Parabolic category $\mathscr{O}$ for classical Lie superalgebras

Volodymyr Mazorchuk

Abstract We compare properties of (the parabolic version of) the BGG category $\mathscr{O}$ for semi-simple Lie algebras with those for classical (not necessarily simple) Lie superalgebras.

1 Introduction

Category $\mathscr{O}$ for semi-simple complex finite dimensional Lie algebras, introduced in [BGG], is a central object of study in the modern representation theory (see [Hu]) with many interesting connections to, in particular, combinatorics, algebraic geometry and topology. This category has a natural counterpart in the super-world and this super version $\tilde{\mathscr{O}}$ of category $\mathscr{O}$ was intensively studied (mostly for some particular simple classical Lie superalgebra) in the last decade, see e.g. [Br1, Br2, Go2, FM2, CLW, CMW] or the recent books [Mu, CW] for details.

The aim of the present paper is to compare some basic but general properties of the category $\mathscr{O}$ in the non-super and super cases for a rather general classical super-setup. We mainly restrict to the properties for which the non-super and super cases can be connected using the usual restriction and induction functors and the biadjunction (up to parity change) between these two functors is our main tool. The paper also complements, extends and gives a more detailed exposition for some results which appeared in [AM, Section 7].

The original category $\mathscr{O}$ has a natural parabolic version which first appeared in [RC]. We start in Section 2 by setting up an elementary approach (using root system geometry) to the definition of a parabolic category $\mathscr{O}$ for classical finite dimensional Lie superalgebras. In Section 3 we define the parabolic category $\widetilde{\mathscr{O}}^{\omega}$ and describe its basic categorical properties, including simple objects and blocks. In Section 4 we

Volodymyr Mazorchuk
Department of Mathematics, Uppsala University, Box. 480, SE-75106, Uppsala, SWEDEN e-mail: mazor@math.uu.se

address the natural stratification on $\widetilde{\mathcal{O}}^{\omega}$ and its consequences, in particular, existence of tilting modules and estimates for the finitistic dimension, proving the following:

Theorem 1. *Category $\widetilde{\mathcal{O}}^{\omega}$ has finite finitistic dimension.*

Finally, in Section 5 we address properties of $\widetilde{\mathcal{O}}^{\omega}$ which are based upon projective-injective modules in this category. This includes an Irving-type theorem describing socular constituents of Verma modules and an analogue of Soergel's Struktursatz. As the last application we prove that category $\widetilde{\mathcal{O}}^{\omega}$ is Ringel self-dual.

Acknowledgment. The research is partially supported by the Swedish Research Council and the Royal Swedish Academy of Sciences.

2 Preliminaries

2.1 Classical Lie superalgebras

We work over $\mathbb{C}$ and set $\mathbb{N} = \{1,2,3,\dots\}$, $\mathbb{Z}_+ = \{0,1,2,3\dots\}$. For a Lie (super)algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ the corresponding enveloping (super)algebra.

Let $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ be a Lie superalgebra over $\mathbb{C}$. From now on we assume that $\mathfrak{g}$ is *classical* in the sense that $\mathfrak{g}_{\overline{0}}$ is a finite dimensional reductive Lie algebra and $\mathfrak{g}_{\overline{1}}$ is a semi-simple finite dimensional $\mathfrak{g}_{\overline{0}}$-module. We do **not** require $\mathfrak{g}$ to be simple. We denote by $\mathfrak{g}$-smod the abelian category of $\mathfrak{g}$-supermodules. Morphisms in $\mathfrak{g}$-smod are homogeneous $\mathfrak{g}$-homomorphisms of degree 0.

Example 1. The general linear superalgebra $\mathfrak{gl}(\mathbb{C}^{m|n})$ of the super vector space $\mathbb{C}^{m|n} = \mathbb{C}^m_{\overline{0}} \oplus \mathbb{C}^n_{\overline{1}}$ with respect to the usual super-commutator of linear operators. Fix the standard bases in $\mathbb{C}^m_{\overline{0}}$ and $\mathbb{C}^n_{\overline{1}}$ and $\mathfrak{gl}(\mathbb{C}^{m|n})$ becomes isomorphic to the superalgebra $\mathfrak{gl}(m|n)$ of $(n+m)\times(n+m)$ matrices naturally divided into $n\times n$, $n\times m$, $m\times n$ and $m\times m$ blocks, under the usual super-commutator of matrices.

Example 2. The subsuperalgebra $\mathfrak{q}_n$ of $\mathfrak{gl}(n|n)$ consisting of all matrices of the form

$$\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right). \tag{1}$$

The even part corresponds to $B = 0$ while the odd part corresponds to $A = 0$.

Example 3. Let $\mathfrak{a}$ be any finite dimensional reductive Lie algebra and V any semi-simple finite dimensional $\mathfrak{a}$-module. Set $\mathfrak{g}_{\overline{0}} := \mathfrak{a}$, $\mathfrak{g}_{\overline{1}} := V$ and $\mathfrak{g}(\mathfrak{a},V) := \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$. Setting $[\mathfrak{g}_{\overline{1}},\mathfrak{g}_{\overline{1}}] = 0$ and considering the natural action of the Lie algebra $\mathfrak{g}_{\overline{0}}$ on the $\mathfrak{g}_{\overline{0}}$-module $\mathfrak{g}_{\overline{1}}$ defines on $\mathfrak{g}(\mathfrak{a},V)$ the structure of a Lie superalgebra, which is called the *generalized Takiff superalgebra* associated with $\mathfrak{a}$ and V. These superalgebras appear in [GM].

2.2 Natural categories of supermodules via restriction

Consider $\mathfrak{g}_{\overline{0}}$ as a purely even Lie superalgebra. Then $\mathfrak{g}_{\overline{0}}$-smod is equivalent to the direct sum of an even and an odd copy of $\mathfrak{g}_{\overline{0}}$-mod in the obvious way:

$$\mathfrak{g}_{\overline{0}}\text{-smod} \cong (\mathfrak{g}_{\overline{0}}\text{-mod})_{\overline{0}} \oplus (\mathfrak{g}_{\overline{0}}\text{-mod})_{\overline{1}}.$$

Further, we have the usual restriction functor

$$\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\overline{0}}} : \mathfrak{g}\text{-smod} \to \mathfrak{g}_{\overline{0}}\text{-smod}.$$

For any subcategory $\mathscr{C}$ in $\mathfrak{g}_{\overline{0}}$-mod we now can define the category $\widetilde{\mathscr{C}}$ (the notation follows [Mu]) as the subcategory in $\mathfrak{g}$-smod consisting of all objects and morphisms which are sent to $(\mathscr{C})_{\overline{0}} \oplus (\mathscr{C})_{\overline{1}}$ by $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\overline{0}}}$.

The functor $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\overline{0}}}$ is exact and has both the left adjoint

$$\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\overline{0}}} := U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\overline{0}})} {}_{-} : \mathfrak{g}_{\overline{0}}\text{-smod} \to \mathfrak{g}\text{-smod}$$

and the right adjoint

$$\mathrm{Coind}^{\mathfrak{g}}_{\mathfrak{g}_{\overline{0}}} := \mathrm{Hom}_{U(\mathfrak{g}_{\overline{0}})}(U(\mathfrak{g}), {}_{-}) : \mathfrak{g}_{\overline{0}}\text{-smod} \to \mathfrak{g}\text{-smod}.$$

Furthermore, by [Go1, Theorem 3.2.3] we have

$$\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\overline{0}}} \cong \Pi^{\dim \mathfrak{g}_{\overline{1}}} \circ \mathrm{Coind}^{\mathfrak{g}}_{\mathfrak{g}_{\overline{0}}}, \tag{2}$$

where Π is the functor which changes the parity (see e.g. [Go1]). If the subcategory $\mathscr{C}$ above is isomorphism-closed and stable under tensoring with the $\mathfrak{g}_{\overline{0}}$-module $\bigwedge \mathfrak{g}_{\overline{1}}$, then $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\overline{0}}}$ maps $\mathscr{C}$ to $\widetilde{\mathscr{C}}$.

