Finitistic dimension and tilting modules for stratified algebras

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Abstract

We calculate the finitistic dimension of certain stratified algebras in terms of the projective dimension of the characteristic tilting module. This includes, in particular, quasi-hereditary algebras, whose Koszul dual is again quasi-hereditary; stratified algebras, which are quotients of quasi-hereditary algebras over complete local commutative rings; and stratified algebras associated with Harish-Chandra bimodules for complex semi-simple finite-dimensional Lie algebras.

1 Introduction and description of results

A fundamental invariant of any algebra and its module category is the *global dimension*, the maximal degree in which cohomology can occur, or, if this happens to be infinite, the *finitistic dimension*, the supremum of the finite projective dimensions occurring in this category; the celebrated *finitistic dimension conjecture* predicts the finitistic dimension to be always finite. *Filtrations* of the algebra and *stratifications* of its module category often can be used to provide *upper bounds* for these dimensions, as is known for quasi-hereditary or standardly stratified algebras; for example, in [AHLU] such an upper bound is established for the finitistic dimension of a standardly stratified algebra in terms of combinatorial properties of the partially ordered set of simple modules, generalizing the well-known upper bound for the global dimension of quasi-hereditary algebras.

Such upper bounds are rarely sharp, and little is known on the precise values of global or finitistic dimension of these classes of algebras. Therefore, in [MP] a more effective upper bound has been proposed for the finitistic dimension of a stratified algebra with a duality and it has been conjectured that this bound gives the *exact value*.

This conjecture has already been verified in several cases; for quasi-hereditary algebras with two simple modules in [Pa1], for many Schur algebras in [Pa1, Pa2], for the BGG-category $\mathcal{O}$ of a semi-simple complex Lie algebra in [MP], and for various classes of quasi-hereditary algebras in [EP]. The present paper contributes general criteria to verify the conjecture as well as several new classes of examples, for which the conjecture holds true. The subsequent paper [MO] proves the conjecture for all quasi-hereditary algebras with a duality preserving isomorphism classes of simples, and even for all properly stratified

algebras with such a duality and for which the characteristic tilting and cotilting modules coincide.

In the present paper we prove two general results on the finitistic dimension of stratified algebras, involving the projective dimension of the characteristic tilting module, and we use these results to compute the finitistic dimensions of various categories of Harish-Chandra bimodules over simple complex finite-dimensional Lie algebras. Our first result, proved in Section 3 is the following:

Theorem 1. *Let A be a quasi-hereditary algebra and $B = \mathrm{Ext}^\bullet_A(L, L)$. Assume that B is quasi-hereditary, modules $\Delta^{(B)}(i) = \mathrm{Ext}^\bullet_A(\Delta(i), L)$ are standard with the natural graded filtration being a Loewy one, and the Loewy length of every costandard B-module equals the Loewy length of the corresponding standard B-module. Then*

$$\mathrm{gl.\,d.}(A) = 2\,\mathrm{dim}_{\mathcal{F}(\Delta)}(A) = 2\,\mathrm{p.\,d.}(T(A)).$$

Assuming additionally that A has a simple preserving duality, we obtain a partial case of the main result from [MO]. However, the methods we use are completely different from those used in [MO]. Our methods are, roughly speaking, a manifestation of the standard fact that B, being quasi-hereditary, has a tilting module.

Our second principal result, proved in Section 4, is the following:

Theorem 2. *Let R be a basic complete local commutative algebra over some field $\Bbbk$, and $\mathfrak{m}$ be the maximal ideal of R. Let further A be a quasi-hereditary algebra over R and I be an ideal of R of finite $\Bbbk$-codimension. Then the algebra A/AI is properly stratified, the algebra $A/A\mathfrak{m}$ is quasi-hereditary, and, moreover, we have*

$$\mathrm{fin.\,dim.}(A/AI) = \mathrm{gl.\,d.}(A/A\mathfrak{m}).$$

As an immediate corollary we obtain:

Corollary 1. *Let A, R, $\mathfrak{m}$ and I be as in Theorem 2. Assume that the global dimension of the quasi-hereditary algebra $A/A\mathfrak{m}$ equals twice the projective dimension of the characteristic tilting module. Then the finitistic dimension of the properly stratified algebra A/AI equals twice the projective dimension of the characteristic tilting module.*

In Section 5 we apply Theorem 1 and Theorem 2 to calculate the finitistic dimension of several categories of Harish-Chandra bimodules over simple complex finite-dimensional Lie algebras. In particular, as one of the corollaries we obtain the following:

Corollary 2. *The finitistic dimension of a regular block of a thick category $\mathcal{O}$, [So], equals twice the projective dimension of the characteristic tilting module in this block.*

Finally, in Section 6 we calculate the finitistic dimension of some parabolic generalizations of the BGG category $\mathcal{O}$ using different methods.

2 General conventions

For a finite-dimensional algebra A over some field $\Bbbk$ and for a primitive idempotent e in A we denote by $L(e)$, $P(e)$ and $I(e)$ the corresponding simple, indecomposable projective and indecomposable injective modules respectively. We denote by $\mathrm{gl.d.}(A)$ the global dimension of A and by $\mathrm{fin.dim.}(A)$ the projectively defined finitistic dimension of A. For an A-module M we denote by $\mathrm{l.l.}(M)$ the Loewy length of M. Sometimes for an A-module M we will write $M^{(A)}$ to emphasize the fact that M is a module over A. Mainly we will use it if the algebra A is not clear from the context.

For two A-modules M and N we define the *trace* $\mathrm{Tr}_M(N)$ as the sum of all images $f(M)$, where $f : M \to N$ is a homomorphism. We remark that, by definition, $\mathrm{Tr}_M(N)$ is a submodule of N.

For a field $\Bbbk$ we denote by $D_{\Bbbk}(\_)$ the functor $\mathrm{Hom}_{\Bbbk}(\_, \Bbbk)$.

By a *duality* on a category we always mean a contravariant exact and involutive equivalence, which preserves isoclasses of simple objects.

If $\mathcal{M}$ is a set of A-modules, we will say, abusing language, that an A-module M is *filtered by modules from* $\mathcal{M}$ if there is a filtration of M, whose subquotients are isomorphic to some modules in $\mathcal{M}$.

Let $\mathcal{A}$ be an abelian category and $\mathcal{M}$ be a set of objects from $\mathcal{A}$. Assume that for every object $M \in \mathcal{A}$ there exists a (possibly infinite) resolution

$$\cdots \to P_1 \to P_0 \to M \to 0, \tag{1}$$

where $P_i \in \mathcal{M}$ for all i. For $M \in \mathcal{A}$ we call the length of a minimal resolution of the form (1) the *$\mathcal{M}$-filtration dimension* of M and denote it by $\dim_{\mathcal{M}} M$. The *$\mathcal{M}$-filtration dimension of* $\mathcal{A}$ is defined as the supremum of the $\mathcal{M}$-filtration dimensions of $M \in \mathcal{A}$.

Assume now that for every object $M \in \mathcal{A}$ there exists a (possibly infinite) coresolution

$$0 \to M \to P_0 \to P_1 \to \dots, \tag{2}$$

where $P_i \in \mathcal{M}$ for all i. For $M \in \mathcal{A}$ we call the length of the minimal coresolution of the form (2) the *$\mathcal{M}$-filtration codimension* of M and denote it by $\mathrm{codim}_{\mathcal{M}} M$. The *$\mathcal{M}$-filtration codimension of* $\mathcal{A}$ is defined as the supremum of the $\mathcal{M}$-filtration codimensions of $M \in \mathcal{A}$.

If A is an associative algebra and $\mathcal{M}$ is a set of A-modules, then the $\mathcal{M}$-filtration (co)dimension of A is defined as the $\mathcal{M}$-filtration (co)dimension of the category A-mod.

