WEIGHT MODULES OVER INFINITE DIMENSIONAL WEYL ALGEBRAS

VYACHESLAV FUTORNY, DIMITAR GRANTCHAROV
AND VOLODYMYR MAZORCHUK

Abstract. We classify simple weight modules over infinite dimensional Weyl algebras and realize them using the action on certain localizations of the polynomial ring. We describe indecomposable projective and injective weight modules and deduce from this a description of blocks of the category of weight modules by quivers and relations. As a corollary we establish Koszulity for all blocks.

2000 MSC: 17B10, 17B65, 16D60

1. Introduction

Weyl algebras are classical objects of study in representation theory that arise naturally both in mathematics and physics and have many important applications. For example, an essential part of the $\mathcal{D}$-module approach to the representation theory of a finite dimensional simple Lie algebra $\mathfrak{g}$ is existence of a natural homomorphism from $U(\mathfrak{g})$ to a finite dimensional (or finite rank) Weyl algebra A_n. In the case of an affine Lie algebra there is a similar homomorphism, but now to an infinite dimensional Weyl algebra A_∞. A realization of the standard Verma $\widehat{\mathfrak{sl}}_2$-modules as A_∞-modules was given by Wakimoto in [Wa] and was later generalized to other types of Verma modules and higher rank affine Lie algebras, see [Co, CF, FF1, FF2, JK, Vo].

The algebra A_∞ can be viewed as an infinite rank generalized Weyl algebra (in the sense of [Ba, BB]) and thus has a natural category of representations consisting of the so-called *weight* modules. Such modules over finite and infinite dimensional Weyl algebras have been extensively studied in the last 20 years. Various constructions and classification results appear in [BB, BBF, GS]. In particular, a partial classification of simple weight A_∞-modules was given in [BBF]. On the other hand, new examples of simple weight A_∞-modules recently appeared in [MZ].

In the present paper we complete classification of simple weight modules over infinite dimensional Weyl algebras. The Weyl algebra is usually defined as the algebra of certain differential operators of a polynomial algebra. This action localizes to Laurent polynomials and then can be twisted to all polynomials with not necessarily integer exponents. We use the latter action both for our main classification result, Theorem 5, and also to give an explicit realization of all simple weight modules in Subsection 3.4. This explicit realization is helpful for numerous reasons. Firstly, it can be used to construct new free field representations of affine Lie algebras and direct limit Lie algebras. Also, one would now expect realizations of these representations in terms of sections of vector bundles on flag varieties. Furthermore, this realization allows us to describe indecomposable projective and injective weight modules, see Subsections 3.2,3.3 and 3.7 which, in turn, leads to an explicit description, via quivers and relations, for all blocks of the category of weight modules. All details of the latter are collected in Section 3 with the main result being Theorem 12. As a consequence of this description we show that all blocks correspond to Koszul algebras, see Proposition 11.

Acknowledgements. D.G. gratefully acknowledges the hospitality and excellent working conditions at the São Paulo University where most of this work was completed. V.F. was supported by the CNPq grant (301743/2007-0) and by the Fapesp grant (2010/50347-9). D.G was supported by the Fapesp grant (2011/21621-8) and by the NSA grant H98230-10-1-0207. V.M. was supported by the Royal Swedish Academy of Sciences and by the Swedish Research Council.

2. Generalities

2.1. **The infinite dimensional Weyl algebra.** Let $\Bbbk$ be an algebraically closed field of characteristic zero and $\mathtt{I}$ be an infinite set satisfying $|I| < |\Bbbk|$ (this assumption will be essential, in particular, for Lemma 1). Consider the commutative $\Bbbk$-algebra B of polynomials in infinitely many variables x_i, $i \in \mathtt{I}$. For $i \in \mathtt{I}$ let $X_i := x_i\cdot$ be the linear operator on B given by multiplication with x_i. Let further $Y_i := \partial_i$ denote the linear operator on B given by the partial derivative with respect to x_i, that is ∂_i is a derivation of B defined by $\partial_i \cdot x_j = \delta_{i,j}$, the Kronecker delta, on the generators.