2.3 Weight (super)modules

Fix some Cartan subalgebra $\mathfrak{h}_{\overline{0}}$ in $\mathfrak{g}_{\overline{0}}$. Since the Lie algebra $\mathfrak{g}_{\overline{0}}$ is reductive, the algebra $\mathfrak{h}_{\overline{0}}$ is commutative and contains the (possibly zero) center of $\mathfrak{g}_{\overline{0}}$. A $\mathfrak{g}_{\overline{0}}$-module V is called a *weight* module (with respect to $\mathfrak{h}_{\overline{0}}$) provided that the action of $\mathfrak{h}_{\overline{0}}$ on V is diagonalizable. Put differently, the module V is weight if we have a decomposition

$$V \cong \bigoplus_{\lambda \in \mathfrak{h}_{\overline{0}}^*} V_\lambda, \quad \text{where} \quad V_\lambda := \{v \in V | h \cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h}_{\overline{0}}\}.$$

The space V_λ is called the *weight space* of V corresponding to a *weight* λ. For a weight module V the *support* of V is the set

$$\mathrm{supp}(V) = \mathrm{supp}_{\mathfrak{h}_{\overline{0}}}(V) := \{\lambda \in \mathfrak{h}_{\overline{0}}^* | V_\lambda \neq 0\}.$$

Denote by $\mathfrak{W}$ the full subcategory in $\mathfrak{g}_{\overline{0}}$-mod consisting of all weight modules. Note that $\mathfrak{W}$ is both isomorphism-closed and closed under the usual tensor product of $\mathfrak{g}_{\overline{0}}$-modules. Furthermore, since $\mathfrak{g}_{\overline{1}}$ is a semi-simple finite dimensional $\mathfrak{g}_{\overline{0}}$-module, we have $\mathfrak{g}_{\overline{1}} \in \mathfrak{W}$ and thus $\bigwedge \mathfrak{g}_{\overline{1}} \in \mathfrak{W}$. This implies that $\mathfrak{W}$ is stable under tensoring with $\bigwedge \mathfrak{g}_{\overline{1}}$.

Now we can consider the corresponding category $\widetilde{\mathfrak{W}}$ of *weight* $\mathfrak{g}$-supermodules and from Subsection 2.2 we obtain that $(\mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}}, \mathrm{Res}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}}, \mathrm{Coind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}})$ restricts to an adjoint triple of functors between $\mathfrak{W}$ and $\widetilde{\mathfrak{W}}$.

Example 4. We have $\mathfrak{g}_{\overline{0}} \in \mathfrak{W}$ and $\mathfrak{g} \in \widetilde{\mathfrak{W}}$, where $\mathfrak{g}_{\overline{0}}$ and $\mathfrak{g}$ are the adjoint module and supermodule, respectively.

2.4 Parabolic and triangular decompositions

This is inspired by [DMP]. Consider the real vector space $\mathscr{H} := \mathbb{R}\mathrm{supp}(\mathfrak{g}_{\overline{0}})$. The set $R_{\overline{0}} := \mathrm{supp}(\mathfrak{g}_{\overline{0}}) \setminus \{0\}$ is a root system in $\mathscr{H}$ and we let W be the corresponding Weyl group and $(\cdot,\cdot)$ the usual W-invariant inner product on $\mathscr{H}$. For a fixed $\omega \in \mathscr{H}$ we have a *parabolic decomposition*

$$\mathfrak{g}_{\overline{0}} = \mathfrak{n}_{\overline{0}}^{\omega,-} \oplus \mathfrak{l}_{\overline{0}}^{\omega} \oplus \mathfrak{n}_{\overline{0}}^{\omega,+} \tag{3}$$

of $\mathfrak{g}_{\overline{0}}$, where

$$\mathfrak{n}_{\overline{0}}^{\omega,-} = \bigoplus_{\substack{\alpha \in R_{\overline{0}} \\ (\alpha,\omega)<0}} (\mathfrak{g}_{\overline{0}})_\lambda, \qquad \mathfrak{l}_{\overline{0}}^{\omega} = \bigoplus_{\substack{\alpha \in R_{\overline{0}} \cup \{0\} \\ (\alpha,\omega)=0}} (\mathfrak{g}_{\overline{0}})_\lambda, \qquad \mathfrak{n}_{\overline{0}}^{\omega,+} = \bigoplus_{\substack{\alpha \in R_{\overline{0}} \\ (\alpha,\omega)>0}} (\mathfrak{g}_{\overline{0}})_\lambda.$$

The subalgebra $\mathfrak{p}_{\overline{0}}^{\omega} := \mathfrak{l}_{\overline{0}}^{\omega} \oplus \mathfrak{n}_{\overline{0}}^{\omega,+}$ is a parabolic subalgebra of $\mathfrak{g}_{\overline{0}}$, $\mathfrak{n}_{\overline{0}}^{\omega,+}$ is the nilpotent radical of $\mathfrak{p}_{\overline{0}}^{\alpha}$ and $\mathfrak{l}_{\overline{0}}^{\omega}$ is the corresponding Levi subalgebra. In the case $(\alpha,\omega) \neq 0$ for all $\alpha \in R_{\overline{0}}$, we have $\mathfrak{l}_{\overline{0}}^{\omega} = \mathfrak{h}_{\overline{0}}$, moreover, $\mathfrak{b}_{\overline{0}}^{\omega} := \mathfrak{h}_{\overline{0}} \oplus \mathfrak{n}_{\overline{0}}^{\omega,+}$ is a Borel subalgebra of $\mathfrak{g}_{\overline{0}}$ and the decomposition (3) is a *triangular decomposition* of $\mathfrak{g}_{\overline{0}}$ in the sense of [MoPi]. For nonzero $\omega_1, \omega_2 \in \mathscr{H}$ say that ω_1 and ω_2 are equivalent if the parabolic decompositions of $\mathfrak{g}_{\overline{0}}$ corresponding to ω_1 and ω_2 coincide. Then the equivalence classes are exactly the (nonzero) facets of the simplicial cone decomposition of $\mathscr{H}$ as described e.g. in [Sa, § 1.2].

Consider the derived Lie algebra $\mathfrak{g}'_{\overline{0}} := [\mathfrak{g}_{\overline{0}}, \mathfrak{g}_{\overline{0}}]$ of $\mathfrak{g}_{\overline{0}}$ and let $\mathfrak{h}'_{\overline{0}} := \mathfrak{g}'_{\overline{0}} \cap \mathfrak{h}_{\overline{0}}$. Set $R'_{\overline{0}} := \mathrm{supp}_{\mathfrak{h}'_{\overline{0}}}(\mathfrak{g}_{\overline{0}}) \setminus \{0\}$ and $\mathscr{H}' := \mathbb{R}R'_{\overline{0}}$. Then the set $R'_{\overline{0}}$ is again a root system in $\mathscr{H}'$ (of the same type as $R_{\overline{0}}$). Moreover, there is the obvious canonical isomorphism $R'_{\overline{0}} \cong R_{\overline{0}}$ of root systems which induces an isomorphism $\mathscr{H}' \cong \mathscr{H}$ of vector spaces. Using this isomorphism we identify $\mathscr{H}'$ and $\mathscr{H}$.

Let $R'_{\overline{1}} := \mathrm{supp}_{\mathfrak{h}'_{\overline{0}}}(\mathfrak{g}_{\overline{1}}) \subset \mathscr{H}'$. Then the same $\omega \in \mathscr{H} = \mathscr{H}'$ leads to the decomposition

$$\mathfrak{g}_{\bar{1}} = \mathfrak{n}_{\bar{1}}^{\omega,-} \oplus \mathfrak{l}_{\bar{1}}^{\omega} \oplus \mathfrak{n}_{\bar{1}}^{\omega,+}, \tag{4}$$

where

$$\mathfrak{n}_{\bar{1}}^{\omega,-} = \bigoplus_{\substack{\alpha\in R_{\bar{1}} \\ (\alpha,\omega)<0}} (\mathfrak{g}_{\bar{1}})_\lambda, \qquad \mathfrak{l}_{\bar{1}}^{\omega} = \bigoplus_{\substack{\alpha\in R_{\bar{1}} \\ (\alpha,\omega)=0}} (\mathfrak{g}_{\bar{1}})_\lambda, \qquad \mathfrak{n}_{\bar{1}}^{\omega,+} = \bigoplus_{\substack{\alpha\in R_{\bar{1}} \\ (\alpha,\omega)>0}} (\mathfrak{g}_{\bar{1}})_\lambda.$$

Setting

$$\mathfrak{n}^{\omega,\pm} := \mathfrak{n}_{\bar{0}}^{\omega,\pm} \oplus \mathfrak{n}_{\bar{1}}^{\omega,\pm} \quad \text{and} \quad \mathfrak{l}^{\omega} := \mathfrak{l}_{\bar{0}}^{\omega} \oplus \mathfrak{l}_{\bar{1}}^{\omega},$$

and combining (3) and (4) we obtain the following *parabolic decomposition* of $\mathfrak{g}$ corresponding to ω:

$$\mathfrak{g} := \mathfrak{n}^{\omega,-} \oplus \mathfrak{l}^{\omega} \oplus \mathfrak{n}^{\omega,+}. \tag{5}$$

Here $\mathfrak{p}^{\omega} := \mathfrak{l}^{\omega} \oplus \mathfrak{n}^{\omega,+}$ is a parabolic subalgebra with the "nilpotent radical" $\mathfrak{n}^{\omega,+}$ and the "Levi subalgebra" $\mathfrak{l}^{\omega}$.

Set $R' := R'_{\bar{0}} \cup R'_{\bar{1}}$. If the only $\alpha \in R'$ satisfying $(\alpha,\omega) = 0$ is $\alpha = 0$, then decomposition (5) is called a *triangular decomposition*. An important difference with the Lie algebra case is that even in the case of a triangular decomposition we might have $\mathfrak{l}^{\omega} \neq \mathfrak{h}_{\bar{0}}$.