3 Quasi-hereditary algebras and their global dimension

3.1 Quasi-hereditary algebras

Let A be a finite-dimensional algebra over some field $\Bbbk$ and $\mathbf{e} = (e_1, \dots, e_n)$ be a linear order on a complete set of pairwise orthogonal primitive idempotents of A. For $i = 1, \dots, n$

we set $L(i) = L(e_i)$, $P(i) = P(e_i)$ and $I(i) = I(e_i)$. Let $P(> i) = \oplus_{j>i} P(j)$ and define $\Delta(i) = P(i)/\operatorname{Tr}_{P(>i)}(P(i))$. Dually, we define $\nabla(i)$ as the intersection of kernels of all morphisms from $I(i)$ to $I(> i) = \oplus_{j>i} I(j)$. The modules $\Delta(i)$ are called *standard* and the modules $\nabla(i)$ are called *costandard*. A filtration, whose subquotients are standard modules, is called a *standard filtration* and a filtration, whose subquotients are costandard modules, is called a *costandard filtration*.

We recall (see [CPS1]) that A is called *quasi-hereditary* provided that for all $i = 1, \dots, n$ the kernel of the canonical surjection $P(i) \twoheadrightarrow \Delta(i)$ is filtered by $\Delta(j)$, $i < j$; and the kernel of the canonical surjection $\Delta(i) \twoheadrightarrow L(i)$ is filtered by $L(j)$, $j < i$. Equivalently, A is quasi-hereditary if the cokernel of the canonical injection $\nabla(i) \hookrightarrow I(i)$ is filtered by $\nabla(j)$, $i < j$; and the cokernel of the canonical injection $L(i) \hookrightarrow \nabla(i)$ is filtered by $L(j)$, $j < i$.

Denote by $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ the full subcategories of A-mod, which consist of all modules having a standard or a costandard filtration respectively. The modules in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ are called *tilting modules* (see [Ri]). Every tilting module is a direct sum of indecomposable tilting modules, the latter being in a natural bijection with simple modules. We denote by $T(i)$ the indecomposable tilting module, whose standard filtration starts with $\Delta(i)$.

We set $L = \oplus_{i=1}^{n} L(i)$ and $T = \oplus_{i=1}^{n} T(i)$. The module T is called the *characteristic tilting module* for A. If there exists a duality on A-mod, then it sends standard modules to costandard modules and preserves tilting modules.

3.2 Global dimensions

In this subsection we compute the global dimension of a quasi-hereditary algebra, whose homological dual is again quasi-hereditary. We do not assume a priori that A has a simple preserving duality, however, we impose a technical condition on the structure of standard and costandard module over the homological dual to A (this can be simplified assuming that A has a simple preserving duality). Our argument is a direct manifestation of the fact that the homological dual to A has a tilting module.

Theorem 3. *Let A be a quasi-hereditary algebra and $B = \operatorname{Ext}_A^\bullet(L, L)$. Assume that*

(a) B is quasi-hereditary;

(b) modules $\Delta^{(B)}(i) = \operatorname{Ext}_A^\bullet(\Delta(i), L)$ are standard B-modules and the natural graded filtration is a Loewy one;

(c) the Loewy length of $\nabla^{(B)}(i)$ and $\Delta^{(B)}(i)$ coincide.

Then

$$\operatorname{gl.d.}(A) = 2 \operatorname{dim}_{\mathcal{F}(\Delta)}(A) = 2 \operatorname{p.d.}(T(A)).$$

Proof. The inequality $\operatorname{gl.d.}(A) \leq 2 \operatorname{dim}_{\mathcal{F}(\Delta)}(A) = 2 \operatorname{p.d.}(T)$ follows from [CZ, Corollary 2.9] and [CZ, Corollary 2.11], so it suffices to show that $\operatorname{gl.d.}(A) \geq 2 \operatorname{p.d.}(T)$.

The algebra B is positively graded in a natural way and $\operatorname{gl.d.}(A)$ coincides with the degree N of the maximal non-zero graded component of B. The zero component of this

grading is semi-simple by Schur's Lemma, which implies that the graded filtration on B, considered as a left module over itself, has semi-simple subquotients. Hence $N \geq \mathrm{l.l.}({}_BB) - 1$ (we have to subtract one as the grading starts in degree zero). Let $T^{(B)}$ denote the characteristic tilting module for B. Since the module ${}_BB$ is a projective generator in B-mod, it has the maximal possible Loewy length and we get $\mathrm{l.l.}({}_BB) - 1 \geq \mathrm{l.l.}(T^{(B)}) - 1$.

Now, every indecomposable summand $T^{(B)}(i)$ of $T^{(B)}$ has a submodule, isomorphic to $\Delta^{(B)}(i)$, and a quotient, isomorphic to $\nabla^{(B)}(i)$. The simple module $L^{(B)}(i)$ is the simple top of $\Delta^{(B)}(i)$ and the simple socle of $\nabla^{(B)}(i)$, moreover, the multiplicity of $L^{(B)}(i)$ in $T^{(B)}(i)$ is 1. Hence $\mathrm{l.l.}(T^{(B)}(i)) \geq (\mathrm{l.l.}(\nabla^{(B)}(i)) + \mathrm{l.l.}(\Delta^{(B)}(i))) - 1$. Now (c) implies $\mathrm{l.l.}(T^{(B)}) - 1 \geq 2\max_i \big(\mathrm{l.l.}(\Delta^{(B)}(i)) - 1\big)$.

Recall that $\Delta^{(B)}(i) = \mathrm{Ext}^\bullet_A(\Delta(i), L)$ by assumption, and that the graded filtration of $\Delta^{(B)}(i)$ is a Loewy filtration. Hence $\mathrm{l.l.}(\Delta^{(B)}(i)) - 1 = \mathrm{p.d.}(\Delta(i))$, which results $\mathrm{l.l.}(T^{(B)}) - 1 \geq 2\max_i(\mathrm{p.d.}(\Delta(i)))$.

Since $T \in \mathcal{F}(\Delta)$ we have $\max_i(\mathrm{p.d.}(\Delta(i))) \geq \mathrm{p.d.}(T)$. Combining all the inequalities above we obtain

$$\begin{aligned}\mathrm{gl.d.}(A) \geq \mathrm{l.l.}({}_BB) - 1 \geq \mathrm{l.l.}(T^{(B)}) - 1 \geq \\ \geq 2\max_i \big(\mathrm{l.l.}(\Delta^{(B)}(i)) - 1\big) \geq 2\max_i(\mathrm{p.d.}(\Delta(i))) \geq 2\,\mathrm{p.d.}(T),\end{aligned}$$

completing the proof. □

Following the proof of Theorem 3 one also obtains the following lower bound for the global dimension of a quasi-hereditary algebra:

Corollary 3. *Let A be a quasi-hereditary algebra and $B = \mathrm{Ext}^\bullet_A(L, L)$. Assume that*

(a) B is quasi-hereditary;

(b) modules $\Delta^{(B)}(i) = \mathrm{Ext}^\bullet_A(\Delta(i), L)$ are standard B-modules and the natural graded filtration is a Loewy one;

(c) modules $\nabla^{(B)}(i) = \mathrm{Hom}_{\Bbbk}\left(\mathrm{Ext}^\bullet_A(\mathrm{Hom}_{\Bbbk}(\nabla(i), \Bbbk), L), \Bbbk\right) = \mathrm{Ext}^\bullet_A(L, \nabla(i))$ are costandard B-modules and the natural graded filtration is a Loewy one;

Then

$$\mathrm{gl.d.}(A) \geq \max_i \big(\mathrm{p.d.}(\Delta(i)) + \mathrm{i.d.}(\nabla(i))\big).$$

3.3 Applications

Corollary 4. *Assume that A is a Koszul quasi-hereditary algebra having a simple preserving duality with modules $\Delta^{(B)}(i) = \mathrm{Ext}^\bullet_A(\Delta(i), L)$ being Koszul and costandard for B. Then* $\mathrm{gl.d.}(A) = 2\dim_{\mathcal{F}(\Delta)}(A) = 2\,\mathrm{p.d.}(T(A))$.

