An equivalence of two categories of $sl(n, \mathbb{C})$-modules

S. König and V. Mazorchuk*

Abstract

We prove that the following two categories of $sl(n, \mathbb{C})$-modules are equivalent: 1) the category of modules with integral support filtered by submodules of Verma modules and complete with respect to Enright's completion functor; 2) the category of all subquotients of modules $F \otimes M$, where F is a finite-dimensional module and M is a fixed simple generic Gelfand-Zetlin module with integral central character. Our proof is based on an explicit construction of an equivalence which, additionally, commutes with translation functors. Finally, we describe some applications of this both to certain generalizations of the category $\mathcal{O}$ and to Gelfand-Zetlin modules.

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1 Introduction

Let $\mathfrak{G}$ denote a semi-simple complex finite-dimensional Lie algebra. Apart from the finite-dimensional $\mathfrak{G}$-modules, categories of $\mathfrak{G}$-modules studied in the literature typically either have nice abstract properties or are accessible to explicit computations. But it is rare to find both properties simultaneously. The aim of this note is to show that a category enjoying the second property, produced from Gelfand-Zetlin modules, is equivalent to one of the first kind, containing certain complete modules.

Both of these categories appeared in the context of a certain generalization $\mathcal{O}(\mathcal{P}, \Lambda)$ of the famous category $\mathcal{O}$ ([BGG]) which had been proposed in [FKM1]. This category $\mathcal{O}(\mathcal{P}, \Lambda)$ is associated with a parabolic subalgebra $\mathcal{P}$ of $\mathfrak{G}$ and a category Λ of modules over the Levi factor $\mathfrak{A}'$ of $\mathcal{P}$ satisfying certain properties (see definition of an admissible category in [FKM1]). It was shown that several nice properties of $\mathcal{O}$, e.g. decomposition into direct sum of module categories over finite-dimensional algebras (which were shown to be left projectively stratified) and BGG-reciprocity, can be generalized. Also in [FKM1] we presented two natural examples of admissible categories. The first one in the case when the semi-simple part $\mathfrak{A}$ of $\mathfrak{A}'$ is isomorphic to $sl(2, \mathbb{C})$ and the second one in the case when Λ is the category "generated" by a fixed simple generic Gelfand-Zetlin module over the Lie algebra $sl(n, \mathbb{C})$ (in this case $\mathfrak{A} \simeq sl(n, \mathbb{C})$). The last example is quite natural and rather big. In fact, generic Gelfand-Zetlin modules form the richest family ($(n(n+1)/2-1)$ parameters) of known simple $sl(n, \mathbb{C})$-modules, with many nice properties (see [DFO1, Chapter 2], [FKM1, Section 11] and [Ma1, Section 6]).

*Alexander von Humboldt fellow at Bielefeld University

To be completely honest, the first example is a special case of the second one, but it differs from all others by the property that in the $sl(2,\mathbb{C})$ case all modules in Λ have finite-dimensional weight spaces. Relying on this property and using some technical tools worked out in [M], in [FKM2] the first mentioned example was investigated in more details. In particular, an analogue of Soergel's combinatorial description for blocks was obtained and a theory of tilting modules was developed. Being optimistic that the approach used in [FKM2] should work in a much more general situation, in [FKM3] we presented another example of an admissible category of $\mathfrak{A}$-modules (for arbitrary $\mathfrak{A}$) with finite-dimensional weight spaces as a (full) subcategory in the classical category $\mathcal{O}$. This category should play a basic role in a conjectural generalization of the results of [FKM2]. The objects of this full subcategory are modules with integral support filtered by submodules of Verma modules and complete with respect to Enright's completion functor. The most non-trivial feature in this example is that the abelian structure on the category is not inherited from that on $\mathcal{O}$. It was also shown in [FKM3] that for corresponding generalizations of $\mathcal{O}$ based on the last example there also exists a theory of tilting modules (including Ringel self-duality) which is an analogue of Soergel's combinatorial description for the classical case.

The above story is not yet completely satisfactory: first – we have a natural example of the admissible category of Gelfand-Zetlin modules but we do not know much about it; second – we have the relatively artificial second example of an admissible category but it has a lot of nice properties. The aim of this paper is to clarify the situation. In fact, we will prove that the two categories above are equivalent. Moreover, we construct an equivalence which commutes with translation functors (the last ones control admissibility of the categories). This allows us to translate all the results from [FKM3] to the second mentioned example from [FKM1]. On the level of admissible categories this also gives us some new information about Gelfand-Zetlin modules. For example, we obtain that indecomposable projectives in the admissible category of Gelfand-Zetlin modules are rigid.