Example 5. Let $\mathfrak{g} = \mathfrak{q}_n$ for $n > 1$, and $\mathfrak{h}_{\bar{0}}$ be the subalgebra of all matrices of the form (1) for which $B = 0$ and A is diagonal. Choose any ω such that $\mathfrak{n}_{\bar{0}}^{\omega,+}$ consists of all matrices of the form (1) for which $B = 0$ and A is upper triangular. Then $\mathfrak{n}_{\bar{0}}^{\omega,-}$ consists of all matrices of the form (1) for which $B = 0$ and A is lower triangular; $\mathfrak{h}_{\bar{1}}$ consists of all matrices of the form (1) for which $A = 0$ and B is diagonal; $\mathfrak{n}_{\bar{1}}^{\omega,+}$ consists of all matrices of the form (1) for which $A = 0$ and B is upper triangular; $\mathfrak{n}_{\bar{1}}^{\omega,-}$ consists of all matrices of the form (1) for which $A = 0$ and B is lower triangular. In this case the "Cartan subalgebra" $\mathfrak{h} := \mathfrak{l}^{\omega}$ is not commutative.

Equivalence classes of elements from $\mathscr{H}'$ which give rise to the same parabolic decomposition of $\mathfrak{g}$ define a simplicial cone decomposition of $\mathscr{H}'$ which refines the one defined for $\mathfrak{g}_{\bar{0}}$ above.

3 Parabolic category $\widetilde{\mathscr{O}}$ and its elementary properties

3.1 Parabolic categories $\mathscr{O}^{\omega}$ and $\widetilde{\mathscr{O}}^{\omega}$

Fix an ω as above and consider the corresponding parabolic decompositions of $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}$, given by (3) and (5), respectively. Denote by $\mathscr{O}^{\omega} = {}^{\mathfrak{g}_{\bar{0}}}\mathscr{O}^{\omega}$ the full subcategory of $\mathfrak{g}_{\bar{0}}$-mod consisting of all modules M which are

- finitely generated,
- decompose into a direct sum of simple finite dimensional $\mathfrak{l}_{\bar{0}}^{\omega}$-modules,

- are locally $\mathfrak{n}_{\overline{0}}^{\omega,+}$-finite in the sense that $\dim(U(\mathfrak{n}_{\overline{0}}^{\omega,+})v) < \infty$ for all $v \in M$.

The category $\mathcal{O}^{\omega}$ is the $\mathfrak{p}_{\overline{0}}^{\omega}$-parabolic version of the BGG category $\mathcal{O}$. The original category $\mathcal{O}$ was defined in [BGG] (it corresponds to the situation when the decomposition (3) is a triangular decomposition), and the parabolic version was defined in [RC]. We also refer to [Hu] for more details. We will drop the superscript $\mathfrak{g}_{\overline{0}}$ if it is clear from the context.

The category $\mathcal{O}^{\omega}$ is isomorphism-closed and stable under tensoring with simple finite dimensional $\mathfrak{g}_{\overline{0}}$-modules. Hence the corresponding category $\widetilde{\mathcal{O}}^{\omega} = {}^{\mathfrak{g}}\widetilde{\mathcal{O}}^{\omega}$ of $\mathfrak{g}$-modules leads us to the nice situation described at the end of Subsection 2.2 (we will drop the superscript $\mathfrak{g}$ if it is clear from the context). Alternatively, the category $\widetilde{\mathcal{O}}^{\omega}$ can be described as the full subcategory of $\mathfrak{g}$-smod consisting of all supermodules M which are

- finitely generated,
- decompose into a direct sum of simple finite dimensional $\mathfrak{l}_{\overline{0}}^{\omega}$-modules,
- are locally $\mathfrak{n}^{\omega,+}$-finite in the sense that $\dim(U(\mathfrak{n}^{\omega,+})v) < \infty$ for all $v \in M$.

Note that it is really $\mathfrak{l}_{\overline{0}}^{\omega}$ and not $\mathfrak{l}^{\omega}$ in the second condition.

3.2 *Elementary categorical properties of $\widetilde{\mathcal{O}}^{\omega}$*

Proposition 1.

(a) *$\widetilde{\mathcal{O}}^{\omega}$ is a Serre subcategory of $\widetilde{\mathfrak{W}}$, in particular, $\widetilde{\mathcal{O}}^{\omega}$ is abelian.*
(b) *Every object of $\widetilde{\mathcal{O}}^{\omega}$ has finite length as a $\mathfrak{g}$-module.*
(c) *$\widetilde{\mathcal{O}}^{\omega}$ has enough projective modules.*
(d) *$\widetilde{\mathcal{O}}^{\omega}$ has enough injective modules.*
(e) *All morphism spaces in $\widetilde{\mathcal{O}}^{\omega}$ are finite dimensional.*
(f) *For every i and any $M,N \in \widetilde{\mathcal{O}}^{\omega}$ we have $\dim \mathrm{Ext}^i_{\widetilde{\mathcal{O}}^{\omega}}(M,N) < \infty$.*

Proof. Claim (a) follows directly from the definitions. To prove claim (b) we just observe that each $M \in \widetilde{\mathcal{O}}^{\omega}$ is in $\mathcal{O}^{\omega}$, when considered as a $\mathfrak{g}_{\overline{0}}$-module. In particular, it has finite length already as a $\mathfrak{g}_{\overline{0}}$-module (see e.g. [RC, Proposition 3.3] or [BGG, Hu]).

Because of claim (b), to prove claims (c) and (d) it is enough to prove that each simple object in $\widetilde{\mathcal{O}}^{\omega}$ has both a projective cover and an injective envelope. We prove the first claim and the second one is proved similarly. Let $L \in \widetilde{\mathcal{O}}^{\omega}$ be simple and let $P \in \mathcal{O}^{\omega}$ be a projective cover of $(\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\overline{0}}} L)_{\overline{0}}$ (see e.g. [RC, Corollary 4.2] or [BGG, Hu] for existence of projective covers in $\mathcal{O}^{\omega}$), which we may assume to be nonzero up to parity change. Then, by adjunction, we have

$$0 \neq \mathrm{Hom}_{\mathfrak{g}_{\overline{0}}}\big(P, (\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\overline{0}}} L)_{\overline{0}}\big) = \mathrm{Hom}_{\mathfrak{g}}\big(\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\overline{0}}} P, L\big).$$

As $\mathrm{Ind}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}}$ is left adjoint to the exact functor $\mathrm{Res}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}}$, the former functor maps projective objects to projective objects. Hence $\mathrm{Ind}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}} P$ is a projective object in $\widetilde{\mathscr{O}}^{\omega}$ which surjects onto L.

Claim (e) follows directly from the definition and the fact that all morphism spaces in $\mathscr{O}^{\omega}$ are finite dimensional (again, see e.g. [RC, Sections 3 and 4] or [BGG, Hu]). Claim (f) follows from claims (c) and (e) considering projective resolutions. □

3.3 Simple objects in $\widetilde{\mathscr{O}}^{\omega}$

For each simple finite-dimensional $\mathfrak{l}_{\bar{0}}^{\omega}$-module V we have the corresponding *generalized Verma module*

$$M(\omega,V) := U(\mathfrak{g}_{\bar{0}}) \bigotimes_{U(\mathfrak{l}_{\bar{0}}^{\omega}\oplus\mathfrak{n}_{\bar{0}}^{\omega,+})} V,$$

where $\mathfrak{n}_{\bar{0}}^{\omega,+}V = 0$. This module lies in $\mathscr{O}^{\omega}$ and has a unique simple quotient denoted $L(\omega,V)$. Let $\mathscr{I}_{\bar{0}}^{\omega,0}$ denote the set of isomorphism classes of simple finite-dimensional $\mathfrak{l}_{\bar{0}}^{\omega}$-modules. As $\mathfrak{l}_{\bar{0}}^{\omega}$ is reductive, the set $\mathscr{I}_{\bar{0}}^{\omega,0}$ is well-understood, see e.g. [Di, Hu]. Furthermore, by [RC, Proposition 3.3], the set

$$\mathscr{I}_{\bar{0}}^{\omega} := \left\{ L(\omega,V) | V \in \mathscr{I}_{\bar{0}}^{\omega,0} \right\}$$

is a full set of representatives of isomorphism classes of simple objects in $\mathscr{O}^{\omega}$ (and in this sense $\mathscr{I}_{\bar{0}}^{\omega,0}$ and $\mathscr{I}_{\bar{0}}^{\omega}$ are canonically identified). Denote by $\mathscr{I}_{\bar{1}}^{\omega}$ an odd copy of $\mathscr{I}_{\bar{0}}^{\omega}$. Now a rough description of simple objects in $\widetilde{\mathscr{O}}^{\omega}$ is given by the following:

