ENDOMORPHISMS OF CELL 2-REPRESENTATIONS

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Abstract. We determine the endomorphism categories of cell 2-representations of fiat 2-categories associated with strongly regular two-sided cells under some natural assumptions. Along the way, we completely describe $\mathcal{J}$-simple fiat 2-categories which have only one two-sided cell $\mathcal{J}$ apart from the identities, under the same conditions as above. For positively graded 2-categories, we show that the additional restrictions are redundant.

1. Introduction and description of the results

Classically, Schur's Lemma asserts that the endomorphism algebra of a simple module (say for a finite dimensional algebra A over some algebraically closed field $\Bbbk$) is isomorphic to $\Bbbk$. It might happen that the algebra A is obtained by decategorifying some 2-category and that the simple module in question is the decategorification of some 2-representation of A. It is then natural to ask whether the assertion of Schur's Lemma is the 1-shadow of some 2-analogue. Put differently, this is a question about the endomorphism category of a 2-representation of some 2-category.

In [MM1] we defined a class of 2-categories, which we call fiat 2-categories, forming a natural 2-analogue of finite dimensional cellular algebras. Examples of fiat 2-categories appear (sometimes in disguise) in e.g. [BG, CR, FKS, KhLa, La, Ro2]. Fiat 2-categories have certain cell 2-representations, which satisfy some natural generalizations of the concept of simplicity for representations of finite dimensional algebras. The main objective of the present paper is to study the endomorphism categories of these cell 2-representations.

We start the paper by extending the 2-setup from [MM1] to accommodate non-strict 2-natural transformations between 2-representations of fiat 2-categories. This is done in Section 2, which also contains all necessary preliminaries. The advantage of our new setup is the fact that 2-natural transformations become closed under isomorphism of functors and under taking inverses of equivalences (see Subsection 2.4).

Cell 2-representations of fiat 2-categories have particularly nice properties for so-called strongly regular cells (see Subsection 2.7) and in the present paper we concentrate on this case. We give two sufficient conditions for the endomorphism category of such a cell 2-representation to be equivalent to $\Bbbk$-mod. The first condition is formulated in terms of the action of endomorphisms of the identity 1-morphisms of our fiat 2-category on a certain generator of the cell 2-representation, see Theorem 4 in Section 3. The second condition is a numerical condition on the decomposition of 1-morphisms under composition which also appears in [MM1, Theorem 43], see Theorem 14 in Section 5.

Under this numerical assumption, we prove two further interesting results. Firstly, we establish 2-fullness for cell 2-representations with respect to the class of 1-morphisms in the two-sided cell, see Corollary 10 in Subsection 4.4. Secondly, we completely describe fiat 2-categories which have only one two-sided cell $\mathcal{J}$ apart from the identities, in the case when our 2-category is $\mathcal{J}$-simple in the sense of [MM2], see Theorem 13 in Subsection 4.6.

We propose various examples in Section 7, including the fiat 2-category of Soergel bimodules acting on the principal block of the BGG category $\mathcal{O}$ and the fiat 2-category associated with the $\mathfrak{sl}_2$-categorification of Chuang and Rouquier. Finally, in Section 8, we introduce the notion of graded fiat 2-categories and show that the numerical condition mentioned above is always satisfied in the positively graded case, see Theorem 23 in Subsection 8.7.

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2. Preliminaries

We denote by $\mathbb{N}$ and $\mathbb{N}_0$ the sets of positive and non-negative integers, respectively.

2.1. **Various 2-categories.** In this paper by a 2-category we mean a strict locally small 2-category (see [Le] for a concise introduction to 2-categories and bicategories). Let $\mathscr{C}$ be a 2-category. We will use $\mathtt{i}, \mathtt{j}, \dots$ to denote objects in $\mathscr{C}$; 1-morphisms in $\mathscr{C}$ will be denoted by $\mathrm{F}, \mathrm{G}, \dots$; 2-morphisms in $\mathscr{C}$ will be denoted by $\alpha, \beta, \dots$. For $\mathtt{i} \in \mathscr{C}$ we will denote by $\mathbb{1}_{\mathtt{i}}$ the corresponding identity 1-morphisms. For a 1-morphism F we will denote by id_{F} the corresponding identity 2-morphisms.

Denote by **Cat** the 2-category of all small categories. Let $\Bbbk$ be an algebraically closed field. Denote by $\mathfrak{A}_{\Bbbk}$ the 2-category whose objects are small $\Bbbk$-linear fully additive categories; 1-morphisms are additive $\Bbbk$-linear functors and 2-morphisms are natural transformations. Denote by $\mathfrak{A}_{\Bbbk}^{f}$ the full 2-subcategory of $\mathfrak{A}_{\Bbbk}$ whose objects are fully additive categories $\mathcal{A}$ such that $\mathcal{A}$ has only finitely many isomorphism classes of indecomposable objects and all morphisms spaces in $\mathcal{A}$ are finite dimensional. We also denote by $\mathfrak{R}_{\Bbbk}$ the full subcategory of $\mathfrak{A}_{\Bbbk}$ containing all objects which are equivalent to A-mod for some finite dimensional associative $\Bbbk$-algebra A.

2.2. **Finitary and fiat 2-categories.** A 2-category $\mathscr{C}$ is called *finitary (over* $\Bbbk$*)*, see [MM1], if the following conditions are satisfied:

- $\mathscr{C}$ has finitely many objects;
- for any $\mathtt{i}, \mathtt{j} \in \mathscr{C}$ we have $\mathscr{C}(\mathtt{i}, \mathtt{j}) \in \mathfrak{A}_{\Bbbk}^{f}$ and horizontal composition is both additive and $\Bbbk$-linear;
- for any $\mathtt{i} \in \mathscr{C}$ the 1-morphism $\mathbb{1}_{\mathtt{i}}$ is indecomposable.

We will call $\mathscr{C}$ *weakly fiat* provided that it has a weak object preserving anti-autoequivalence $*$ and for any 1-morphism $\mathrm{F} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$ there exist 2-morphisms $\alpha : \mathrm{F} \circ \mathrm{F}^* \to \mathbb{1}_{\mathtt{j}}$ and $\beta : \mathbb{1}_{\mathtt{i}} \to \mathrm{F}^* \circ \mathrm{F}$ such that $\alpha_{\mathrm{F}} \circ_1 \mathrm{F}(\beta) = \mathrm{id}_{\mathrm{F}}$ and $\mathrm{F}^*(\alpha) \circ_1 \beta_{\mathrm{F}^*} = \mathrm{id}_{\mathrm{F}^*}$. If $*$ is involutive, then $\mathscr{C}$ is called *fiat*, see [MM1].

2.3. **2-representations.** From now on $\mathscr{C}$ will denote a finitary 2-category. By a 2-representation of $\mathscr{C}$ we mean a strict 2-functor from $\mathscr{C}$ to either $\mathfrak{A}_{\Bbbk}$ (additive 2-representation), $\mathfrak{A}_{\Bbbk}^{f}$ (finitary 2-representation), or $\mathfrak{R}_{\Bbbk}$ (abelian 2-representation). In this paper we define the 2-categories of 2-representations of $\mathscr{C}$ extending the setup (from the one in [MM1, MM2]) by considering non-strict 2-natural transformations between two 2-representations $\mathbf{M}$ and $\mathbf{N}$. Such a 2-natural transformation Ψ consists of the following data: a map, which assigns to every $\mathtt{i} \in \mathscr{C}$ a functor $\Psi_{\mathtt{i}} : \mathbf{M}(\mathtt{i}) \to \mathbf{N}(\mathtt{i})$, and for any 1-morphism $\mathrm{F} \in \mathscr{C}(\mathtt{i},\mathtt{j})$ a natural isomorphism $\eta_{\mathrm{F}} = \eta_{\mathrm{F}}^{\Psi} : \Psi_{\mathtt{j}} \circ \mathbf{M}(\mathrm{F}) \to \mathbf{N}(\mathrm{F}) \circ \Psi_{\mathtt{i}}$, where naturality means that for any $\mathrm{G} \in \mathscr{C}(\mathtt{i},\mathtt{j})$ and any $\alpha : \mathrm{F} \to \mathrm{G}$ we have

$$\eta_{\mathrm{G}} \circ_1 (\mathrm{id}_{\Psi_{\mathtt{j}}} \circ_0 \mathbf{M}(\alpha)) = (\mathbf{N}(\alpha) \circ_0 \mathrm{id}_{\Psi_{\mathtt{i}}}) \circ_1 \eta_{\mathrm{F}}.$$

In other words, the left diagram on the following picture commutes up to η_{F} while the right diagram commutes (compare with [Kh, Subsection 2.2]):

$$\begin{tikzcd}
\mathbf{M}(\mathtt{i}) \arrow[rr, "\mathbf{M}(\mathrm{F})"] \arrow[d, "\Psi_{\mathtt{i}}"'] & & \mathbf{M}(\mathtt{j}) \arrow[d, "\Psi_{\mathtt{j}}"] \arrow[dll, phantom, "\swarrow \eta_{\mathrm{F}}"] \\
\mathbf{N}(\mathtt{i}) \arrow[rr, "\mathbf{N}(\mathrm{F})"'] & & \mathbf{N}(\mathtt{j})
\end{tikzcd}
\qquad
\begin{tikzcd}
\Psi_{\mathtt{j}} \circ \mathbf{M}(\mathrm{F}) \arrow[r, "\eta_{\mathrm{F}}"] \arrow[d, "\mathrm{id}_{\Psi_{\mathtt{j}}} \circ_0 \mathbf{M}(\alpha)"'] & \mathbf{N}(\mathrm{F}) \circ \Psi_{\mathtt{i}} \arrow[d, "\mathbf{N}(\alpha) \circ_0 \mathrm{id}_{\Psi_{\mathtt{i}}}"] \\
\Psi_{\mathtt{j}} \circ \mathbf{M}(\mathrm{G}) \arrow[r, "\eta_{\mathrm{G}}"] & \mathbf{N}(\mathrm{G}) \circ \Psi_{\mathtt{i}}
\end{tikzcd}$$

Moreover, the isomorphisms η should satisfy

(1) $$\eta_{\mathrm{F} \circ_0 \mathrm{G}} = (\mathrm{id}_{\mathbf{N}(\mathrm{F})} \circ_0 \eta_{\mathrm{G}}) \circ_1 (\eta_{\mathrm{F}} \circ_0 \mathrm{id}_{\mathbf{M}(\mathrm{G})})$$

for all composable 1-morphisms F and G.

Given two 2-natural transformations Ψ and Φ as above, a modification $\theta : \Psi \to \Phi$ is a map which assigns to each $\mathtt{i} \in \mathscr{C}$ a natural transformation $\theta_{\mathtt{i}} : \Psi_{\mathtt{i}} \to \Phi_{\mathtt{i}}$ such that for any $\mathrm{F}, \mathrm{G} \in \mathscr{C}(\mathtt{i},\mathtt{j})$ and any $\alpha : \mathrm{F} \to \mathrm{G}$ we have

(2) $$\eta_{\mathrm{G}}^{\Phi} \circ_1 (\theta_{\mathtt{j}} \circ_0 \mathbf{M}(\alpha)) = (\mathbf{N}(\alpha) \circ_0 \theta_{\mathtt{i}}) \circ_1 \eta_{\mathrm{F}}^{\Psi}.$$

Proposition 1. *Together with non-strict 2-natural transformations and modifications as defined above, 2-representations of $\mathscr{C}$ form a 2-category*

Our notation for these 2-categories is $\mathscr{C}$-amod in the case of additive representations and $\mathscr{C}$-afmod in the case of finitary representations. To define the 2-category $\mathscr{C}$-mod for abelian representations we additionally assume that all $\Psi_{\mathtt{i}}$ are right exact (this assumption is missing in [MM1]).

Proof. To check that these are 2-categories, we have to verify that (strict) composition of non-strict 2-natural transformations is a non-strict 2-natural transformation and that both horizontal and vertical compositions of modifications are modifications. The first fact follows by defining

$$\eta_{\mathrm{F}}^{\Psi' \circ \Psi} := (\eta_{\mathrm{F}}^{\Psi'} \circ_0 \mathrm{id}_{\Psi_{\mathtt{i}}}) \circ_1 (\mathrm{id}_{\Psi'_{\mathtt{j}}} \circ_0 \eta_{\mathrm{F}}^{\Psi})$$

and then checking (1) (which is a straightforward computation). Since the diagrams

$$\begin{tikzcd}
\Psi'_{\mathtt{j}} \circ \Psi_{\mathtt{j}} \circ \mathbf{M}(\mathrm{F}) \arrow[r, "\mathrm{id}_{\Psi'_{\mathtt{j}}} \circ_0 \theta_{\mathtt{j}} \circ_0 \mathrm{id}_{\mathbf{M}(\mathrm{F})}"] \arrow[d, "\mathrm{id}_{\Psi'_{\mathtt{j}}} \circ_0 \eta^{\Psi}_{\mathrm{F}}"'] & \Psi'_{\mathtt{j}} \circ \Phi_{\mathtt{j}} \circ \mathbf{M}(\mathrm{F}) \arrow[r, "\theta'_{\mathtt{j}} \circ_0 \mathrm{id}_{\Phi_{\mathtt{j}}} \circ_0 \mathrm{id}_{\mathbf{M}(\mathrm{F})}"] \arrow[d, "\mathrm{id}_{\Psi'_{\mathtt{j}}} \circ_0 \eta^{\Phi}_{\mathrm{F}}"'] & \Phi'_{\mathtt{j}} \circ \Phi_{\mathtt{j}} \circ \mathbf{M}(\mathrm{F}) \arrow[d, "\mathrm{id}_{\Phi'_{\mathtt{j}}} \circ_0 \eta^{\Phi}_{\mathrm{F}}"] \\
\Psi'_{\mathtt{j}} \circ \mathbf{N}(\mathrm{F}) \circ \Psi_{\mathtt{i}} \arrow[r, "\mathrm{id}_{\Psi'_{\mathtt{j}}} \circ_0 \mathrm{id}_{\mathbf{N}(\mathrm{F})} \circ_0 \theta_{\mathtt{i}}"] \arrow[d, "\eta^{\Psi'}_{\mathrm{F}} \circ_0 \mathrm{id}_{\Psi_{\mathtt{i}}}"'] & \Psi'_{\mathtt{j}} \circ \mathbf{N}(\mathrm{F}) \circ \Phi_{\mathtt{i}} \arrow[r, "\theta'_{\mathtt{j}} \circ_0 \mathrm{id}_{\mathbf{N}(\mathrm{F})} \circ_0 \mathrm{id}_{\Phi_{\mathtt{i}}}"] \arrow[d, "\eta^{\Psi'}_{\mathrm{F}} \circ_0 \mathrm{id}_{\Phi_{\mathtt{i}}}"'] & \Phi'_{\mathtt{j}} \circ \mathbf{N}(\mathrm{F}) \circ \Phi_{\mathtt{i}} \arrow[d, "\eta^{\Phi'}_{\mathrm{F}} \circ_0 \mathrm{id}_{\Phi_{\mathtt{i}}}"] \\
\mathbf{K}(\mathrm{F}) \circ \Psi'_{\mathtt{i}} \circ \Psi_{\mathtt{i}} \arrow[r, "\mathrm{id}_{\mathbf{K}(\mathrm{F})} \circ_0 \mathrm{id}_{\Psi'_{\mathtt{i}}} \circ_0 \theta_{\mathtt{i}}"] & \mathbf{K}(\mathrm{F}) \circ \Psi'_{\mathtt{i}} \circ \Phi_{\mathtt{i}} \arrow[r, "\mathrm{id}_{\mathbf{K}(\mathrm{F})} \circ_0 \theta'_{\mathtt{i}} \circ_0 \mathrm{id}_{\Phi_{\mathtt{i}}}"] & \mathbf{K}(\mathrm{F}) \circ \Phi'_{\mathtt{i}} \circ \Phi_{\mathtt{i}}
\end{tikzcd}$$

