotent. That $\mathrm{Nil}(\{\varnothing\}) = \{e\}$ has nilpotency class 1 is obvious. Let $X \neq \varnothing$. The same arguments as the ones used in Lemma 12 prove that the nilpotency class of $\mathrm{Nil}(X)$ is at most $|X|$. Let $X = \{a_{i_1}, \dots, a_{i_k}\}$ and $i_1 < i_2 < \cdots < i_k$.

**Lemma 26.** *The element $w = a_{i_1} a_{i_2} \cdots a_{i_k}$ has order $k$.*

*Proof.* From Lemma 12 we have that the order of $w$ is at most $k$, so we have to prove that $w^l$ is not an idempotent for any $l < k$. Observe that, obviously, the subsemigroup of $\mathrm{K}_n$, generated by $a_{i_1}, a_{i_2}, \dots, a_{i_k}$ is isomorphic to $\mathrm{K}_k$ via $a_{i_j} \mapsto a_j$. Hence, without loss of generality, we may assume $X = \{1, \dots, n\}$.

By a direct calculation we have that the matrix $\psi_n(a_1 a_2 \cdots a_n)$ is an upper triangular matrix with zero diagonal, whose all element above the diagonal equal 1. Hence $\psi_n(a_1 a_2 \cdots a_n)$ is nilpotent of nilpotency class exactly $n$. The claim follows. $\square$

From Lemma 26 we obtain that the nilpotency class of $\mathrm{Nil}(X)$ is exactly $|X|$. This proves (i).

Let $S$ be a maximal nilpotent subsemigroup of $\mathrm{K}_n$ and $f \in S$ be the corresponding zero element. Then $f = e_X$ for some $X \subset \{1, \dots, n\}$ by Proposition 11. Since for every element $x$ from $S$ we then should have $x^k = e_X$ for some $k$, from Lemma 12 we obtain $S \subset \mathrm{Nil}(X)$, Now (ii) follows from (i). The statement (iii) is now obvious. $\square$

# 9 Isolated and completely isolated subsemigroups of $\mathrm{K}_n$

Let $S$ be a semigroup. Recall that a subsemigroup, $T \subset S$, is called *isolated* provided that for all $x \in S$ the inclusion $x^l \in T$ for some $l \in \mathbb{N}$ implies $x \in T$. A subsemigroup, $T \subset S$, is called *completely isolated* provided that $xy \in T$ implies $x \in T$ or $y \in T$ for all $x, y \in S$.

**Proposition 27.** *(i) The map $\mathfrak{c}$ induces a bijection between isolated subsemigroups of $\mathrm{K}_n$ and subsemigroups of $(2^{\{1,\dots,n\}}, \cup)$. In particular, the minimal isolated subsemigroups of $\mathrm{K}_n$ are $\mathrm{Nil}(X)$, $X \subset \{1, \dots, n\}$.*

*(ii) The map $\mathfrak{c}$ induces a bijection between completely isolated subsemigroups of $\mathrm{K}_n$ and completely isolated subsemigroups of $(2^{\{1,\dots,n\}}, \cup)$.*

*Proof.* Let $S$ be an isolated subsemigroup of $\mathrm{K}_n$. Then $\mathfrak{c}(S) = T$ is a subsemigroup of $(2^{\{1,\dots,n\}}, \cup)$, which is obviously isolated since $(2^{\{1,\dots,n\}}, \cup)$ consists of idempotents. That $S = \mathfrak{c}^{-1}(T)$ follows from [MT, Proposition 4]. On the other hand, for any subsemigroup $T$ of $(2^{\{1,\dots,n\}},\cup)$ the set $\mathfrak{c}^{-1}(T)$ is a subsemigroup of $\mathrm{K}_n$ and hence is isolated since $T$ is isolated. This proves (i). (ii) follows easily from (i). $\square$

# 10 Deletion properties

In this section we establish two combinatorial properties of $\mathrm{K}_n$, which will be used later on during the study of linear representations of $\mathrm{K}_n$. However, we think that these properties are rather remarkable and interesting on their own.

To simplify the notation we set $f = e_{\{2,3,\dots,n\}}$. Our *first deletion property* is the following statement:

**Proposition 28.** *Let $v, w \in \mathrm{W}(\{a_2,\dots,a_n\})$ be canonical and different. Then $va_1f \neq wa_1f$.*

*Proof.* Take the word $va_1f \in \mathrm{W}(\{a_2,\dots,a_n\})$. This word does not have to be canonical. However, we can use Lemma 1 (maybe several times) to reduce it to the unique canonical form given by Theorem 6. Since $v$ is assumed to be canonical, on the first step we can apply Lemma 1 only to some subword, $a_i\alpha a_i$, of $va_1f$, where the left $a_i$ is a letter of $v$ and the right $a_i$ is a letter of $f$. This means that $a_1$ is a letter of $\alpha$, and therefore only Lemma 1(i) can be applied. Thus the new word will have the form $va_1\beta$, where $\beta$ is obtained from $f$ by the deletion of one of the letters. The main point is that the left-hand side $v$ remains the same. Now, applying the same argument inductively, we obtain that the canonical form of $va_1f$ will by $va_1\gamma$, where $\gamma$ is a quasi-subword of $f$.

The same argument shows that the canonical form of $wa_1f$ will have the form $wa_1\gamma'$, where $\gamma'$ is a quasi-subword of $f$. Since $a_1$ does not occur in both $v$ and $w$ by assumption, and $v \neq w$, we obtain that $va_1\gamma \neq wa_1\gamma'$. The statement now follows from Theorem 6. $\square$

The *second deletion property* is the following more tricky statement (and is perhaps the deepest result of our paper):

**Proposition 29.** *Let $w, v, u \in \mathrm{W}(\{a_2,\dots,a_n\})$ be canonical. Assume that $v \neq u$ and both $wa_1v$ and $wa_1u$ are canonical. Then $wv \neq wu$, $wva_1 \neq wua_1$ and $wva_1f \neq wua_1f$.*

*Proof.* We first prove that $wv \neq wu$. Assume this is not the case, that is assume that $wv = wu$. To proceed we will need some preparation.