Proof. Since A is Koszul, so is B. The modules $\Delta^{(B)}(i)$ have simple heads and are graded as Koszul modules. Hence, by [BGS, Proposition 2.4.1], the graded filtration of this module is the radical filtration and thus a Loewy filtration. Now the statement follows from Theorem 3. □

Recall from [ADL] that a Koszul quasi-hereditary algebra is called *standard Koszul* provided that both left and right standard modules are Koszul.

Corollary 5. *Assume that A is a standard Koszul quasi-hereditary algebra with duality. Then* $\mathrm{gl.\,d.}(A) = 2\dim_{\mathcal{F}(\Delta)}(A) = 2\,\mathrm{p.\,d.}(T)$.

Proof. Since A is standard Koszul, we get from [ADL, Theorem 2] that B is quasi-hereditary, Koszul with standard modules having the necessary form. The result now follows from Corollary 4. □

Corollary 6. *([MP, Theorem 2]) Let A be the quasi-hereditary algebra of an indecomposable block of the BGG category $\mathcal{O}$, [BGG], or the parabolic category $\mathcal{O}_S$ of Rocha-Caridi, [Ro]. Then* $\mathrm{gl.\,d.}(A) = 2\dim_{\mathcal{F}(\Delta)}(A) = 2\,\mathrm{p.\,d.}(T)$.

Proof. By [BGS] and [Ba] the algebra A is standard Koszul and the result follows from Corollary 5. □

4 Properly stratified algebras and their finitistic dimension

4.1 Properly stratified algebras

Let A be an algebra over some field $\Bbbk$ and let $\mathbf{e} = (e_1, \dots, e_n)$ be a linear order on a complete set of pairwise orthogonal primitive idempotents of A. We keep all the notation from Subsection 3.1. For $i = 1, \dots, n$ we also define $\overline{\Delta}(i)$ to be the maximal quotient of $\Delta(i)$ such that $[\overline{\Delta}(i) : L(i)] = 1$, and $\overline{\nabla}(i)$ to be the maximal submodule of $\nabla(i)$ such that $[\overline{\nabla}(i) : L(i)] = 1$. The modules $\overline{\Delta}(i)$ are called *proper standard* modules and the modules $\overline{\nabla}(i)$ are called *proper costandard* modules. A filtration, whose subquotients are proper standard modules, is called a *proper standard filtration* and a filtration, whose subquotients are proper costandard modules is called a *proper costandard filtration.*

Following [CPS3] we call A *standardly stratified* with respect to $\mathbf{e}$ provided that the kernel of the canonical surjection $P(i) \twoheadrightarrow \Delta(i)$ is filtered by $\Delta(j)$, $i < j$. Note that the original definition from [CPS3] is more general as it uses not a linear order on $\mathbf{e}$ but rather a pre-order.

Following [Dl] we call A *properly stratified* with respect to $\mathbf{e}$ provided that A is standardly stratified and $\Delta(i)$ has a proper standard filtration for all $i = 1, \dots, n$. Moreover, it is easy to see that any proper standard filtration of $\Delta(i)$ contains only $\overline{\Delta}(i)$ (with the same i). The following lemma gives several equivalent conditions, which guarantee that an algebra is properly stratified, see [Dl].

Lemma 1. *The following conditions are equivalent:*

1. *A is properly stratified.*
2. *Both A and A^{opp} are standardly stratified.*
3. *$\nabla(i)$ is filtered by $\overline{\nabla}(i)$ for all $i = 1, \dots, n$ and, moreover, the cokernel of the canonical injection $\nabla(i) \hookrightarrow I(i)$ is filtered by $\nabla(j)$, $i < j$.*

Denote by $\mathcal{F}(\overline{\Delta})$, and $\mathcal{F}(\overline{\nabla})$ the full subcategories of A-mod, which consist of all modules, having a proper standard, or a proper costandard, filtration respectively. The modules in $\mathcal{F}(\Delta) \cap \mathcal{F}(\overline{\nabla})$ are called *tilting modules* and the modules in $\mathcal{F}(\overline{\Delta}) \cap \mathcal{F}(\nabla)$ are called *cotilting modules*, see [AHLU]. Every tilting (resp. cotilting) module is a direct sum of indecomposable tilting (cotilting) modules, the latter being in a natural bijection with simple modules. We denote by $T(i)$ the indecomposable tilting module, whose standard filtration starts with a submodule, isomorphic to $\Delta(i)$; and by $C(i)$ the indecomposable cotilting module, whose costandard filtration ends with a quotient, isomorphic to $\nabla(i)$.

We set $T = \oplus_{i=1}^{n} T(i)$ and $C = \oplus_{i=1}^{n} C(i)$. The module T is called the *characteristic tilting module* for A and the module $C(A)$ is called the *characteristic cotilting module* for A. A properly stratified algebra is quasi-hereditary if and only if $\mathrm{gl.d.}(A) < \infty$. If A is quasi-hereditary then $T(i) \simeq C(i)$ for all i. If there exists a duality on A-mod, which preserves isomorphism classes of simple modules, then it sends standard modules to costandard modules, proper standard modules to proper costandard modules and tilting modules to cotilting modules.

4.2 Properly stratified algebras over complete local commutative rings

Let R be a complete local commutative ring with maximal ideal $\mathfrak{m}$ and let A be an algebra over R, by which we mean that R is contained in the center of A and A is a free left (and hence right) R-module of finite rank. In particular, this ensures that the algebra $A/A\mathfrak{m}$ is a finite-dimensional algebra over $R/\mathfrak{m}$. We will say that the algebra A is *quasi-hereditary over R* provided that the algebra $A/A\mathfrak{m}$ is quasi-hereditary over $R/\mathfrak{m}$. By [CPS2, Section 3] in our setup the fact that A is quasi-hereditary over R is equivalent to the fact that A is quasi-hereditary over R in the more general sense of [CPS2, Definition 3.2]. We remark that, by [CPS2, Corollary 3.4], the algebra A is quasi-hereditary over R if and only if the algebra A^{opp} is quasi-hereditary over $R^{\mathrm{opp}} = R$.

We will use quasi-hereditary algebras over complete local commutative rings to construct new quasi-hereditary and properly stratified algebras over fields (in particular, this explains why we have chosen a more restrictive setup, for which, however, the definition is much easier). Until the end of this section we fix an algebra A, which is quasi-hereditary over a complete local commutative ring R. Let $\mathbf{e} = (e_1, \dots, e_n)$ be a complete list of pairwise orthogonal primitive idempotents in $A/A\mathfrak{m}$. By the definition of a quasi-hereditary algebra, there exists a linear order on $\mathbf{e}$, which we assume to be given by the natural order

on the set of indexes, such that the algebra $A/A\mathfrak{m}$ is quasi-hereditary over $R/\mathfrak{m}$ with respect to this order. Completeness of R allows us to lift all idempotents e_i to idempotents $\tilde{e}_i \in A$. For $i = 1, \dots, n$ we define $P(i) = A\tilde{e}_i$, $P(> i) = \oplus_{j>i} A\tilde{e}_j$, $Q(i) = \tilde{e}_i A$, $Q(> i) = \oplus_{j>i} \tilde{e}_j A$, and set

$$\Delta(i) = P(i)/\operatorname{Tr}_{P(>i)} P(i), \qquad \Delta^\circ(i) = Q(i)/\operatorname{Tr}_{Q(>i)} Q(i).$$

Starting from a quasi-hereditary algebra over a complete local commutative ring, one can construct new quasi-hereditary and properly stratified algebras over fields in the following way:

Proposition 1. *Assume that R is a complete local commutative algebra over a field $\Bbbk$, $\mathfrak{m}$ is the maximal ideal of R, and A is a quasi-hereditary algebra over R, as above. Let I be an ideal in R of finite codimension over $\Bbbk$. Then*

(a) the algebra $B = A/AI$ is finite-dimensional and properly stratified over $\Bbbk$;