Proposition 2. *Let L be a simple object in $\widetilde{\mathscr{O}}^{\omega}$. Then there is $V \in \mathscr{I}_{\bar{0}}^{\omega,0}$ such that L is a quotient of $\mathrm{Ind}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}} L(\omega,V)$ up to parity change.*

Proof. As $\mathrm{Res}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}} L \in \mathscr{O}^{\omega}$ and each object in $\mathscr{O}^{\omega}$ has finite length, $\mathrm{Res}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}} L$ has a simple subobject, which is isomorphic to $L(\omega,V)$ for some $V \in \mathscr{I}_{\bar{0}}^{\omega,0}$ by the above (and which we may assume to be even up to parity change). By adjunction, we have

$$0 \neq \mathrm{Hom}_{\mathfrak{g}_{\bar{0}}}\big(L(\omega,V), (\mathrm{Res}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}} L)_{\bar{0}}\big) = \mathrm{Hom}_{\mathfrak{g}}\big(\mathrm{Ind}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}} L(\omega,V), L\big).$$

and the claim follows. □

We denote by $\mathscr{I}^{\omega}$ the set of isomorphism classes of simple objects in $\widetilde{\mathscr{O}}^{\omega}$. Define a binary relation $\Omega \subset \mathscr{I}^{\omega} \times (\mathscr{I}_{\bar{0}}^{\omega} \cup \mathscr{I}_{\bar{1}}^{\omega})$ (here the last union is automatically disjoint) by $(L, L(\omega,V)) \in \Omega$ if $L(\omega,V)$ is isomorphic to a submodule of $\mathrm{Res}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}} L$. Then Ω is *finitary* in the sense that for each $L \in \mathscr{I}^{\omega}$ the set

$$\{L(\omega,V)\in\mathscr{I}^{\omega}_{\overline{0}}\cup\mathscr{I}^{\omega}_{\overline{1}}|(L,L(\omega,V))\in\Omega\}$$

is non-empty and finite, moreover, for each $L(\omega,V)\in\mathscr{I}^{\omega}_{\overline{0}}\cup\mathscr{I}^{\omega}_{\overline{1}}$ the set

$$\{L\in\mathscr{I}^{\omega}|(L,L(\omega,V))\in\Omega\}$$

is non-empty and finite. Unfortunately, in the general case Ω is not a function in any direction. Anyway, $\mathscr{I}^{\omega}$ can be considered as a "finite cover" of $\mathscr{I}^{\omega}_{\overline{0}}$ in some sense. Put differently, the set $\mathscr{I}^{\omega}$ is only "finitely more complicated" than the very well-understood set $\mathscr{I}^{\omega}_{\overline{0}}$. An alternative description of $\mathscr{I}^{\omega}$ which uses $\mathfrak{l}^{\omega}$ will be given in Proposition 6.

3.4 Blocks of $\widetilde{\mathscr{O}}^{\omega}$

Let $\sim$ be the minimal equivalence relation on $\mathscr{I}^{\omega}_{\overline{0}}$ which contains all pairs $(L,L')\in\mathscr{I}^{\omega}_{\overline{0}}\times\mathscr{I}^{\omega}_{\overline{0}}$ such that $\mathrm{Ext}^1_{\mathscr{O}^{\omega}}(L,L')\neq 0$. For an equivalence class $\mathscr{X}\in\mathscr{I}^{\omega}_{\overline{0}}/\sim$ let $\mathscr{O}^{\omega}(\mathscr{X})$ denote the Serre subcategory of $\mathscr{O}^{\omega}$ generated by simples in $\mathscr{X}$. Then we have the usual decomposition

$$\mathscr{O}^{\omega}\cong\bigoplus_{\mathscr{X}\in\mathscr{I}^{\omega}_{\overline{0}}/\sim}\mathscr{O}^{\omega}(\mathscr{X})$$

into a direct sum of indecomposable subcategories, called *blocks* of $\mathscr{O}^{\omega}$. Every equivalence class $\mathscr{X}$ is finite as there are only finitely many (up to isomorphism) simple highest weight $\mathfrak{g}_{\overline{0}}$-modules for each given central character, see [Di, Chapter 7] for details.

Let $\approx$ be the minimal equivalence relation on $\mathscr{I}^{\omega}$ which contains all pairs $(L,L')\in\mathscr{I}^{\omega}\times\mathscr{I}^{\omega}$ such that $\mathrm{Ext}^1_{\widetilde{\mathscr{O}}^{\omega}}(L,L')\neq 0$. For an equivalence class $\mathscr{X}\in\mathscr{I}^{\omega}/\approx$ let $\widetilde{\mathscr{O}}^{\omega}(\mathscr{X})$ denote the Serre subcategory of $\widetilde{\mathscr{O}}^{\omega}$ generated by simples in $\mathscr{X}$. Then we have the decomposition

$$\widetilde{\mathscr{O}}^{\omega}\cong\bigoplus_{\mathscr{X}\in\mathscr{I}^{\omega}/\approx}\widetilde{\mathscr{O}}^{\omega}(\mathscr{X})$$

into a direct sum of indecomposable subcategories, called *blocks* of $\widetilde{\mathscr{O}}^{\omega}$. For example, an explicit description of blocks for $\mathfrak{g}=\mathfrak{gl}(m|n)$ can be found in [CMW].

Proposition 3. *Each $\mathscr{X}\in\mathscr{I}^{\omega}/\approx$ is at most countable.*

Proof. Let $L\hookrightarrow N\twoheadrightarrow L'$ be a non-split extension in $\widetilde{\mathscr{O}}^{\omega}(\mathscr{X})$ with L,L' simple. Take some $\lambda\in\mathrm{supp}(N)$ such that $N_\lambda\neq L_\lambda$. Then $N=U(\mathfrak{g})N_\lambda$ and it follows that $\mathrm{supp}(N)\subset\lambda+\mathbb{Z}R'$, in particular, we have both $\mathrm{supp}(L)\subset\lambda+\mathbb{Z}R'$ and $\mathrm{supp}(L')\subset\lambda+\mathbb{Z}R'$. Therefore $\mathrm{supp}(N')\subset\lambda+\mathbb{Z}R'$ for any $N'\in\widetilde{\mathscr{O}}^{\omega}(\mathscr{X})$. Note that $\lambda+\mathbb{Z}R'$ is an at most countable set.

Simples in $\mathscr{O}^\omega$ are classified by their highest weight (see [RC, Proposition 3.3]), in particular, there are only at most countably many simple objects in $\mathscr{O}^\omega$ with support in $\lambda + \mathbb{Z}R'$. Now each simple in $\widetilde{\mathscr{O}}^\omega(\mathscr{X})$ has, as a $\mathfrak{g}_{\overline{0}}$-module and up to parity change, some simple submodule $L(\omega, V)$ from $\mathscr{O}^\omega$ with support in $\lambda + \mathbb{Z}R'$ and hence is a quotient of $\mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}} L(\omega, V)$ (see Proposition 2). The module $\mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}} L(\omega, V)$ has finite length by Proposition 1(b). Since we have only at most countably many $L(\omega, V)$ to start with, the claim follows. $\square$

For each $L \in \mathscr{I}^\omega$ fix an indecomposable projective cover $P(L)$ of L (which exists by Proposition 1(c)). For $\mathscr{X} \in \mathscr{I}^\omega/\approx$ let $\widetilde{\mathscr{P}}^\omega(\mathscr{X})$ denote the full subcategory of $\widetilde{\mathscr{O}}^\omega$ with objects $P(L)$, $L \in \mathscr{X}$. From Propositions 1 it follows that the $\mathbb{C}$-linear category $\widetilde{\mathscr{P}}^\omega(\mathscr{X})$ has the following properties:

- for each $L \in \mathscr{X}$ we have $\mathrm{Hom}_{\widetilde{\mathscr{P}}^\omega(\mathscr{X})}(P(L), P(L')) \neq 0$ for at most finitely many $L' \in \mathscr{X}$;
- for each $L \in \mathscr{X}$ we have $\mathrm{Hom}_{\widetilde{\mathscr{P}}^\omega(\mathscr{X})}(P(L'), P(L)) \neq 0$ for at most finitely many $L' \in \mathscr{X}$.

We also set $\widetilde{\mathscr{P}}^\omega = \bigcup\limits_{\mathscr{X} \in \mathscr{I}^\omega/\approx} \widetilde{\mathscr{P}}^\omega(\mathscr{X})$.