**Lemma 30.** *Let $\alpha, \beta \in \mathrm{W}(\{a_2, \dots, a_n\})$ be canonical and assume that $\alpha a_1 \beta$ is canonical as well. Then the canonical form of $\alpha\beta$ is obtained from $\alpha\beta$ by deleting some letters of the word $\alpha$ using Lemma 1*(ii)*. Moreover, the reduction process can be organized such that on every step the new letter which we delete is placed to the left with respect to the letter, deleted on the previous step.*

*Proof.* We proceed inductively on the number of deletions. Assume that $a_i\gamma a_i$ is a subword of $\alpha\beta$, to which we can apply Lemma 1. Since $\alpha a_1\beta$ was canonical, we obtain that $a_i\gamma a_i = a_i\gamma'\gamma'' a_i$, where $a_i\gamma'$ is a suffix of $\alpha$ and $\gamma'' a_i$ is a prefix of $\beta$. Since $a_i\gamma' a_1\gamma'' a_i$, as a subword of a canonical word, was canonical itself, the word $\gamma'\gamma''$ must contain some $a_j$ with $j > i$. Hence we can only apply Lemma 1(ii) to $a_i\gamma a_i$ and thus have to delete some letter from $\alpha$. We can of course always start with the rightmost letter of $\alpha$, which can be deleted.

Since we delete the rightmost possible letter, the rest of the word, which is to the right of this letter, has to be canonical. This part is not affected by our deletion, so it remains canonical. On the other hand, since we have used Lemma 1(ii), the right neighbor of our letter should have bigger index. So, if our deletion creates possibilities for new deletions, for these new possibilities we can only use Lemma 1(ii) (this is the same argument as in the previous paragraph). In particular, it follows that new letters which can be deleted can appear only to the left. Moreover, the same argument as above shows that if our deletion creates some new letters which can be deleted, it is again only Lemma 1(ii) which can be used. Therefore, we can again always choose the new rightmost letter and proceed inductively, completing the proof. □

From Lemma 30 we obtain that the canonical form $\mathrm{can}(wu)$ is obtained from $wu$ by deleting some letters from $w$, and the canonical form $\mathrm{can}(wv)$ is obtained from $wv$ by deleting some letters from $w$. In particular, $wu = wv$ implies $\mathrm{can}(wu) = \mathrm{can}(wv)$. Without loss of generality we may assume $\mathfrak{l}(u) \leq \mathfrak{l}(v)$. Then the above observations imply that $v = u'u$ (as a word) for some word $u'$. In particular, if $\mathfrak{l}(u) = \mathfrak{l}(v)$, we already get a contradiction, proving that $wv \neq wu$ in this case.

Hence now we can assume that $\mathfrak{l}(u) < \mathfrak{l}(v)$ and that $v = u'u$ for some non-empty word $u'$. Now we are going to make some analysis of $wu$ and $wv$, which we tried to illustrate on Figure 1. It will be convenient for us to distinguish the *symbols* $\{a_1, \dots, a_n\}$ of our alphabet from the *letters* of a given word (this word will, in fact, be the word $w$). So, in the rest of the proof by a *letter* of some word we will mean a symbol of the alphabet together with the position in the word (so different letters can correspond

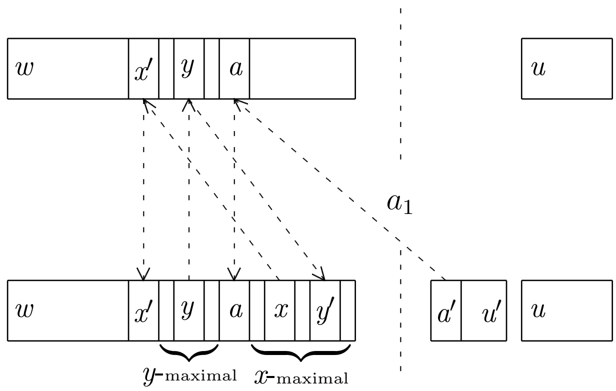


Figure 1: Analysis in the proof of Proposition 29.

to the same symbol). For example, the word $a_1a_2a_3a_1$ is written using only three different symbols, but it contains four different letters (the first letter is the symbol $a_1$ staying in position one and the fourth letter is the the symbol $a_1$ staying in position four). We will use $a, x, y, v$ to denote the letters of the words we will work with.

Let $a'$ be the leftmost letter of the non-empty word $u'$. Let $a_i$ be the corresponding symbol. By Lemma 30, the letter $a'$ survives in $\mathrm{can}(wv)$. Since $\mathfrak{l}(u) \leq \mathfrak{l}(v)$, the corresponding letter of $\mathrm{can}(wu) = \mathrm{can}(wv)$ comes from $w$, say from some letter $a$ (this should be one of the occurrences of $a_i$ in $w$). Since $wa_1v$ was canonical and $a$ is the leftmost letter of $v$, there should exist a symbol, $a_j$, in $w$ to the right of $a$ such that $j > i$. We can choose the maximal possible $j$ and let $x$ be the rightmost occurrence of $a_j$ in $w$ to the right of our letter $a$. All letters in $w$ to the right of $x$ (if any) have smaller indicies. In $wv$ these letters are followed by $a'$, which also has smaller index. Hence it is not possible to delete this $x$ using Lemma 1(ii). From Lemma 30 we obtain that $x$ survives in $\mathrm{can}(wv)$.