(b) the standard B modules are exactly

$$\Delta^{(B)}(i) = \Delta(i)/\Delta(i)I \simeq \Delta(i) \otimes_R R/I, \qquad i = 1, \dots, n;$$

(c) the costandard B modules are exactly

$$\nabla^{(B)}(i) = D_\Bbbk\big(\Delta^\circ(i)/I\Delta^\circ(i)\big) \simeq D_\Bbbk\big(R/I \otimes_R \Delta^\circ(i)\big), \qquad i = 1, \dots, n;$$

(d) the algebra B is quasi-hereditary if and only if $I = \mathfrak{m}$;

(e) the proper standard B modules are exactly

$$\overline{\Delta}^{(B)}(i) = \Delta(i)/\Delta(i)\mathfrak{m} \simeq \Delta(i) \otimes_R \Bbbk, \qquad i = 1, \dots, n;$$

(f) the proper costandard B modules are exactly

$$\overline{\nabla}^{(B)}(i) = D_\Bbbk\big(\Delta^\circ(i)/\mathfrak{m}\Delta^\circ(i)\big) \simeq D_\Bbbk\big(\Bbbk \otimes_R \Delta^\circ(i)\big), \qquad i = 1, \dots, n.$$

Proof. We are going to prove all the assertions by induction on n, the number of isomorphism classes of simple A-modules. Since A is free over R of finite rank and R is central, the algebra A/AI is free over R/I of finite rank and R/I is central in A/AI. In particular, since R/I is finite dimensional over $\Bbbk$ and A/AI is free over R/I of finite rank, we get that A/AI is finite dimensional over $\Bbbk$.

If $n = 1$, the algebra A/AI is automatically local and hence properly stratified with simple proper standard and proper costandard modules, projective standard modules and injective costandard modules. All the assertions of the proposition are obvious in this case.

Let us now prove the induction step. As A is free of finite rank over R, we have that $P(n)$ is free over R of finite rank as well. In particular, $\Delta(n) \otimes_R R/I$ is free over R/I of finite rank.

To proceed we will need the following lemma, which we will also use later in this section:

Lemma 2. *Let $I \subset J$ be proper ideals in R of finite codimension and*

$$F_J^I : A/AI\text{−mod} \to A/AJ\text{−mod}$$

denote the functor $F_J^I(M) = M/J(M)$. Let further M, N be A/AI-modules, projective over R/I, and $f : M \to N$ be such that $F_J^I(f) : F_J^I(M) \to F_J^I(N)$ is a monomorphism. Then f is a monomorphism.

This will follow from the following standard statement.

Lemma 3. *Let Λ be a local finite-dimensional algebra over some field, P and Q be two free Λ-modules of finite rank, and $f : P \to Q$ be a homomorphism. Then the following statements are equivalent:*

(i) f is injective.

(ii) $f(P)$ is a direct summand of Q.

(iii) f induces a monomorphism $\hat{f} : P/\mathrm{Rad}(P) \to Q/\mathrm{Rad}(Q)$.

Proof. (iii) says that $P/\mathrm{Rad}(P)$ is a direct summand and thus the lift f of $\hat{f}$ is a split monomorphism, implying both (i) and (ii). Given (i) and assuming that (iii) is wrong we find a generating element x in the top of P, such that $f(x) \in \mathrm{Rad}(Q)$. This means that the free submodule $X \subset P$, generated by x is mapped to the radical of Q. Comparing Loewy lengths implies that the highest non-vanishing power of the radical of X must be in the kernel, contradicting (i). □

Proof of Lemma 2. For an R/J-module M set $G(M) = M/\mathfrak{m}M$. From Lemma 3 it follows that the monomorphism $F_J^I(f)$ induces the monomorphism $G\left(F_J^I(f)\right) : G\left(F_J^I(M)\right) \to G\left(F_J^I(N)\right)$. Again by Lemma 3, the map f, which is a lift of $G\left(F_J^I(f)\right)$, must be a monomorphism as well. □

We are going to apply Lemma 2 to the situation $J = \mathfrak{m}$. Recall that the algebra $A/A\mathfrak{m}$ is quasi-hereditary by the definition. This implies that the trace of $\Delta^{(A/A\mathfrak{m})}(n)$ in each $P^{(A/A\mathfrak{m})}(i)$ is isomorphic to $\Delta^{(A/A\mathfrak{m})}(n)^{k_i}$ for some non-negative integer k_i. Fix some monomorphism $g : \Delta^{(A/A\mathfrak{m})}(n)^{k_i} \hookrightarrow P^{(A/A\mathfrak{m})}(i)$. Composing g with the canonical projection $\mathfrak{i}_1 : \Delta^{(A/AI)}(n)^{k_i} \to \Delta^{(A/A\mathfrak{m})}(n)^{k_i}$ we obtain a map from $\Delta^{(A/AI)}(n)^{k_i}$, which lifts, because of the projectivity of $\Delta^{(A/AI)}(n)^{k_i}$, to a map $\hat{g} : \Delta^{(A/AI)}(n)^{k_i} \to P^{(A/AI)}(i)$, making the following diagram commutative:

$$\begin{array}{ccc} \Delta^{(A/AI)}(n)^{k_i} & \xrightarrow{\hat{g}} & P^{(A/AI)}(i) \\ \downarrow{\scriptstyle \mathfrak{i}_1} & & \downarrow{\scriptstyle \mathfrak{i}_2} \\ \Delta^{(A/A\mathfrak{m})}(n)^{k_i} & \xrightarrow{g} & P^{(A/A\mathfrak{m})}(i), \end{array}$$

where $\mathfrak{i}_2 : P^{(A/AI)}(i) \to P^{(A/A\mathfrak{m})}(i)$ is the canonical projection. From Lemma 2 we obtain that $\hat{g}$ is injective, which implies that the trace of $\Delta^{(A/AI)}(n)$ in $P^{(A/AI)}(i)$ is isomorphic

to $\Delta^{(A/AI)}(n)^{k_i}$. Because of the left-right symmetry of the definition, analogous results are also true for right modules.

Factoring out Ae_nA we now have by induction that $\Delta^{(A/AI)}(i)$ is a standard module over A/AI and that the module $D_{\Bbbk}\left((\Delta^\circ)^{(A/AI)}(i)\right)$ is a costandard module over A/AI, and, moreover, that A/AI is properly stratified (with respect to the same order on $\mathbf{e}$). This proves the first three statements of the proposition.

Choosing a Jordan-Hölder series, $0 = V_0 \subset V_1 \subset \cdots \subset V_k = R/I$, of R/I as an R-module and applying $\Delta^{(A)}(j)\otimes_{R-}$ we obtain a filtration of $\Delta^{(A)}(j)\otimes_R R/I$ with subquotients isomorphic to $\Delta^{(A)}(j)\otimes_R \Bbbk$. This implies that the modules $\Delta^{(A)}(j)\otimes_R \Bbbk$ are proper standard modules for A/AI, proving the fifth statement. The sixth statement follows by the left-right symmetry.

Finally, the algebra A/AI is quasi-hereditary if and only if the proper standard and the standard modules for A/AI coincide. Comparing proper standard and standard modules, which we have already described above, we conclude that this is the case if and only if $I = \mathfrak{m}$. This completes the proof. □

4.3 Comparing finitistic dimensions

Let $\Bbbk$ be an algebraically closed field, R a local commutative algebra over $\Bbbk$ and A a quasi-hereditary algebra over R. Let further $\mathfrak{m}$ be the maximal ideal of R and $I \subset \mathfrak{m}$ be a proper ideal of R of finite codimension over $\Bbbk$. The ultimate goal of this subsection is to prove the following theorem

Theorem 4.

$$\text{fin. dim.}(A/AI) = \text{gl. d.}(A/A\mathfrak{m}).$$

Our main tool for the proof of this theorem will be the functor $F_J^I : A/AI\text{-mod} \to A/AJ\text{-mod}$ (see Lemma 2 above), defined by

$$F_J^I(M) = M/JM = M \otimes_{R/RI} R/RJ,$$

where $J \subset \mathfrak{m}$ is a proper ideal of R containing I.