$$\begin{tikzcd}
\Psi_{\mathtt{j}} \circ \mathbf{M}(\mathrm{F}) \arrow[r, "\theta_{\mathtt{j}} \circ_0 \mathbf{M}(\alpha)"] \arrow[d, "\eta^{\Psi}_{\mathrm{F}}"'] & \Phi_{\mathtt{j}} \circ \mathbf{M}(\mathrm{G}) \arrow[r, "\tau_{\mathtt{j}} \circ_0 \mathrm{id}_{\mathbf{M}(\mathrm{G})}"] \arrow[d, "\eta^{\Phi}_{\mathrm{G}}"'] & \Sigma_{\mathtt{j}} \circ \mathbf{M}(\mathrm{G}) \arrow[d, "\eta^{\Sigma}_{\mathrm{G}}"] \\
\mathbf{N}(\mathrm{F}) \circ \Psi_{\mathtt{i}} \arrow[r, "\mathbf{N}(\alpha) \circ_0 \theta_{\mathtt{i}}"] & \mathbf{N}(\mathrm{G}) \circ \Phi_{\mathtt{i}} \arrow[r, "\mathrm{id}_{\mathbf{N}(\mathrm{G})} \circ_0 \tau_{\mathtt{i}}"] & \mathbf{N}(\mathrm{G}) \circ \Sigma_{\mathtt{i}}
\end{tikzcd}$$

commute, the latter two facts also follow. □

2.4. **Properties of 2-natural transformations.** Let $\mathbf{M}$ and $\mathbf{N}$ be two 2-representations of $\mathscr{C}$ and $\Psi : \mathbf{M} \to \mathbf{N}$ a 2-natural transformation. Given, for every $\mathtt{i} \in \mathscr{C}$, a functor $\Phi_{\mathtt{i}}$ and an isomorphism $\xi_{\mathtt{i}} : \Phi_{\mathtt{i}} \to \Psi_{\mathtt{i}}$, define, for every 1-morphism $\mathrm{F} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$

$$\eta^{\Phi}_{\mathrm{F}} := (\mathrm{id}_{\mathbf{N}(\mathrm{F})} \circ_0 \xi_{\mathtt{i}}^{-1}) \circ_1 \eta^{\Psi}_{\mathrm{F}} \circ_1 (\xi_{\mathtt{j}} \circ_0 \mathrm{id}_{\mathbf{M}(\mathrm{F})}).$$

Then it is straightforward to check that this extends Φ to a 2-natural transformation.

Proposition 2. *Let $\mathbf{M}$ and $\mathbf{N}$ be two 2-representations of $\mathscr{C}$ and $\Psi : \mathbf{M} \to \mathbf{N}$ a 2-natural transformation. Assume that for every $\mathtt{i} \in \mathscr{C}$ the functor $\Psi_{\mathtt{i}}$ is an equivalence. Then there exists an inverse 2-natural transformation.*

Proof. For any $\mathtt{i} \in \mathscr{C}$ choose an inverse equivalence $\Phi_{\mathtt{i}}$ of $\Psi_{\mathtt{i}}$. Let

$$\xi_{\mathtt{i}} : \mathrm{Id}_{\mathbf{M}(\mathtt{i})} \to \Phi_{\mathtt{i}} \circ \Psi_{\mathtt{i}} \quad \text{and} \quad \zeta_{\mathtt{i}} : \Psi_{\mathtt{i}} \circ \Phi_{\mathtt{i}} \to \mathrm{Id}_{\mathbf{N}(\mathtt{i})}$$

be some isomorphisms. Define

$$\eta^{\Phi}_{\mathrm{F}} := \big((\mathrm{id}_{\Phi_{\mathtt{j}} \circ \mathbf{N}(\mathrm{F})} \circ_0 \zeta_{\mathtt{i}}) \circ_1 (\mathrm{id}_{\Phi_{\mathtt{j}}} \circ_0 \eta^{\Psi}_{\mathrm{F}} \circ_0 \mathrm{id}_{\Phi_{\mathtt{i}}}) \circ_1 (\xi_{\mathtt{j}} \circ_0 \mathrm{id}_{\mathbf{M}(\mathrm{F}) \circ \Phi_{\mathtt{i}}}) \big)^{-1}.$$

It is obvious that this produces a natural transformation, but we have to check that

$$\eta^{\Phi}_{\mathrm{F} \circ \mathrm{G}} = (\mathrm{id}_{\mathbf{N}(\mathrm{F})} \circ_0 \eta^{\Phi}_{\mathrm{G}}) \circ_1 (\eta^{\Phi}_{\mathrm{F}} \circ_0 \mathrm{id}_{\mathbf{M}(\mathrm{G})}). \tag{3}$$

This follows from commutativity of the diagram

$$
\begin{tikzcd}[column sep=tiny]
 & & & \mathbf{M}(\mathrm{F})\mathbf{M}(\mathrm{G})\Phi_{\mathtt{i}} \arrow[dl] \arrow[dr] & & & \\
 & & \Phi_{\mathtt{k}}\Psi_{\mathtt{k}}\mathbf{M}(\mathrm{F})\mathbf{M}(\mathrm{G})\Phi_{\mathtt{i}} \arrow[dl] \arrow[dr] & & \mathbf{M}(\mathrm{F})\Phi_{\mathtt{j}}\Psi_{\mathtt{j}}\mathbf{M}(\mathrm{G})\Phi_{\mathtt{i}} \arrow[dl] \arrow[dr] & & \\
 & \Phi_{\mathtt{k}}\mathbf{N}(\mathrm{F})\Psi_{\mathtt{j}}\mathbf{M}(\mathrm{G})\Phi_{\mathtt{i}} \arrow[dr] & & \Phi_{\mathtt{k}}\Psi_{\mathtt{k}}\mathbf{M}(\mathrm{F})\Phi_{\mathtt{j}}\Psi_{\mathtt{j}}\mathbf{M}(\mathrm{G})\Phi_{\mathtt{i}} \arrow[dl] \arrow[dr] & & \mathbf{M}(\mathrm{F})\Phi_{\mathtt{j}}\mathbf{N}(\mathrm{G})\Psi_{\mathtt{i}}\Phi_{\mathtt{i}} \arrow[dl] \arrow[dr] & \\
 & & \Phi_{\mathtt{k}}\mathbf{N}(\mathrm{F})\Psi_{\mathtt{j}}\Phi_{\mathtt{j}}\Psi_{\mathtt{j}}\mathbf{M}(\mathrm{G})\Phi_{\mathtt{i}} \arrow[dl] \arrow[dr] & & \Phi_{\mathtt{k}}\Psi_{\mathtt{k}}\mathbf{M}(\mathrm{F})\Phi_{\mathtt{j}}\mathbf{N}(\mathrm{G})\Psi_{\mathtt{i}}\Phi_{\mathtt{i}} \arrow[dl] \arrow[dr] & & \mathbf{M}(\mathrm{F})\Phi_{\mathtt{j}}\mathbf{N}(\mathrm{G}) \arrow[dl] \\
 & \Phi_{\mathtt{k}}\mathbf{N}(\mathrm{F})\Psi_{\mathtt{j}}\mathbf{M}(\mathrm{G})\Phi_{\mathtt{i}} \arrow[dr] & & \Phi_{\mathtt{k}}\mathbf{N}(\mathrm{F})\Psi_{\mathtt{j}}\Phi_{\mathtt{j}}\mathbf{N}(\mathrm{G})\Psi_{\mathtt{i}}\Phi_{\mathtt{i}} \arrow[dl] \arrow[dr] & & \Phi_{\mathtt{k}}\Psi_{\mathtt{k}}\mathbf{M}(\mathrm{F})\Phi_{\mathtt{j}}\mathbf{N}(\mathrm{G}) \arrow[dl] & \\
 & & \Phi_{\mathtt{k}}\mathbf{N}(\mathrm{F})\mathbf{N}(\mathrm{G})\Psi_{\mathtt{i}}\Phi_{\mathtt{i}} \arrow[dr] & & \Phi_{\mathtt{k}}\mathbf{N}(\mathrm{F})\Psi_{\mathtt{j}}\Phi_{\mathtt{j}}\mathbf{N}(\mathrm{G}) \arrow[dl] & & \\
 & & & \Phi_{\mathtt{k}}\mathbf{N}(\mathrm{F})\mathbf{N}(\mathrm{G}) & & &
\end{tikzcd}
$$

where the maps are the obvious ones (each of the maps has exactly one component of the form ξ, ζ or η^{Ψ} and identities elsewhere). Commutativity of all squares is immediate. Then reading along the right border gives (the inverse of) the right hand side of (3). Computing (the inverse of) the left hand side of (3) directly, using the definition of η^{Φ} and property (1) of $\eta^{\Psi}_{\mathrm{F}\circ\mathrm{G}}$, gives the left border of the diagram, after noting that the third and fourth morphism in this path compose to the identity on $\Phi_{\mathtt{k}}\mathbf{N}(\mathrm{F})\Psi_{\mathtt{j}}\mathbf{M}(\mathrm{G})\Phi_{\mathtt{i}}$ by adjunction. Therefore (3) holds and this extends Φ to a 2-natural transformation. $\square$

In this scenario we will say that the 2-representations $\mathbf{M}$ and $\mathbf{N}$ are *equivalent*.

2.5. **Abelianization and identities.** Denote by $\overline{\cdot} : \mathscr{C}\text{-afmod} \to \mathscr{C}\text{-mod}$ the abelianization 2-functor defined as in [MM2, Subsection 4.2]: for $\mathbf{M} \in \mathscr{C}\text{-afmod}$ and $\mathtt{i} \in \mathscr{C}$, the category $\overline{\mathbf{M}}(\mathtt{i})$ consists of all diagrams of the form $X \xrightarrow{\alpha} Y$, where $X, Y \in \mathbf{M}(\mathtt{i})$ and α is a morphism in $\mathbf{M}(\mathtt{i})$. Morphisms in $\overline{\mathbf{M}}(\mathtt{i})$ are commutative squares modulo factorization of the right downwards arrow using a homotopy. The 2-action of $\mathscr{C}$ on $\overline{\mathbf{M}}(\mathtt{i})$ is defined component-wise.

For any 2-representation $\mathbf{M}$ of $\mathscr{C}$ and any non-negative integer k, we denote by $\spadesuit_k$ the 2-natural transformation from $\mathbf{M}$ to $\mathbf{M}$ given by assigning to each $\mathtt{i} \in \mathscr{C}$ the functor

$$\underbrace{\mathrm{Id}_{\mathbf{M}(\mathtt{i})} \oplus \mathrm{Id}_{\mathbf{M}(\mathtt{i})} \oplus \cdots \oplus \mathrm{Id}_{\mathbf{M}(\mathtt{i})}}_{k\,\text{summands}}$$

and defining $\eta_{\mathrm{F}}^{\spadesuit_k}$ as $\mathrm{id}_{\mathrm{F}} \oplus \cdots \oplus \mathrm{id}_{\mathrm{F}}$ (again with k summands).

2.6. **Principal 2-representations and additive subrepresentations.** For $\mathtt{i} \in \mathscr{C}$ we denote by $\mathbb{P}_{\mathtt{i}}$ the principal 2-representation $\mathscr{C}(\mathtt{i}, {}_{-}) \in \mathscr{C}$-afmod. For any $\mathbf{M} \in \mathscr{C}$-amod we have the usual Yoneda Lemma (see [Le, Subsection 2.1] and compare to [MM2, Lemma 9]):

Lemma 3.

$$\mathrm{Hom}_{\mathscr{C}\text{-amod}}(\mathbb{P}_{\mathtt{i}}, \mathbf{M}) \cong \mathbf{M}(\mathtt{i}). \tag{4}$$

Proof. Let $\Psi : \mathbb{P}_{\mathtt{i}} \to \mathbf{M}$ be a 2-natural transformation and set $X := \Psi_{\mathtt{i}}(\mathbb{1}_{\mathtt{i}})$. Denote by $\Phi : \mathbb{P}_{\mathtt{i}} \to \mathbf{M}$ the unique strict 2-natural transformation sending $\mathbb{1}_{\mathtt{i}}$ to X (see [MM2, Lemma 9]). Then, for any 1-morphism $\mathrm{F} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$, we have the natural isomorphism

$$(\theta_{\mathtt{j}})_{\mathrm{F}} := (\eta^{\Psi}_{\mathrm{F}})_{\mathbb{1}_{\mathtt{i}}} : \Psi_{\mathtt{j}}(\mathrm{F}) \to \mathbf{M}(\mathrm{F})\, \Psi_{\mathtt{i}}(\mathbb{1}_{\mathtt{i}}) = \mathbf{M}(\mathrm{F})\, X = \Phi_{\mathtt{j}}(\mathrm{F}).$$

This gives us an (invertible) modification θ from Ψ to Φ and the claim follows. $\square$

Given $\mathbf{M} \in \mathscr{C}$-mod and $X \in \mathbf{M}(\mathtt{i})$ for some $\mathtt{i} \in \mathscr{C}$, define $\mathbf{M}_X \in \mathscr{C}$-afmod by restricting $\mathbf{M}$ to the full subcategories $\mathrm{add}(\mathrm{F}\, X)$, where F runs through the set of all 1-morphisms in $\mathscr{C}(\mathtt{i}, \mathtt{j})$, $\mathtt{j} \in \mathscr{C}$.

2.7. **The multisemigroup of $\mathscr{C}$ and cells.** The set $\mathcal{S}[\mathscr{C}]$ of isomorphism classes of indecomposable 1-morphisms in $\mathscr{C}$ has the natural structure of a multisemigroup induced by horizontal composition, see [MM2, Subsection 3.1] (see also [KM] for more details on multisemigroups). Let $\leq_L$, $\leq_R$ and $\leq_J$ denote the natural left, right and two-sided orders on $\mathcal{S}[\mathscr{C}]$, respectively. For example, $\mathrm{F} \leq_L \mathrm{G}$ means that for some 1-morphism H the composition $\mathrm{H} \circ \mathrm{F}$ contains a direct summand isomorphic to G. Equivalence classes with respect to $\leq_L$ are called *left cells*. Right and two-sided cells are defined analogously. Cells correspond exactly to Green's equivalence classes for the multisemigroup $\mathcal{S}[\mathscr{C}]$.

A two-sided cell $\mathcal{J}$ is called *regular* if different left (right) cells in $\mathcal{J}$ are not comparable with respect to the left (right) order. A two-sided cell $\mathcal{J}$ is called *strongly regular* if it is regular and, moreover, the intersection of any left and any right cell inside $\mathcal{J}$ consists of exactly one element.

Given a left cell $\mathcal{L}$, there exists an $\mathtt{i}_{\mathcal{L}} \in \mathscr{C}$ such that every 1-morphism $\mathrm{F} \in \mathcal{L}$ belongs to $\mathscr{C}(\mathtt{i}_{\mathcal{L}}, \mathtt{j})$ for some $\mathtt{j} \in \mathscr{C}$. Similarly, given a right cell $\mathcal{R}$, there exists a $\mathtt{j}_{\mathcal{R}} \in \mathscr{C}$ such that every 1-morphism $\mathrm{F} \in \mathcal{R}$ belongs to $\mathscr{C}(\mathtt{i}, \mathtt{j}_{\mathcal{R}})$ for some $\mathtt{i} \in \mathscr{C}$.