Since $\mathrm{can}(wv) = \mathrm{can}(wu)$, the letter $x$ forces the existence of some letter $x'$ (representing the same symbol $a_j$ as the letter $x$) to the left of $a$, which survives in $\mathrm{can}(wu)$ and corresponds there to the letter $x$ in $\mathrm{can}(wv)$. Since $w$ was canonical, between $x'$ and $x$ in $w$ there should exist some symbol $a_k$ such that $k > j$. Since $j$ is the maximal possible index to the right of $a$, this symbol $a_k$ appears in $w$ between $x'$ and $a$. We again take $k$ the maximal possible and let $y$ be the rightmost occurrence of $a_k$ between $x'$ and $a$. Then, by definion, $k$ is bigger than the index of all other symbols in $w$ to the right of $y$. The letter $a$ survives in $\mathrm{can}(wu)$, which implies that one can not use Lemma 1(ii) to delete $y$ in $wu$. Hence $y$ survives in $\mathrm{can}(wu)$ between $x'$ and $a$.

Since $\mathrm{can}(wv) = \mathrm{can}(wu)$, this $y$ should correspond to some occurrence of $a_k$ to the right of $x$. However, this contradicts to the choice of $x$, which was supposed to have the maximal possible index in $w$ to the right of $a$. The obtained contradiction proves that $wv = wu$ is not possible, that is the first inequality of our statement.

Since $wv \neq wu$, the canonical forms $\alpha$ and $\beta$ of $wv$ and $wu$ respectively are different. As $wv, wu \in \mathrm{W}(\{a_2, \dots, a_n\})$ we obtain that $\alpha a_1$ and $\beta a_1$ are both canonical and hence different. This proves the inequality $wva_1 \neq wua_1$. From Proposition 28 we also obtain $\alpha a_1 f \neq \beta a_1 f$, which proves the inequality $wva_1 f \neq wua_1 f$. This completes the proof. □

# 11 Linear representations of $\mathrm{K}_n$

For a commutative ring, $\mathbf{R}$, we denote by $\mathbf{R}\mathrm{K}_n$ the semigroup algebra of $\mathrm{K}_n$ over $\mathbf{R}$ and by $\overline{\mathbf{R}\mathrm{K}}_n$ the quotient of $\mathbf{R}\mathrm{K}_n$ modulo the ideal, generated by the zero element $e_{\{1,2,\dots,n\}}$.

## 11.1 Faithful representations of $\mathrm{K}_n$

We start with the following observation:

**Proposition 31.** *Let $\rho$ be a faithful linear representation of $\mathrm{K}_n$ over some field. Then $\dim \rho \geq n$.*

*Proof.* In the proof of Theorem 25 we saw that the element $a_1 a_2 \cdots a_n$ is a nilpotent element of nilpotency class exactly $n$. Since $e_{\{1,2,\dots,n\}}$ is the zero element in $\mathrm{K}_n$, factoring, if necessary, the image of $\rho(e_{\{1,2,\dots,n\}})$ out, we may assume that $\rho(e_{\{1,2,\dots,n\}}) = 0$. If $\rho$ is faithful, the matrix $\rho(a_1 \cdots a_n)$ must then be a nilpotent matrix of nilpotency class exactly $n$. Obviously, such matrix exists only if $\dim \rho \geq n$. □

As we have already mentioned in Remark 19, Kiselman's representation of $\mathrm{K}_n$ is not faithful for $n = 4$ (and hence for all $n > 4$ either). Let now $\mathbb{K}$ be a field. From Proposition 18 we have $\psi_n(e_{\{1,2,\dots,n\}}) = 0$ and hence $\psi_n$ is a representation of $\overline{\mathbb{K}\mathrm{K}}_n$ as well. We continue with the following observation about faithfulness:

**Proposition 32.** *The indecomposable projective cover of Kiselman's representation of $\overline{\mathbb{K}\mathrm{K}}_n$ in $\mathbb{K}^n$ is faithful as a representation of $\mathrm{K}_n$.*

*Proof.* Set $\pi_1 = e - a_n \in \mathbb{K}\mathrm{K}_n$, $\pi_2 = a_n - a_n a_{n-1} \in \mathbb{K}\mathrm{K}_n, \dots, \pi_{n-1} = a_n a_{n-1} \cdots a_3 - a_n a_{n-1} \cdots a_2 \in \mathbb{K}\mathrm{K}_n$, $\pi_n = a_n a_{n-1} \cdots a_2$. By a direct calculation using the formulae from Section 5 one obtains that for $i = 1, \dots, n$ the matrix $\psi_n(\pi_i)$ is the diagonal matrix $D_i$, whose diagonal is the vector $(0, \dots, 0, 1, 0, \dots, 0)$, where the element 1 stays on the $i$-th place.

First we claim that the vector $v = (0, 0, \dots, 0, 1)^t$ generates Kiselman's representation. Indeed, $A_1 v = (1, 1, \dots, 1, 0)^t$ and hence, acting on $A_1 v$ by $D_i$, $i = 1, \dots, n-1$, we produce all elements from the standard basis of $\mathbb{K}^n$.

From Proposition 11 we know that $\pi_n = e_{\{2,3,\dots,n\}}$ is an idempotent. Furthermore, $\psi_n(\pi_n)v = v$ and hence $\overline{\mathbb{K}\mathrm{K}}_n \pi_n$ is a projective cover of Kiselman's representation.

Every element of $\mathrm{K}_n$ can be written as either $w$ or $w a_1 v$, where $w, v \in \mathrm{W}(\{a_2, \dots, a_n\})$. From Remark 16 it follows that $\pi_n w = w \pi_n = \pi_n v = v \pi_n = \pi_n$. Hence for any $\alpha \in \mathrm{K}_n$ we have

$$\pi_n \alpha \pi_n = \begin{cases} \pi_n, & a_1 \text{ is not a letter of } \alpha; \\ e_{\{1,2,\dots,n\}}, & \text{otherwise.} \end{cases}$$

Hence $\pi_n \mathbb{K}\mathrm{K}_n \pi_n$ has dimension two and a monomial basis, consisting of $\pi_n$ and $e_{\{1,2,\dots,n\}}$. Factoring out the zero element $e_{\{1,2,\dots,n\}}$ we get a copy of the ground field since $\pi_n$ is an idempotent. Thus $\pi_n \overline{\mathbb{K}\mathrm{K}}_n \pi_n$ is a local algebra. Hence $\pi_n$ is a primitive idempotent of $\overline{\mathbb{K}\mathrm{K}}_n$, which implies that the $\overline{\mathbb{K}\mathrm{K}}_n$-module $\overline{\mathbb{K}\mathrm{K}}_n \pi_n$ is indecomposable.