Lemma 4. *(a) F_J^I is left adjoint to the natural inclusion A/AJ-mod $\subset A/AI$-mod. In particular, F_J^I is right exact.*

(b) F_J^I sends indecomposable projectives to indecomposable projectives.

(c) Let $P, P' \in A/AI$-mod be two indecomposable projectives and let $f : P \to P'$ be a morphism, which is not an isomorphism. Then $F(f)$ is not an isomorphism either.

Proof. If $M \in A/AI$-mod and $N \in A/AJ$-mod then every $f : M \to N$ must annihilate MJ and the first statement follows.

Let now $P \in A/AI$-mod be an indecomposable projective module. Then the top of this module belongs already to $A/A\mathfrak{m}$-mod and hence is not annihilated by F_J^I. This implies

that $F_J^I(P) = P/(PJ) \neq 0$ has simple top and hence is indecomposable. Moreover, by the adjointness from the first statement we have

$$\mathrm{Hom}_{A/AI\text{-mod}}\big(P, N\big) = \mathrm{Hom}_{A/AJ\text{-mod}}\big(F_J^I(P), N\big)$$

for all $N \in A/AJ$-mod. In particular, the functor $\mathrm{Hom}_{A/AJ\text{-mod}}\big(F_J^I(P), \_\big)$ is exact and hence $F_J^I(P)$ is projective in A/AI-mod. This proves the second statement.

To prove the third statement we note that the right exactness of F_J^I implies exactness of

$$F_J^I(P) \stackrel{F_J^I(f)}{\to} F_J^I(P') \to F_J^I\big(\mathrm{Coker}(f)\big) \to 0.$$

The obvious inequality $F_J^I\big(\mathrm{Coker}(f)\big) \neq 0$ now implies the third statement and completes the proof. □

For a proper ideal $J \subset R$ of finite codimension we denote by $\mathcal{M}(J)$ the full subcategory of the category of A/AJ-modules, whose objects are all modules M, which are projective over R/J. We also denote by $\mathcal{P}^{<\infty}(J)$ the full subcategory in the category of all A/AJ-modules, whose objects are all modules M of finite projective dimension. For $m \in \{0, 1, 2, \dots\}$ we let $\mathcal{P}^{(m)}(J)$ denote the full subcategory in $\mathcal{P}^{<\infty}(J)$, whose objects are all modules M such that p.d.$(M) = m$. For $m \in \{0, 1, 2, \dots\}$ we let $\mathcal{P}^{(<m)}(J)$ denote the full subcategory in $\mathcal{P}^{<\infty}(J)$, whose objects are all modules M such that p.d.$(M) < m$.

Lemma 5. *(a) F_J^I sends $\mathcal{M}(I)$ to $\mathcal{M}(J)$.*

(b) F_J^I is exact on bounded exact complexes of modules from $\mathcal{M}(I)$.

Proof. The first statement is obvious. If $\mathcal{C}^\bullet$ is a bounded exact complex of modules from $\mathcal{M}(I)$, then the fact that R is local implies that $\mathcal{C}^\bullet$, viewed as a complex of R-modules, is a direct sum of trivial complexes of the form

$$\dots \to 0 \to M \stackrel{\mathrm{Id}}{\longrightarrow} M \to 0 \to \dots. \tag{3}$$

The lemma now follows from the fact that the application of $\_ \otimes_{R/RI} R/RJ$ to (3) produces an exact complex. □

Lemma 6. *(a) F_J^I sends $\mathcal{P}^{(m)}(I)$ to $\mathcal{P}^{(m)}(J)$, moreover, for any non-zero $M \in \mathcal{P}^{(m)}(I)$ the module $F_J^I(M)$ is non-zero as well.*

(b) F_J^I sends $\mathcal{P}^{(<\infty)}(I)$ to $\mathcal{P}^{(<\infty)}(J)$.

Proof. Let M be an A/AI-module of projective dimension m and

$$0 \to P_m \to \dots P_1 \to P_0 \to M \to 0$$

be a minimal projective resolution of M. From Proposition 1 it follows that all projective A/AI-modules are projective over R/RI. In particular, M is R/RI-projective as well.

But $\_ \otimes_{R/RI} R/RJ$ sends non-zero projective R/RI-modules to non-zero projective R/RJ-modules. In particular, $F_J^I(M) \neq 0$ as soon as M is non-zero. Using the second statement of Lemma 5 we obtain that the sequence

$$0 \to F_J^I(P_k) \to \dots F_J^I(P_1) \to F_J^I(P_0) \to F_J^I(M) \to 0 \tag{4}$$

is again exact. From Lemma 4 it follows that (4) is a minimal projective resolution of the A/AJ-module $M \otimes_{R/RI} R/J$. In particular, the projective dimension of $F_J^I(M)$ equals m. This proves the first statement and the second statement follows from the first one. □

We let $\mathtt{i}_M : M \twoheadrightarrow F_J^I(M)$ denote the canonical projection, which is a natural transformation from the identity functor to the composition of F_J^I with the natural inclusion A/AJ-mod $\subset A/AI$-mod.

Proposition 2. *Let m be a non-negative integer.*

(a) The restriction of F_J^I to $\mathcal{P}^{(<m)}(I)$ is full.

(b) The image of the restriction of F_J^I to $\mathcal{P}^{(m)}(I)$ is dense in $\mathcal{P}^{(m)}(J)$.

Proof. We prove both statements together by induction in m.

For $m = 0$ the second statement follows from the second statement of Lemma 4. Further, if P and Q are projective over A/AI and $f : F_J^I(P) \to F_J^I(Q)$, then the map $f \circ \mathtt{i}_P : P \to F_J^I(Q)$ lifts to a map $\hat{f} : P \to Q$ by projectivity of P. This implies that F_J^I is full on projective modules.

Let us now prove the induction step. Let $M \in \mathcal{P}^{(m)}(J)$. Then there exist a projective A/AJ-module P, $Q \in \mathcal{P}^{(m-1)}(J)$, and a monomorphism $f : Q \to P$ such that $M \cong \mathrm{Coker}(f)$. By induction, there exist a projective A/AI-module $\hat{P}$, a module $\hat{Q} \in \mathcal{P}^{(m-1)}(J)$, and a morphism $\hat{f} : \hat{Q} \to \hat{P}$ such that $P \cong F_J^I(\hat{P})$, $Q \cong F_J^I(\hat{Q})$ and $f = F_J^I(\hat{f})$. By Lemma 2, the morphism $\hat{f}$ in injective. Hence we can consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \hat{Q} & \xrightarrow{\hat{f}} & \hat{P} & \longrightarrow & \mathrm{Coker}(\hat{f}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathtt{i}_{\hat{Q}}} & & \downarrow{\scriptstyle \mathtt{i}_{\hat{P}}} & & \downarrow{\scriptstyle \mathtt{i}_{\mathrm{Coker}(\hat{f})}} & & \\
0 & \longrightarrow & Q & \xrightarrow{f} & P & \longrightarrow & F_J^I(\mathrm{Coker}(\hat{f})) & \longrightarrow & 0.
\end{array}$$

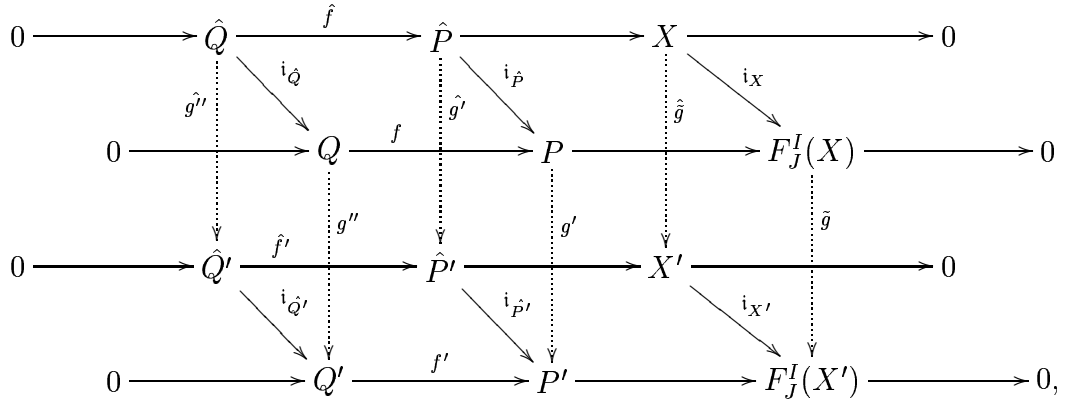
Since $\hat{f}$ in injective, both $\hat{Q}$ and $\hat{P}$ are R/I-projective and R is local, we get that $\mathrm{Coker}(\hat{f})$ is R/I-projective as well. Hence, using the second statement of Lemma 5, we obtain that the bottom row of the diagram is exact. In particular, $M \cong F_J^I(\mathrm{Coker}(\hat{f}))$, which proves the second statement.