The paper is organized as follows: in Sections 2 and 3 we introduce our main objects – the admissible category of complete modules having a quasi Verma flag and the admissible category of Gelfand-Zetlin modules respectively. Our main result is presented in Section 4 and proved in Section 5. Section 4 also contains some applications of the main result. For more details on admissible categories and corresponding generalizations of $\mathcal{O}$ we refer the reader to [FKM1, FKM2, FKM3].

2 Complete modules having a quasi Verma flag

Let $\mathfrak{A}$ be a semi-simple complex finite-dimensional Lie algebra with a fixed triangular decomposition $\mathfrak{A} = \mathfrak{N}_- \oplus \mathfrak{H} \oplus \mathfrak{N}_+$ and let $\mathcal{O} = \mathcal{O}(\mathfrak{A})$ be the corresponding category of finitely-generated, $\mathfrak{H}$-diagonalizable and locally $\mathfrak{N}_+$-finite modules (see [BGG]). A module, M, from $\mathcal{O}$ is said to have a *quasi Verma flag* if there is a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ such that each M_i/M_{i-1} is a (non-zero) submodule in a Verma module. As each Verma module has a simple socle, which is itself a Verma module, the length of a quasi Verma flag does not depend on the choice of a flag above and equals the number of simple Verma subquotients of M.

Fix a Weyl-Chevalley basis in $\mathfrak{A}$. With any simple root α we can associate an *elementary Enright completion*, r_α, defined as the composition of the following three functors ([E, De, M]). The first one is an induction from $U = U(\mathfrak{A})$ to the Ore localization U_α of U with respect to the powers of $X_{-\alpha}$. The second one is the restriction back to U and the last one is taking the locally X_α-finite part. Clearly $r_\alpha : \mathcal{O} \to \mathcal{O}$ and $r_\alpha \circ r_\alpha = r_\alpha$. It is known that (on objects which are torsion-free over $\mathfrak{N}_-$) the functors r_α satisfy the braid relations (main result in [De]), hence for any element w of the Weyl group W we can define the corresponding composition r_w. Now by *Enright completion* we will mean $r = r_{w_0}$, where w_0 is the longest element in W.

A module, M, from $\mathcal{O}$ is said to be *complete* if $r(M) = M$. Our first main object of interest is the full subcategory $\mathcal{K}$ of $\mathcal{O}$ which consists of all complete modules having a quasi Verma flag and integral support ([FKM3, Section 4]).

Let $M, N \in \mathcal{K}$ and $f : M \to N$ be a homomorphism. Define $\mathrm{coker}(f)$ as $r(N/r(f(M)))$. Then the usual kernels of homomorphisms and the cokernels just defined endow $\mathcal{K}$ ([FKM3, Lemma 13]) with an abelian structure. Moreover, tensoring with a finite-dimensional module is an exact endofunctor on $\mathcal{K}$ with respect to this abelian structure ([FKM3, Proposition 2]). Altogether this means that $\mathcal{K}$ is an *admissible category*, that is, $\mathcal{K}$ is an abelian category, is a full subcategory in the category of all $\mathfrak{A}$-modules, all modules in $\mathcal{K}$ are finitely generated and tensoring with a finite-dimensional module is an exact endofunctor on $\mathcal{K}$. We note, that the abelian structure on $\mathcal{K}$ is not inherited from that on $\mathcal{O}$.

Remark: In the recent paper [KM] the authors have studied this abelian structure and explained how it is related to the abelian structure of category $\mathcal{O}$.

3 Gelfand-Zetlin modules

From now on we assume that $\mathfrak{A} \simeq sl(n, \mathbb{C})$ and X_i, Y_i, $i = 1, 2, \dots, n-1$ denote the canonical generators of $\mathfrak{A}$. By a *tableau* we mean a doubly indexed complex vector $[s] = (s_{i,j})_{i=1,2,\dots,n}^{j=1,2,\dots,i}$. We also denote by $[\delta^{i,j}]$ the Kronecker tableau.

Fix a tableau $[l]$ satisfying the following conditions:

- $l_{i,j} - l_{i,k} \notin \mathbb{Z}$ for all $i = 1, 2, \dots, n-1$, $1 \leq j < k \leq i$;
- $l_{i,j} - l_{i+1,k} \notin \mathbb{Z}$ for all $i = 1, 2, \dots, n-1$, $j = 1, 2, \dots i$, $k = 1, 2, \dots, i+1$.