To complete the proof we have just to show that the corresponding representation of $\mathrm{K}_n$ is faithful. By definition, the module $\overline{\mathbb{K}\mathrm{K}}_n \pi_n$ has a monomial basis, which consists of all non-zero elements from the left principal ideal of $\mathrm{K}_n$, generated by $\pi_n$. In particular, we have the basis elements $\pi_n$ and $a_1 \pi_n$ (note that $a_1 \pi_n$ is a canonical word).

If $w, v \in \mathrm{W}(\{a_2, \dots, a_n\})$ are different and canonical, then $w a_1 \pi_n \neq v a_1 \pi_n$ by Proposition 28. The elements $w a_1 \pi_n$ and $v a_1 \pi_n$ are linearly independent in $\mathbb{K}\mathrm{K}_n \pi_n$, in particular, they are different. Therefore the elements $w$ and $v$ from $\mathrm{K}_n$ are represented by different linear operators on $\mathbb{K}\mathrm{K}_n \pi_n$.

If $u, v, w \in \mathrm{W}(\{a_2, \dots, a_n\})$ are canonical, then $u \pi_n = \pi_n$ and $v a_1 w \pi_n = v a_1 \pi_n \neq \pi_n$. Hence the elements $u$ and $v a_1 w$ from $\mathrm{K}_n$ are represented by different linear operators on $\overline{\mathbb{K}\mathrm{K}}_n \pi_n$.

Let $w_1 a_1 v_1$ and $w_2 a_1 v_2$ be two different elements from $\mathrm{K}_n$, written in the canonical form. In particular, $w_1, w_2, v_1, v_2 \in \mathrm{W}(\{a_2, \dots, a_n\})$ and are canonical. If $w_1 \neq w_2$, we have $w_1 a_1 v_1 \pi_n = w_1 a_1 \pi_n$ and $w_2 a_1 v_2 \pi_n = w_2 a_1 \pi_n$ (since $\pi_n$ is the zero element with respect to $a_j$, $j > 1$). Moreover, from Proposition 28 we get $w_1 a_1 \pi_n \neq w_2 a_1 \pi_n$. Both $w_1 a_1 \pi_n$ and $w_2 a_1 \pi_n$ are basis elements of $\overline{\mathbb{K}\mathrm{K}}_n\pi_n$, which implies that the elements $w_1a_1v_1$ and $w_2a_1v_2$ are represented by different linear operators on $\overline{\mathbb{K}\mathrm{K}}_n\pi_n$.

Assume now that $w_1 = w_2 = w$. Then $v_1 \neq v_2$ and we have $w_1a_1v_1a_1\pi_n = w_1v_1a_1\pi_n$ and $w_2a_1v_2a_1\pi_n = w_2v_2a_1\pi_n$ using Lemma 1(ii). From Proposition 29 we get $w_1v_1a_1\pi_n \neq w_2v_2a_1\pi_n$. Both $w_1v_1a_1\pi_n$ and $w_2v_2a_1\pi_n$ are basis elements of $\overline{\mathbb{K}\mathrm{K}}_n\pi_n$, which implies that the elements $w_1a_1v_1$ and $w_2a_1v_2$ are represented by different linear operators on $\overline{\mathbb{K}\mathrm{K}}_n\pi_n$. Hence the representation of $\mathrm{K}_n$ on $\overline{\mathbb{K}\mathrm{K}}_n\pi_n$ is faithful. $\square$

The ideas from the proof of Proposition 32 can be used to construct a huge family of faithful $n$-dimensional representations of $\mathrm{K}_n$. Consider the polynomial ring $\mathbb{Z}[\xi_{i,j} : 1 \leq i < j \leq n]$. Define the following representation of $\mathrm{K}_n$ by $n \times n$-matrices over $\mathbb{Z}[\xi_{i,j} : 1 \leq i < j \leq n]$:

$$\kappa_n : a_{n-i+1} \mapsto \begin{pmatrix} 1 & 0 & \dots & 0 & \xi_{1,i} & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \xi_{2,i} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \xi_{i-1,i} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

where the $i$-th row is zero and the $i$-th column equals $(\xi_{1,i}, \dots, \xi_{i-1,i}, 0, \dots, 0)^t$.

**Proposition 33.** *The representation $\kappa_n$ is faithful.*

*Proof.* We proceed by induction on $n$. For $n = 1, 2$ the statement is easily checked by a direct calculation.

Let $w, u \in \mathrm{W}(\{a_2, \dots, a_n\})$ be different and canonical. The semigroup generated by $a_2, \dots, a_n$ is obviously isomorphic to $\mathrm{K}_{n-1}$ under the map $a_i \mapsto a_{i-1}$. Let us denote this isomorphism by $F$. Then the first $n - 1$ rows and the first $n - 1$ columns of $\kappa_n(w)$ and $\kappa_n(u)$ are exactly the matrices $\kappa_{n-1}(F(w))$ and $\kappa_{n-1}(F(u))$ respectively. By induction we have $\kappa_{n-1}(F(w)) \neq \kappa_{n-1}(F(u))$ and hence $\kappa_n(w) \neq \kappa_n(u)$.

Let $u, v, w \in \mathrm{W}(\{a_2, \dots, a_n\})$ be canonical. Then the last diagonal element of $u$ is 1 while the last diagonal element of $va_1w$ is 0. Hence $\kappa_n(u) \neq \kappa_n(va_1w)$.