2.8. **Cell 2-representations.** Let $\mathcal{L}$ be a left cell and $\mathtt{i} = \mathtt{i}_{\mathcal{L}}$. Consider $\overline{\mathbb{P}}_{\mathtt{i}}$. For an indecomposable 1-morphism F in some $\mathscr{C}(\mathtt{i}, \mathtt{j})$ denote by L_{F} the unique simple top of the indecomposable projective module $0 \to \mathrm{F}$ in $\overline{\mathbb{P}}_{\mathtt{i}}(\mathtt{j})$. By [MM1, Proposition 17], there exists a unique $\mathrm{G}_{\mathcal{L}} \in \mathcal{L}$ (called the *Duflo involution* in $\mathcal{L}$) such that the indecomposable projective module $0 \to \mathbb{1}_{\mathtt{i}}$ has a unique quotient N such that the simple socle of N is isomorphic to $L_{\mathrm{G}_{\mathcal{L}}}$ and $\mathrm{F}\, N/L_{\mathrm{G}_{\mathcal{L}}} = 0$ for any $\mathrm{F} \in \mathcal{L}$. Set $Q := \mathrm{G}_{\mathcal{L}}\, L_{\mathrm{G}_{\mathcal{L}}}$. Then the additive 2-representation $\mathbf{C}_{\mathcal{L}} := \left(\overline{\mathbb{P}}_{\mathtt{i}}\right)_Q$ is called the *additive cell* 2-representation of $\mathscr{C}$ associated to $\mathcal{L}$. The abelianization $\overline{\mathbf{C}}_{\mathcal{L}}$ of $\mathbf{C}_{\mathcal{L}}$ is called the *abelian cell* 2-representation of $\mathscr{C}$ associated to $\mathcal{L}$. For $\mathrm{F} \in \mathcal{L}$ we set $P_{\mathrm{F}} := \mathrm{F}\, L_{\mathrm{G}_{\mathcal{L}}}$, which we also identify with the indecomposable projective object $0 \to \mathrm{F}\, L_{\mathrm{G}_{\mathcal{L}}}$ in $\overline{\mathbf{C}}_{\mathcal{L}}$.

3. First sufficient condition for 2-Schur's lemma

3.1. **The claim.** In this section we prove the following:

Theorem 4. *Let $\mathscr{C}$ be a fiat 2-category, $\mathcal{J}$ a strongly regular two-sided cell of $\mathscr{C}$ and $\mathcal{L}$ a left cell in $\mathcal{J}$. Set $\mathtt{i} = \mathtt{i}_{\mathcal{L}}$ and $\mathrm{G} = \mathrm{G}_{\mathcal{L}}$. Assume that the natural map*

$$\begin{array}{ccc} \mathrm{End}_{\mathscr{C}}(\mathbb{1}_{\mathtt{i}}) & \longrightarrow & \mathrm{End}_{\mathbf{C}_{\mathcal{L}}}(P_{\mathrm{G}}) \\ \varphi & \mapsto & \mathbf{C}_{\mathcal{L}}(\varphi)_{P_{\mathrm{G}}} \end{array} \tag{5}$$

is surjective. Then any endomorphism of $\mathbf{C}_{\mathcal{L}}$ is isomorphic to $\spadesuit_k$ for some k (in the category $\mathrm{End}_{\mathscr{C}\text{-afmod}}(\mathbf{C}_{\mathcal{L}})$). Similarly, any endomorphism of $\overline{\mathbf{C}}_{\mathcal{L}}$ is isomorphic to $\spadesuit_k$ for some k (in the category $\mathrm{End}_{\mathscr{C}\text{-mod}}(\overline{\mathbf{C}}_{\mathcal{L}})$).

3.2. **Annihilators of various objects in $\overline{\mathbf{C}}_{\mathcal{L}}$.** For any 2-representation $\mathbf{M}$ of $\mathscr{C}$ and $X \in \mathbf{M}(\mathtt{j})$ for some $\mathtt{j}$, let $\mathrm{Ann}_{\mathscr{C}}(X)$ denote the left 2-ideal of $\mathscr{C}$ consisting of all 2-morphisms α which annihilate X. The key observation to prove Theorem 4 is the following:

Lemma 5. *Under the assumption of Theorem 4, if $X \in \overline{\mathbf{C}}_{\mathcal{L}}(\mathtt{i})$ is such that $\mathrm{Ann}_{\mathscr{C}}(X) \supset \mathrm{Ann}_{\mathscr{C}}(L_{\mathrm{G}})$, then $X \in \mathrm{add}(L_{\mathrm{G}})$.*

Proof. Let $\mathrm{F} \in \mathcal{L}$ be different from G. Then $\mathrm{F}^* L_{\mathrm{F}} \neq 0$ by [MM1, Lemma 15]. At the same time, from the fact that $\mathcal{J}$ is strongly simple it follows that $\mathrm{F}^* \notin \mathcal{L}$. Therefore $\mathrm{F}^* L_{\mathrm{G}} = 0$ by [MM1, Lemma 15]. Hence $\mathrm{id}_{\mathrm{F}^*} \in \mathrm{Ann}_{\mathscr{C}}(L_{\mathrm{G}})$ and at the same time $\mathrm{id}_{\mathrm{F}^*} \notin \mathrm{Ann}_{\mathscr{C}}(L_{\mathrm{F}})$.

Since F^* is exact, the previous paragraph implies that for any X satisfying $\mathrm{Ann}_{\mathscr{C}}(X) \supset \mathrm{Ann}_{\mathscr{C}}(L_{\mathrm{G}})$, every simple subquotient of X is isomorphic to L_{G}. Assume now that X is indecomposable such that there is a short exact sequence

$$0 \to L_{\mathrm{G}} \to X \to L_{\mathrm{G}} \to 0.$$

Then there is a short exact sequence $K \hookrightarrow P_{\mathrm{G}} \twoheadrightarrow X$ and an endomorphism of P_{G} which induces a non-trivial nilpotent endomorphism of X. From (5), it follows that the natural map

$$\begin{array}{ccc} \mathrm{End}_{\mathscr{C}}(\mathbb{1}_{\mathtt{i}}) & \longrightarrow & \mathrm{End}_{\mathbf{C}_{\mathcal{L}}}(X) \\ \varphi & \mapsto & \mathbf{C}_{\mathcal{L}}(\varphi)_{X} \end{array}$$

is surjective. Let $\alpha \in \mathrm{End}_{\mathscr{C}}(\mathbb{1}_{\mathtt{i}})$ be a 2-morphism which produces a non-trivial nilpotent endomorphism of X. Then $\alpha \notin \mathrm{Ann}_{\mathscr{C}}(X)$ while $\alpha^2 \in \mathrm{Ann}_{\mathscr{C}}(X)$. At the same time, $\mathrm{End}_{\mathscr{C}}(\mathbb{1}_{\mathtt{i}})$ is a local finite dimensional $\Bbbk$-algebra (see Subsection 2.2), and hence α is either nilpotent or invertible. But α cannot be invertible as α^2 annihilates X. Therefore, α is nilpotent. This implies that $\alpha \in \mathrm{Ann}_{\mathscr{C}}(L_{\mathrm{G}})$ as any nonzero endomorphism of L_{G} is invertible by Schur's lemma.

Finally, if Y is an indecomposable module, every simple subquotient of which is isomorphic to L_{G}, then Y has a subquotient X as in the previous paragraph. Therefore $\mathrm{Ann}_{\mathscr{C}}(L_{\mathrm{G}}) \not\subset \mathrm{Ann}_{\mathscr{C}}(Y)$. The claim of the lemma follows. □

3.3. **Proof of Theorem 4.** Let $\Psi \in \mathrm{End}_{\mathscr{C}\text{-mod}}(\overline{\mathbf{C}}_{\mathcal{L}})$. By Lemma 5, we have $\Psi_{\mathtt{i}}(L_{\mathrm{G}}) \cong L_{\mathrm{G}}^{\oplus k}$ for some non-negative integer k. Now for any $\mathrm{F} \in \mathcal{L}$ we have an isomorphism

$$\Psi_{\mathtt{j}}(P_{\mathrm{F}}) = \Psi_{\mathtt{j}}(\mathrm{F}\, L_{\mathrm{G}}) \cong \mathrm{F}\, L_{\mathrm{G}}^{\oplus k} \cong P_{\mathrm{F}}^{\oplus k},$$

natural in F. As $\Psi_{\mathtt{j}}$ is right exact, every indecomposable projective is of the form P_{F}, and 2-morphisms in $\mathscr{C}$ surject onto homomorphisms between indecomposable projectives (see [MM1, Subsection 4.5]), we have that $\Psi_{\mathtt{j}}$ is isomorphic to $\mathrm{Id}_{\overline{\mathbf{C}}_{\mathcal{L}}(\mathtt{j})}^{\oplus k}$.

Clearly, k does not depend on $\mathtt{j}$. Now we repeat the argument from the proof of Lemma 3. We have the natural isomorphisms

$$(\theta_{\mathtt{j}})_{\mathrm{F}\,L_{\mathrm{G}}} := (\eta^{\Psi}_{\mathrm{F}})_{L_{\mathrm{G}}} : \Psi_{\mathtt{j}} \circ \overline{\mathbf{C}}_{\mathcal{L}}(\mathrm{F})\, L_{\mathrm{G}} \to \overline{\mathbf{C}}_{\mathcal{L}}(\mathrm{F}) \circ (\spadesuit_k)_{\mathtt{i}}\, L_{\mathrm{G}} = \overline{\mathbf{C}}_{\mathcal{L}}(\mathrm{F})\, L_{\mathrm{G}}^{\oplus k},$$

which give us an invertible modification θ from Ψ to $\spadesuit_k$. This proves the abelian part of Theorem 4.

To prove the additive part we just note that any $\Psi \in \mathrm{End}_{\mathscr{C}\text{-mod}}(\mathbf{C}_{\mathcal{L}})$ abelianizes to $\overline{\Psi} \in \mathrm{End}_{\mathscr{C}\text{-mod}}(\overline{\mathbf{C}}_{\mathcal{L}})$. Now the additive claim of Theorem 4 follows from the abelian claim by restricting to projective modules. □

4. Description of $\mathcal{J}$-simple fiat 2-categories

4.1. **Definition of 2-full 2-representations.** Let $\mathscr{C}$ be a finitary category and $\mathbf{M}$ a 2-representation of $\mathscr{C}$. We will say that $\mathbf{M}$ is 2-*full* provided that for any 1-morphisms $\mathrm{F}, \mathrm{G} \in \mathscr{C}$ the representation map

$$\mathrm{Hom}_{\mathscr{C}}(\mathrm{F}, \mathrm{G}) \to \mathrm{Hom}_{\mathfrak{X}}(\mathbf{M}(\mathrm{F}), \mathbf{M}(\mathrm{F})), \tag{6}$$

where $\mathfrak{X} \in \{\mathfrak{A}_{\Bbbk}, \mathfrak{A}^f_{\Bbbk}, \mathfrak{R}_{\Bbbk}\}$ is the target 2-category of $\mathbf{M}$, is surjective. In other words, 2-morphisms in $\mathscr{C}$ surject onto the space of natural transformations between functors.

If $\mathcal{J}$ is a 2-sided cell of $\mathscr{C}$, we will say that $\mathbf{M}$ is $\mathcal{J}$-2-*full* provided that for any 1-morphisms $\mathrm{F}, \mathrm{G} \in \mathcal{J}$ the representation map (6) is surjective.

4.2. **The 2-category associated with $\mathcal{J}$.** Let now $\mathscr{C}$ be a fiat 2-category and $\mathcal{J}$ a two-sided cell in $\mathscr{C}$. Let $\mathcal{L}$ be a left cell of $\mathcal{J}$, $\mathrm{G} := \mathrm{G}_{\mathcal{L}}$ and $\mathtt{i} := \mathtt{i}_{\mathcal{L}}$. Let $\mathscr{J}$ be the unique maximal 2-ideal of $\mathscr{C}$ which does not contain id_{F} for any $\mathrm{F} \in \mathcal{J}$ (see [MM2, Theorem 15]). Then the quotient 2-category $\mathscr{C}/\mathscr{J}$ is $\mathcal{J}$-simple (see [MM2, Subsection 6.2]). Denote by $\mathscr{C}^{(\mathcal{J})}$ the 2-full 2-subcategory of $\mathscr{C}/\mathscr{J}$ generated by $\mathbb{1}_{\mathtt{i}_{\mathcal{L}}}$ and all $\mathrm{F} \in \mathcal{J}$ (and closed with respect to isomorphism of 1-morphisms). We will call $\mathscr{C}^{(\mathcal{J})}$ the *$\mathcal{J}$-simple 2-category associated to $\mathcal{J}$.*

The cell 2-representation $\mathbf{C}_{\mathcal{L}}$ of $\mathscr{C}$ factors over $\mathscr{C}/\mathscr{J}$ by [MM2, Theorem 19] and hence restricts to a 2-representation of $\mathscr{C}^{(\mathcal{J})}$. Assume now that $\mathcal{J}$ is strongly regular. Then, by [MM1, Proposition 32], $\mathcal{J}$ remains a strongly regular two-sided cell in $\mathscr{C}^{(\mathcal{J})}$. Moreover, using [MM2, Subsection 6.5], the restriction of $\mathbf{C}_{\mathcal{L}}$ to $\mathscr{C}^{(\mathcal{J})}$ is equivalent to the corresponding cell 2-representation of $\mathscr{C}^{(\mathcal{J})}$. For $\mathrm{F} \in \mathcal{J}$ denote by m_{F} the multiplicity of G in $\mathrm{F}^*\mathrm{F}$.

For the remainder of this section we fix a strongly regular cell $\mathcal{J}$, assume that $\mathscr{C} = \mathscr{C}^{(\mathcal{J})}$ and that the function $\mathrm{F} \mapsto m_{\mathrm{F}}$ is constant on right cells of $\mathcal{J}$.

4.3. **Detecting 2-fullness.** We consider the cell 2-representation $\mathbf{M} := \overline{\mathbf{C}}_{\mathcal{L}}$. We start our analysis with the following observation:

Proposition 6. *For $\mathrm{F} \in \mathcal{J}$ and $\mathtt{j} \in \mathscr{C}$ consider the representation map*

$$\mathrm{Hom}_{\mathscr{C}}(\mathrm{F}, \mathbb{1}_{\mathtt{j}}) \to \mathrm{Hom}_{\mathfrak{R}_{\Bbbk}}(\mathbf{M}(\mathrm{F}), \mathbf{M}(\mathbb{1}_{\mathtt{j}})). \tag{7}$$

If this map is surjective for $\mathrm{F} = \mathrm{G}$ and $\mathtt{j} = \mathtt{i}$, then it is surjective for any F and $\mathtt{j}$.

Note that both sides of (7) are empty unless $\mathrm{F} \in \mathscr{C}(\mathtt{j}, \mathtt{j})$. As usual, to simplify notation we will use the module notation and write $\mathrm{F}\, X$ instead of $\mathbf{M}(\mathrm{F})(X)$.

Proof. Let $\mathrm{H}, \mathrm{K} \in \mathcal{L}$ and assume that $\mathrm{H}, \mathrm{K} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$. By strong regularity of $\mathcal{J}$ we have $\mathrm{HK}^* = a\mathrm{F}$ for some $\mathrm{F} \in \mathcal{J}$ and $a \in \mathbb{N}$, moreover, if we vary H and K, we can obtain any $\mathrm{F} \in \mathcal{J}$ in this way. To see that $\mathrm{HK}^* \neq 0$, one evaluates HK^* on L_{K} obtaining $\mathrm{K}^* L_{\mathrm{K}} = P_{\mathrm{G}}$ (by [MM1, Corollary 38(a)]), and $\mathrm{H}P_{\mathrm{G}} \neq 0$ since $\mathrm{H}L_{\mathrm{G}} = P_{\mathrm{H}} \neq 0$.