Consider the additive abelian group $\Bbbk^{\mathtt{I}}$ consisting of all vectors $\mathbf{v} = (v_i)_{i\in\mathtt{I}}$ with coefficients from $\Bbbk$. Let $\mathbb{Z}_f^{\mathtt{I}}$ denote the subgroup of all integral vectors with at most finitely many nonzero coefficients. For $\mathbf{v} \in \mathbb{Z}_f^{\mathtt{I}}$ we denote by $\mathbf{x}^{\mathbf{v}}$ the monomial $\prod_{i\in\mathtt{I}} x_i^{v_i} \in B$. The monomials $\{\mathbf{x}^{\mathbf{v}} : \mathbf{v} \in \mathbb{Z}_f^{\mathtt{I}}\}$ form the *standard basis* in B (over $\Bbbk$).

The *infinite Weyl algebra* $A = A_{\Bbbk,I}$ is the subalgebra in $\mathrm{End}_{\Bbbk}(B)$ generated by all X_i and Y_i, $i \in \mathtt{I}$. It is easy to check that these generators satisfy the following relations:

$$\begin{array}{llll} X_iX_j = X_jX_i, & \forall i,j; & Y_iY_j = Y_jY_i, & \forall i,j; \\ X_iY_j = Y_jX_i, & i \neq j; & X_iY_i = Y_iX_i - \mathrm{Id}, & \forall i, \end{array}$$

where Id denotes the identity linear transformation. It is easy to check that these relations give a presentation for A.

2.2. **A maximal commutative subalgebra.** For $i \in \mathtt{I}$ set $t_i := X_iY_i \in A$ and denote by A_0 the subalgebra of A generated by all t_i's. From the presentation of A it follows that A_0 is a commutative subalgebra of A and that the generators t_i are algebraically independent.

For $\mathbf{v} \in \mathbb{Z}_f^{\mathtt{I}}$ consider the element

$$\mathbf{X}_{\mathbf{v}} := \prod_{i:v_i>0} X_i^{v_i} \prod_{i:v_i<0} Y_i^{-v_i}.$$

From the presentation of A it follows that A is free both as a left and as a right A_0-module with basis $\{\mathbf{X}_{\mathbf{v}} : \mathbf{v} \in \mathbb{Z}_f^{\mathtt{I}}\}$. Since none of $\mathbf{X}_{\mathbf{v}}$, $\mathbf{v} \neq 0$, commutes with all elements of A_0, it follows that A_0 is a maximal commutative subalgebra in A.

For $\mathbf{p} \in \Bbbk^{\mathtt{I}}$ denote by $\mathfrak{m}_{\mathbf{p}}$ the maximal ideal in A_0 generated by $t_i - p_i$, $i \in \mathtt{I}$.

Lemma 1. *Every maximal ideal in A_0 has the form $\mathfrak{m}_{\mathbf{p}}$ for some $\mathbf{p} \in \Bbbk^{\mathtt{I}}$.*

Proof. Let $\mathfrak{m}$ be a maximal ideal in A_0. Then $A_0/\mathfrak{m}$ is a field extending $\Bbbk$. As $\Bbbk$ is algebraically closed, we either have that the extension $\Bbbk \hookrightarrow A_0/\mathfrak{m}$ is an isomorphism, in which case $\mathfrak{m} = \mathfrak{m}_{\mathbf{p}}$ for some $\mathbf{p} \in \Bbbk^{\mathtt{I}}$, or this extension is purely transcendental. In the latter case take any $z \in (A_0/\mathfrak{m}) \setminus \Bbbk$. Then the elements $(z-c)^{-1}$, $c \in \Bbbk$, are linearly independent over $\Bbbk$ which means that the $\Bbbk$-dimension of $A_0/\mathfrak{m}$ is at least $|\Bbbk|$. On the other hand, the $\Bbbk$-dimension of A_0 is $|I| < |\Bbbk|$, a contradiction. The claim follows. $\square$