Let $w_1, w_2, v_1, v_2 \in \mathrm{W}(\{a_2, \dots, a_n\})$ be canonical. Assume that $w_1 \neq w_2$ and that $w_1a_1v_1$ and $w_2a_1v_2$ are also canonical. Recall that $\pi_n = a_n \cdots a_2$. As in the proof of Proposition 32 we have $w_1a_1v_1\pi_n = w_1a_1\pi_n$ and $w_2a_1v_2\pi_n =$

$w_2 a_1 \pi_n$. Further

$$\kappa_n(a_1\pi_n) =: \begin{pmatrix} 0 & 0 & \dots & 0 & \xi_{1,n} \\ 0 & 0 & \dots & 0 & \xi_{2,n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \xi_{n-1,n} \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Since $w_1 \neq w_2$, by induction we, similarly to the arguments above, derive that the matrices $M_1$ and $M_2$, formed by the first $n-1$ rows and the first $n-1$ columns of the matrices $\kappa_n(w_1)$ and $\kappa_n(w_2)$ respectively, are different. Since $w_1, w_2 \in \mathrm{W}(\{a_2, \dots, a_n\})$, the coefficients of these matrices do not contain $\xi_{i,n}$ for all $i$. Now observe that $\xi_{1,n}, \dots, \xi_{n-1,n}$ are linearly independent (over $\mathbb{Z}[\xi_{i,j} : 1 \leq i < j \leq n-1]$) elements of $\mathbf{R}$. From the definition of the matrix multiplication we get that the last columns in the matrices $\kappa_n(w_1 a_1 \pi_n)$ and $\kappa_n(w_2 a_1 \pi_n)$ will be different. Hence $\kappa_n(w_1 a_1 v_1 \pi_n) \neq \kappa_n(w_2 a_1 v_1 \pi_n)$ and therefore $\kappa_n(w_1 a_1 v_1) \neq \kappa_n(w_2 a_1 v_1)$.

Finally, let us assume that $w, u, v \in \mathrm{W}(\{a_2, \dots, a_n\})$ are canonical and such that $wa_1u$ and $wa_1v$ are canonical and different. By Lemma 1(ii) we have $wa_1ua_1\pi_n = wua_1\pi_n$ and $wa_1va_1\pi_n = wva_1\pi_n$. Moreover, from Proposition 33 we have $wu \neq wv$. The same arguments as in the previous paragraph show that the last columns in the matrices $\kappa_n(wua_1\pi_n)$ and $\kappa_n(wua_1\pi_n)$ will be different. Hence $\kappa_n(wa_1u) \neq \kappa_n(wa_1v)$. This completes the proof. □

As an immediate corollary we obtain the following statement, which, together with Proposition 31, was announced in [Go, Theorem 4]:

**Theorem 34.** $\mathrm{K}_n$ *has a faithful representation by* $n \times n$ *matrices with non-negative integer coefficients.*

*Proof.* By Proposition 33, the representation $\kappa_n$ is faithful. For every pair $\{\alpha, \beta\}$ of different elements from $\mathrm{K}_n$ we have $\kappa_n(\alpha) \neq \kappa_n(\beta)$, hence there exist $i_{\{\alpha,\beta\}}$ and $j_{\{\alpha,\beta\}}$ such that the $(i_{\{\alpha,\beta\}}, j_{\{\alpha,\beta\}})$-entry of $\kappa_n(\alpha)$ is different from the $(i_{\{\alpha,\beta\}}, j_{\{\alpha,\beta\}})$-entry of $\kappa_n(\beta)$. These entries are polynomials with integer coefficients, so this condition can be written as the condition "some non-zero polynomial in $\xi_{i,j}$ is not equal to zero". Since $\mathrm{K}_n$ is finite by Theorem 3, the faithfullness of $\kappa_n$ gives us a finite number of polynomial inequalities. Since the set $\mathbb{N}^{n(n-1)/2}$ is Zariski dense in $\mathbb{Q}^{n(n-1)/2}$, we will get that there are infinitely many collections of $n_{i,j} \in \mathbb{N}$, $1 \leq i < j \leq n$, such that after the evaluation $\xi_{i,j} \to n_{i,j}$ all our inequalities are still satisfied. This means that there are infinitely many collections of $n_{i,j} \in \mathbb{N}$, $1 \leq i < j \leq n$, such that after the evaluation $\xi_{i,j} \to n_{i,j}$ we obtain a faithful representation of $\mathrm{K}_n$ with non-negative integer coefficients. This completes the proof. □

Following the proof of Proposition 33 one can in fact explicitly present a collection of $n_{i,j}$, such that after the evaluation $\xi_{i,j} \to n_{i,j}$ one obtains a faithful representation of $\mathrm{K}_n$ with non-negative integer coefficients. Define two sequences, $\mathrm{m}_i$ and $\mathrm{l}_i$, $i \geq 1$, recursively as follows: $\mathrm{m}_1 = \mathrm{l}_1 = 1$, $\mathrm{m}_i = \mathrm{l}_{i-1} + 1$, $\mathrm{l}_i = i^{2^i} \mathrm{m}_i^{i2^i}$, $i \geq 2$.

**Proposition 35.** *Denote by $\kappa'_n$ the representation of $\mathrm{K}_n$ with non-negative integer coefficients, obtained from $\kappa_n$ via the evaluation $\xi_{i,j} \to \mathrm{m}_j^i$.*

*(i) $\kappa'_n$ is faithfull.*

*(ii) For every $w \in \mathrm{K}_n$ each entry of the matrix $\kappa'_n(w)$ is smaller than $\mathrm{l}_n$.*

*Proof.* We prove this by the simultaneous induction on $n$. For $n = 2$ both statements are easily checked by a direct calculation. Since $\mathrm{m}_n^i > \mathrm{l}_j$ for all $i \geq 1$ and $j < n$ by construction, the maximal possible entry appearing in the matrix $\kappa'_n(a_i)$, $i \leq n$, is $\mathrm{m}_n^{n-1} < \mathrm{m}_n^n$. From Corollary 2(iii) it follows that every element from $\mathrm{K}_n$ can be written as a product of at most $2^n$ generators. It is easy to see that then the maximal possible entry of such product is smaller than $n^{2^n}(\mathrm{m}_n^n)^{2^n}$. The induction step for (ii) is now completed by comparing this with the definition of $\mathrm{l}_n$.