Similarly, we have $\mathrm{K}^*\mathrm{H} = b\mathrm{G}$ for some $b \in \mathbb{N}$ since $\mathrm{K}^*\mathrm{H}$ is in the same left cell as H (which is $\mathcal{L}$) and the same right cell as K^* (which is $\mathcal{L}^*$), and $\mathcal{L} \cap \mathcal{L}^* = \{\mathrm{G}\}$ since $\mathcal{J}$ is strongly regular. Using the involution $*$ we have

$$\mathrm{Hom}_{\mathscr{C}}(\mathrm{H}, \mathrm{K}) \cong \mathrm{Hom}_{\mathscr{C}}(\mathrm{K}^*, \mathrm{H}^*).$$

By adjunction, we have

$$\mathrm{Hom}_{\mathscr{C}}(\mathrm{H}, \mathrm{K}) \cong b\mathrm{Hom}_{\mathscr{C}}(\mathrm{G}, \mathbb{1}_{\mathtt{i}}), \qquad \mathrm{Hom}_{\mathscr{C}}(\mathrm{K}^*, \mathrm{H}^*) \cong a\mathrm{Hom}_{\mathscr{C}}(\mathrm{F}, \mathbb{1}_{\mathtt{j}}). \tag{8}$$

Evaluating $\mathrm{Hom}_{\mathscr{C}}(\mathrm{H}, \mathrm{K})$ at L_{G} (which is surjective by [MM1, Subsection 4.5]) and using adjunction, we get

$$\mathrm{Hom}_{\mathbf{M}(\mathtt{j})}(\mathrm{H}\, L_{\mathrm{G}}, \mathrm{K}\, L_{\mathrm{G}}) \cong b\mathrm{Hom}_{\mathbf{M}(\mathtt{i})}(\mathrm{G}\, L_{\mathrm{G}}, L_{\mathrm{G}}).$$

As $\mathrm{G}\, L_{\mathrm{G}} \cong P_{\mathrm{G}}$, the space $\mathrm{Hom}_{\mathbf{M}(\mathtt{i})}(\mathrm{G}\, L_{\mathrm{G}}, L_{\mathrm{G}})$ is one-dimensional, and thus

$$b = \dim \mathrm{Hom}_{\mathbf{M}(\mathtt{j})}(\mathrm{H}\, L_{\mathrm{G}}, \mathrm{K}\, L_{\mathrm{G}}) \tag{9}$$

On the other hand, evaluating $\mathrm{Hom}_{\mathscr{C}}(\mathrm{K}^*, \mathrm{H}^*)$ at a multiplicity free direct sum L of all simple modules in $\mathbf{M}(\mathtt{j})$ and using adjunction, we have

$$\mathrm{Hom}_{\mathbf{M}(\mathtt{i})}(\mathrm{K}^*\, L, \mathrm{H}^*\, L) \cong a\mathrm{Hom}_{\mathbf{M}(\mathtt{j})}(\mathrm{F}\, L, L). \tag{10}$$

By [MM1, Lemma 12], $\mathrm{K}^*\, L_{\mathrm{Q}} \neq 0$ for a direct summand L_{Q} of L, labeled by $\mathrm{Q} \in \mathcal{L}$, implies that K is in the same right cell as Q. Strong regularity implies $\mathrm{Q} = \mathrm{K}$ and by [MM1, Corollary 38(a)], we have $\mathrm{K}^*\, L \cong P_{\mathrm{G}}$. Similarly $\mathrm{H}^*\, L \cong P_{\mathrm{G}}$ and the left hand side of (10) is isomorphic to $\mathrm{End}_{\mathbf{M}(\mathtt{i})}(P_{\mathrm{G}})$.

As F is a direct summand of HK^*, again L_{K} is the only simple module which is not annihilated by F. By [MM1, Theorem 43], the module $\mathrm{F}\, L_{\mathrm{K}}$ is an indecomposable projective in $\mathbf{M}(\mathtt{j})$, namely P_{H}. This means that $\dim \mathrm{Hom}_{\mathbf{M}(\mathtt{j})}(\mathrm{F}\, L, L) = 1$ and hence

$$a = \dim \mathrm{End}_{\mathbf{M}(\mathtt{i})}(P_{\mathrm{G}}). \tag{11}$$

To proceed we need the following claim:

Lemma 7. *Let A be a finite dimensional $\Bbbk$-algebra and $e, f \in A$ primitive idempotents. Assume that* F *is an exact endofunctor of* A-mod *such that* $\mathrm{F}\, L_f \cong Ae$ *and* $\mathrm{F}\, L_g = 0$ *for any simple* $L_g \not\cong L_f$*. Then* F *is isomorphic to the functor* F' *given by tensoring with the bimodule* $Ae \otimes_{\Bbbk} fA$ *and, moreover,*

$$\mathrm{Hom}_{\mathfrak{R}_{\Bbbk}}(\mathrm{F}, \mathrm{Id}_{A\text{-mod}}) \cong \mathrm{Hom}_A(Ae, Af).$$

Proof. Let L be a multiplicity free sum of all simple A-modules. As $\mathrm{F}\, L_f$ has simple top L_e, it follows that F is a quotient of F', which gives us a surjective natural transformation $\alpha : \mathrm{F}' \to \mathrm{F}$. Further, $\mathrm{F}\, L \cong \mathrm{F}'\, L$, meaning that α is an isomorphism when evaluated on simple modules. Using induction on the length of a module and the 3-Lemma we obtain that α is an isomorphism, which proves the first claim. The second claim follows by adjunction. □

From Lemma 7 and surjectivity of (7) for G, we get

$$\dim \mathrm{Hom}_{\mathscr{C}}(\mathrm{G}, \mathbb{1}_{\mathtt{i}}) = \dim \mathrm{End}_{\mathbf{M}(\mathtt{i})}(P_{\mathrm{G}}).$$

Using (8), (9) and Lemma 7, we have

$$\begin{aligned}\dim \mathrm{Hom}_{\mathscr{C}}(\mathrm{H}, \mathrm{K}) &= \dim \mathrm{Hom}_{\mathbf{M}(\mathtt{j})}(\mathrm{H}\, L_{\mathrm{G}}, \mathrm{K}\, L_{\mathrm{G}}) \cdot \dim \mathrm{End}_{\mathbf{M}(\mathtt{i})}(P_{\mathrm{G}})\\ &= \dim \mathrm{Hom}_{\mathbf{M}(\mathtt{j})}(P_{\mathrm{H}}, P_{\mathrm{K}}) \cdot \dim \mathrm{End}_{\mathbf{M}(\mathtt{i})}(P_{\mathrm{G}}).\end{aligned}$$

On the other hand, using (8) and (11) we have

$$\dim \mathrm{Hom}_{\mathscr{C}}(\mathrm{K}^*, \mathrm{H}^*) = \dim \mathrm{Hom}_{\mathscr{C}}(\mathrm{F}, \mathbb{1}_{\mathtt{j}}) \cdot \dim \mathrm{End}_{\mathbf{M}(\mathtt{i})}(P_{\mathrm{G}}).$$

As $\mathscr{C}$ is $\mathcal{J}$-simple, $\dim \mathrm{Hom}_{\mathscr{C}}(\mathrm{F}, \mathbb{1}_{\mathtt{j}}) \leq \dim \mathrm{Hom}_{\mathfrak{R}_\Bbbk}(\mathbf{M}(\mathrm{F}), \mathbf{M}(\mathbb{1}_{\mathtt{j}}))$ and the latter by Lemma 7 is equal to $\dim \mathrm{Hom}_{\mathbf{M}(\mathtt{j})}(P_{\mathrm{H}}, P_{\mathrm{K}})$. Dividing through by $\dim \mathrm{End}_{\mathbf{M}(\mathtt{i})}(P_{\mathrm{G}})$ yields

$$\begin{aligned}\dim \mathrm{Hom}_{\mathbf{M}(\mathtt{j})}(P_{\mathrm{H}}, P_{\mathrm{K}}) &= \dim \mathrm{Hom}_{\mathscr{C}}(\mathrm{F}, \mathbb{1}_{\mathtt{j}})\\ &\leq \dim \mathrm{Hom}_{\mathfrak{R}_\Bbbk}(\mathbf{M}(\mathrm{F}), \mathbf{M}(\mathbb{1}_{\mathtt{j}}))\\ &= \dim \mathrm{Hom}_{\mathbf{M}(\mathtt{j})}(P_{\mathrm{H}}, P_{\mathrm{K}})\end{aligned}$$

and hence

$$\dim \mathrm{Hom}_{\mathscr{C}}(\mathrm{F}, \mathbb{1}_{\mathtt{j}}) = \dim \mathrm{Hom}_{\mathfrak{R}_\Bbbk}(\mathbf{M}(\mathrm{F}), \mathbf{M}(\mathbb{1}_{\mathtt{j}})).$$

Injectivity of the representation map, which follows from $\mathcal{J}$-simplicity of $\mathscr{C}$, now implies surjectivity and hence the statement of the proposition. □

Proposition 8. *Let* $\mathrm{H}, \mathrm{K} \in \mathscr{C}(\mathtt{j}, \mathtt{k}) \cap \mathcal{J}$. *If the representation map* (7) *is surjective for* $\mathrm{F} = \mathrm{G}$ *and* $\mathtt{i} = \mathtt{j}$, *then the representation map*

$$\mathrm{Hom}_{\mathscr{C}}(\mathrm{H}, \mathrm{K}) \to \mathrm{Hom}_{\mathfrak{R}_\Bbbk}(\mathbf{M}(\mathrm{H}), \mathbf{M}(\mathrm{K}))$$

is surjective.

Proof. As $\mathcal{J}$ is strongly regular, we have $\mathrm{K}^*\mathrm{H} = \mathrm{Q}^{\oplus m}$ for some $m \in \mathbb{N}_0$, where Q is in the intersection of the left cell of H and the right cell of K^*. We have the commutative diagram

$$\begin{tikzcd}\mathrm{Hom}_{\mathscr{C}}(\mathrm{H}, \mathrm{K}) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}_{\mathscr{C}}(\mathrm{K}^*\mathrm{H}, \mathbb{1}_{\mathtt{j}}) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}_{\mathscr{C}}(\mathrm{Q}, \mathbb{1}_{\mathtt{j}})^{\oplus m} \arrow[d]\\ \mathrm{Hom}_{\mathfrak{R}_\Bbbk}(\mathbf{M}(\mathrm{H}), \mathbf{M}(\mathrm{K})) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathfrak{R}_\Bbbk}(\mathbf{M}(\mathrm{K}^*\mathrm{H}), \mathrm{Id}_{\mathbf{M}(\mathtt{j})}) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathfrak{R}_\Bbbk}(\mathbf{M}(\mathrm{Q}), \mathrm{Id}_{\mathbf{M}(\mathtt{j})})^{\oplus m}\end{tikzcd}$$

where the vertical arrows are the representation maps, the left horizontal arrows are isomorphisms given by adjunction, and the right horizontal arrows are isomorphisms given by additivity. Then the rightmost vertical arrow is an isomorphism by Proposition 6 and $\mathcal{J}$-simplicity of $\mathscr{C}$. This implies that all vertical arrows are isomorphisms and the claim follows. □

4.4. Cell 2-representations are $\mathcal{J}$-2-full.

Theorem 9. *The cell 2-representation* $\mathbf{M} := \overline{\mathbf{C}}_{\mathcal{L}}$ *is* $\mathcal{J}$*-2-full.*

Proof. Thanks to Proposition 8, we have only to show that the representation map (7) is surjective for $\mathrm{F} = \mathrm{G}$ and $\mathtt{i} = \mathtt{j}$. In order to show this it suffices, by Lemma 7 and $\mathcal{J}$-simplicity of $\mathscr{C}$, to show that

$$\dim \mathrm{Hom}_{\mathscr{C}}(\mathrm{G}, \mathbb{1}_{\mathtt{i}}) = \dim \mathrm{End}_{\mathbf{M}(\mathtt{i})}(P_{\mathrm{G}}).$$

By Lemma 7 and $\mathcal{J}$-simplicity of $\mathscr{C}$, we have

$$\dim \mathrm{Hom}_{\mathscr{C}}(\mathrm{G}, \mathbb{1}_{\mathtt{i}}) \leq \dim \mathrm{End}_{\mathbf{M}(\mathtt{i})}(P_{\mathrm{G}}).$$

Recall from [MM1, Proposition 17] that there is a unique submodule K of the indecomposable projective module $0 \to \mathbb{1}_{\mathtt{i}}$ in $\overline{\mathbb{P}}_{\mathtt{i}}(\mathtt{i})$ which has simple top L_{G} and such that the quotient of the projective by K is annihilated by G. We denote by β some 2-morphism from G to $\mathbb{1}_{\mathtt{i}}$ which gives rise to a surjection from $0 \to \mathrm{G}$ to K in $\overline{\mathbb{P}}_{\mathtt{i}}(\mathtt{i})$. Then the $\mathrm{End}_{\mathscr{C}}(\mathrm{G})$-module $\mathrm{Hom}_{\mathscr{C}}(\mathrm{G}, \mathbb{1}_{\mathtt{i}})$ has simple top and β is a representative for this simple top.

Let A be a basic finite dimensional associative $\Bbbk$-algebra such that $\mathbf{M}(\mathtt{i}) \cong A$-mod. Let $1 = \sum_{i=1}^{n} e_i$ be a decomposition of $1 \in A$ into a sum of pairwise orthogonal primitive idempotents. We assume that $e = e_1$ is a primitive idempotent corresponding to L_{G}. From Lemma 7, we have that the functor $\mathbf{M}(\mathrm{G})$ is isomorphic to tensoring with $Ae \otimes_{\Bbbk} eA$. Clearly, $\mathbf{M}(\mathbb{1}_{\mathtt{i}})$ is isomorphic to tensoring with A.

Since $\mathcal{J}$ is strongly regular, Duflo involutions in $\mathcal{J} \cap \mathscr{C}(\mathtt{i}, \mathtt{i})$ are in bijection with $\{e_1, e_2, \dots, e_n\}$. Let G_i be the Duflo involution corresponding to e_i. Similarly to the existence of β, there is a β_i for each i, which we can put into the 2-morphism

$$\gamma := (\beta_1, \beta_2, \dots, \beta_n) : \bigoplus_i \mathrm{G}_i \to \mathbb{1}_{\mathtt{i}}.$$

The cokernel $\mathrm{Coker}(\gamma)$, as an object of $\overline{\mathbb{P}}_{\mathtt{i}}$, is annihilated by all 1-morphisms in $\mathcal{J}$. This implies that $\mathbf{M}(\mathrm{Coker}(\gamma))$ annihilates L_{F} for every $\mathrm{F} \in \mathcal{L}$ and hence $\mathbf{M}(\mathrm{Coker}(\gamma)) = 0$ by right exactness of $\mathbf{M}(\mathrm{Coker}(\gamma))$. From this we derive that $\mathbf{M}(\gamma)$ is surjective and hence we can choose β and the above identifications of functors with bimodules such that $\mathbf{M}(\beta)$ is the multiplication map $Ae \otimes_{\Bbbk} eA \to A$.

In order to show that $\dim \mathrm{Hom}_{\mathscr{C}}(\mathrm{G}, \mathbb{1}_{\mathtt{i}}) \geq \dim \mathrm{End}_{\mathbf{M}(\mathtt{i})}(P_{\mathrm{G}})$, we show that no $\varphi \in \mathrm{End}_{\mathscr{C}}(\mathrm{G})$ that induces a nonzero endomorphism of P_{G} when evaluated at L_{G}, is sent to zero under composition with β.