Let now $M' \in \mathcal{P}^{(<m)}(J)$ be some other module and $g : M \to M'$ be a morphism. Again let P' be a projective cover of M' and $Q' \in \mathcal{P}^{(<m-1)}(J)$ be the corresponding kernel. Using

standard homological arguments, there exist $g' : P \to P'$ and $g'' : Q \to Q'$, making the following diagram with exact rows commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q & \xrightarrow{f} & P & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g''} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & Q' & \xrightarrow{f'} & P' & \longrightarrow & M' & \longrightarrow & 0. \end{array}$$

Now we can use the inductive assumptions to obtain the following commutative diagram with exact rows:



where $X = \mathrm{Coker}(\hat{f})$, $X' = \mathrm{Coker}(\hat{f}')$. Identifying M with $F_J^I(X)$ and M' with $F_J^I(X')$ as in the previous part of the proof, we get $F_J^I(\hat{\tilde{g}}) = \tilde{g} = g$. The remark that $X, X' \in \mathcal{P}^{(<m)}(I)$ completes the proof. □

Proof of Theorem 4. Let $J = \mathfrak{m}$. Then the first statement of Lemma 6 implies the inequality $\mathrm{fin.dim.}(A/AI) \leq \mathrm{gl.d.}(A/A\mathfrak{m})$ and the second statement of Proposition 2 gives the converse inequality. □

Corollary 7. *(a) F_J^I sends modules with standard filtrations to modules with standard filtrations and is full and dense on these modules.*

(b) F_J^I sends tilting modules to tilting modules and is full and dense on these modules.

Proof. That F_J^I sends standard modules to standard modules was shown in Proposition 1. Recall that all standard modules have finite projective dimension by [AHLU, Proposition 2.2]. Since the category of modules with standard filtrations is closed under taking kernels of epimorphisms, and all projective modules have standard filtrations, the arguments of Proposition 2 work for this category as well. This proves the first statement.

Because of the first statement, to complete the proof it is enough to show that F_J^I sends tilting modules to tilting modules, moreover, it is enough to show that for the characteristic tilting A/AI-module T we have $\mathrm{Ext}^1_{A/AJ}\left(\Delta^{(A/AJ)}, F_J^I(T)\right) = 0$. If this would be wrong, there would exist a non-split extension, say M, of $F_J^I(T)$ by $\Delta^{(A/AJ)}$. Using statement (a), the module M lifts to an extension of T by $\Delta^{(A/AI)}$. The latter must split as T is a tilting module, a contradiction. This completes the proof. □

One more immediate corollary of Theorem 4 and the second statement of Corollary 7 is the following:

Corollary 8. *Let A, R, $\mathfrak{m}$ and I be as in Theorem 2. Assume that the global dimension of the quasi-hereditary algebra $A/A\mathfrak{m}$ equals twice the projective dimension of the characteristic tilting module. Then the finitistic dimension of the properly stratified algebra A/AI equals twice the projective dimension of the characteristic tilting module.*

In the setup of this Section one can also obtain some additional information about homomorphisms between projective and tilting modules, which can be compared with [So, Theorem 5]. For projective modules, the statement is trivial: if A, R, $\mathfrak{m}$ and I are as in Theorem 2, and M is any A/AI-module, free over R/I, then $\mathrm{Hom}_{A/AI}(A/AI, M) = M$ is obviously free over R/I, and hence $\mathrm{Hom}_{A/AI}(P, M)$ is R/I-free for any projective A/AI-module P. Homomorphism between tilting modules in this case are also R/I-free, as follows from the following statement.

Proposition 3. *Let A, R, $\mathfrak{m}$ and I be as in Theorem 2. Let T be an A/AI-module having a standard filtration, and Q be an A/AI-module having a proper costandard filtration. Assume that Q is R/I-free. Then $\mathrm{Hom}_{A/AI}\big(T, Q\big)$ is a free R/I-module*

Proof. We prove the statement by induction in p. d.$(T) < \infty$ (see [AHLU, Proposition 2.2]). If T is projective, the statement follows from the remark above. Let now $0 \to M \to P \to T \to 0$ be an exact sequence with projective P. From [Dl, Theorem 5(v)] it follows that M has a standard filtration. The same statement also implies that the induced sequence

$$0 \to \mathrm{Hom}_{A/AI}\big(T, Q\big) \to \mathrm{Hom}_{A/AI}\big(P, Q\big) \to \mathrm{Hom}_{A/AI}\big(M, Q\big) \to 0$$

is exact. Since both $\mathrm{Hom}_{A/AI}\big(P, Q\big)$ and $\mathrm{Hom}_{A/AI}\big(M, Q\big)$ are R/I-free by induction, we get that $\mathrm{Hom}_{A/AI}\big(T, Q\big)$ is R/I-free as well since the algebra R/I is local and finite-dimensional over $\Bbbk$. □

5 Application to the category of Harish-Chandra bimodules

5.1 Setup for Lie algebras

For a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ the universal enveloping algebra of $\mathfrak{a}$ and by $Z(\mathfrak{a})$ the center of $U(\mathfrak{a})$.

Let $\mathfrak{g}$ be a semi-simple finite-dimensional complex Lie algebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Denote by W the Weyl group of $\mathfrak{g}$. Let ρ be one half of the sum of all positive roots. We denote by $\cdot$ the *dot-action* of W on $\mathfrak{h}^*$, defined by $w \cdot \lambda = w(\lambda + \rho) - \rho$. For $\lambda \in \mathfrak{h}^*$ we set $W_\lambda = \{w \in W \mid w \cdot \lambda = \lambda\}$.

We recall that the Harish-Chandra isomorphism with respect to the triangular decomposition above (see [Di, Section 7.4]) induces a bijection between the maximal ideals in $Z(\mathfrak{g})$ and dominant weights $\lambda \in \mathfrak{h}^*$. For a dominant $\lambda \in \mathfrak{h}^*$ we denote by $\mathfrak{m}_\lambda$ the corresponding maximal ideal in $Z(\mathfrak{g})$.

5.2 Harish-Chandra bimodules

For a $\mathfrak{g}-\mathfrak{g}$-bimodule M we define the adjoint action of $\mathfrak{g}$ on M via $g \cdot m = gm - mg$, $g \in \mathfrak{g}$, $m \in M$, and denote by M^{ad} the resulting $\mathfrak{g}$-module.