Let $S([l])$ be the set of all tableaux $[t]$ satisfying the following conditions:

- $t_{n,j} = l_{n,j}$ for all $j = 1, 2, \dots, n$;
- $t_{i,j} - l_{i,j} \in \mathbb{Z}$ for all $i = 1, 2, \dots, n-1$, $j = 1, 2, \dots i$.

Let $V([l])$ be the complex vector space with basis $S([l])$. It is known ([DFO1]) that the formulae

$$X_i[t] = \sum_{j=1}^{i} - \frac{\prod_k (t_{i+1,k} - t_{i,j})}{\prod_{k \neq j} (t_{i,k} - t_{i,j})} ([t] + [\delta^{i,j}]), \quad Y_i[t] = \sum_{j=1}^{i} \frac{\prod_k (t_{i-1,k} - t_{i,j})}{\prod_{k \neq j} (t_{i,k} - t_{i,j})} ([t] - [\delta^{i,j}])$$

define on $V([l])$ the structure of a simple $\mathfrak{A}$-module.

Fix $[l]$ as above such that all $l_{n,j}$ are integers. The second main object of our interest will be the category $\mathcal{F} = \mathcal{F}([l])$, which is a full subcategory in the category of all $\mathfrak{A}$-modules and consists of all subquotients of all modules of the form $F \otimes V([l])$, where F is a finite-dimensional $\mathfrak{A}$-module. By definition, $\mathcal{F}$ is an abelian category with the natural abelian structure. By [FKM1, Lemma 4], the module $F \otimes V([l])$ is of finite length and belongs to $\mathcal{F}$. With respect to the standard abelian structure, $F \otimes \_$ is exact and hence $\mathcal{F}$ is an admissible category.

This information is enough to state the main result of the paper, but our proof will require some additional information about Gelfand-Zetlin modules and it is natural to present it in this Section.

Consider a natural chain of subalgebras $sl(2,\mathbb{C}) \subset sl(3,\mathbb{C}) \subset \cdots \subset sl(n,\mathbb{C})$, where $sl(k,\mathbb{C})$ is generated by X_i, Y_i, $i = 1, 2, \dots, k-1$. Then we have the corresponding chain $U(sl(2,\mathbb{C})) \subset \cdots \subset U(sl(n,\mathbb{C}))$ in U. Denote by Γ the algebra generated by $\mathfrak{H}$ and all centers $Z(sl(k,\mathbb{C}))$, $k = 2, 3 \dots, n$. It is known that Γ is a polynomial algebra in $(n(n+1)/2) - 1$ variables. An $\mathfrak{A}$-module M will be called a *Gelfand-Zetlin module* if M decomposes into a direct sum of non-isomorphic finite-dimensional Γ-modules ([DFO1]). For example, each basis element $[t]$ of $V([l])$ generates a one-dimensional Γ-submodule in $V([l])$, moreover, the characters of $Z(sl(k,\mathbb{C}))$ acting on $[t]$ can be computed in terms of certain symmetric functions in $t_{k,i}$, $i = 1, 2, \dots, k$ ([DFO1, Corollary 23]). By our choice of $[l]$, these characters are distinct for distinct $[t] \in S([l])$. Additionally, for the Cartan elements $H_i = [X_i, Y_i]$ one has

$$H_k[t] = \left(2\sum_{i=1}^{k} t_{k,i} - \sum_{i=1}^{k-1} t_{k-1,i} - \sum_{i=1}^{k+1} t_{k+1,i} \right) [t]$$

([DFO1, Proposition 21]).

Tableaux naturally parametrize simple Γ-modules. A Gelfand-Zetlin module is said to have *strong tableaux realization* if any simple Γ-submodule of it is parametrized by a tableau, which does not have integer differences in the i-th row for all $i = 1, 2, \dots, n-1$ (compare with [Ma2]). Each simple Gelfand-Zetlin module having a strong tableaux realization can be constructed in the same way as modules $V([l])$ ([DFO1, Section 2.3] or [Ma2, Section 4]).

Originally, the module $V([l])$ was constructed first for the algebra $gl(n,\mathbb{C})$ and then it becomes an $sl(n,\mathbb{C})$-module by restriction. Hence it may happen that $V([l])$ and $V([s])$ are isomorphic even if $[l] \neq [s]$. The corresponding criterion can be formulated as follows.