To prove (i) we just follow the proof of Proposition 33. It is easy to see that the only thing we have to verify is that, given two different matrices $\kappa'_{n-1}(F(w))$ and $\kappa'_{n-1}(F(v))$, the rightmost columns of the matrices $\kappa_n(wa_1\pi_n)$ and $\kappa_n(va_1\pi_n)$ are different. These columns are linear combinations of $\mathrm{m}_n^i$, $i = 1, \dots, n-1$ with coefficients from the matrices $\kappa'_{n-1}(F(w))$ and $\kappa'_{n-1}(F(u))$. By induction, all such coefficients do not exceed $\mathrm{l}_{n-1}$, which is strictly smaller than $\mathrm{m}_n$ by definition. It follows that two such linear combinations with different collections of such coefficients will be different. This completes the proof. $\square$

## 11.2 Irreducible representations and the structure of $\mathbb{K}\mathrm{K}_n$

Let $\mathbb{K}$ be a field. For any $X \subset \{1, 2, \dots, n\}$ we define the map $\rho_X : \mathrm{K}_n \to \mathbb{K}$ as follows:

$$\rho_X(w) = \begin{cases} 1, & \mathfrak{c}(w) \subset X; \\ 0, & \text{otherwise.} \end{cases}$$

**Proposition 36.** *(i) For any $X \subset \{1, 2, \dots, n\}$ the map $\rho_X$ gives an irreducible representation of $\mathbb{K}\mathrm{K}_n$.*

*(ii) Representations $\rho_X$, $X \subset \{1, 2, \dots, n\}$, are pairwise non-equivalent and constitute an exhaustive list of irreducible representations of $\mathbb{K}\mathrm{K}_n$. In particular, $\mathbb{K}\mathrm{K}_n$ has $2^n$ non-equivalent irreducible representations.*

*(iii) $\rho_X$ is a representation of $\overline{\mathbb{K}\mathrm{K}_n}$ if and only if $X \neq \{1, 2, \dots, n\}$. In particular, $\overline{\mathbb{K}\mathrm{K}_n}$ has $2^n - 1$ non-equivalent irreducible representations.*

*Proof.* Fix $X \subset \{1, 2, \dots, n\}$. For $i \in \{1, \dots, n\}$ define $\overline{\rho}_X(a_i)$ to be 1 if $i \in X$ and 0 otherwise. It is straightforward to check that this assignment satisfies the defining relations (1.2) of $\mathrm{K}_n$. Hence it extends uniquely to a representation of $\mathrm{K}_n$. From the definition of $\mathfrak{c}$ one immediately obtains that this extension is the map $\rho_X$. The representation $\rho_X$ is irreducible since it is one-dimensional. This proves (i).

Let $X$ and $Y$ be different subsets of $\{1, \dots, n\}$. Withour loss of generality we may assume that $X \setminus Y \neq \varnothing$. Let $i \in X \setminus Y$. Then $\rho_X(a_i) = 1$ and $\rho_Y(a_i) = 0$. Hence $\rho_X$ and $\rho_Y$ are not equivalent. In particular, we have $2^n$ non-equivalent irreducible representations of $\mathbb{K}\mathrm{K}_n$. However, from Proposition 11 we know that $\mathrm{K}_n$ has $2^n$ idempotents, and from Theorem 22 we know that all Green's relations on $\mathrm{K}_n$ are trivial. Hence, Munn's Theorem (see for example [CP, Theorem 5.33]) gives us that $\mathbb{K}\mathrm{K}_n$ has exactly $2^n$ non-equivalent irreducible representations. This proves (ii). (iii) follows immediately from (i), (ii) and a direct calculation. This completes the proof. □

**Corollary 37.** *The algebra $\mathbb{K}\mathrm{K}_n$ is basic.*

*Proof.* From Proposition 36(ii) we have that all simple $\mathbb{K}\mathrm{K}_n$-modules are one-dimensional. This implies the statement. □

Since we now know all irreducible representations of $\mathbb{K}\mathrm{K}_n$, it is a natural question to determine the decomposition of the regular module into a direct sum of indecomposable projectives, that is to find a decomposition of the unit element of $\mathbb{K}\mathrm{K}_n$ into a direct sum of pairwise orthogonal primitive idempotents.

Let $X \subset \{1, \dots, n\}$. Assume that $X = \{i_1, \dots, i_s\}$, where $i_1 > i_2 > \dots > i_s$; and $\{1, \dots, n\} \setminus X = \{j_1, \dots, j_t\}$, where $j_1 < j_2 < \dots < j_t$. Set

$$e_X^{(n)} = a_{i_1} a_{i_2} \cdots a_{i_s} (e - a_{j_1})(e - a_{j_2}) \cdots (e - a_{j_t}) \in \mathbb{K}\mathrm{K}_n.$$

**Proposition 38.** *(i)*

$$\begin{aligned}\{e_X^{(n)} : X \subset \{1, \dots, n\}\} = a_n \{e_Y^{(n-1)} : Y \subset \{1, \dots, n-1\}\} \cup \\ \cup \{e_Y^{(n-1)} : Y \subset \{1, \dots, n-1\}\}(e - a_n).\end{aligned}$$

*(ii)* *For every* $X \subset \{1,\dots,n\}$ *the element* $e_X^{(n)}$ *is a primitive idempotent of* $\mathbb{K}\mathrm{K}_n$.