3. Weight modules

3.1. **Weight A-modules.** Let M be an A-module and $\mathfrak{m}$ be a maximal ideal in A_0. An element $x \in M$ is called *weight* if $\mathfrak{m} \cdot x = 0$. The module M is called a *weight* module if it has a basis consisting of weight elements. By Lemma 1, the letter is equivalent to $M = \bigoplus_{\mathbf{p} \in \Bbbk^{\mathtt{I}}} M_{\mathbf{p}}$, where

$$M_{\mathbf{p}} := \{x \in M : t_i \cdot x = p_i x, \forall i \in I\}$$

is the *weight space* of *weight* $\mathbf{p}$. For a weight module M the set of all $\mathbf{p} \in \Bbbk^{\mathtt{I}}$ such that $M_{\mathbf{p}} \neq 0$ is called the *support* of M and denoted $\mathrm{supp}(M)$.

Denote by $\mathfrak{W}$ the full subcategory in A-mod, the category of all A-modules, consisting of all weight A-modules. For $\mathbf{p} \in \Bbbk^{\mathtt{I}}$ we denote by $\mathfrak{W}_{\mathbf{p}}$ the full subcategory of $\mathfrak{W}$ consisting of all M satisfying $\mathrm{supp}(M) \subset \mathbf{p} + \mathbb{Z}_f^{\mathtt{I}}$. Clearly, $\mathfrak{W}_{\mathbf{p}} = \mathfrak{W}_{\mathbf{m}}$ if and only if $\mathbf{m} \in \mathbf{p} + \mathbb{Z}_f^{\mathtt{I}}$. Further, we have the usual decomposition

$$\mathfrak{W} \cong \bigoplus_{\xi \in \Bbbk^{\mathtt{I}}/\mathbb{Z}_f^{\mathtt{I}}} \mathfrak{W}_{\mathbf{p}_\xi},$$

where $\mathbf{p}_\xi \in \xi$ is some fixed representative.

3.2. **Projective weight A-modules.** For $\mathbf{p} \in \Bbbk^{\mathtt{I}}$ denote by $\mathfrak{V}_{\mathbf{p}}$ the full subcategory in A_0-mod consisting of all A_0-modules N such that $\mathfrak{m}_{\mathbf{p}} \cdot N = 0$. The category $\mathfrak{V}_{\mathbf{p}}$ is semisimple and has a unique (up to isomorphism) simple object $\Bbbk_{\mathbf{p}} := A_0/\mathfrak{m}_{\mathbf{p}}$ which is also projective.

The restriction functor $\mathrm{Res}_{\mathbf{p}} : \mathfrak{W} \to \mathfrak{V}_{\mathbf{p}}$ is obviously exact and is right adjoint to the induction functor

$$\mathrm{Ind}_{\mathbf{p}} := A \otimes_{A_0} {}_{-} : \mathfrak{V}_{\mathbf{p}} \to \mathfrak{W}.$$

The functor $\mathrm{Ind}_{\mathbf{p}}$ is exact as A is free over A_0, moreover, being left adjoint to an exact functor, the functor $\mathrm{Ind}_{\mathbf{p}}$ maps projective objects to projective objects. Therefore $P(\mathbf{p}) := \mathrm{Ind}_{\mathbf{p}}\Bbbk_{\mathbf{p}} \cong A/A\mathfrak{m}_{\mathbf{p}}$ is projective in $\mathfrak{W}$. Moreover, by adjunction, for any $M \in \mathfrak{W}$ we have a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{W}}(P(\mathbf{p}), M) \cong M_{\mathbf{p}}. \tag{3.1}$$

Denote by σ_i the automorphism of A_0 given by

$$\sigma_i(t_j) = \begin{cases} t_j, & i \neq j; \\ t_i + 1, & i = j. \end{cases}$$

Let H be the group generated by σ_i, $i \in \mathtt{I}$. Identifying maximal ideals of A_0 with elements in $\Bbbk^{\mathtt{I}}$ yields a canonical identification $H \cong \mathbb{Z}_f^{\mathtt{I}}$. If M is a weight A-module, then the defining relations of A imply that for any $\mathbf{p} \in \Bbbk^{\mathtt{I}}$ we have $X_i M_{\mathbf{p}} \subset M_{\sigma_i(\mathbf{p})}$ and $Y_i M_{\mathbf{p}} \subset M_{\sigma_i^{-1}(\mathbf{p})}$.