In order to see this, let $\varphi \in \mathrm{End}_{\mathscr{C}}(\mathrm{G})$ be such that $\mathbf{M}(\varphi) \in eAe \otimes eAe$ is not killed under the map $eAe \otimes eAe \twoheadrightarrow eAe \otimes eAe/\mathrm{Rad}(eAe) \cong eAe$. In other words, writing $\mathbf{M}(\varphi) = \sum_j (\psi_j \otimes (c_j e + r_j))$ for some $c_j \in \Bbbk, r_j \in \mathrm{Rad}(eAe)$, and where ψ_j runs over a basis of eAe, chosen in accordance with radical powers, we have that $\psi := \sum_j c_j \psi_j$ is nonzero in eAe. Then $\mathbf{M}(\beta \circ \varphi) = \psi + (\sum_j c_j \psi_j r_j) \in eAe$. As $\psi \in \mathrm{Rad}^k(eAe)$ implies $\psi_j \in \mathrm{Rad}^k(eAe)$ for all ψ_j such that $c_j \neq 0$, the summand $\sum_j c_j \psi_j r_j$ is in $\mathrm{Rad}^{k+1}(eAe)$ and hence $\mathbf{M}(\beta \circ \varphi) \in \mathrm{Hom}_{\mathfrak{R}_{\Bbbk}}(\mathbf{M}(\mathrm{G}), \mathbf{M}(\mathbb{1}_{\mathtt{i}}))$ is nonzero. Therefore $\beta \circ \varphi \in \mathrm{Hom}_{\mathscr{C}}(\mathrm{G}, \mathbb{1}_{\mathtt{i}})$ is nonzero for any $\varphi \in \mathrm{End}_{\mathscr{C}}(\mathrm{G})$ that is not killed by evaluation at L_{G}. By surjectivity of the map from $\mathrm{End}_{\mathscr{C}}(\mathrm{G})$ onto $\mathrm{End}_{\mathbf{M}(\mathtt{i})}(P_{\mathrm{G}})$ given by evaluation at L_{G} (see [MM1, Subsection 4.5]), this implies

$$\dim \mathrm{Hom}_{\mathscr{C}}(\mathrm{G}, \mathbb{1}_{\mathtt{i}}) \geq \dim \mathrm{End}_{\mathbf{M}(\mathtt{i})}(P_{\mathrm{G}})$$

and completes the proof of the proposition. □

Corollary 10. *Assume that $\mathscr{C}$ is any fiat 2-category and $\mathcal{J}$ is a strongly regular 2-sided cell of $\mathscr{C}$ such that the function $\mathrm{F} \mapsto m_{\mathrm{F}}$ is constant on right cells of $\mathcal{J}$. Then for any left cell $\mathcal{L}$ in $\mathcal{J}$ the cell 2-representation $\overline{\mathbf{C}}_{\mathcal{L}}$ is $\mathcal{J}$-2-full.*

Proof. This follows directly from Theorem 9 and [MM1, Corollary 33]. □

4.5. **Construction of $\mathcal{J}$-simple 2-categories $\mathscr{C}^{(\mathcal{J})}$.** Let $n \in \mathbb{N}$ and $A := (A_1, A_2, \dots, A_n)$ be a collection of pairwise non-isomorphic, basic, connected, weakly symmetric finite dimensional associative $\Bbbk$-algebras. For $i \in \{1, 2, \dots, n\}$ choose some small category $\mathcal{C}_i$ equivalent to A_i-mod, and let Z_i denote the center of A_i. Set $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n)$. Denote by $\mathscr{C}_{\mathcal{C}}$ the 2-full fiat 2-subcategory of $\mathfrak{R}_{\Bbbk}$ with objects $\mathcal{C}_i$, which is closed under isomorphisms of 1-morphisms and generated by functors that are isomorphic to tensoring with projective A_i-A_j bimodules.

We identify Z_i with $\mathrm{End}_{\mathscr{C}_\mathcal{C}}(\mathbb{1}_{\mathcal{C}_i})$ and denote by Z_i' the subalgebra of Z_i generated by $\mathrm{id}_{\mathbb{1}_{\mathcal{C}_i}}$ and all elements which factor through 1-morphisms given by tensoring with projective A_i-A_i bimodules.

Remark 11. In general, $Z_i' \neq Z_i$. For example, let $n = 1$ and $A = A_1 = \Bbbk[x]/(x^3)$. Then $Z = Z_1 = A$ while Z_1' is the linear span of 1 and x^2 in Z. Indeed, we have only one projective bimodule $A \otimes_\Bbbk A$, which has Loewy length 5 and unique Loewy filtration. As A has Loewy length 3, any nonzero composition $A \to A \otimes_\Bbbk A \to A$ must map the top of A to the socle of A. It is easy to check that the composition of the unique (up to scalar) injection $A \hookrightarrow A \otimes_\Bbbk A$ and the unique (up to scalar) surjection $A \otimes_\Bbbk A \twoheadrightarrow A$ is nonzero.

Choose subalgebras X_i in Z_i containing Z_i' and let $X = (X_1, X_2, \dots, X_n)$. Consider the additive 2-subcategory $\mathscr{C}_{\mathcal{C},X}$ of $\mathscr{C}_\mathcal{C}$ defined as follows: $\mathscr{C}_{\mathcal{C},X}$ has the same objects and the same 1-morphisms as $\mathscr{C}_\mathcal{C}$; all 2-morphism spaces between indecomposable 1-morphisms in $\mathscr{C}_{\mathcal{C},X}$ are the same as for $\mathscr{C}_\mathcal{C}$ except for $\mathrm{End}_{\mathscr{C}_{\mathcal{C},X}}(\mathbb{1}_{\mathcal{C}_i}) := X_i$.

Lemma 12. *The* 2*-category* $\mathscr{C}_{\mathcal{C},X}$ *is well-defined and fiat.*

Proof. To prove that $\mathscr{C}_{\mathcal{C},X}$ is well-defined we have to check that it is closed under both horizontal and vertical composition of 2-morphisms. That it is closed under vertical composition follows directly from the fact that X_i is a subalgebra. To check that it is closed under horizontal composition, we first observe that if $\mathbb{1}_{\mathcal{C}_i}$ appears (up to isomorphism) as a direct summand of $\mathrm{F} \circ \mathrm{G}$ for some indecomposable 1-morphisms F and G, then both F and F are isomorphic to $\mathbb{1}_{\mathcal{C}_i}$. For $x, y \in X_i$, we have

$$\begin{array}{ccccccc} A & \xrightarrow{\sim} & A \otimes_A A & \xrightarrow{x \otimes y} & A \otimes_A A & \xrightarrow{\sim} & A \\ 1 & \mapsto & 1 \otimes 1 & \mapsto & x \otimes y & \mapsto & xy \end{array}$$

from which the claim follows, again using that X_i is a subalgebra.

To prove that $\mathscr{C}_{\mathcal{C},X}$ is fiat we have to check that it contains all adjunction morphisms. The adjunction morphism from $\mathbb{1}_{\mathcal{C}_i}$ to $\mathbb{1}_{\mathcal{C}_i}$ is $\mathrm{id}_{\mathbb{1}_{\mathcal{C}_i}}$ and thus contained in $\mathscr{C}_{\mathcal{C},X}$. All other adjunction morphisms are between $\mathbb{1}_{\mathcal{C}_i}$ and direct sums of indecomposable 1-morphisms none of which is isomorphic to $\mathbb{1}_{\mathcal{C}_i}$ and therefore contained in $\mathscr{C}_{\mathcal{C},X}$ by definition. □

4.6. **Description of $\mathcal{J}$-simple 2-categories $\mathscr{C}^{(\mathcal{J})}$.** Now we are ready to prove the main result of this section, which gives a description, up to biequivalence, of fiat 2-categories that are "simple" in some sense.

Theorem 13. *Let* $\mathscr{C} = \mathscr{C}^{(\mathcal{J})}$ *be a skeletal fiat* $\mathcal{J}$*-simple* 2*-category. Assume that* $\mathcal{J}$ *is strongly regular and that the function* $\mathrm{F} \mapsto m_\mathrm{F}$ *is constant on right cells of* $\mathcal{J}$*. Then* $\mathscr{C}$ *is biequivalent to* $\mathscr{C}_{\mathcal{C},X}$ *for appropriate* $\mathcal{C}$ *and* X*.*

Proof. Let $\mathcal{L}$ be a left cell in $\mathcal{J}$ and $\mathbf{M} := \overline{\mathbf{C}}_\mathcal{L}$ be the corresponding cell 2-representation. Set $\mathcal{C}_i := \mathbf{M}(\mathtt{i})$ and let A_i be a basic algebra such that A_i-mod is equivalent to $\mathbf{M}(\mathtt{i})$. Let Z_i be the center of A_i which we identify with $\mathrm{End}_{\mathfrak{R}_\Bbbk}(\mathbb{1}_{\mathbf{M}(\mathtt{i})})$. Set $X_i := \mathbf{M}(\mathrm{End}_\mathscr{C}(\mathbb{1}_\mathtt{i})) \subset Z_i$. Then the representation map $\mathbf{M}$ is a 2-functor from $\mathscr{C}$ to $\mathscr{C}_{\mathcal{C},X}$, which is a biequivalence by Theorem 9, $\mathcal{J}$-simplicity of $\mathscr{C}$ and construction of X. □

5. Second sufficient condition for 2-Schur's lemma

Here we give a different version of Theorem 4.

Theorem 14. *Let $\mathscr{C}$ be a fiat 2-category and $\mathcal{J}$ a strongly regular two-sided cell of $\mathscr{C}$ such that the function $\mathrm{F} \mapsto m_{\mathrm{F}}$ is constant on right cells of $\mathcal{J}$. Let $\mathcal{L}$ be a left cell of $\mathcal{J}$. Then any endomorphism of $\mathbf{C}_{\mathcal{L}}$ is isomorphic to $\spadesuit_k$ for some k (in the category $\mathrm{End}_{\mathscr{C}\text{-afmod}}(\mathbf{C}_{\mathcal{L}})$). Similarly, any endomorphism of $\overline{\mathbf{C}}_{\mathcal{L}}$ is isomorphic to $\spadesuit_k$ for some k (in the category $\mathrm{End}_{\mathscr{C}\text{-mod}}(\overline{\mathbf{C}}_{\mathcal{L}})$).*

Proof. We follow the proof of Theorem 4 described in Section 3. What we need is the analogue of Lemma 5 in the new situation. More precise, we have to prove that given a non-split short exact sequence

$$0 \to L_{\mathrm{G}} \to X \to L_{\mathrm{G}} \to 0$$

in $\overline{\mathbf{C}}_{\mathcal{L}}(\mathtt{i})$, the obvious inclusion $\mathrm{Ann}_{\mathscr{C}}(X) \subset \mathrm{Ann}_{\mathscr{C}}(L_{\mathrm{G}})$ is strict.

As in Subsection 4.4, $\overline{\mathbf{C}}_{\mathcal{L}}(\mathtt{i})$ is equivalent to A-mod for some finite dimensional associative $\Bbbk$-algebra A and the functor $\overline{\mathbf{C}}_{\mathcal{L}}(\mathrm{G})$ can be identified with tensoring with $Ae \otimes_{\Bbbk} eA$ for some primitive idempotent $e \in A$. By Theorem 9, this identification is fully faithful on 2-morphisms. Clearly,

$$\mathrm{Ann}_{\mathscr{C}}(L_{\mathrm{G}}) \cap \mathrm{End}_{A \otimes_{\Bbbk} A^{\mathrm{op}}}(Ae \otimes_{\Bbbk} eA) = eAe \otimes_{\Bbbk} \mathrm{Rad}(eAe).$$

At the same time, as X is a non-split self-extension of L_{G}, we have

$$\mathrm{Ann}_{\mathscr{C}}(X) \cap \mathrm{End}_{A \otimes_{\Bbbk} A^{\mathrm{op}}}(Ae \otimes_{\Bbbk} eA) = eAe \otimes_{\Bbbk} U,$$

where U is a proper subalgebra of $\mathrm{Rad}(eAe)$ (since $eA \otimes_A X = eX = X$ as a vector space). The rest of the proof follows precisely the proof of Theorem 4. $\square$

6. The second layer of 2-Schur's lemma

6.1. **Endomorphisms of the identity functor.** So far we have only determined the *objects* in the endomorphism category of a cell 2-representation (Theorems 4 and 14) up to isomorphism. Now we would like to describe morphisms in this category.

Proposition 15. *Let $\mathscr{C}$ be a fiat 2-category, $\mathcal{J}$ a strongly regular two-sided cell of $\mathscr{C}$ and $\mathcal{L}$ a left cell in $\mathcal{J}$. For any $k \in \mathbb{N}$, consider $\spadesuit_k \in \mathrm{End}_{\mathscr{C}\text{-mod}}(\overline{\mathbf{C}}_{\mathcal{L}})$ (or $\spadesuit_k \in \mathrm{End}_{\mathscr{C}\text{-mod}}(\mathbf{C}_{\mathcal{L}})$). Then there are isomorphisms*

$$\mathrm{End}_{\mathrm{End}_{\mathscr{C}\text{-mod}}(\overline{\mathbf{C}}_{\mathcal{L}})}(\spadesuit_k) \cong \mathrm{Mat}_{k\times k}(\Bbbk) \quad \text{and} \quad \mathrm{End}_{\mathrm{End}_{\mathscr{C}\text{-mod}}(\mathbf{C}_{\mathcal{L}})}(\spadesuit_k) \cong \mathrm{Mat}_{k\times k}(\Bbbk).$$

Proof. We prove the statement for $\overline{\mathbf{C}}_{\mathcal{L}}$, the other case being analogous. For $\mathtt{i} \in \mathscr{C}$, let $A_{\mathtt{i}}$ be a finite dimensional associative $\Bbbk$-algebra such that $\overline{\mathbf{C}}_{\mathcal{L}}(\mathtt{i})$ is equivalent to $A_{\mathtt{i}}$-mod. Let $\theta : \spadesuit_k \to \spadesuit_k$ be a modification. As endomorphisms of $\mathrm{Id}_{\overline{\mathbf{C}}_{\mathcal{L}}(\mathtt{i})}$ can be identified with the center $Z_{\mathtt{i}}$ of $A_{\mathtt{i}}$, we can view $\theta_{\mathtt{i}}$ as an element of $\mathrm{Mat}_{k\times k}(Z_{\mathtt{i}})$.

First consider the case $k = 1$. Clearly, scalars belong to the endomorphism ring of $\spadesuit_1$. We would like to show that the radical of $Z_{\mathtt{i}}$ does not. Let e be a primitive idempotent of $A_{\mathtt{i}}$. From [MM1, Corollary 38(b)] it follows that there is $\mathrm{F} \in \mathcal{J}$ such that $\overline{\mathbf{C}}_{\mathcal{L}}(\mathrm{F})$ can be described by tensoring with a direct sum of bimodules of the form $A_{\mathtt{i}}e \otimes_{\Bbbk} eA_{\mathtt{i}}$. The action of $\spadesuit_1$ on $\overline{\mathbf{C}}_{\mathcal{L}}(\mathtt{i})$ is described as tensoring with $A_{\mathtt{i}}$, and the isomorphism $\eta_{\mathrm{F}}^{\spadesuit_1}$ is a direct sum of morphisms

$$A_{\mathtt{i}} \otimes_{A_{\mathtt{i}}} A_{\mathtt{i}}e \otimes_{\Bbbk} eA_{\mathtt{i}} \cong A_{\mathtt{i}}e \otimes_{\Bbbk} eA_{\mathtt{i}} \otimes_{A_{\mathtt{i}}} A_{\mathtt{i}}$$

sending $1\otimes e\otimes e$ to $e\otimes e\otimes 1$.