Lemma 1. *$V([l]) \simeq V([s])$ if and only if there exist $x \in \mathbb{C}$ and permutations $\sigma_i \in \Sigma_i$, $i = 1, 2 \dots, n$ such that $l_{n,i} - x = s_{n,\sigma_n(i)}$ for all $i = 1, 2, \dots, n$ and $l_{k,i} - x - s_{k,\sigma_k(i)} \in \mathbb{Z}$ for all $k = 1, 2, \dots, n-1$, $i = 1, 2, \dots, k$.*

Proof. Follows from [DFO1, Section 2.1]. □

Finally, we note that the k-th row of the tableau $[t]$ has a precise representation theoretical meaning. It gives us the parameter (highest weight plus the half sum of all positive roots, see [D]) of the Verma module (over $gl(k,\mathbb{C})$) having the same central character (with respect to $gl(k,\mathbb{C})$) as $[t]$, see [Ma1, Section 6.4] for details.

4 The main result and some applications

Now we are ready to state our main result.

Theorem 1. *Let $\mathfrak{A} = sl(n,\mathbb{C})$ and $\mathcal{K}$, $\mathcal{F}$ be defined as above. The categories $\mathcal{K}$ and $\mathcal{F}$ are equivalent, moreover, this equivalence can be chosen such that it preserves the natural decomposition of both $\mathcal{K}$ and $\mathcal{F}$ with respect to the central characters and commutes with* *translation functors. In particular, the categories $\mathcal{F}([l])$ and $\mathcal{F}([l'])$ for different $[l]$ and $[l']$, both satisfying the conditions from Section 3, are equivalent.*

We note that in the case $l_{i,j} - l'_{i,j} \in \mathbb{Z}$ the equivalence between $\mathcal{F}([l])$ and $\mathcal{F}([l'])$ follows already from [FKM1, Section 11].

We give the proof in the next Section. Now we will only describe one important tool which we will use. As it was originally defined in [M] we will call it the *Mathieu twist.* Fix a simple root α. Then, according to [M, Lemma 4.3], the algebra U_α, defined in the Section 2, has a one-parameter family θ_x, $x \in \mathbb{C}$ of automorphisms such that $\theta_x(u) = X_{-\alpha}^x u X_{-\alpha}^{-x}$ for all $u \in U_\alpha$, $x \in \mathbb{Z}$ and the map $x \mapsto \theta_x(u)$ is polynomial in x for all $u \in U_\alpha$. Now for a simple α and $x \in \mathbb{C}$ we define the *Mathieu twist functor* m_α^x as the composition of the following three functors. The first one is $U_\alpha \otimes_U -$; the second one is the twist by θ_x and the last one is the restriction to U. We will denote by l_α the functor of taking the locally X_α-finite part of an $\mathfrak{A}$-module. Thus $r_\alpha = l_{\mathfrak{A}} \circ m_\alpha^0$.

The main result has the following applications.

Corollary 1. *The indecomposable projective object in $\mathcal{F}_0$ is rigid, that is, its socle series coincides with the radical series.*

Proof. First we prove that the indecomposable projective P in $\mathcal{K}_0$ is rigid. Let W_i denote the set of elements in W which have length i. Then by induction on i we have that $\mathrm{soc}^i(P) \simeq r(\oplus_{w\in W_{i-1}} M(w\cdot 0))$ and $\mathrm{rad}^i(P) \simeq r(\oplus_{w\in W_{|\Delta_+|-i+1}} M(w\cdot 0))$. This means that the socle and the radical filtrations of P coincide and hence P is rigid. Now everything follows from Theorem 1. □

We remark that the argument in the previous proof fully determines the cohomology in the categories $\mathcal{K}_0$ and $\mathcal{F}_0$. In fact, it is easy to read off all possible self–extensions of the respective simple objects.

Corollary 2. *The indecomposable projective object in $\mathcal{F}_0$ is injective as well.*

Proof. By Corollary 1, this projective object has a simple socle. Hence its injective envelope is indecomposable. Having the same dimension as vector spaces, these two objects must coincide. □

Let $\mathfrak{G}$, $\mathcal{P}$ and $\mathfrak{A}$ be as in Section 1. Denote by $\mathcal{O}(\mathcal{P},\mathcal{F})$ (resp. $\mathcal{O}(\mathcal{P},\mathcal{K})$) the full subcategory of the category of $\mathfrak{G}$-modules, which consists of all finitely generated modules, which are weight modules with respect to the center of the Levi factor of $\mathcal{P}$, locally finite with respect to the nilpotent radical of $\mathcal{P}$ and which can be decomposed into a direct sum of modules from $\mathcal{F}$ (resp. $\mathcal{K}$), when viewed as $\mathfrak{A}$-modules.