Let $\widetilde{\mathscr{P}}^\omega(\mathscr{X})^{\mathrm{op}}$ be the category, opposite to $\widetilde{\mathscr{P}}^\omega(\mathscr{X})$. Consider the category $\widetilde{\mathscr{P}}^\omega(\mathscr{X})^{\mathrm{op}}$-fmod of finite dimensional $\widetilde{\mathscr{P}}^\omega(\mathscr{X})^{\mathrm{op}}$-modules, that is the category of $\mathbb{C}$-linear functors

$$\mathrm{F} : \widetilde{\mathscr{P}}^\omega(\mathscr{X})^{\mathrm{op}} \to \mathbb{C}\text{-mod}$$

satisfying the condition $\sum\limits_{L \in \mathscr{X}} \dim(\mathrm{F}(P(L))) < \infty$. Now from the standard abstract nonsense (see e.g. [Ga]), we have:

Proposition 4. *For $\mathscr{X} \in \mathscr{I}^\omega/\approx$ the categories $\widetilde{\mathscr{O}}^\omega(\mathscr{X})$ and $\widetilde{\mathscr{P}}^\omega(\mathscr{X})^{\mathrm{op}}$-fmod are equivalent.*

3.5 Duality

The category $\mathscr{O}^\omega$ has the standard *simple preserving duality* $\star$, that is a contravariant anti-equivalence which preserves isomorphism classes of simple objects (see [Hu, Subsection 3.2] for details). This duality lifts to $\widetilde{\mathscr{O}}^\omega$ in the obvious way (and will also be denoted by $\star$), however, because of (2), simple modules are preserved by the lifted duality $\star$ only up to a possible parity change. We note also that some simple modules in $\widetilde{\mathscr{O}}^\omega$ might be stable under Π (that is, isomorphic, in $\widetilde{\mathscr{O}}^\omega$, to their parity changed counterparts). We will not need any explicit criterion for when $\star$ preserves simples strictly or only up to parity change, we refer the reader to [Fr3] for the $\mathfrak{q}_n$-example.

4 Stratification

4.1 Standard and proper standard objects

The superalgebra $\mathfrak{l}^{\omega}$ is a classical Lie superalgebra in the sense of Subsection 2.1. The category ${}^{\mathfrak{l}^{\omega}_{\bar{0}}}\mathscr{O}^0$ is just the category of semi-simple finite dimensional $\mathfrak{l}^{\omega}_{\bar{0}}$-modules. Consider now the category ${}^{\mathfrak{l}^{\omega}}\widetilde{\mathscr{O}}^0$ which has all the properties described in Proposition 1. Furthermore, we have:

Proposition 5. *Projective and injective modules in* ${}^{\mathfrak{l}^{\omega}}\widetilde{\mathscr{O}}^0$ *coincide.*

Proof. It is enough to show that the indecomposable projective cover of each simple object in ${}^{\mathfrak{l}^{\omega}}\widetilde{\mathscr{O}}^0$ is injective and that the indecomposable injective envelope of each simple object in ${}^{\mathfrak{l}^{\omega}}\widetilde{\mathscr{O}}^0$ is projective. We will prove the first claim an the second is proved similarly. From the proof of Proposition 1(c) it follows that the indecomposable projective cover of each simple object in ${}^{\mathfrak{l}^{\omega}}\widetilde{\mathscr{O}}^0$ is a direct summand of a module of the form $\mathrm{Ind}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}} P$, where P is projective in ${}^{\mathfrak{l}^{\omega}_{\bar{0}}}\mathscr{O}^0$. The latter category is semi-simple and hence P is also injective. Now (2) implies that, up to parity change, the module $\mathrm{Ind}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}} P$ is isomorphic to $\mathrm{Coind}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}} P$ where P is injective. Then the module $\mathrm{Coind}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}} P$ is injective as $\mathrm{Coind}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}}$ is right adjoint to an exact functor (and thus sends injective modules to injective modules). The claim follows. □

Denote by ${}^{\mathfrak{l}}\mathscr{I}^{\omega}$ the set of isomorphism classes of simple objects in ${}^{\mathfrak{l}^{\omega}}\widetilde{\mathscr{O}}^0$. For $V \in {}^{\mathfrak{l}}\mathscr{I}^{\omega}$ denote by $\hat{V}$ the indecomposable projective cover of V in ${}^{\mathfrak{l}^{\omega}}\widetilde{\mathscr{O}}^0$ (note that the module $\hat{V}$ is also injective by Proposition 5 but it does not have to coincide with the indecomposable injective envelope of V). Set $\mathfrak{n}^{\omega,+}V = \mathfrak{n}^{\omega,+}\hat{V} = 0$. Define the *proper standard* or *generalized Verma* $\mathfrak{g}$-module

$$\overline{\Delta}(V) := U(\mathfrak{g}) \bigotimes_{U(\mathfrak{l}^{\omega}\oplus\mathfrak{n}^{\omega,+})} V$$

and the *standard* $\mathfrak{g}$-module

$$\Delta(V) := U(\mathfrak{g}) \bigotimes_{U(\mathfrak{l}^{\omega}\oplus\mathfrak{n}^{\omega,+})} \hat{V}.$$

Since the parabolic induction from $\mathfrak{l}^{\omega} \oplus \mathfrak{n}^{\omega,+}$ to $\mathfrak{g}$ is exact, Proposition 1(b) implies that each standard module has a finite filtration whose subquotients are proper standard modules (these subquotients do **not** have to be isomorphic one to the other).

Proposition 6. *Let* $V \in {}^{\mathfrak{l}}\mathscr{I}^{\omega}$.

(*a*) *The module* $\overline{\Delta}(V)$ *has simple top denoted by* $L(V)$.
(*b*) *The module* $L(V)$ *is also the simple top of* $\Delta(V)$.
(*c*) *The set* $\{L(V) | V \in {}^{\mathfrak{l}}\mathscr{I}^{\omega}\}$ *is a full set of representatives of isomorphism classes of simple objects in* $\widetilde{\mathscr{O}}^{\omega}$.

Proof. For $\lambda, \mu \in \mathfrak{h}_{\overline{0}}^*$ write $\lambda \leq_\omega \mu$ if and only if $\mu - \lambda \in \mathbb{Z}_+\mathrm{supp}(\mathfrak{l}^\omega \oplus \mathfrak{n}^{\omega,+})$. Now for $\lambda \in \mathrm{supp}(V)$ the unique maximal submodule of $\overline{\Delta}(V)$ is the sum of all submodules M of $\overline{\Delta}(V)$ which satisfy the following condition: $\mu \in \mathrm{supp}(M)$ implies $\mu <_\omega \lambda$. This implies claim (a) and claim (c) follows from the definition of $\widetilde{\mathscr{O}}^\omega$ and the universal property of induced modules. Claim (b) follows from claim (a) and definitions. We refer the reader to [Di, Chapter 7] for similar properties of the usual Verma modules written with all details. $\square$

Proposition 6(c) allows us to canonically identify ${}^{\mathfrak{l}}\mathscr{I}^\omega$ and $\mathscr{I}^\omega$. From the proof of Proposition 6(a) it follows that the simple top $L(V)$ has composition multiplicity one in $\overline{\Delta}(V)$.

4.2 Stratified structure

Theorem 2. *Each projective module in $\widetilde{\mathscr{O}}^\omega$ has a* standard filtration*, that is a filtration whose subquotients are isomorphic to standard modules.*

Proof. This claim is proved similarly to e.g. [FKM, Proposition 3] or [Fr3, Theorem 12]. As mentioned above, each projective object in $\widetilde{\mathscr{O}}^\omega$ is a direct summand of a module of the form $\mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}} P$ where P is projective in $\mathscr{O}^\omega$. Similarly to [BGG] one shows that existence of a standard filtration is and additive property, that is inherited by all direct summands. Each projective in $\mathscr{O}^\omega$ has a Verma filtration, that is a filtration whose subquotients are isomorphic to Verma modules. Hence it is enough to show that each module of the form $\mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}} M(\lambda)$, where $M(\lambda)$ is a usual Verma module, has a standard filtration. The induction from $\mathfrak{g}_{\overline{0}}$ to $\mathfrak{g}$ can be factorized via $\mathfrak{l}^\omega$. From Proposition 5 it thus follows that the module $\mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}} M(\lambda)$, when considered as an $\mathfrak{l}^\omega$-module, is a direct sum of projective-injective modules in ${}^{\mathfrak{l}^\omega}\widetilde{\mathscr{O}}^0$. Moreover, by construction, the module $\mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}} M(\lambda)$ is free of finite rank over $U(\mathfrak{n}^{\omega,-})$. Take an $\mathfrak{l}^\omega$-direct summand N of $\mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}} M(\lambda)$ of maximal possible weight (with respect to the order $\leq_\omega$ introduced in the previous subsection). From the universal property of induced modules and the fact that $\mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}} M(\lambda)$ is free over $U(\mathfrak{n}^{\omega,-})$ it follows that $U(\mathfrak{g})N$ is a direct sum of standard modules. Furthermore, $U(\mathfrak{g})N$, when considered as an $\mathfrak{l}^\omega$-module, is a direct sum of projective-injective objects in ${}^{\mathfrak{l}^\omega}\widetilde{\mathscr{O}}^0$. This implies that $U(\mathfrak{g})N$ is a direct summand of $\mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}} M(\lambda)$ as an $\mathfrak{l}^\omega$-module. Now the proof is completed by induction with respect to $\leq_\omega$. $\square$

Combination of Proposition 6 and Theorem 2 means that each $\widetilde{\mathscr{P}^\omega}(\mathscr{X})$ is weakly properly stratified in the sense of [Fr1] (in particular, it is standardly stratified in the sense of [CPS]).