The group $\mathbb{Z}_f^{\mathtt{I}}$ acts freely on $\Bbbk^{\mathtt{I}}$. Moreover, elements of $\mathbb{Z}_f^{\mathtt{I}}$ naturally index the free basis of A as a right A_0-module. It follows that

$$P(\mathbf{p}) \cong \bigoplus_{\sigma \in \mathbb{Z}_f^{\mathtt{I}}} P(\mathbf{p})_{\sigma(\mathbf{p})},$$

that all $\sigma(\mathbf{p})$ are different and all $P(\mathbf{p})_{\sigma(\mathbf{p})}$ are one-dimensional over $\Bbbk$.

Proposition 2. *For any $\mathbf{p} \in \Bbbk^{\mathtt{I}}$ we have:*

(*a*) *$P(\mathbf{p})$ is indecomposable and has simple top $L(\mathbf{p})$,*
(*b*) *$\dim_{\Bbbk} L(\mathbf{p})_{\mathbf{p}} = 1$ and $L(\mathbf{p})$ is the unique (up to isomorphism) simple module with this property.*

Proof. Using (3.1), we have $\mathrm{End}_{\mathfrak{W}}(P(\mathbf{p})) \cong P(\mathbf{p})_{\mathbf{p}} \cong \Bbbk$. This implies claim (a). As $P(\mathbf{p})$ is generated by $P(\mathbf{p})_{\mathbf{p}}$, any proper submodule of $P(\mathbf{p})$ does not intersect $P(\mathbf{p})_{\mathbf{p}}$. On the other hand, let N denote the sum of all submodules of $P(\mathbf{p})$ which do not intersect $P(\mathbf{p})_{\mathbf{p}}$. Then N is a proper submodule of $P(\mathbf{p})$ and hence is the unique maximal submodule. Then $P(\mathbf{p})/N$ is the unique simple quotient of $P(\mathbf{p})$ and $(P(\mathbf{p})/N)_{\mathbf{p}} \neq 0$ implying claim (b). $\square$

Note that $L(\mathbf{0})$ is isomorphic to the defining representation B of A, where all component of $\mathbf{0}$ are zero.

3.3. **Involution.** Denote by $\mathfrak{W}^f$ the full subcategory of $\mathfrak{W}$ consisting of all modules with finite dimensional weight spaces. From the previous subsection we have that both $P(\mathbf{p})$ and $L(\mathbf{p})$ belong to $\mathfrak{W}^f$ for all $\mathbf{p} \in \Bbbk^{\mathtt{I}}$.

The algebra A has the standard involutive anti-automorphism $\diamond$ defined on the generators via $X_i^\diamond = Y_i$. Note that $\diamond$ fixes A_0 pointwise. The category $\mathfrak{W}^f$ has the standard restricted duality $\star$ defined as follows: for $M \in \mathfrak{W}^f$ we have $M^\star := \bigoplus_{\mathbf{p}\in\Bbbk^{\mathtt{I}}} \mathrm{Hom}_{\Bbbk}(M_{\mathbf{p}}, \Bbbk)$ with the left action of A defined via $\diamond$, and with the obvious action on morphisms. As $\diamond$ fixes A_0 pointwise, we have $\mathrm{supp}(M^\star) = \mathrm{supp}(M)$ which implies that $L(\mathbf{p})^\star \cong L(\mathbf{p})$. It follows that $P(\mathbf{p})^\star$ is the injective envelope of $L(\mathbf{p})$.