Corollary 3. *$\mathcal{O}(\mathcal{P},\mathcal{F})$ and $\mathcal{O}(\mathcal{P},\mathcal{K})$ are blockwise equivalent. In particular, Soergel's endomorphism Theorem, Soergel's double centralizer property and Ringel self-duality are true for the principal block of $\mathcal{O}(\mathcal{P},\mathcal{F})$ and Soergel's character formulae for tilting modules is true for $\mathcal{O}(\mathcal{P},\mathcal{F})$ (see [FKM3] for precise formulations).*

Proof. The statement about the existence of equivalence follows from Theorem 1 and Lemma 9 (the last to be proved in Section 5). The rest follows from [FKM3]. □

5 Proof of the main result

First we recall some known properties of $\mathcal{K}$ and $\mathcal{F}$.

Lemma 2. *With respect to the action of the center of $U(\mathfrak{A})$, both $\mathcal{F}$ and $\mathcal{K}$ decompose into a direct sum of module categories over local algebras.*

Proof. For $\mathcal{F}$ this is proved in [FKM1, Section 11], for $\mathcal{K}$ this is proved in [FKM3, Lemma 15, Lemma 16]. □

Lemma 3. *Fix $\chi \in Z(\mathfrak{A})^*$. Then the lengths of the indecomposable projectives in $\mathcal{F}_\chi$ and in $\mathcal{K}_\chi$ are the same and this common number coincides with the number of non-isomorphic Verma modules (over $\mathfrak{A}$) having central character χ.*

Proof. Let l be the number of non-isomorphic Verma modules over $\mathfrak{A}$ having central character χ. That the length of the indecomposable projective in $\mathcal{F}_\chi$ equals l is proved in [FKM1, Section 12]. For $\mathcal{K}_\chi$ it can be shown as follows. The indecomposable projective in K_χ is the big projective module $P(\lambda)$, where λ belongs to the closure of the antidominant Weyl chamber ([FKM3, Corollary 3]). Moreover, its length in K_χ coincides with the composition multiplicity $(P(\lambda) : L(\lambda))$. As $L(\lambda)$ is a simple socle of each Verma module in $\mathcal{O}_\chi$, the last number equals the length of any Verma flag of $P(\lambda)$. By BGG-reciprocity and the mentioned description of the socles of Verma modules, each Verma module from $\mathcal{O}_\chi$ occurs exactly once in any Verma flag of $P(\lambda)$. Hence, the length of $P(\lambda)$ in $\mathcal{K}_\chi$, which coincides with the length of a (quasi) Verma flag of $P(\lambda)$, equals l. □

Lemma 4. *Assume that we have already constructed an exact functor, f, from $\mathcal{F}$ to $\mathcal{K}$, which commutes with $F\otimes\_$ for any finite-dimensional F, faithful on morphisms and sends (for each χ) the simple from $\mathcal{F}_\chi$ to the simple from $\mathcal{K}_\chi$ inducing an isomorphism on the endomorphism rings. Then f is the desired equivalence, which proves Theorem 1.*

Proof. Denote by $\hat{\chi}$ the central integral character of a simple-projective Verma module. For this $\hat{\chi}$, the simple module in $\mathcal{F}_{\hat{\chi}}$ (or $\mathcal{K}_{\hat{\chi}}$) coincides with the corresponding indecomposable projective. As $F \otimes \_$ is exact on both $\mathcal{F}$ and $\mathcal{K}$ and f commutes with $F \otimes \_$, we get that f sends the indecomposable projective from $\mathcal{F}_\chi$ to the indecomposable projective in $\mathcal{K}_\chi$ for any χ. As f is exact, it sends simples to simples. All $\mathcal{F}_\chi$ and $\mathcal{K}_\chi$ are module categories over local algebras. Moreover, f acts blockwise, so it is enough to prove that $f : \mathcal{F}_\chi \to \mathcal{K}_\chi$ is an equivalence. But the lengths of the indecomposable projectives in $\mathcal{F}_\chi$ and $\mathcal{K}_\chi$ coincide by Lemma 3. Since f preserves the endomorphism ring of a simple and is exact, we derive that f is full on morphisms and the final statement follows from the exactness of f. □

By Lemma 4, to prove Theorem 1 we need only to construct an exact functor from $\mathcal{F}$ to $\mathcal{K}$, which commutes with all $F \otimes \_$, is faithful on morphisms and sends simples from $\mathcal{F}$ to simples in $\mathcal{K}$ preserving the central character and the endomorphism ring of any simple. We will construct this functor composing several m_α^x and l_α. Hence, in the next step we review some properties of these functors.