*(iii)* $e_X^{(n)} e_Y^{(n)} = 0$ *if* $X \neq Y$.

*(iv)* $e = \sum_{X\subset\{1,\dots,n\}} e_X^{(n)}$.

*Proof.* If $n \in X$, from the definition of $e_X^{(n)}$ we have $e_X^{(n)} = a_n e_{X\setminus\{n\}}^{(n-1)}$. If $n \notin X$, from the definition of $e_X^{(n)}$ we have $e_X^{(n)} = e_X^{(n-1)}(e - a_n)$. This proves (i).

Now we prove the rest by a simultaneous induction on $n$. For $n = 1$ the statements (ii), (iii) and (iv) are obvious.

Let $Y \subset \{1,\dots,n-1\}$. Then

$$\begin{array}{lcl} a_n e_Y^{(n-1)} a_n e_Y^{(n-1)} & = & \text{(by Lemma 1(i))} \\ a_n e_Y^{(n-1)} e_Y^{(n-1)} & = & \text{(by inductive assumption)} \\ a_n e_Y^{(n-1)}. & & \end{array}$$

Analogously, using Lemma 1(i) and the inductive assumption, we have

$$\begin{array}{lr} e_Y^{(n-1)}(e-a_n) e_Y^{(n-1)}(e-a_n) & = \\ e_Y^{(n-1)} e_Y^{(n-1)} - e_Y^{(n-1)} a_n e_Y^{(n-1)} - e_Y^{(n-1)} e_Y^{(n-1)} a_n + e_Y^{(n-1)} a_n e_Y^{(n-1)} a_n & = \\ e_Y^{(n-1)} e_Y^{(n-1)} - e_Y^{(n-1)} a_n e_Y^{(n-1)} - e_Y^{(n-1)} e_Y^{(n-1)} a_n + e_Y^{(n-1)} a_n e_Y^{(n-1)} & = \\ e_Y^{(n-1)} - e_Y^{(n-1)} a_n & = \\ e_Y^{(n-1)}(e-a_n). & \end{array}$$

Hence all $e_X^{(n)}$ are idempotents.

Let $Y, Z \subset \{1,\dots,n-1\}$. Then, using Lemma 1(i) and the inductive assumption, we compute:

$$a_n e_Y^{(n-1)} a_n e_Z^{(n-1)} = a_n e_Y^{(n-1)} e_Z^{(n-1)} = 0;$$
$$a_n e_Y^{(n-1)} e_Z^{(n-1)}(e-a_n) = 0;$$
$$e_Z^{(n-1)}(e-a_n) a_n e_Y^{(n-1)} = 0.$$

Finally,

$$\begin{aligned} & e_Y^{(n-1)}(e-a_n) e_Z^{(n-1)}(e-a_n) = \\ & \quad = e_Y^{(n-1)} e_Z^{(n-1)} - e_Y^{(n-1)} a_n e_Z^{(n-1)} - e_Y^{(n-1)} e_Z^{(n-1)} a_n + e_Y^{(n-1)} a_n e_Z^{(n-1)} a_n = \\ & \qquad\qquad = -e_Y^{(n-1)} a_n e_Z^{(n-1)} + e_Y^{(n-1)} a_n e_Z^{(n-1)} = 0. \end{aligned}$$

Hence the idempotents $e_X^{(n)}$, $X \subset \{1,\dots,n\}$, are pairwise orthogonal.

Further, using (i) and the inductive assumption we have

$$\sum_{X\subset\{1,\dots,n\}} e_X^{(n)} = a_n\left(\sum_{Y\subset\{1,\dots,n-1\}} e_Y^{(n-1)}\right) + \left(\sum_{Y\subset\{1,\dots,n-1\}} e_Y^{(n-1)}\right)(e-a_n) =$$
$$= a_n + (e-a_n) = e.$$

By the definition of $e_X^{(n)}$, the element $e_X^{(n)}$ is a linear combination of different canonical monomials. Hence $e_X^{(n)} \neq 0$ in $\mathbb{K}\mathrm{K}_n$. Now since the number of different $e_X^{(n)}$'s is $2^n$, the statement about the primitivity of $e_X^{(n)}$'s follows from Proposition 36(ii) and Corollary 37. This completes the proof. $\square$

**Corollary 39.** *Let $X \subset \{1,2,\dots,n\}$. Then $\mathbb{K}\mathrm{K}_n e_X^{(n)}$ is the projective cover of $\rho_X$.*

*Proof.* It is a straightforward calculation that $\rho_X(e_X^{(n)}) = 1$. The claim follows. $\square$

**Remark 40.** One easily checks that the simple subquotients of Kiselman's representation of $\mathbb{K}\mathrm{K}_n$ are $\rho_X$, where $|\{1,2,\dots,n\}\setminus X| = 1$, each occurring with multiplicity one.

As one more immediate corollary we obtain the following very surprising result, which once more emphasizes the importance of Kiselman's representation and shows that Proposition 32 is fairly remarkable:

**Corollary 41.** *Let $X \subset \{1,2,\dots,n\}$ be such that $X \neq \{2,3,\dots,n\}$. Then the projective module $\mathbb{K}\mathrm{K}_n e_X^{(n)}$ is* not *a faithful representation of $\mathrm{K}_n$.*

*Proof.* The statement is obvious in the case $X = \{1,2,\dots,n\}$, so we may assume $X \neq \{1,2,\dots,n\}$. Set $w = e_{\{2,3,\dots,n\}} - e_{\{1,2,\dots,n\}} \in \mathbb{K}\mathrm{K}_n$. It is certainly enough to show that $w\mathbb{K}\mathrm{K}_n e_X^{(n)} = 0$ (which means that the different elements $e_{\{2,3,\dots,n\}}$ and $e_{\{1,2,\dots,n\}}$ are represented by the same linear transformations on $\mathbb{K}\mathrm{K}_n e_X^{(n)}$). For $v \in \mathrm{W}(\{a_1,\dots,a_n\})$ we have

$$wv = \begin{cases} w, & v \text{ does not contain } a_1;\\ e_{\{1,2,\dots,n\}}, & \text{otherwise.}\end{cases}$$