Let $0\neq z\in e\mathrm{Rad}(Z_{\mathtt{i}})e$. Then applying z after η sends $1\otimes e\otimes e$ to $e\otimes e\otimes z$, which is identified with $e\otimes z$ in $A_{\mathtt{i}}e\otimes_{\Bbbk} eA_{\mathtt{i}}$. Applying z before η sends $1\otimes e\otimes e$ to $z\otimes e\otimes 1$, which is identified with $z\otimes e$ in $A_{\mathtt{i}}e\otimes_{\Bbbk} eA_{\mathtt{i}}$. We have $e\otimes z\neq z\otimes e$ as $z\in e\mathrm{Rad}(Z_{\mathtt{i}})e$.

Now consider arbitrary k. From the above it follows that we can view $\theta_{\mathtt{i}}$ as an element of $\mathrm{Mat}_{k\times k}(\Bbbk)$ (here $\Bbbk\cong Z_{\mathtt{i}}/\mathrm{Rad}(Z_{\mathtt{i}})$). That every element $M\in \mathrm{Mat}_{k\times k}(\Bbbk)$ indeed defines an element of $\mathrm{End}_{\mathrm{End}_{\mathscr{C}\text{-mod}}(\overline{\mathbf{C}}_{\mathcal{L}})}(\spadesuit_k)$ can be seen from the commutative diagram

$$\begin{tikzcd}
A^{\oplus k}\otimes_A Ae\otimes_{\Bbbk} eA \arrow[r, "\eta_k"] \arrow[d, "M\otimes\mathrm{id}"] & Ae\otimes_{\Bbbk} eA\otimes_A A^{\oplus k} \arrow[d, "\mathrm{id}\otimes M"] \\
A^{\oplus k}\otimes_A Ae\otimes_{\Bbbk} eA \arrow[r, "\eta_k"] & Ae\otimes_{\Bbbk} eA\otimes_A A^{\oplus k}
\end{tikzcd}$$

where $A:=A_{\mathtt{i}}$ and η_k is the diagonal $k\times k$-matrix with η on the diagonal. This completes the proof. $\square$

6.2. The endomorphism category of a cell representations.

Theorem 16. *Let $\mathscr{C}$ be a fiat 2-category, $\mathcal{J}$ a strongly regular two-sided cell of $\mathscr{C}$ and $\mathcal{L}$ a left cell in $\mathcal{J}$. Assume that one of the following holds:*

(*i*) *The natural map* (5) *is surjective.*

(*ii*) *The function $\mathrm{F}\mapsto m_{\mathrm{F}}$ is constant on right cells of $\mathcal{J}$.*

Then both categories $\mathrm{End}_{\mathscr{C}\text{-mod}}(\overline{\mathbf{C}}_{\mathcal{L}})$ and $\mathrm{End}_{\mathscr{C}\text{-amod}}(\mathbf{C}_{\mathcal{L}})$ are equivalent to $\Bbbk$-mod.

Proof. This follows directly from Theorems 4 and 14 and Proposition 15. $\square$

7. Examples

7.1. Category $\mathcal{O}$ in type A.

Consider the simple complex Lie algebra $\mathfrak{g}=\mathfrak{sl}_n$ with the standard triangular decomposition $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ and a small category $\mathcal{O}_0$ equivalent to the principal block of the BGG-category $\mathcal{O}$ for $\mathfrak{g}$ (see [Hu]). Let $\mathscr{S}$ be the 2-category of projective functors associated to $\mathcal{O}_0$ as in [MM1, Subsection 7.1]. Indecomposable 1-morphisms in $\mathscr{S}$ are in natural bijection with elements of the symmetric group S_n (the Weyl group of $\mathfrak{g}$) and left, right and two-sided cells are Kazhdan-Lusztig right, left and two-sided cells, respectively. As shown in [MM1, Subsection 7.1], all two-sided cells are strongly regular and satisfy the numerical condition in Theorem 16(ii). Hence Theorem 16 completely describes the endomorphism category of all cell 2-representations for $\mathscr{S}$ (the latter were first constructed in [MS2]). As cell 2-representations corresponding to the same two-sided cell are equivalent (see [MS2, MM1]), it follows that this equivalence is unique (as a functor) up to isomorphism of functors. In [MS2], equivalence of cell 2-representations corresponding to the same two-sided cell was obtained using Arkhipov's twisting functors and the fact that they naturally commute with projective functors, see [AS]. Our present result shows that the shadows of Arkhipov's twisting functors act, on a cell 2-representation, simply as a direct sum of the identity.

We also would like to note that in this example we can also apply Theorem 4. A very special feature of S_n is that every two-sided Kazhdan-Lusztig cell of S_n contains the longest element $w:=w_0^P$ in some parabolic subgroup P in S_n. Then w is the Duflo involution in its Kazhdan-Lusztig right cell and hence the corresponding projective in the cell 2-representation is isomorphic to $\theta_w L_w$. From [MS1, Theorem 6.3] it follows that the center of $\mathcal{O}_0$ surjects onto the endomorphism algebra of $\theta_w L_w$ and hence we can apply Theorem 4.

7.2. **Category $\mathcal{O}$ in type B_2.** Consider the previous example for $\mathfrak{g}$ of type B_2. Let W be the Weyl group of type B_2 with elements $\{e, s, t, st, ts, sts, tst, stst\}$ (here $s^2 = t^2 = e$ and $stst = tsts$). We have the 2-category $\mathscr{S}$ with 1-morphisms θ_w, $w \in W$. Cells are again given by Kazhdan-Lusztig combinatorics, the two-sided cells are $\mathcal{J}_e = \{e\}, \mathcal{J}_{s,t} = \{s, t, st, ts, sts, tst\}$ and $\mathcal{J}_{stst} = \{stst\}$. The middle cell splits into two left cells $\mathcal{L}_1 = \{s, st, sts\}$ and $\mathcal{L}_2 = \{t, ts, tst\}$ (recall that our left cells are Kazhdan-Lusztig's right cells and vice versa) as shown in the following picture:

	$\mathcal{L}_1$	$\mathcal{L}_2$
$\mathcal{L}_1^*$	$\{s, sts\}$	$\{ts\}$
$\mathcal{L}_2^*$	$\{st\}$	$\{t, tst\}$.

Since strong regularity fails, we cannot apply Theorem 16 and, indeed, it turns out that the cell 2-representation $\overline{\mathbf{C}}_{\mathcal{L}_1}$ has more endomorphisms than just the identity, as we now show.

For $w \in \mathcal{L}_i$, $i = 1, 2$, set $L_w := L_{\theta_w}$. Let T_s and T_t be Arkhipov's twisting functors corresponding to s and t. Starting from $\overline{\mathbf{C}}_{\mathcal{L}_1}$ we apply T_s, project onto $\overline{\mathbf{C}}_{\mathcal{L}_2}$, apply T_t and project onto $\overline{\mathbf{C}}_{\mathcal{L}_1}$. This maps L_s to $L_s \oplus L_{sts}$. As twisting functors naturally commute with projective functors, it follows that $\mathrm{Ann}_{\mathscr{S}}(L_s) = \mathrm{Ann}_{\mathscr{S}}(L_{sts})$ and hence mapping L_s to L_{sts} extends to an endomorphism of $\overline{\mathbf{C}}_{\mathcal{L}_1}$ which is clearly not isomorphic to the identity functor.

7.3. **$\mathfrak{sl}_2$-categorification.** Consider the 2-category $\mathscr{B}_n$ associated with the $\mathfrak{sl}_2$-categorification of Chuang and Rouquier (see [CR]) as described in detail in [MM2, Subsection 7.1]. This is a fiat 2-category with strongly regular cells satisfying the numerical condition in Theorem 16(ii). Hence Theorem 16 completely describes endomorphisms for each cell 2-representation of $\mathfrak{sl}_2$ (compare [CR, Proposition 5.26]). However, we would like to point out that in the case of $\mathscr{B}_n$ describing the endomorphism category for cell 2-representations is much easier (than e.g. for the example in Subsection 7.1). Indeed, as explained in [MM2, Subsection 7.1], each two-sided cell of $\mathscr{B}_n$ has a left cell with Duflo involution G such that, in the corresponding cell 2-representation, the simple module L_{G} is projective (the corresponding Duflo involution has the form $\mathbb{1}_{\mathtt{i}}$). Due to this, any endomorphism of the cell 2-representation maps L_{G} to a direct sum of copies of L_{G} and is uniquely determined by the image of L_{G} up to isomorphism.

7.4. **A non-symmetric local algebra.** In this subsection we describe an example for which the condition in Theorem 16(i) is not satisfied, while the condition in Theorem 16(ii) is. Let $A := \Bbbk\langle x, y\rangle/(x^2, y^2, xy + yx)$ and $\mathcal{C}$ be a small category equivalent to A-mod. The center Z of A is the linear span of 1 and xy. Consider the fiat 2-category $\mathscr{C}_{\mathcal{C},Z}$. This category has two two-sided cells, one consisting of the identity and the other one, say $\mathcal{J}$, consisting of the 1-morphism G given by tensoring with $A \otimes_{\Bbbk} A$. Then G is the Duflo involution in $\mathcal{J}$ and the corresponding cell 2-representation is equivalent to the defining 2-representation. Therefore, the projective module P_{G} is isomorphic to ${}_AA$. Since A is not commutative, Z does not surject on the endomorphism algebra of P_{G}. Hence the condition in Theorem 16(i) is not satisfied. On the other hand, the condition in Theorem 16(ii) is satisfied as explained in [MM1, Subsection 7.3].

8. The numerical condition in the graded case

In this section, by *graded* we always mean $\mathbb{Z}$-graded.

8.1. **2-categories with free $\mathbb{Z}$-action.** Let $\mathscr{A}$ be 2-category. Assume that, for each $\mathtt{i},\mathtt{j}\in\mathscr{A}$, we are given an automorphism $(\cdot)_1$ of $\mathscr{A}(\mathtt{i},\mathtt{j})$. For $k\in\mathbb{Z}$, set $(\cdot)_k := (\cdot)_1^k$ and, for $\mathrm{F}\in\mathscr{A}(\mathtt{i},\mathtt{j})$, set $\mathrm{F}_k := (\mathrm{F})_k$. We will say that this datum defines a *free* action of $\mathbb{Z}$ on $\mathscr{A}$ provided that, for any $\mathrm{F}\in\mathscr{A}(\mathtt{i},\mathtt{j})$, the equality $\mathrm{F}_k=\mathrm{F}_m$ implies $k=m$ and, moreover, for any composable 1-morphisms F and G, we have

$$\mathrm{F}_k\circ\mathrm{G}_m = (\mathrm{F}\circ\mathrm{G})_{k+m}. \tag{12}$$

Example 17. Let A be a graded, connected, weakly symmetric finite dimensional associative $\Bbbk$-algebra and $\mathcal{C}$ a small category equivalent to the category A-gmod of finite dimensional graded A-modules. The algebra $A\otimes_{\Bbbk}A^{\mathrm{op}}$ inherits the structure of a graded algebra from A. Let $\langle 1\rangle$ denote the functor which shifts the grading such that $(M\langle 1\rangle)_i = M_{i+1}$, $i\in\mathbb{Z}$. Consider the 2-category $\mathscr{C}_{\mathcal{C}}$ defined as follows: It has one object (which we identify with $\mathcal{C}$), its 1-morphisms are closed under isomorphism of functors and are generated by $\langle\pm 1\rangle$ and functors induced by tensoring with projective A-A-bimodules (the latter are naturally graded), its 2-morphisms are natural transformations of functors (which correspond to homogeneous bimodule morphisms of degree zero). The group $\mathbb{Z}$ acts on $\mathscr{C}_{\mathcal{C}}$ by shifting the grading and this is free in the above sense.

8.2. **Graded fiat 2-categories.** Assume that $\mathscr{A}$ is a 2-category equipped with a free action of $\mathbb{Z}$. Assume further that $\mathscr{A}$ satisfies the following conditions:

- $\mathscr{A}$ has finitely many objects;
- for any $\mathtt{i},\mathtt{j}\in\mathscr{A}$, we have $\mathscr{A}(\mathtt{i},\mathtt{j})\in\mathfrak{A}_{\Bbbk}$ and horizontal composition is both additive and $\Bbbk$-linear;
- the set of $\mathbb{Z}$-orbits on isomorphism classes of indecomposable objects in $\mathscr{A}(\mathtt{i},\mathtt{j})$ is finite;
- all spaces of 2-morphisms are finite dimensional;
- for each 1-morphism F, there are only finitely many indecomposable 1-morphisms G (up to isomorphism) such that $\mathrm{Hom}_{\mathscr{A}}(\mathrm{F},\mathrm{G})\neq 0$;
- for each 1-morphism F, there are only finitely many indecomposable 1-morphisms G (up to isomorphism) such that $\mathrm{Hom}_{\mathscr{A}}(\mathrm{G},\mathrm{F})\neq 0$;
- for any $\mathtt{i}\in\mathscr{C}$ the 1-morphism $\mathbb{1}_{\mathtt{i}}$ is indecomposable;
- $\mathscr{A}$ has a weak object preserving involution and adjunction morphisms.

We will call such $\mathscr{A}$ *pro-fiat*.

Define the quotient 2-category $\mathscr{C}=\mathscr{A}/\mathbb{Z}$ to have the same objects as $\mathscr{A}$, and as morphism categories the categorical quotients $\mathscr{C}(\mathtt{i},\mathtt{j}) := \mathscr{A}(\mathtt{i},\mathtt{j})/\mathbb{Z}$. Recall that objects of $\mathscr{A}(\mathtt{i},\mathtt{j})/\mathbb{Z}$ are orbits of $\mathbb{Z}$ acting on objects of $\mathscr{A}(\mathtt{i},\mathtt{j})$ (for $\mathrm{F}\in\mathscr{A}(\mathtt{i},\mathtt{j})$, we will denote the corresponding orbit by $\mathrm{F}_\bullet$) and, for $\mathrm{F},\mathrm{G}\in\mathscr{A}(\mathtt{i},\mathtt{j})$, the space $\mathrm{Hom}_{\mathscr{C}}(\mathrm{F}_\bullet,\mathrm{G}_\bullet)$ is the quotient of $\bigoplus_{k,l\in\mathbb{Z}}\mathrm{Hom}_{\mathscr{A}(\mathtt{i},\mathtt{j})}(\mathrm{F}_k,\mathrm{G}_l)$ modulo the subspace generated by the expressions $\alpha - \alpha_l$ for $l \in \mathbb{Z}$. Horizontal composition in $\mathscr{C}$ is induced by the one in $\mathscr{A}$ in the natural way (which is well-defined due to (12)). We denote by $\Omega : \mathscr{A} \to \mathscr{C}$ the projection 2-functor.

Thanks to our assumptions on $\mathscr{A}$, the 2-category $\mathscr{C}$ is a fiat 2-category. We will say that $\mathscr{C}$ is a *graded fiat* 2-category. If we fix a representative F_s in each $\mathrm{F}_\bullet$, then, by construction, the category $\mathscr{C}(\mathtt{i},\mathtt{j})$ becomes graded (in the sense that for any 1-morphisms $\mathrm{F}_\bullet, \mathrm{G}_\bullet$ we have

$$\mathrm{Hom}_{\mathscr{C}}(\mathrm{F}_\bullet, \mathrm{G}_\bullet) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}^i_{\mathscr{C}}(\mathrm{F}_\bullet, \mathrm{G}_\bullet),$$

where G_t is our fixed representative for $\mathrm{G}_\bullet$ and $\mathrm{Hom}^i_{\mathscr{C}}(\mathrm{F}_\bullet, \mathrm{G}_\bullet) = \mathrm{Hom}_{\mathscr{A}}(\mathrm{F}_s, \mathrm{G}_{t+i})$, vertical composition being additive on degrees). We will say that this grading is *positive* provided that the following condition is satisfied: for any indecomposable 1-morphisms $\mathrm{F}_\bullet, \mathrm{G}_\bullet \in \mathscr{C}$, the inequality $\mathrm{Hom}^i_{\mathscr{C}}(\mathrm{F}_\bullet, \mathrm{G}_\bullet) \neq 0$ implies $i > 0$ unless $\mathrm{F}_\bullet = \mathrm{G}_\bullet$. In the latter case we require $\mathrm{End}^0_{\mathscr{C}}(\mathrm{F}_\bullet) = \Bbbk\,\mathrm{id}_{\mathrm{F}_\bullet}$.