Using $\star$ we define *proper costandard* modules as $\star$-duals of proper standard modules. We also define *costandard* modules as $\star$-duals of standard modules. Then every costandard module has a *proper costandard filtration*, that is a filtration whose subquotients are isomorphic to proper costandard modules. The $\star$-dual of Theorem 2 says that each injective module in $\widetilde{\mathscr{O}}^{\omega}$ has a *costandard filtration*, that is a filtration whose subquotients are isomorphic to costandard modules.

The $\star$-dual of Proposition 6 says that all costandard and proper costandard modules have simple socle. For $V \in {}^{\mathfrak{l}}\mathscr{I}^{\omega}$ we denote by $\nabla(V)$ and $\overline{\nabla}(V)$ the costandard and proper costandard modules with simple socle $L(V)$, respectively. We denote by $\mathscr{F}(\Delta)$ the full subcategory of $\widetilde{\mathscr{O}}^{\omega}$ consisting of all modules having a standard filtration and define $\mathscr{F}(\overline{\Delta})$, $\mathscr{F}(\nabla)$ and $\mathscr{F}(\overline{\nabla})$ similarly.

By standard arguments (see e.g. [Fr1, Fr2]), the fact that $\widetilde{\mathscr{P}^{\omega}}(\mathscr{X})$ is weakly properly stratified is equivalent to the following homological orthogonality:

Corollary 1. *For $V, V' \in {}^{\mathfrak{l}}\mathscr{I}^{\omega}$ we have*

$$\mathrm{Ext}^i_{\mathscr{O}}(\Delta(V), \overline{\nabla}(V')) \cong \begin{cases} \mathbb{C}, & \text{if } V \cong V' \text{ and } i = 0; \\ 0, & \text{otherwise}. \end{cases}$$

Using $\star$ one obtains a similar homological orthogonality between proper standard and costandard modules. As a consequence of this, for $N \in \mathscr{F}(\Delta)$ and $V \in {}^{\mathfrak{l}}\mathscr{I}^{\omega}$ the number of occurrences of $\Delta(V)$ as a subquotient of a standard filtration of N does not depend on the choice of the filtration and will be denoted by $(N : \Delta(V))$.

Weakly proper stratification of $\widetilde{\mathscr{P}^{\omega}}(\mathscr{X})$ also implies the following standard characterization of modules with (proper) (co)standard filtration (see [Ri, Fr1]):

Corollary 2.

(a) $\mathscr{F}(\Delta) = \{M \in \widetilde{\mathscr{O}}^{\omega} | \mathrm{Ext}^i_{\widetilde{\mathscr{O}}^{\omega}}(M, \overline{\nabla}(V)) = 0$ *for any* $V \in {}^{\mathfrak{l}}\mathscr{I}^{\omega}$ *and* $i > 0\}$.
(b) $\mathscr{F}(\nabla) = \{M \in \widetilde{\mathscr{O}}^{\omega} | \mathrm{Ext}^i_{\widetilde{\mathscr{O}}^{\omega}}(\overline{\Delta}(V), M) = 0$ *for any* $V \in {}^{\mathfrak{l}}\mathscr{I}^{\omega}$ *and* $i > 0\}$.
(c) $\mathscr{F}(\overline{\Delta}) = \{M \in \widetilde{\mathscr{O}}^{\omega} | \mathrm{Ext}^i_{\widetilde{\mathscr{O}}^{\omega}}(M, \nabla(V)) = 0$ *for any* $V \in {}^{\mathfrak{l}}\mathscr{I}^{\omega}$ *and* $i > 0\}$.
(d) $\mathscr{F}(\overline{\nabla}) = \{M \in \widetilde{\mathscr{O}}^{\omega} | \mathrm{Ext}^i_{\widetilde{\mathscr{O}}^{\omega}}(\Delta(V), M) = 0$ *for any* $V \in {}^{\mathfrak{l}}\mathscr{I}^{\omega}$ *and* $i > 0\}$.

For a simple L we denote by $[N : L]$ the composition multiplicity of L in N. Another standard corollary is the following *BGG-reciprocity*:

Corollary 3. *For $V, V' \in {}^{\mathfrak{l}}\mathscr{I}^{\omega}$ we have*

$$(P(V) : \Delta(V')) = [\overline{\nabla}(V') : L(V)].$$

For example, description of the stratified structure of the category $\mathscr{O}$ for the queer Lie superalgebra $\mathfrak{q}_n$ can be found with all details in [Fr3].

4.3 Tilting modules

An object in $\widetilde{\mathscr{O}}^{\omega}$ is called a *tilting* or *cotilting* module if it belongs to $\mathscr{F}(\Delta) \cap \mathscr{F}(\overline{\nabla})$ or $\mathscr{F}(\nabla) \cap \mathscr{F}(\overline{\Delta})$, respectively. We have the following standard description of (co)tilting modules (see [Ri, Fr1, Ma2, MT]):

Proposition 7.

(a) *Each (co)tilting module is a direct sum of indecomposable (co)tilting modules.*
(b) *For every $V \in {}^{\mathfrak{l}}\mathscr{I}^{\omega}$ there is a unique (up to isomorphism) indecomposable tilting module $T(V)$ such that $\Delta(V) \hookrightarrow T(V)$ and the cokernel of this embedding has a standard filtration.*
(c) *$T(V)$ is also cotilting.*

Proof. From Corollary 2 it follows that all categories $\mathscr{F}(\Delta)$, $\mathscr{F}(\overline{\nabla})$, $\mathscr{F}(\nabla)$ and $\mathscr{F}(\overline{\Delta})$ are fully additive. This implies claim (a). Uniqueness of $T(V)$ follows from Corollary 1 by standard arguments, e.g. as in [Ri]. To prove existence, recall that it is well-known, see e.g. [Hu], that $\mathscr{O}^{\omega}$ has tilting modules, that is $\star$-self-dual modules with (generalized) Verma flag. Inducing these up to $\mathfrak{g}$ gives $\star$-self-dual (up to parity change) modules with standard filtration. Now existence of $T(V)$ follows by tracking the highest weight (with respect to $\leq_{\omega}$), which proves claim (b). Furthermore, by construction, all these induced modules belong to the category $\mathscr{F}(\Delta)\cap\mathscr{F}(\overline{\nabla})\cap\mathscr{F}(\nabla)\cap\mathscr{F}(\overline{\Delta})$, which proves claim (c). $\square$

The standard useful property of tilting modules (see [Ri, Fr1]) is that every module with standard filtration has a finite coresolution by tilting modules. Furthermore, every module with a proper costandard filtration has a (possibly infinite) resolution by tilting modules.

For each $V \in {}^{\mathfrak{l}}\mathscr{I}^{\omega}$ we fix some $T(V)$ as given by Proposition 7(b). Denote by $\widetilde{\mathscr{T}}^{\omega}$ the full subcategory of $\widetilde{\mathscr{O}}^{\omega}$ with objects $T(V)$, $V \in {}^{\mathfrak{l}}\mathscr{I}^{\omega}$. Similarly, for $\mathscr{X} \in {}^{\mathfrak{l}}\mathscr{I}^{\omega}/\approx$ we denote by $\widetilde{\mathscr{T}}^{\omega}(\mathscr{X})$ the full subcategory of $\widetilde{\mathscr{O}}^{\omega}$ with objects $T(V)$, $V \in \mathscr{X}$.

4.4 Finitistic dimension

As an application of tilting module in the subsection we obtain a bound for the finitistic dimension of $\widetilde{\mathscr{O}}^{\omega}$ which, in particular, implies Theorem 1. Let $\mathscr{C}$ be an abelian category with enough projectives. The *global* dimension $\mathrm{gl.dim}(\mathscr{C})$ is defined as the supremum of projective dimensions $\mathrm{p.dim}(X)$ taken over all objects $X \in \mathscr{C}$. The *finitistic* dimension $\mathrm{fin.dim}(\mathscr{C})$ is defined as the supremum of $\mathrm{p.dim}(X)$ taken over all objects in $X \in \mathscr{C}$ for which $\mathrm{p.dim}(X) < \infty$. It is known that the global dimension of the category $\mathscr{O}^{\omega}$ is finite and hence coincides with the finitistic dimension of $\mathscr{O}^{\omega}$ (finiteness follows from [RC] and [So90] and explicit bounds and different interpretations can be found in [Ma1, KKM, MaPa, MO, FM1]). For $\widetilde{\mathscr{O}}^{\omega}$ we have:

Theorem 3.

$$\mathrm{fin.dim}(\widetilde{\mathscr{O}}^{\omega}) = 2\cdot \max_{V\in {}^{\mathfrak{l}}\mathscr{I}^{\omega}} \mathrm{p.dim}(T(V)) \leq \mathrm{gl.dim}(\mathscr{O}^{\omega}).$$

Proof. Let us prove the inequality first. We start with the claim that all injective modules in $\widetilde{\mathcal{O}}^{\omega}$ have finite projective dimension. Indeed, this is obviously true for injectives in $\mathcal{O}^{\omega}$. Given a finite projective resolution of an injective I in $\mathcal{O}^{\omega}$, we can induce this resolution up to $\widetilde{\mathcal{O}}^{\omega}$ and obtain a finite projective resolution of $\mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}} I$ in $\widetilde{\mathcal{O}}^{\omega}$. As any injective in $\widetilde{\mathcal{O}}^{\omega}$ is a direct summand of some $\mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}} I$, we have our claim. Moreover, as a bonus we even have that $\max\limits_{V\in{}^{\mathfrak{l}}\mathscr{I}^{\omega}} \mathrm{p.dim}(I(V)) \leq \mathrm{gl.dim}(\mathcal{O}^{\omega})$.