Denote by $\mathcal{H}$ the full subcategory of the category of $\mathfrak{g}-\mathfrak{g}$-bimodules, whose objects are the finitely generated $\mathfrak{g}-\mathfrak{g}$-bimodules M for which M^{ad} is a direct sum of finite dimensional $\mathfrak{g}$-modules. The objects of $\mathcal{H}$ are called *Harish-Chandra bimodules*. Let λ, μ be dominant and $m, n \in \mathbb{N} \cup \{\infty\}$. Denote by ${}^{n}_{\lambda}\mathcal{H}^{m}_{\mu}$ the full subcategory of $\mathcal{H}$, which consists of all $\mathfrak{g}-\mathfrak{g}$-bimodules $M \in \mathcal{H}$, satisfying the following condition: $\mathfrak{m}^{n}_{\lambda} M = M\mathfrak{m}^{m}_{\mu} = 0$ (where $\mathfrak{m}^{\infty}_{\lambda} M = 0$ and $M\mathfrak{m}^{\infty}_{\mu} = 0$ means that the left action of $\mathfrak{m}_\lambda$ and, respectively, the right action of $\mathfrak{m}_\mu$ on M is locally nilpotent). We refer the reader to [Ja] and [BG] for all undefined notions, notation, and more details on Harish-Chandra bimodules.

5.3 Harish-Chandra bimodules via quasi-hereditary algebras over local rings

From [BG, Section 5] it follows that the category ${}^{\infty}_{\lambda}\mathcal{H}^{m}_{\mu}$ has enough projective modules for every $m \in \mathbb{N}$. Let A_m denote the corresponding basic associative algebra, that is the unique (up to an isomorphism) basic finite-dimensional associative algebra, whose module category is equivalent to ${}^{\infty}_{\lambda}\mathcal{H}^{m}_{\mu}$. For $M \in {}^{\infty}_{\lambda}\mathcal{H}^{m}_{\mu}$ the bimodule $M/M\mathfrak{m}^{m-1}_{\mu}$ belongs to ${}^{\infty}_{\lambda}\mathcal{H}^{m-1}_{\mu}$, which defines a full and dense functor $F^{m}_{m-1} : {}^{\infty}_{\lambda}\mathcal{H}^{m}_{\mu} \to {}^{\infty}_{\lambda}\mathcal{H}^{m-1}_{\mu}$ via $F^{m}_{m+1}(M) = M/M\mathfrak{m}^{m-1}_{\mu}$. This functor induces an algebra epimorphism $A_m \twoheadrightarrow A_{m-1}$ and thus we can define the inverse limit algebra $A = \varprojlim A_m$ (see [So, Section 5]).

The right action of $Z(\mathfrak{g})$ on ${}^{\infty}_{\lambda}\mathcal{H}^{m}_{\mu}$ induces a homomorphism $Z(\mathfrak{g}) \to A_m$, whose image we denote by R_m. The functors F^{m}_{m-1} induce surjections $R_m \twoheadrightarrow R_{m-1}$ and one obtains that the inverse limit algebra $R = \varprojlim R_m$ is a subalgebra of A. It follows from the definition that R is commutative. Let R' denote the completion of $Z(\mathfrak{g})$ with respect to $\mathfrak{m}_\mu$. By the definition of ${}^{\infty}_{\lambda}\mathcal{H}^{m}_{\mu}$, R is even a homomorphic image of R' and, considering the action of $Z(\mathfrak{g})$ on the bimodule $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}^{m}_{\mu}$, we obtain that the epimorphism $R' \twoheadrightarrow R$ is in fact an isomorphism. The ring R is thus local and complete with the image $\mathfrak{m}$ of $\mathfrak{m}_\mu$ being the maximal ideal of R.

Proposition 4. *Assume that μ is regular. Then:*

(a) A is a quasi-hereditary algebra over R.

(b) For any $m \in \mathbb{N}$ we have $A/A\mathfrak{m}^{m} \simeq A_m$.

Proof. The second statement follows from the definition of A, so we prove the first one. By [BG, Section 5], the bimodule $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}^{m}_{\mu}$ is projective in ${}^{\infty}_{\lambda}\mathcal{H}^{m}_{\mu}$ and all other projective bimodules in ${}^{\infty}_{\lambda}\mathcal{H}^{m}_{\mu}$ are direct summands in some (left) translations of $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}^{m}_{\mu}$. Kostant's Theorem (that $U(\mathfrak{g})$ is a free $Z(\mathfrak{g})$-module) implies that $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}^{m}_{\mu}$ is free over $R/\mathfrak{m}^{m}_{\mu}$. Since any left translation of a bimodule commutes with the right action of the

center on a bimodule, we obtain that all projective bimodules in ${}^{\infty}_{\lambda}\mathcal{H}^m_\mu$ are free over $R/\mathfrak{m}^m_\mu$. Taking the inverse limit we obtain that A is free as an R-module.

That the algebra A_1 is quasi-hereditary (if μ is regular) is well-known. For example, this follows from [BG, Section 5] and [BGG]. This completes the proof. □

5.4 Finitistic dimension of ${}^{\infty}_{\lambda}\mathcal{H}^m_\mu$

In this subsection we work under the assumptions of Proposition 4, that is, we assume that μ is regular.

Corollary 9. *(a) For any $m \in \mathbb{N}$ the algebra A_m is properly stratified.*

(b) The algebra A_m is quasi-hereditary if and only if $m = 1$.

(c) For any ideal I in R of finite codimension the algebra A/AI is properly stratified.

Proof. Follows from Proposition 4 and Proposition 1. □

Corollary 10. *For any ideal I of finite codimension in R the finitistic dimension of A/AI equals the global dimension of A_1. In particular, the finitistic dimension of ${}^{\infty}_{\lambda}\mathcal{H}^m_\mu$ equals the global dimension of ${}^{\infty}_{\lambda}\mathcal{H}^1_\mu$.*

Proof. Follows from Proposition 4 and Theorem 4. □

Corollary 11. *For any ideal I of finite codimension in R the finitistic dimension of A/AI equals twice the projective dimension of the characteristic tilting module in A/AI.*

Proof. Let x denote the finitistic dimension of A_m, y denote the global dimension of A_1, z denote twice the projective dimension of the characteristic tilting module in A_m, t denote twice the projective dimension of the characteristic tilting module in A_1. We have $x = y$ by Corollary 10, $z = t$ by Proposition 2 and Corollary 7, and $y = t$ by [BG, theorem 5.9] and Corollary 6. Hence $x = z$ completing the proof. □

6 Finitistic dimension of some other categories of Harish-Chandra bimodules via translation functors

The aim of this section is to prove the following result:

Theorem 5. *Let $\lambda \in \mathfrak{h}^*$ be dominant and integral. Then the following numbers are equal:*

(i) The global dimension of ${}^{\infty}_{\lambda}\mathcal{H}^1_0$.

(ii) The finitistic dimension of ${}^{\infty}_{\lambda}\mathcal{H}^1_0$.

(iii) Twice the projective dimension of the characteristic tilting module in ${}^{\infty}_{\lambda}\mathcal{H}^1_0$.

(iv) The finitistic dimension of ${}^{\infty}_{\lambda}\mathcal{H}^1_{\lambda}$.

(v) Twice the projective dimension of the characteristic tilting module in ${}^{\infty}_{\lambda}\mathcal{H}^1_{\lambda}$.

(vi) The finitistic dimension of ${}^{\infty}_{0}\mathcal{H}^1_{\lambda}$.

(vii) Twice the projective dimension of the characteristic tilting module in ${}^{\infty}_{0}\mathcal{H}^1_{\lambda}$.

Note that [BG, Theorem 5.9] asserts that the category ${}^{\infty}_{\lambda}\mathcal{H}^1_{0}$ from (i), (ii) and (iii) is equivalent to an integral block of the BGG category $\mathcal{O}$, [BGG]; the category ${}^{\infty}_{0}\mathcal{H}^1_{\lambda}$ from (vi) and (vii) is equivalent to a regular block of a parabolic generalization of $\mathcal{O}$ from [FKM2]; and the category ${}^{\infty}_{\lambda}\mathcal{H}^1_{\lambda}$ from (iv) and (v) is equivalent to a singular block of this parabolic generalization of $\mathcal{O}$ from [FKM2]. In particular, all these categories are equivalent to module categories of quasi-hereditary or properly stratified algebras. Moreover, [BG, Theorem 5.9] also gives a full and faithful embedding $\mathfrak{i}$ of ${}^{\infty}_{\lambda}\mathcal{H}^1_{\lambda}$ into ${}^{\infty}_{\lambda}\mathcal{H}^1_{0}$, which sends projective modules to projective modules and tilting modules to titling modules.