Example 18. Let $D = \Bbbk[x]/(x^2)$ with x in degree 2 and consider $\mathscr{C}_{\mathcal{C}}$ as in Example 17 for some $\mathcal{C}$ equivalent to D-gmod. Choosing the representatives $\mathrm{Id}_{D\text{-gmod}}$ and $(D \otimes_\Bbbk D \otimes_D {}_-)\langle 1 \rangle$ makes $\mathscr{C}_{\mathcal{C}}/\mathbb{Z}$ into a positively graded 2-category.

8.3. **From 2-representations of $\mathscr{A}$ to 2-representations of $\mathscr{C}$.** Let $\mathscr{A}$ be a pro-fiat 2-category and $\mathscr{C} := \mathscr{A}/\mathbb{Z}$. Let $\mathbf{M}$ be a 2-representation of $\mathscr{A}$ and $\mathtt{i} \in \mathscr{A}$. Then the group $\mathbb{Z}$ acts (strictly) on $\mathbf{M}(\mathtt{i})$ via isomorphisms $\mathbb{1}_{\mathtt{i},k}$, $k \in \mathbb{Z}$. We call $\mathbf{M}$ *pro-graded* if this action is free (i.e. the stabilizer of every object is trivial) for every $\mathtt{i}$.

Let $\mathbf{M}$ be a pro-graded 2-representation of $\mathscr{A}$. We define a 2-representation $\underline{\mathbf{M}}$ of $\mathscr{C}$ as follows: For $\mathtt{i} \in \mathscr{C}$, we set $\underline{\mathbf{M}}(\mathtt{i}) := \mathbf{M}(\mathtt{i})/\mathbb{Z}$, that is objects of $\underline{\mathbf{M}}(\mathtt{i})$ are orbits of $\mathbb{Z}$ acting on objects of $\mathbf{M}(\mathtt{i})$ (for $Q \in \mathbf{M}(\mathtt{i})$, we will denote the corresponding orbit by (Q)). For $\mathrm{F} \in \mathscr{A}(\mathtt{i},\mathtt{j})$ and $Q \in \mathbf{M}(\mathtt{i})$, we define $\underline{\mathbf{M}}(\mathrm{F}_\bullet)\,(Q) := (\mathbf{M}(\mathrm{F})\,Q)$ while, for $f : Q \to P$, mapping the class $\hat{f} : (Q) \to (P)$ to the class

$$\widehat{\mathbf{M}(\mathrm{F})f} : (\mathbf{M}(\mathrm{F})\,Q) \to (\mathbf{M}(\mathrm{F})\,P)$$

defines the action of $\underline{\mathbf{M}}(\mathrm{F}_\bullet)$ on morphisms (this is well-defined because of the strictness of our $\mathbb{Z}$-action). Functoriality of $\underline{\mathbf{M}}(\mathrm{F}_\bullet)$ follows directly from the definition. Each $\alpha : \mathrm{F} \to \mathrm{G}$ induces a morphism from $\mathrm{F}_\bullet$ to $\mathrm{G}_\bullet$ and we define

$$\underline{\mathbf{M}}(\alpha)_{(Q)} : \underline{\mathbf{M}}(\mathrm{F}_\bullet)\,(Q) \to \underline{\mathbf{M}}(\mathrm{G}_\bullet)\,(Q)$$

as the class of $\mathbf{M}(\alpha)_Q : \mathbf{M}(\mathrm{F})\,Q \to \mathbf{M}(\mathrm{G})\,Q$. This extends to all 2-morphisms by additivity. It follows directly from the definitions that $\underline{\mathbf{M}}$ becomes a 2-representation of $\mathscr{C}$.

8.4. **Functoriality of $\underline{\cdot}$.** Unfortunately, $\underline{\cdot}$ is not a 2-functor between the 2-categories of 2-representations of $\mathscr{A}$ and $\mathscr{C} = \mathscr{A}/\mathbb{Z}$. However, it turns out to be a 2-functor on a suitably defined subcategory of 2-representations of $\mathscr{A}$. Define the 2-category $\mathscr{A}$-pgamod as follows: objects are pro-graded additive 2-representations of $\mathscr{A}$; 1-morphisms are 2-natural transformations satisfying the condition that $\eta_{\mathbb{1}_{\mathtt{i},n}}$ is the identity map for all $\mathtt{i}$ and n (that is, our 2-natural transformations commute *strictly* with all shifts of the identity); 2-morphisms are modifications. This clearly forms a 2-subcategory in the category of additive 2-representations of $\mathscr{A}$.

Proposition 19. *The operation $\underline{\cdot}$ defines a 2-functor from $\mathscr{A}$-pgamod to $\mathscr{C}$-amod.*

Proof. Let $\mathbf{M},\mathbf{N}\in\mathscr{A}$-pgamod and $\Psi\in\mathrm{Hom}_{\mathscr{A}\text{-pgamod}}(\mathbf{M},\mathbf{N})$. Define $\underline{\Psi}:\underline{\mathbf{M}}\to\underline{\mathbf{N}}$ by $\underline{\Psi}_{\mathtt{i}}\,(Q):=(\Psi_{\mathtt{i}}\,Q)$. This is well defined as $\Psi_{\mathtt{i}}$ commutes strictly with the action of $\mathbb{1}_{\mathtt{i},n}$ and each element in (Q) is obtained by applying some $\mathbb{1}_{\mathtt{i},n}$ to Q. We have to check commutativity of the diagram

$$\begin{tikzcd}
\underline{\Psi}_{\mathtt{j}}\circ\underline{\mathbf{M}}(\mathrm{F}_\bullet) \arrow[r, "\eta_{\mathrm{F}_\bullet}"] \arrow[d, "\mathrm{id}_{\underline{\Psi}_{\mathtt{j}}}\circ_0\underline{\mathbf{M}}(\alpha)"'] & \underline{\mathbf{N}}(\mathrm{F}_\bullet)\circ\underline{\Psi}_{\mathtt{i}} \arrow[d, "\underline{\mathbf{N}}(\alpha)\circ_0\mathrm{id}_{\underline{\Psi}_{\mathtt{i}}}"] \\
\underline{\Psi}_{\mathtt{j}}\circ\underline{\mathbf{M}}(\mathrm{G}_\bullet) \arrow[r, "\eta_{\mathrm{G}_\bullet}"] & \underline{\mathbf{N}}(\mathrm{G}_\bullet)\circ\underline{\Psi}_{\mathtt{i}}
\end{tikzcd}$$

for any $\alpha:\mathrm{F}\to\mathrm{G}$ in $\mathscr{A}$ (here $\eta_{\mathrm{F}_\bullet}$ is the class of η_{F} and similarly for $\eta_{\mathrm{G}_\bullet}$). To check commutativity of this diagram, we have to evaluate it at any object and it is straightforward to check commutativity there using strict commutativity of Ψ with shifts of the identity. Condition (1) for $\eta_{\mathrm{F}_\bullet}$ is automatic. This verifies the first level of 2-functoriality.

For a modification $\theta:\Psi\to\Phi$ in $\mathscr{A}$-pgamod, we define $\underline{\theta}$ by $\underline{\theta}_{\mathtt{i},(Q)}:=\widehat{\theta_{\mathtt{i},Q}}$. We have to check (2), that is commutativity of the diagram

$$\begin{tikzcd}
\underline{\Psi}_{\mathtt{j}}\circ\underline{\mathbf{M}}(\mathrm{F}_\bullet) \arrow[r, "\eta^{\Psi}_{\mathrm{F}_\bullet}"] \arrow[d, "\underline{\theta}_{\mathtt{j}}\circ_0\underline{\mathbf{M}}(\alpha)"'] & \underline{\mathbf{N}}(\mathrm{F}_\bullet)\circ\underline{\Psi}_{\mathtt{i}} \arrow[d, "\underline{\mathbf{N}}(\alpha)\circ_0\underline{\theta}_{\mathtt{i}}"] \\
\underline{\Phi}_{\mathtt{j}}\circ\underline{\mathbf{M}}(\mathrm{G}_\bullet) \arrow[r, "\eta^{\Phi}_{\mathrm{G}_\bullet}"] & \underline{\mathbf{N}}(\mathrm{G}_\bullet)\circ\underline{\Phi}_{\mathtt{i}}
\end{tikzcd}$$

which again follows by evaluating it at any object and using strict commutativity of Ψ and Φ with shifts of the identity. □

8.5. **Principal and cell 2-representations of $\mathscr{A}$.** For $\mathtt{i}\in\mathscr{A}$, consider the principal 2-representation $\mathbb{P}^{\mathscr{A}}_{\mathtt{i}}$ of $\mathscr{A}$.

Proposition 20. *The 2-representations $\underline{\mathbb{P}^{\mathscr{A}}_{\mathtt{i}}}$ and $\mathbb{P}_{\mathtt{i}}$ of $\mathscr{C}$ are equivalent.*

Proof. First we note that $\mathbb{P}^{\mathscr{A}}_{\mathtt{i}}$ is pro-graded by definition. For $\mathtt{j}\in\mathscr{C}$, the orbits of $\mathbb{Z}$ on $\underline{\mathbb{P}^{\mathscr{A}}_{\mathtt{i}}}(\mathtt{j})$ coincide with the fibers of Ω on $\mathscr{C}(\mathtt{i},\mathtt{j})$. The equivalence is then defined by mapping the fiber to its image under Ω. □

Directly from the definitions, we have that $\underline{(\overline{\mathbf{M}})}=\overline{(\underline{\mathbf{M}})}$ for any 2-representation $\mathbf{M}$ of $\mathscr{A}$. Consider the 2-representation $\overline{\mathbb{P}^{\mathscr{A}}_{\mathtt{i}}}$. By definition, each $\overline{\mathbb{P}^{\mathscr{A}}_{\mathtt{i}}}(\mathtt{j})$ is a length category with enough projective objects. For any $\mathtt{j}$, there is a bijection between isomorphism classes of simple objects in $\overline{\mathbb{P}}_{\mathtt{i}}(\mathtt{j})$ and $\mathbb{Z}$-orbits on isomorphism classes of simple objects in $\overline{\mathbb{P}^{\mathscr{A}}_{\mathtt{i}}}(\mathtt{j})$.

The 2-functor Ω induces a bijection between left, right and two-sided cells of $\mathscr{A}$ and $\mathscr{C}$. Let $\mathcal{L}$ be a left cell in $\mathscr{C}$ and G a 1-morphism in $\mathscr{A}$ such that $\mathrm{G}_\bullet$ is the Duflo involution in $\mathcal{L}$. Setting $Q:=\mathrm{G}\,L_{\mathrm{G}}$ as in Subsection 2.8, we consider the 2-representation $\mathbf{C}^{\mathscr{A}}_{\mathcal{L}}:=(\overline{\mathbb{P}^{\mathscr{A}}_{\mathtt{i}}}(\mathtt{j}))_Q$. We leave it to the reader to check that this is the cell 2-representation of $\mathscr{A}$ associated with $\Omega^{-1}(\mathcal{L})$.

Proposition 21. *The 2-representations $\underline{\mathbf{C}^{\mathscr{A}}_{\mathcal{L}}}$ and $\mathbf{C}_{\mathcal{L}}$ of $\mathscr{C}$ are equivalent.*

Proof. The fact that $\mathbf{C}^{\mathscr{A}}_{\mathcal{L}}$ is pro-graded follows from the definition of $\mathbf{C}^{\mathscr{A}}_{\mathcal{L}}$ and the fact that $\mathbb{P}^{\mathscr{A}}_{\mathtt{i}}$ is pro-graded. Similarly to Proposition 20, the equivalence is induced by Ω. □

8.6. **Graded adjunctions.** Let $\mathscr{A}$ be a pro-fiat 2-category and $\mathscr{C} := \mathscr{A}/\mathbb{Z}$. Let $\mathcal{L}$ be a strongly regular left cell of $\mathscr{C}$ and $\mathtt{i} := \mathtt{i}_{\mathcal{L}}$. We assume that we have chosen some representatives in $\mathbb{Z}$-orbits such that the induced grading on $\mathscr{C}$ is positive. We also assume that $\mathbb{1}_{\mathtt{i},\bullet}$ is represented by the identity 1-morphism $\mathbb{1}_{\mathtt{i},0}$ in $\mathscr{A}(\mathtt{i},\mathtt{i})$. Let $\mathrm{G}_\bullet$ be the Duflo involution for $\mathcal{L}$ and let G be its chosen representative in $\mathscr{A}(\mathtt{i},\mathtt{i})$.

We have $\mathrm{Hom}_{\mathscr{C}}(\mathrm{G}_\bullet, \mathbb{1}_{\mathtt{i},\bullet}) \neq 0$ by [MM1, Proposition 17] and hence it makes sense to define $\mathbf{a}$ as the smallest integer such that

$$\mathrm{Hom}^{\mathbf{a}}_{\mathscr{C}}(\mathrm{G}_\bullet, \mathbb{1}_{\mathtt{i},\bullet}) = \mathrm{Hom}_{\mathscr{A}}(\mathrm{G}_{-\mathbf{a}}, \mathbb{1}_{\mathtt{i},0}) \neq 0.$$

Consider the cell 2-representation $\mathbf{C}_{\mathcal{L}}$ of $\mathscr{C}$. By Proposition 21, we have a positive grading on $\mathbf{C}_{\mathcal{L}}(\mathtt{i})$. Denote by $\mathbf{l}$ the maximal $i \in \mathbb{Z}$ such that $\mathrm{End}^i(P_{\mathrm{G}_\bullet}) \neq 0$.

Lemma 22. *We have* $\mathrm{G}^* \cong \mathrm{G}_{\mathbf{l}-2\mathbf{a}}$.

Proof. As $\mathrm{G}^*_\bullet \cong \mathrm{G}_\bullet$, we have $\mathrm{G}^* \cong \mathrm{G}_x$ for some $x \in \mathbb{Z}$. As in [MM1, Subsection 4.7], we denote by Δ the unique quotient of $0 \to \mathbb{1}_{\mathtt{i},0}$ which has simple socle $L_{\mathrm{G}_{-\mathbf{a}}}$. We compute:

$$\begin{array}{rcl} 0 & \neq & \mathrm{Hom}(\mathrm{G}\,\mathbb{1}_{\mathtt{i},0}, L_{\mathrm{G}}) \\ & \subset & \mathrm{Hom}(\mathrm{G}\,\Delta, L_{\mathrm{G}}) \\ & = & \mathrm{Hom}(\mathrm{G}\,L_{\mathrm{G}_{-\mathbf{a}}}, L_{\mathrm{G}}) \\ & = & \mathrm{Hom}(L_{\mathrm{G}_{-\mathbf{a}}}, \mathrm{G}_x\,L_{\mathrm{G}}) \\ & = & \mathrm{Hom}(L_{\mathrm{G}_{-\mathbf{a}}}, \mathrm{G}_{x+\mathbf{a}}\,L_{\mathrm{G}_{-\mathbf{a}}}). \end{array}$$

Here the third line follows from the fact that G annihilates all subquotients of Δ apart from $L_{\mathrm{G}_{-\mathbf{a}}}$ (see [MM1, Proposition 17]), and the fourth line uses adjunction. The module $\mathrm{G}_{x+\mathbf{a}}\,L_{\mathrm{G}_{-\mathbf{a}}}$ has simple socle $L_{\mathrm{G}_{x+\mathbf{a}-\mathbf{l}}}$. Therefore, the inequality $\mathrm{Hom}(L_{\mathrm{G}_{-\mathbf{a}}}, \mathrm{G}_{x+\mathbf{a}}\,L_{\mathrm{G}_{-\mathbf{a}}}) \neq 0$ means that $-\mathbf{a} = x + \mathbf{a} - \mathbf{l}$, that is $x = \mathbf{l} - 2\mathbf{a}$. □

8.7. **The numerical condition.** Now we are ready to formulate our main result in this section.