Fix a simple root, α, and denote by $\mathfrak{A}(\alpha)$ the $sl(2,\mathbb{C})$ subalgebra of $\mathfrak{A}$ associated with α. Let Λ_α denote the full subcategory of the category of all finitely-generated $\mathfrak{A}(\alpha)$-modules, which consists of all direct summands of the modules $F \otimes M$, where F is finite-dimensional and M is a finitely generated weight module with one-dimensional weight spaces and such that $X_{-\alpha}$ acts bijectively on M. Let $\mathcal{M}_\alpha$ denote the full subcategory of the category of all finitely generated $\mathfrak{A}$-modules, which consists of all modules M, that can be decomposed into a direct sum of modules from Λ_α when viewed as $\mathfrak{A}(\alpha)$-modules. It is easy to see that Λ_α (resp. $\mathcal{M}_\alpha$) inherits an abelian structure from the category of all $\mathfrak{A}(\alpha)$ (resp. $\mathfrak{A}$) -modules (see [FKM2, Remark 1]).

Let Λ^α denote the full subcategory of the category of all finitely generated $\mathfrak{A}(\alpha)$-modules, which consists of complete modules having a quasi-Verma flag from the corresponding category $\mathcal{O}$. Let $\mathcal{M}^\alpha$ denote the full subcategory of the category of all finitely-generated $\mathfrak{A}$-modules, which consists of all modules M, that can be decomposed into a direct sum of modules from Λ^α when viewed as $\mathfrak{A}(\alpha)$-modules. Λ^α has a natural abelian structure with usual kernels and cokernels defined for $f : M \to N$ as $r_\alpha(N/r_\alpha(M))$ ([FKM3, Lemma 13]). In a natural way, this abelian structure can be extended to $\mathcal{M}^\alpha$. In the following Lemma we will refer to this abelian structure on $\mathcal{M}^\alpha$.

Lemma 5. *$l_\alpha : \mathcal{M}_\alpha \to \mathcal{M}^\alpha$ is exact, commutes with $F \otimes \_$ for any finite-dimensional F and faithful on morphisms.*

Proof. All properties can be checked on the $\mathfrak{A}(\alpha)$-level, where they are trivial (see also [FKM2, Lemma 3,5] for a much more general situation). □

Lemma 6. *For any $x \in \mathbb{C}$ we have that $m_\alpha^x : \mathcal{M}_\alpha \to \mathcal{M}_\alpha$ is exact and faithful on morphisms.*

Proof. Follows directly from the definitions of m_α^x and $\mathcal{M}_\alpha$. □

Lemma 7. *Let $\mathcal{M}_\alpha^{GZ}$ denote the full subcategory of $\mathcal{M}_\alpha$ consisting of all Gelfand-Zetlin modules having strong tableaux realization. Then for any $x \in \mathbb{C}$ the functor m_α^x sends $\mathcal{M}_\alpha^{GZ}$ into itself and its restriction to this category commutes with $F \otimes \_$ for any finite-dimensional F.*

We note, that the category $\mathcal{M}_\alpha$ does not contain all Gelfand-Zetlin modules (even not those having a strong tableaux realization) if α is not the first simple root. We also note that the statement can be easily extended to $m_\alpha^x : (l_\alpha(\mathcal{M}_\alpha^{GZ})) \to \mathcal{M}_\alpha^{GZ}$.