We will prove this theorem using (left) translation functors $T^{\mu}_{\lambda} : {}^{\infty}_{\lambda}\mathcal{H}^1_{\nu} \to {}^{\infty}_{\mu}\mathcal{H}^1_{\nu}$, and we refer the reader to [Ja, 4.12] for the precise definition and properties of these functors. We will only need that translation functors are exact, send projective modules to projective modules, and tilting modules to tilting modules. In particular, if $M = T^{\mu}_{\lambda}(N)$, we automatically get $\mathrm{p.d.}(M) \leq \mathrm{p.d.}(N)$. To prove Theorem 5 we will need several lemmas.

Lemma 7. *Let A be a finite-dimensional algebra over some field and I be an injective cogenerator of $A-$*mod. *Assume that* $\mathrm{p.d.}(I) < \infty$ *and* $\mathrm{fin.dim.}(A) < \infty$. *Then* $\mathrm{fin.dim.}(A) = \mathrm{p.d.}(I)$.

Proof. Let M be an A-module such that $\mathrm{p.d.}(M) = \mathrm{fin.dim.}(A) < \infty$, and $0 \to M \to Q \to N \to 0$ be an exact sequence with injective Q. Now $\mathrm{p.d.}(Q) < \infty$ implies $\mathrm{p.d.}(N) < \infty$ and hence $\mathrm{p.d.}(N) \leq \mathrm{p.d.}(M)$ by the choice of M. Applying $\mathrm{Hom}_A(\_, L)$ we hence obtain a surjection from $\mathrm{Ext}^{\mathrm{p.d.}(M)}_A(Q, L)$ to $\mathrm{Ext}^{\mathrm{p.d.}(M)}_A(M, L) \neq 0$ in the long exact sequence. This implies $\mathrm{p.d.}(Q) \geq \mathrm{p.d.}(M)$ and proves our statement. □

Lemma 8. *Let $\mathcal{C}$ be one of the categories ${}^{\infty}_{\lambda}\mathcal{H}^1_{\lambda}$, ${}^{\infty}_{0}\mathcal{H}^1_{\lambda}$ or ${}^{\infty}_{\lambda}\mathcal{H}^1_{0}$. Then the finitistic dimension of $\mathcal{C}$ equals the projective dimension of the dominant costandard module in $\mathcal{C}$.*

Proof. First we note that $\mathrm{fin.dim.}(\mathcal{C})$ is finite as $\mathcal{C}$ is equivalent to the module category of a properly stratified algebra.

Further, the tilting modules in $\mathcal{C}$ are produced by translating the standard tilting module. The last one is self-dual and hence cotilting, implying that all tilting modules in $\mathcal{C}$ are cotilting. Further, all tilting modules have finite projective dimension, hence all cotilting modules have finite projective dimension. All cotilting modules have a costandard filtration and thus, by induction, all costandard modules have finite projective dimension. But all injective modules have a costandard filtration, implying that all injective modules have finite projective dimension. From Lemma 7 it now follows that the finitistic dimension of $\mathcal{C}$ equals the projective dimension of an injective cogenerator of $\mathcal{C}$.

Translating the dominant costandard module produces new modules with costandard filtrations, which surject onto the original one. Since translation does not increase the projective dimension, we get that the kernels of all these surjections have projective dimensions less than or equal to the projective dimension of the dominant costandard module. Since all costandard modules can appear via an iteration of this process, an induction implies that the projective dimension of the dominant costandard module is the maximal one among all projective dimensions of all costandard modules.

Finally, since injective modules have costandard filtrations, their projective dimensions cannot be bigger than the maximum of the projective dimensions of all costandard modules. This, together with Lemma 7, implies our statement. □

Corollary 12. *The projective dimension of an injective cogenerator of any of the categories* ${}^{\infty}_{\lambda}\mathcal{H}^1_{\lambda}$, ${}^{\infty}_{0}\mathcal{H}^1_{\lambda}$ *and* ${}^{\infty}_{\lambda}\mathcal{H}^1_{0}$ *is finite.*

Lemma 9. *The numbers in* (ii), (iv) *and* (vi) *of Theorem 5 are equal.*

Proof. The categories $\mathfrak{i}\,({}^{\infty}_{\lambda}\mathcal{H}^1_{\lambda})$ and ${}^{\infty}_{\lambda}\mathcal{H}^1_{0}$ share the same dominant costandard module and hence Lemma 8 implies (ii)=(iv).

Finally, there is a translation functor, which sends the dominant costandard module from ${}^{\infty}_{\lambda}\mathcal{H}^1_{\lambda}$ to the dominant costandard module in ${}^{\infty}_{0}\mathcal{H}^1_{\lambda}$. Analogously, there is a translation functor, which sends the dominant costandard module from ${}^{\infty}_{0}\mathcal{H}^1_{\lambda}$ to a direct sum of several copies of the dominant costandard module in ${}^{\infty}_{\lambda}\mathcal{H}^1_{\lambda}$. This implies that these two modules must have the same projective dimension and hence (iv)=(vi). This completes the proof. □

Proof of Theorem 5. First we remark that (i)=(ii) since ${}^{\infty}_{\lambda}\mathcal{H}^1_{0}$ is equivalent to the module category of a quasi-hereditary algebra. Further, the equality (ii)=(iii) is the statement of Corollary 6 (see also [MP, Theorem 2]). Using $\mathfrak{i}$ we also have (v)$\leq$(iii).

Now, Corollary 6 implies (iii)=(ii), Lemma 9 implies (ii)=(iv), and [MP, Theorem 1] gives (iv)$\leq$(v). Hence (v)=(iii).

Finally, let us prove (v)=(vii). Since translations from ${}^{\infty}_{0}\mathcal{H}^1_{\lambda}$ to ${}^{\infty}_{\lambda}\mathcal{H}^1_{\lambda}$ produce all tilting modules we obtain (v)$\leq$(vii).

Let now μ be integral with stabilizer $w_0W_\lambda w_0$, where w_0 is the longest element in the Weyl group. Since conjugation with the longest element sends simple reflections to simple reflections, it defines an involutive automorphism of the Dynkin diagram of $\mathfrak{g}$, which gives rise to an automorphism φ of $\mathfrak{g}$, which preserves the Borel subalgebra. This automorphism induces an equivalence between the categories ${}^{\infty}_{\lambda}\mathcal{H}^1_{0}$ and ${}^{\infty}_{\mu}\mathcal{H}^1_{0}$. In particular, it follows that these categories have the same global dimension. Since the simple tilting module in both categories is unique by [Di, Proposition 7.6.3], it follows that simple tilting modules in ${}^{\infty}_{\lambda}\mathcal{H}^1_{0}$ and ${}^{\infty}_{\mu}\mathcal{H}^1_{0}$ have the same projective dimension. Translating the simple tilting module from ${}^{\infty}_{\mu}\mathcal{H}^1_{0}$ to ${}^{\infty}_{0}\mathcal{H}^1_{0}$ produces a tilting module in ${}^{\infty}_{0}\mathcal{H}^1_{0}$, which, in fact, is the standard tilting module in ${}^{\infty}_{0}\mathcal{H}^1_{\lambda}$. This implies (vii)$\leq$(v) and thus (v)=(vii). This completes the proof. □

Various formulae for the global dimension of ${}^{\infty}_{\lambda}\mathcal{H}^1_{0}$ can be found in [MP]. Moreover, [MP, Theorem 1] immediately implies the following:

Corollary 13. *For an integral λ we have:*

$$\text{fin. dim.}\left({}^{\infty}_{0}\mathcal{H}^{1}_{\lambda}\right) = 2\dim_{\mathcal{F}(\Delta)}\left({}^{\infty}_{0}\mathcal{H}^{1}_{\lambda}\right).$$

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