Theorem 23. *Let $\mathscr{A}$ be a pro-fiat 2-category and assume that $\mathscr{C} := \mathscr{A}/\mathbb{Z}$ is positively graded. Assume that $\mathcal{J}$ is a strongly regular two-sided cell in $\mathscr{C}$. Then $\mathcal{J}$ satisfies the condition in Theorem 16*(ii)*.*

Proof. Let $\mathcal{L}$ and $\mathcal{L}'$ be two different left cells in $\mathcal{J}$, set $\mathtt{i} := \mathtt{i}_{\mathcal{L}}$ and let $\mathrm{G} \in \mathcal{L}$ be the Duflo involution. Let F be the unique element in the intersection of $\mathcal{L}'$ and the right cell of G. Consider the cell 2-representations $\overline{\mathbf{C}}_{\mathcal{L}}$ and $\overline{\mathbf{C}}_{\mathcal{L}'}$ of $\mathscr{C}$. Let A and B be positively graded associative algebras describing $\overline{\mathbf{C}}_{\mathcal{L}}(\mathtt{i})$ and $\overline{\mathbf{C}}_{\mathcal{L}'}(\mathtt{i})$, respectively. Let e be a primitive idempotent of A corresponding to L_{G} and e' a primitive idempotent of B corresponding to L_{F}. By [MM1, Corollary 38(a)] and Lemma 7, the functor $\overline{\mathbf{C}}_{\mathcal{L}}(\mathrm{G})$ can be described as $Ae \otimes_{\Bbbk} eA$. By [MM1, Corollary 38(b)] and Lemma 7, the functor $\overline{\mathbf{C}}_{\mathcal{L}'}(\mathrm{G})$ can be described as $(Be' \otimes_{\Bbbk} e'B)^{\oplus k}$ for some $k \in \mathbb{N}$. To prove the claim we need to show that $k = 1$.

Lemma 24. *There is a nonzero 2-natural transformation from $\overline{\mathbf{C}}_{\mathcal{L}}$ to $\overline{\mathbf{C}}_{\mathcal{L}'}$ sending L_{G} to L_{F}.*

Proof. To prove this we have to show that $\mathrm{Ann}_{\mathscr{C}}(L_{\mathrm{G}}) \subset \mathrm{Ann}_{\mathscr{C}}(L_{\mathrm{F}})$. Choose a 1-morphism K and a 2-morphism $\alpha : \mathrm{K} \to \mathrm{G}$ with the following properties:

- each indecomposable direct summand H of K satisfies $\mathrm{H} \not\leq_J \mathcal{J}$;

- if an indecomposable direct summand H of K is isomorphic to G, then the restriction of α to H is not an isomorphism;
- for any indecomposable 1-morphism H satisfying $\mathrm{H} \not\leq_J \mathcal{J}$ and any 2-morphism $\beta : \mathrm{H} \to \mathrm{G}$ (which is not an isomorphism if $\mathrm{H} \cong \mathrm{G}$), β factors through α.

We claim that the cokernel of the evaluation of $\mathrm{K} \overset{\alpha}{\to} \mathrm{G}$ at L_{F} is nonzero. Let H be an indecomposable direct summand of K and $\mathrm{H} \overset{\alpha_{\mathrm{H}}}{\to} \mathrm{G}$ the restriction of α to H. If $\mathrm{H} \not\sim_L \mathrm{G}$, then $\mathrm{H}\, L_{\mathrm{F}} = 0$ by [MM1, Lemma 12]. If $\mathrm{H} \sim_L \mathrm{G}$ and $\mathrm{H} \ncong \mathrm{G}$, then the top of $\mathrm{H}\, L_{\mathrm{F}}$ is different from the top of $\mathrm{G}\, L_{\mathrm{F}}$ by strong regularity of $\mathcal{J}$. Finally, if $\mathrm{H} \cong \mathrm{G}$, then α_{H} is not an isomorphism by our second assumption, which implies that its evaluation at L_{F} is not an isomorphism either. The claim follows.

Let Q be the functor isomorphic to the cokernel of $\overline{\mathbf{C}}_{\mathcal{L}'}(\alpha)$. It is nonzero by the previous paragraph. Let $\mathrm{K}' \overset{\alpha'}{\to} \mathrm{G}$ be a projective presentation of L_{G} and Q' be the functor isomorphic to the cokernel of $\overline{\mathbf{C}}_{\mathcal{L}'}(\alpha')$. We claim that $\mathrm{Q} \cong \mathrm{Q}'$. From the definitions, we have a natural surjection $\mathrm{Q} \twoheadrightarrow \mathrm{Q}'$ with kernel K. By construction, for an indecomposable 1-morphism H, existence of a nonzero homomorphism from $\overline{\mathbf{C}}_{\mathcal{L}'}(\mathrm{H})$ to K implies $\mathrm{H} <_J \mathcal{J}$. Therefore $\mathrm{K} = 0$ by [MM1, Lemma 42] and hence $\mathrm{Q} \cong \mathrm{Q}' \neq 0$. Now [MM1, Theorem 24] implies existence of a nonzero 2-natural transformation Φ from $\overline{\mathbf{C}}_{\mathcal{L}}$ to $\overline{\mathbf{C}}_{\mathcal{L}'}$. Setting $X := \Phi(L_{\mathrm{G}}) \neq 0$, we have the inclusion $\mathrm{Ann}_{\mathscr{C}}(L_{\mathrm{G}}) \subset \mathrm{Ann}_{\mathscr{C}}(X)$.

Let $\mathrm{F}' \in \mathcal{L}'$ be different from F. Then there is $\mathrm{G}' \in \mathcal{J} \setminus \mathcal{L}$ and $\mathrm{G}'\, L_{\mathrm{F}'} \neq 0$. This implies that $\mathrm{Ann}_{\mathscr{C}}(L_{\mathrm{G}}) \not\subset \mathrm{Ann}_{\mathscr{C}}(L_{\mathrm{F}'})$ and hence every composition subquotient of X is isomorphic to L_{F}. Taking a composition subquotient can only increase the annihilator and hence $\mathrm{Ann}_{\mathscr{C}}(L_{\mathrm{G}}) \subset \mathrm{Ann}_{\mathscr{C}}(X) \subset \mathrm{Ann}_{\mathscr{C}}(L_{\mathrm{F}})$. The claim of the lemma follows. □

From Lemma 24, we get a nonzero 2-natural transformation from $\overline{\mathbf{C}}_{\mathcal{L}}$ to $\overline{\mathbf{C}}_{\mathcal{L}'}$ and, from the discussion before the lemma, we have that it thus maps the functor $Ae \otimes_{\Bbbk} eA$ to the functor $(Be' \otimes_{\Bbbk} e'B)^{\oplus k}$. Now we lift the whole picture to $\mathscr{A}$. We identify 1-morphisms in $\mathscr{C}$ with their chosen representatives in $\mathscr{A}$ (consistent with Subsection 8.6).

Abusing notation, let $\mathcal{L}$ and $\mathcal{L}'$ be the left cells in $\mathscr{A}$ corresponding to $\mathcal{L}$ and $\mathcal{L}'$ under Ω. Then, considering $\overline{\mathbf{C}}^{\mathscr{A}}_{\mathcal{L}}$ and $\overline{\mathbf{C}}^{\mathscr{A}}_{\mathcal{L}'}$, the above gives us a nonzero 2-natural transformation Φ from $\overline{\mathbf{C}}^{\mathscr{A}}_{\mathcal{L}}$ to $\overline{\mathbf{C}}^{\mathscr{A}}_{\mathcal{L}'}$ mapping L_{G} to L_{F}. We can also identify $\overline{\mathbf{C}}^{\mathscr{A}}_{\mathcal{L}}(\mathtt{i})$ and $\overline{\mathbf{C}}^{\mathscr{A}}_{\mathcal{L}'}(\mathtt{i})$ with A-gmod and B-gmod, respectively (see Proposition 21). The functor $\overline{\mathbf{C}}^{\mathscr{A}}_{\mathcal{L}}(\mathrm{G})$ is given by tensoring with $Ae \otimes_{\Bbbk} eA\langle \mathbf{a} \rangle$.

Let K denote an indecomposable functor on $\overline{\mathbf{C}}^{\mathscr{A}}_{\mathcal{L}'}(\mathtt{i})$ such that

$$\Phi(\mathrm{G}) \cong \bigoplus_{i \in \mathbb{Z}} \mathrm{K}_i^{\oplus k_i},$$

where $k_i \neq 0$ implies $i \leq 0$ and $k_0 \neq 0$ (note that we have $k = \sum_i k_i$). We set $\psi(t) := \sum_{i \in \mathbb{Z}} k_i t^{-i}$

Now we would like to evaluate adjunction morphisms at simple modules. Fix some adjunction morphisms

$$\eta : \mathrm{G}^* \circ \mathrm{G} \to \mathbb{1}_{\mathtt{i}} \quad \text{and} \quad \zeta : \mathbb{1}_{\mathtt{i}} \to \mathrm{G} \circ \mathrm{G}^*.$$

Let n_i denote the graded composition multiplicity $(P_{\mathrm{G}} : L_{\mathrm{G}}\langle -i\rangle)$ and define $\chi_{\mathrm{G}}(t) := \sum_{i\in\mathbb{Z}} n_i t^i$. We have

$$\mathrm{G}\circ\mathrm{G} \cong \bigoplus_{i\in\mathbb{Z}} \mathrm{G}_i^{\oplus n_i + \mathbf{a}}$$

and therefore

$$\mathrm{G}^*\circ\mathrm{G} \cong \mathrm{G}\circ\mathrm{G}^* \cong \bigoplus_{i\in\mathbb{Z}} \mathrm{G}_i^{\oplus n_i + \mathbf{l}-\mathbf{a}}.$$

As the evaluation of η at L_{G} is nonzero (it is the image of the identity morphism on P_{G}), we obtain that η induces a nonzero map from the unique direct summand $\mathrm{G}_{-\mathbf{a}}$ of $\mathrm{G}^*\circ\mathrm{G}$ to $\mathbb{1}_{\mathtt{i}}$. All other direct summands of $\mathrm{G}^*\circ\mathrm{G}$ are annihilated by this evaluation. Similarly, ζ induces a nonzero map from $\mathbb{1}_{\mathtt{i}}$ to the unique direct summand $\mathrm{G}_{\mathbf{l}-\mathbf{a}}$ of $\mathrm{G}\circ\mathrm{G}^*$. Composition of these two maps gives a nonzero map from $\mathrm{G}_{-\mathbf{a}}$ to $\mathrm{G}_{\mathbf{l}-\mathbf{a}}$, that is a map of degree $\mathbf{l}$.

Let $\mathbf{l}'$ be the maximal i such that $(e'Be')_i \neq 0$. Exactly the same arguments applied to K say that any adjunctions

$$\eta' : \mathrm{K}^*\circ\mathrm{K} \to \mathbb{1}_{\mathtt{i}} \quad \text{and} \quad \zeta' : \mathbb{1}_{\mathtt{i}} \to \mathrm{K}\circ\mathrm{K}^*$$

produce, when evaluated at L_{F}, a non-zero map from K to $\mathrm{K}\langle\mathbf{l}'\rangle$, that is a map of degree $\mathbf{l}'$. By additivity of adjunctions, for a direct sum of shifts of K we get a direct sum of the corresponding shifts of such maps (all of degree $\mathbf{l}'$). As Φ is a 2-natural transformation, it sends the adjunction morphisms η and ζ to some adjunction morphism for a direct sum of shifts of K (with k summands). Since grading is preserved, we obtain $\mathbf{l} = \mathbf{l}'$.

Denote by p_i the graded composition multiplicity $(e'Be' : L_{\mathrm{F}}\langle -i\rangle)$ and define $\chi_{\mathrm{F}}(t) := \sum_{i\in\mathbb{Z}} p_i t^i$.

Lemma 25. *We have* $\chi_{\mathrm{G}}(t) = \chi_{\mathrm{F}}(t)\psi(t)$.

Proof. This follows by comparing the linear transformations induced by the exact functors $\overline{\mathbf{C}}_{\mathcal{L}}^{\mathscr{A}}(\mathrm{G})$ and $\overline{\mathbf{C}}_{\mathcal{L}'}^{\mathscr{A}}(\mathrm{G})$ on the Grothendieck groups of A-gmod and B-gmod, respectively. □

Note that $\mathbf{l}$ is the degree of $\chi_{\mathrm{G}}(t)$ while $\mathbf{l}'$ is the degree of $\chi_{\mathrm{F}}(t)$. Therefore, $\mathbf{l} = \mathbf{l}'$ implies that the degree of $\psi(t)$ is zero. As $n_0 = 1$, we have $k_0 = 1$ and thus $k = 1$, which completes the proof. □

Remark 26. The main result (Theorem 43) in [MM1] and Theorems 9, (13) and (14) as well as Corollary 10 in the present paper are proved for strongly regular cells under the assumption that condition Theorem 16(ii) is satisfied. Theorem 23 shows that for positively graded fiat 2-categories this assumption is superfluous.

Conjecture 27. Theorem 23 is true without the grading assumption.

References

[AS] H. Andersen, C. Stroppel; Twisting functors on $\mathcal{O}$. Represent. Theory **7** (2003), 681–699.

[BG] J. Bernstein, S. Gelfand; Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras. Compositio Math. **41** (1980), no. 2, 245–285.

[CR] J. Chuang, R. Rouquier; Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification. Ann. of Math. (2) **167** (2008), no. 1, 245–298.

[FKS] I. Frenkel, M. Khovanov, C. Stroppel; A categorification of finite-dimensional irreducible representations of quantum $\mathfrak{sl}_2$ and their tensor products. Selecta Math. (N.S.) **12** (2006), no. 3-4, 379–431.

[Hu] J. Humphreys; Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$. Graduate Studies in Mathematics, **94**. American Mathematical Society, Providence, RI, 2008.

[Kh] O. Khomenko; Categories with projective functors. Proc. London Math. Soc. (3) **90** (2005), no. 3, 711–737.

[KhLa] M. Khovanov, A. Lauda; A categorification of A quantum $\mathfrak{sl}_n$. Quantum Topol. **1** (2010), 1–92.

[KM] G. Kudryavtseva, V. Mazorchuk; On multisemigroups. Preprint arXiv:1203.6224.

[La] A. Lauda; A categorification of quantum $\mathfrak{s}l(2)$. Adv. Math. **225** (2010), no. 6, 3327–3424.

[Le] T. Leinster; Basic bicategories. Preprint arXiv:math/9810017.

[MM1] V. Mazorchuk, V. Miemietz; Cell 2-representations of finitary 2-categories; Compositio Math. **147** (2011), 1519–1545.

[MM2] V. Mazorchuk, V. Miemietz; Additive versus abelian 2-representations of fiat 2-categories. Preprint arxiv:1112.4949.

[MS1] V. Mazorchuk, C. Stroppel; Translation and shuffling of projectively presentable modules and a categorification of a parabolic Hecke module. Trans. Amer. Math. Soc. **357** (2005), no. 7, 2939–2973.

[MS2] V. Mazorchuk, C. Stroppel; Categorification of (induced) cell modules and the rough structure of generalised Verma modules. Adv. Math. **219** (2008), no. 4, 1363–1426.

[Ro2] R. Rouquier; 2-Kac-Moody algebras. Preprint arXiv:0812.5023.

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