Proof. Let $X_\alpha = X_i$ for some $i \in \{1, 2, \dots, n-1\}$. By exactness of m_α^x we have only to prove the statement for simple objects from $\mathcal{M}_\alpha^{GZ}$. Let M be a simple object in $\mathcal{M}_\alpha^{GZ}$. As Y_i acts bijectively on M, there is a finite number of Γ-weight generators v_j, $j \in J$ such that the corresponding tableaux $[t_{k,l}^j]$ satisfy the following condition: $t_{i-1,l}^j = t_{i,l}^j$ for all $l = 1, 2 \dots, i-1$. Hence $X_{i-1}v_j = 0$ for all j. Moreover, $Xv_j = 0$ for any X corresponding to a positive root having X_{i-1} as a summand and all other summands of the form X_k, $k < i-1$. Y_i commutes with all $Z(sl(k, \mathbb{C}))$, $k \neq i$, and with $(H_\alpha)^\perp$. By polynomiality of Mathieu's twist, m_α^x sends an H_α-weight vector of weight y to an H_α-weight vector of weight $y + 2x$. Now let $c \in Z(sl(i, \mathbb{C}))$ and $cv_j = y_j v_j$. From $X_{i-1}v_i = 0$ we get $[X_{i-1}, X_{i-2}]V_i = 0$, $[[X_{i-1}, X_{i-2}], X_{i-3}]v_j = 0$ and so on. Thus we can apply the generalized Harish-Chandra homomorphism ([DFO2] or [Ma1, Section 3.3]). We obtain $c_1 \in Z(sl(i-1, \mathbb{C}))$ and $H \in S(\mathfrak{H})$ such that $m_\alpha^x(c)v_j = m_\alpha^x(c_1 + H)v_j$. We conclude that the images of all v_j are Γ-weight vectors, thus implying $m_\alpha^x(M) \in \mathcal{M}_\alpha^{GZ}$. Moreover, one sees that $m_\alpha^x(M)$ is a simple object of $\mathcal{M}_\alpha^{GZ}$, which can be precisely computed in terms of $[t_{k,l}^j]$ and x. Now the statement about $F \otimes \_$ follows from calculations in [FKM1, Lemma 4]. □

Lemma 8. *Both functors l_α and m_α^x respect the action of the center. In particular, they respect (generalized) central characters.*

Proof. Obvious. $\square$

Lemma 9. *Both l_α and m_α^x commute with parabolic inductions.*

Proof. For l_α this follows from Lemma 5 and for m_α^x from [FKM3, Lemma 4]. $\square$

Lemma 10. *Let $\mathcal{P}$ be a parabolic subalgebra of a semi-simple Lie algebra $\mathfrak{G}$ and V be a simple module over the Levi factor of $\mathcal{P}$ which is turned into a $\mathcal{P}$-module via the trivial action of the nilradical. Then any endomorphism of the module $U(\mathfrak{G}) \otimes_{U(\mathcal{P})} V$ (which usually is called a generalized Verma module associated with $\mathcal{P}$ and V) is scalar.*

Proof. $U(\mathfrak{G}) \otimes_{U(\mathcal{P})} V$ is generated by V and any endomorphism of $U(\mathfrak{G}) \otimes_{U(\mathcal{P})} V$ sends the unique copy of V in $U(\mathfrak{G}) \otimes_{U(\mathcal{P})} V$ into itself. Now the statement follows from [D, Proposition 2.6.5]. $\square$

By virtue of Lemmas above we need only to find a composition of different m_α^x and l_α which sends simples from $\mathcal{F}$ to simples in $\mathcal{K}$ preserving their endomorphism rings. These endomorphism rings equal $\mathbb{C}$ in the case of $\mathcal{F}$ by [D, Proposition 2.6.5]. We note, that simple objects in $\mathcal{K}$ are not simple $\mathfrak{A}$-modules but coincide with projective Verma modules (which are quite far from being simple $\mathfrak{A}$-modules) which also have $\mathbb{C}$ as endomorphism ring.

Using usual induction together with exactness of parabolic induction and Lemma 9, it is sufficient to construct a composition of different m_α^x and l_α which sends simples from $\mathcal{F}$ to the generalized Verma modules over $\mathfrak{A}$ induced from simple Gelfand-Zetlin modules having strong tableaux realization over the parabolic subalgebra with simple Levi part generated by X_i, Y_i, $i > 1$. In fact, on each step the endomorphism ring will be preserved by Lemma 10 and iterating this process inductively we will end up with a Verma module from $\mathcal{O}$. As both m_α^x and l_α respect the central character, the result will be in the correct block of $\mathcal{O}$. As we will also see later, on each step we will obtain a module, complete with respect to some r_α, where α is a simple root, so that the final module will be complete. This will prove our Theorem. Using the integer shift of tableaux one can also see that it is sufficient to construct such a composition for one fixed simple module from $\mathcal{F}$.