Similarly one shows that all projective modules in $\widetilde{\mathcal{O}}^{\omega}$ have finite injective dimension. Now we claim that every $M \in \widetilde{\mathcal{O}}^{\omega}$ has finite projective dimension if and only if it has finite injective dimension. By symmetry, it is enough to prove the "if" statement. Let

$$0 \to I_0 \to I_1 \to \cdots \to I_k \to 0$$

be an injective resolution of M. Each I_s, $s = 0,\dots,k$, has a finite projective resolution by the above. Substituting each I_s by its projective resolution (using the iterated cone construction, see e.g. [MO]), we get a finite complex of projective modules with unique non-zero homology M concentrated in position 0. Deleting all trivial direct summands we obtain a finite projective resolution of M and hence M has finite projective dimension.

Next we claim that $\mathrm{fin.dim}(\widetilde{\mathcal{O}}^{\omega}) = \max\limits_{V\in{}^{\mathfrak{l}}\mathscr{I}^{\omega}} \mathrm{p.dim}(I(V))$. Note that the right hand side is bounded by $\mathrm{gl.dim}(\mathcal{O}^{\omega})$ by the above. Set $N := \max\limits_{V\in{}^{\mathfrak{l}}\mathscr{I}^{\omega}} \mathrm{p.dim}(I(V))$. Assume that $X \in \widetilde{\mathcal{O}}^{\omega}$ is such that $\mathrm{p.dim}(X) > N$. Consider a short exact sequence

$$X \hookrightarrow I \twoheadrightarrow Y \tag{6}$$

Then, because of the dimension shift in the long exact sequence obtained by applying to (6) the functor $\mathrm{Hom}({}_{-}, L(V))$, $V \in {}^{\mathfrak{l}}\mathscr{I}^{\omega}$, we get $\mathrm{p.dim}(Y) > \mathrm{p.dim}(X)$. At the same time, X has finite injective dimension by the above and hence the injective dimension of Y is strictly smaller then the injective dimension of X. Proceeding inductively we get an injective module of projective dimension greater than N, a contradiction. This completes the proof of the inequality $\mathrm{fin.dim}(\widetilde{\mathcal{O}}^{\omega}) \leq \mathrm{gl.dim}(\mathcal{O}^{\omega})$.

Now let us prove the equality

$$\max_{V\in{}^{\mathfrak{l}}\mathscr{I}^{\omega}} \mathrm{p.dim}(I(V)) = 2\cdot \max_{V\in{}^{\mathfrak{l}}\mathscr{I}^{\omega}} \mathrm{p.dim}(T(V)). \tag{7}$$

First we claim that the right hand side of (7) is finite. Indeed, it is finite in the case of $\mathcal{O}^{\omega}$. Having a projective resolution of a tilting module in $\mathcal{O}^{\omega}$, we can induce this resolution up to $\widetilde{\mathcal{O}}^{\omega}$ and get a projective resolution of the induced tilting module. Since, by the highest weight argument, each tilting module is a direct summand of an induced tilting module, we have our claim. Now the proof is completed as in [MO, Theorem 1]. $\square$

We note one important difference between $\widetilde{\mathcal{O}}^{\omega}$ and $\mathcal{O}^{\omega}$, namely the fact addressed in Subsection 3.4 that blocks of $\mathcal{O}^{\omega}$ are described by finite dimensional associative algebras while blocks of $\widetilde{\mathscr{O}}^{\omega}$ are described, in general, only by infinite dimensional associative algebras with local units. This fact makes Theorem 3 non-trivial and, to some extend, surprising.

5 Projective-injective modules and their applications

5.1 Irving-type theorems

Category $\mathscr{O}^{\omega}$ has a lot of projective-injective modules with remarkable properties, see [Ir] and [So90]. These extend to $\widetilde{\mathscr{O}}^{\omega}$ as follows. For $V \in {}^{\mathfrak{l}}\mathscr{I}^{\omega}$ let us denote by $I(V)$ the indecomposable injective envelope of $L(V)$ in $\widetilde{\mathscr{O}}^{\omega}$.

Theorem 4. *Let $V \in {}^{\mathfrak{l}}\mathscr{I}^{\omega}$. Then the following assertions are equivalent:*

(i) *$P(V)$ is injective.*
(ii) *$P(V)$ is isomorphic to $I(V)$ up to parity change.*
(iii) *$L(V)$ occurs in the socle of a projective-injective module in $\widetilde{\mathscr{O}}$.*
(iv) *$L(V)$ occurs in the top of a projective-injective module in $\widetilde{\mathscr{O}}$.*
(v) *$L(V)$ occurs in the socle of some standard module.*
(vi) *$L(V)$ occurs in the socle of some proper standard module.*

Proof. Equivalence of claims (iii) and (iv) follows by applying $\star$. Equivalence of claims (v) and (vi) follows from Proposition 5 and the fact that every standard module has a proper standard filtration. Claim (ii) obviously implies claim (iii). Each projective-injective module has a standard filtration and hence claim (iii) implies claim (v). That claim (ii) implies claim (i) is obvious and the reverse application would follow from the fact that claim (iv) implies claim (ii). It is left to prove that claim (v) implies claim (ii).

Assume claim (v). The module $\Delta(V)$, when restricted to $\mathfrak{g}_{\bar{0}}$, has a Verma flag. Therefore, by the main result of [Ir], we have $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}}\Delta(V) \hookrightarrow I$, where I is projective-injective in $\mathscr{O}^{\omega}$. Adjunction gives a non-zero map $\Delta(V) \to \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}} I$. This map is injective as it is non-zero when restricted to the socle (which is not annihilated by the induction). The module $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}} I$ is projective since I is projective and $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}}$ is left adjoint to an exact functor. The module $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}} I$ is injective since I is injective and $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}}$ is right adjoint to an exact functor by (2). This proves claim (iii). Note that every projective module is tilting and hence self-dual with respect to $\star$. Now claim (ii) follows applying $\star$. $\square$

5.2 Dominance dimension

The next application of projective modules is the dominance dimension property, described for $\mathscr{O}^{\omega}$ in [KSX, St].

Proposition 8. *Every projective P in $\widetilde{\mathcal{O}}^{\omega}$ admits a two step coresolution*

$$0 \to P \to X_1 \to X_2,$$

where both X_1 and X_2 are projective-injective.

Proof. This property is obviously additive. From the proof of Proposition 1 it follows that every projective in $\widetilde{\mathcal{O}}^{\omega}$ is a direct summand of a module induced from a projective module in $\mathcal{O}^{\omega}$. Induction is exact and preserves both projective and injective modules (the latter because of (2)). Hence the claim follows from the corresponding property of $\mathcal{O}^{\omega}$, see [KSX, St]. $\square$

5.3 Soergel's Struktursatz

Denote by $\widetilde{\mathcal{Q}}^{\omega}$ the full subcategory of $\widetilde{\mathcal{P}}^{\omega}$ whose objects are both projective and injective in $\widetilde{\mathcal{O}}^{\omega}$. As a corollary from Proposition 8, we have the following analogue of Soergel's *Struktursatz*, see [So90], for $\widetilde{\mathcal{O}}^{\omega}$.

Theorem 5. *The bifunctor*

$$\Phi := \mathrm{Hom}_{\widetilde{\mathcal{P}}^{\omega}}({}_-,{}_-) : (\widetilde{\mathcal{P}}^{\omega})^{\mathrm{op}} \times \widetilde{\mathcal{Q}}^{\omega} \to \mathbb{C}\text{-mod}$$

induces a functor $\tilde{\Phi} : (\widetilde{\mathcal{P}}^{\omega})^{\mathrm{op}} \to \widetilde{\mathcal{Q}}^{\omega}$-mod and the latter functor is full and faithful.