Hence for any $x \in \mathbb{K}\mathrm{K}_n$ we have $wx = \alpha w + \beta e_{\{1,2,\dots,n\}}$ for some $\alpha,\beta \in \mathbb{K}$. Therefore

$$wxe_X^{(n)} = \alpha we_X^{(n)} + \beta e_{\{1,2,\dots,n\}}e_X^{(n)} = \alpha e_{\{2,3,\dots,n\}}^{(n)}e_X^{(n)} + \beta e_{\{1,2,\dots,n\}}^{(n)}e_X^{(n)} = 0$$

by Proposition 38(iii). The claim follows. $\square$

One can now say even more about the structure of $\mathbb{K}\mathrm{K}_n$, in particular, giving an independent explanation for Corollary 41:

**Proposition 42.** *The algebra $\mathbb{K}\mathrm{K}_n$ is directed in the sense that there exists a linear order, $\prec$, on the set $\{X : X \subset \{1, 2, \dots, n\}\}$ such that*

$$\mathrm{Hom}_{\mathbb{K}\mathrm{K}_n}(\mathbb{K}\mathrm{K}_n e_X^{(n)}, \mathbb{K}\mathrm{K}_n e_Y^{(n)}) = 0$$

*provided that $Y \prec X$. In particular, the algebra $\mathbb{K}\mathrm{K}_n$ is quasi-hereditary with respect to $\prec$ with projective standard modules.*

*Proof.* Let us prove directness by induction on $n$. For $n = 1$ the statement is obvious. To prove the induction step we consider the projective modules $P_1 = \mathbb{K}\mathrm{K}_n a_n$ and $P_2 = \mathbb{K}\mathrm{K}_n(e - a_n)$. Obviously $\mathbb{K}\mathrm{K}_n \cong P_1 \oplus P_2$.

Observe that for any $x \in \mathrm{K}_n$, using Lemma 1(i), we have

$$a_n x(e - a_n) = a_n x - a_n x a_n = a_n x - a_n x = 0.$$

Hence $\mathrm{Hom}_{\mathbb{K}\mathrm{K}_n}(P_1, P_2) = 0$.

The endomorphism algebra of $P_1$ is the opposite of the algebra $B = a_n \mathbb{K}\mathrm{K}_n a_n$. This algebra is the linear span of the set $\{a_n x a_n : x \in \mathrm{K}_n\}$. Using Lemma 1(i), every element from the latter set can be written as $a_n y$, where $y \in \mathrm{K}_{n-1}$, moreover all such elements are obviously linearly independent. It follows that $a_n y \mapsto y$ induces an isomorphism of $B$ onto $\mathbb{K}\mathrm{K}_{n-1}$. By the inductive assumption we obtain that $B$ is directed.

The endomorphism algebra of $P_2$ is the opposite of the algebra $C = (e - a_n)\mathbb{K}\mathrm{K}_n(e - a_n)$. This algebra is the linear span of the set $\{(e - a_n)x(e - a_n) : x \in \mathrm{K}_n\}$. Note that

$$(e - a_n)x(e - a_n) = x - a_n x - x a_n + a_n x a_n = x - x a_n$$

by Lemma 1(i). In particular, if $x$ contains $a_n$, then from Lemma 1(i) it follows that $(e - a_n)x(e - a_n) = x - x a_n = x - x = 0$. This means that $C$ has the following basis: $\{(e - a_n)x(e - a_n) : x \in \mathrm{K}_{n-1}\}$ and one immediately checks that $(e - a_n)x(e - a_n) \mapsto x$ induces an isomorphism from $C$ onto $\mathbb{K}\mathrm{K}_{n-1}$. By the inductive assumption we obtain that $C$ is directed as well.

So, the endomorphism algebras of both $P_1$ and $P_2$ are directed and $\mathrm{Hom}_{\mathbb{K}\mathrm{K}_n}(P_1, P_2) = 0$. It follows that $\mathbb{K}\mathrm{K}_n$ is directed, as asserted.

That a directed algebra is quasi-hereditary with projective standard modules follows immediately from the definition of quasi-hereditary algebras, see for example [DR]. This completes the proof. $\square$

We would like to finish with the following easy corollary from the above results:

**Corollary 43.** $|\mathrm{K}_n| = 2|\mathrm{K}_{n-1}| + \dim_{\mathbb{K}}(e - a_n)\mathbb{K}\mathrm{K}_n a_n$.

*Proof.* Using the proof of Proposition 42 we have

$$\begin{aligned}|\mathrm{K}_n| = \dim_{\mathbb{K}} \mathbb{K}\mathrm{K}_n = \dim_{\mathbb{K}} a_n\mathbb{K}\mathrm{K}_n a_n + \dim_{\mathbb{K}}(e - a_n)\mathbb{K}\mathrm{K}_n a_n + \\ + \dim_{\mathbb{K}} a_n\mathbb{K}\mathrm{K}_n(e - a_n) + \dim_{\mathbb{K}}(e - a_n)\mathbb{K}\mathrm{K}_n(e - a_n) = \\ \dim_{\mathbb{K}} B + \dim_{\mathbb{K}}(e - a_n)\mathbb{K}\mathrm{K}_n a_n + 0 + \dim_{\mathbb{K}} C = \\ 2\dim_{\mathbb{K}} \mathbb{K}\mathrm{K}_{n-1} + \dim_{\mathbb{K}}(e - a_n)\mathbb{K}\mathrm{K}_n a_n = 2|\mathrm{K}_{n-1}| + \dim_{\mathbb{K}}(e - a_n)\mathbb{K}\mathrm{K}_n a_n.\end{aligned}$$

□

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