So, fix some $V([l]) \in \mathcal{F}$ such that the upper row of $[l]$ defines the projective Verma module in $\mathcal{O}$ (this means that the entries of the row decrease). The only Y_i acting bijectively on $V([l])$ is Y_1. Let $X_\alpha = X_1$. Then, clearly, $V([l]) \in \mathcal{M}_\alpha^{GZ}$ and from the proof of Lemma 7 it follows that $m_\alpha^x(V([l])) \simeq V([s])$, where $s_{i,j} = l_{i,j}$, $i > 1$ and $s_{1,1} = l_{1,1} + 2x$. Choose x such that $l_{1,1} + 2x = l_{2,1}$ and consider the module $M_1 = l_\alpha(m_\alpha^x(V([l])))$. It is generated by a Γ-weight vector corresponding to the tableau $[s]$ as above. Let us show that Y_2 acts bijectively on M_1. Indeed, any tableau $[p]$ appearing as a basis element in M_1 satisfies the condition $p_{2,1} - p_{1,1} \in \mathbb{Z}_+$ because of the local nilpotency of X_1. Assume that $p_{2,1} = p_{1,1}$ and consider the set P of all tableaux obtained from $[p]$ by integer shift of $p_{2,2}$. Applying Y_2 to any tableau from P and using Gelfand-Zetlin fomulae we see that we can reduce either $p_{2,1}$ or $p_{2,2}$ by 1, but $p_{2,1} - 1 < p_{1,1}$ and hence in fact we can only reduce $p_{2,2}$. This means that Y_2 sends any tableau from P into a (non-zero by Gelfand-Zetlin formulae) multiple of another tableau. From this we obtain that Y_2 acts bijectively on the subspace generated by P, moreover this subspace is a simple dense module over $\mathfrak{A}(\beta)$ (see [M, Section 4.3]). Letting Y_1 act on all tableaux with $p_{2,1} = p_{1,1}$ we will obtain all basis elements of M_1. This means that M_1 is generated by a direct sum of simple dense $\mathfrak{A}(\beta)$-modules. From the fact that $U(\mathfrak{A})$ is a direct sum of finite-dimensional $\mathfrak{A}(\beta)$-modules under adjoint action we get that, as an $\mathfrak{A}(\beta)$-module, M_1 is a direct sum of subquotients of the modules $V \otimes F$, where V is simple dense and F is finite-dimensional. By [FKM1, Section 10] this means that Y_2 acts bijectively on M_1 and $M_1 \in \mathcal{M}_\beta^{GZ}$. Hence we are allowed to apply m_β^x.

Again from the proof of Lemma 7 one gets that this is equivalent to changing $s_{2,2}$, which can be choosen arbitrarily, for example equal to $s_{3,1}$. Now we can apply m_α^x and make $s_{1,1}$ equal to $s_{2,2} = s_{3,1}$. Again applying m_β^x we can achieve $s_{2,1} = s_{3,2}$. As our tableaux are defined up to permutations of the elements in each row, we can have $s_{1,1} = s_{2,1} = s_{3,1}$ and $s_{2,2} = s_{3,2}$. Now it is clear that proceeding with other simple roots as above we will be able to arrive at a module N generated by a Γ-weight element v corresponding to the tableau $[t]$ defined as follows: $t_{n,i} = l_{n,i}$ for all i, $t_{n-1,i} = l_{n-1,i}$ for all $i > 1$, $t_{i,1} = t_{n,1}$ for all i and $t_{k,i} = t_{n-1,i}$ for all $k < n$. By Lemma 5, this module will be automatically r_α-complete, i.e. $r_\alpha(N) = N$. (Hence, at the very end of the induction process we get a complete module.) But one also has $X_\gamma v = 0$ for any positive root γ containing α. From this we get that N is isomorphic to a generalized Verma module induced from a simple Gelfand-Zetlin module, $\hat{N}$, over the parabolic subalgebra with simple Levi part generated by X_i, Y_i, $i > 1$ (see [Ma2, Section 8]). From $t_{n-1,i} - t_{n-1,j} \notin \mathbb{Z}$ we have that $\hat{N}$ has a strong tableaux realization. Now induction completes our proof.

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Steffen König, Department of Mathematics and Computer Science, University of Leicester, University Road, Leicester, LE1 7RH, England
email: `sck5@mcs.le.ac.uk`
home page: `http://www.mcs.le.ac.uk/~skoenig/`

Volodymyr Mazorchuk, Department of Mathematics, Uppsala University, Box. 480, SE-75106, Uppsala, Sweden
e-mail: `mazor@math.uu.se`
home page: `http://www.math.uu.se/~mazor/`