Proof. Mutatis mutandis the proof of [AM, Theorem 4.4]. $\square$

5.4 Ringel self-duality

Our final application of the above is the following statement about Ringel self-duality of $\widetilde{\mathcal{O}}^{\omega}$:

Theorem 6. *For any $\mathscr{X} \in {}^{\mathfrak{l}}\mathscr{I}^{\omega}/\approx$ the categories $\widetilde{\mathcal{T}}^{\omega}(\mathscr{X})$ and $\widetilde{\mathcal{P}}^{\omega}(\mathscr{X})$ are canonically isomorphic.*

Proof. For simplicity, we prove the claim in the case when ω is such that the decomposition (3) is a triangular decomposition. The general case can be dealt with using e.g. the approach of [MS, Subsection 10.4].

Simples in the category $\mathcal{O}^{\omega}$ which occur as socles of projective-injective modules are exactly the simple objects of maximal Gelfand-Kirillov dimension (this follows from [Ir, Proposition 4.3] and [Ja, Kapitel 8]). Furthermore, for ω as described above simple modules in $\mathcal{O}^{\omega}$ of maximal Gelfand-Kirillov dimension are tilting. Recall that the functor $\mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}}$ is given by a tensor product (over $\mathbb{C}$) with a finite dimensional vector space and hence preserves Gelfand-Kirillov dimension. Therefore, applying $\mathrm{Ind}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}}$ and taking into account that it maps projective-injective modules to projective-injective modules, we get that simples which occur as socles of projective-injective modules $\widetilde{\mathscr{O}}^{\omega}$ are again exactly the simple objects of maximal Gelfand-Kirillov dimension.

Given two modules M and N, the *trace* of M in N is the sum of images of all homomorphisms from M to N. For a projective module $P \in \mathscr{O}^{\omega}$ denote by P' the trace in P of all projective injective modules in $\mathscr{O}^{\omega}$. As mentioned in the previous paragraph, for our choice of ω the socle of a dominant Verma module in $\mathscr{O}^{\omega}$ (which is also projective) is a tilting module. Using translation functors we obtain that P' is a tilting module for any projective P. Applying $\mathrm{Ind}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}}$ and using the previous paragraph, we get the same property for $\widetilde{\mathscr{O}}^{\omega}$.

For any $P_1, P_2 \in \widetilde{\mathscr{P}}^{\omega}(\mathscr{X})$ any homomorphism from P_1 to P_2 restricts to a homomorphism from P_1' to P_2'. From Theorem 5 it follows that this restriction is, in fact, an isomorphism. This implies that P_1' is an indecomposable tilting module and that $P_1' \cong P_2'$ if and only if $P_1 \cong P_2$. Taking into account the previous paragraph, renaming P into P' defines an isomorphism from $\widetilde{\mathscr{P}}^{\omega}(\mathscr{X})$ to $\widetilde{\mathscr{T}}^{\omega}(\mathscr{X})$. This completes the proof. $\square$

References

[AM] H.H. Andersen, V. Mazorchuk; Category $\mathscr{O}$ for quantum groups. Preprint arxiv:1105.5500, to appear in JEMS.

[BGG] I. Bernstein, I. Gelfand, S. Gelfand; A certain category of $\mathfrak{g}$-modules. Funkcional. Anal. i Prilozen. **10** (1976), no. 2, 1–8.

[Br1] J. Brundan; Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{gl}(m|n)$. J. Amer. Math. Soc. **16** (2003), no. 1, 185–231.

[Br2] J. Brundan; Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{q}(n)$. Adv. Math. **182** (2004), no. 1, 28–77.

[CLW] S.-J. Cheng, N. Lam, W. Wang; Brundan-Kazhdan-Lusztig conjecture for general linear Lie superalgebras. Preprint arXiv:1203.0092.

[CMW] S.-J. Cheng, V. Mazorchuk, W. Wang; Equivalence of blocks for the general linear Lie superalgebra. Preprint arXiv:1301.1204.

[CW] S.-J. Cheng, W. Wang; Dualities and Representations of Lie Superalgebras. Graduate Studies in Mathematics **144**. American Math. Soc. 2012.

[CPS] E. Cline, B. Parshall, L. Scott; Stratifying endomorphism algebras. Mem. Amer. Math. Soc. **124** (1996), no. 591.

[DMP] I. Dimitrov, O. Mathieu, I. Penkov; On the structure of weight modules. Trans. Amer. Math. Soc. **352** (2000), no. 6, 2857–2869.

[Di] J. Dixmier; Enveloping algebras. Revised reprint of the 1977 translation. Graduate Studies in Mathematics, **11**. American Mathematical Society, Providence, RI, 1996.

[Fr1] A. Frisk; Two-step tilting for standardly stratified algebras. Algebra Discrete Math. **2004**, no. 3, 38–59.

[Fr2] A. Frisk; Dlab's theorem and tilting modules for stratified algebras. J. Algebra **314** (2007), no. 2, 507–537.

[Fr3] A. Frisk; Typical blocks of the category $\mathscr{O}$ for the queer Lie superalgebra. J. Algebra Appl. **6** (2007), no. 5, 731–778.

[FM1] A. Frisk, V. Mazorchuk; Properly stratified algebras and tilting. Proc. London Math. Soc. (3) **92** (2006), no. 1, 29–61.

[FM2] A. Frisk, V. Mazorchuk; Regular strongly typical blocks of $\mathscr{O}_{\mathfrak{q}}$. Comm. Math. Phys. **291** (2009), no. 2, 533–542.

[FKM] V. Futorny, S. König, V. Mazorchuk; Categories of induced modules and standardly stratified algebras. Algebr. Represent. Theory **5** (2002), no. 3, 259–276.

[Ga] P. Gabriel; Indecomposable representations. II. Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971), pp. 81104. Academic Press, London, 1973.

[Go1] M. Gorelik; On the ghost centre of Lie superalgebras. Ann. Inst. Fourier (Grenoble) **50** (2000), no. 6, 1745–1764.

[Go2] M. Gorelik; Strongly typical representations of the basic classical Lie superalgebras. J. Amer. Math. Soc. **15** (2002), no. 1, 167–184

[GM] J. Greenstein and V.Mazorchuk; On Koszul duality for generalized Takiff algebras and superalgebras. To appear.

[Hu] J.E. Humphreys; Representations of semisimple Lie algebras in the BGG category $\mathscr{O}$. Graduate Studies in Math. **94**. American Mathematical Society, Providence, RI, 2008.

[Ir] R. Irving; Projective modules in the category $\mathscr{O}_S$: self-duality. Trans. Amer. Math. Soc. **291** (1985), no. 2, 701–732.

[Ja] J. C. Jantzen; Einhüllende Algebren halbeinfacher Lie-Algebren. Ergebnisse der Mathematik und ihrer Grenzgebiete **3**. Springer-Verlag, Berlin, 1983.

[KKM] O. Khomenko, S. Koenig, V. Mazorchuk; Finitistic dimension and tilting modules for stratified algebras. J. Algebra **286** (2005), no. 2, 456–475.

[KSX] S. König, I. Slungård, C. Xi; Double centralizer properties, dominant dimension, and tilting modules. J. Algebra **240** (2001), no. 1, 393–412.

[Ma1] V. Mazorchuk; On finitistic dimension of stratified algebras. Algebra Discrete Math. **2004**, no. 3, 77–88.

[Ma2] V. Mazorchuk; Koszul duality for stratified algebras II. Standardly stratified algebras. J. Aust. Math. Soc. **89** (2010), no. 1, 23–49.

[MaPa] V. Mazorchuk, A. Parker; On the relation between finitistic and good filtration dimensions. Comm. Algebra **32** (2004), no. 5, 1903–1916.

[MS] V. Mazorchuk, C. Stroppel; Categorification of (induced) cell modules and the rough structure of generalised Verma modules. Adv. Math. **219** (2008), no. 4, 1363–1426.

[MO] V. Mazorchuk, S. Ovsienko; Finitistic dimension of properly stratified algebras. Adv. Math. **186** (2004), no. 1, 251–265.

[MT] V. Miemietz, W. Turner; Homotopy, homology, and GL_2. Proc. Lond. Math. Soc. (3) **100** (2010), no. 2, 585–606.

[Mu] I. Musson; Lie superalgebras and enveloping algebras. Graduate Studies in Mathematics, **131**. American Mathematical Society, Providence, RI, 2012.

[MoPi] R. Moody, A. Pianzola; Lie algebras with triangular decompositions. Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1995.

[Ri] C.M. Ringel; The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. Math. Z. **208** (1991), no. 2, 209–223.

[RC] A. Rocha-Caridi; Splitting criteria for $\mathfrak{g}$-modules induced from a parabolic and the Bernsten-Gelfand-Gelfand resolution of a finite-dimensional, irreducible $\mathfrak{g}$-module. Trans. Amer. Math. Soc. **262** (1980), no. 2, 335–366.

[Sa] K. Saito; Polyhedra Dual to the Weyl Chamber. Publ. RIMS, Kyoto Univ. **40** (2004), 1337–1384.

[So90] W. Soergel; Kategorie $\mathscr{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe. J. Amer. Math. Soc. **3** (1990), no. 2, 421–445.

[St] C. Stroppel; Category $\mathscr{O}$: quivers and endomorphism rings of projectives. Represent. Theory **7** (2003), 322–345.