3.4. **Realization via polynomial action.** The defining action of A on B admits the following obvious generalization. For $\mathbf{p} \in \Bbbk^{\mathtt{I}}$ set $\mathbf{x}^{\mathbf{p}} := \prod_{i\in\mathtt{I}} x_i^{p_i}$ and let $B(\mathbf{p})$ be the linear span of $\mathbf{x}^{\mathbf{m}}$, $\mathbf{m} \in \mathbf{p} + \mathbb{Z}_f^{\mathtt{I}}$. Define the action of X_i on $B(\mathbf{p})$ by multiplication with x_i and the action of Y_i on $B(\mathbf{p})$ by partial derivative with respect to x_i. It is straightforward to check that this defines on $B(\mathbf{p})$ the structure of a weight A-module, that $\mathrm{supp}(B(\mathbf{p})) = \mathbf{p} + \mathbb{Z}_f^{\mathtt{I}}$ and that all nonzero weight spaces of $B(\mathbf{p})$ are one-dimensional.

Proposition 3. (*a*) *Let $\mathbf{p} \in \Bbbk^{\mathtt{I}}$ be such that $p_i \in \mathbb{Z}$ implies $p_i \in \{0, 1, 2, \dots\}$ for all i. Then $B(\mathbf{p})$ and $B(\mathbf{p})^\star$ are the injective envelope and the projective cover of $L(\mathbf{p})$, respectively.*

(*b*) *Let $\mathbf{p} \in \Bbbk^{\mathtt{I}}$ be such that $p_i \in \mathbb{Z}$ implies $p_i \in \{-1, -2, \dots\}$ for all i. Then $B(\mathbf{p})$ and $B(\mathbf{p})^\star$ are the projective cover and the injective envelope of $L(\mathbf{p})$, respectively.*

Proof. We prove claim (b). Claim (a) is proved similarly. Let $\mathbf{p} \in \Bbbk^{\mathtt{I}}$ be such that $p_i \in \mathbb{Z}$ implies $p_i \in \{-1, -2, \dots\}$ for all i. As $\star$ is a duality, it is enough to show that $B(\mathbf{p}) \cong P(\mathbf{p})$. By construction, $B(\mathbf{p})_{\mathbf{p}} \neq 0$ which gives a nonzero homomorphism $P(\mathbf{p}) \to B(\mathbf{p})$ by (3.1). This map is easily seen to be injective. Comparing the characters of $P(\mathbf{p})$ and $B(\mathbf{p})$, it follows that this map is bijective. □

Consider the set

$$S := \{\mathbf{m} \in \mathbf{p} + \mathbb{Z}_f^{\mathtt{I}} : \forall i\ p_i \in \{0, 1, 2, \dots\} \Rightarrow m_i \in \{0, 1, 2, \dots\}\}$$

and let $N := \bigoplus_{\mathbf{m}\in S} B(\mathbf{p})_{\mathbf{m}}$. Then N is obviously a submodule of $B(\mathbf{p})$. Consider the set

$$S' := \{\mathbf{m} \in S : \forall i\ p_i \in \{-1, -2, \dots\} \Rightarrow m_i \in \{0, 1, 2, \dots\}\}$$

and let $N' := \bigoplus_{\mathbf{m}\in S'} B(\mathbf{p})_{\mathbf{m}}$. Then N' is obviously a submodule of N.

3.5. **Simple weight modules.** For $\mathbf{p} \in \Bbbk^{\mathtt{I}}$ denote by $\overline{\mathbf{p}}$ the set of all $\mathbf{k} \in \mathbf{p} + \mathbb{Z}_f^{\mathtt{I}}$ which satisfy the following conditions for all $i \in \mathtt{I}$:

$$\begin{array}{lll} p_i \in \{-1,-2,-3,\dots\} & \text{implies} & k_i \in \{-1,-2,-3,\dots\}; \\ p_i \in \{0,1,2,3,\dots\} & \text{implies} & k_i \in \{0,1,2,3,\dots\}. \end{array}$$

In the notation from the previous subsection we have $\mathrm{supp}(N/N') = \overline{\mathbf{p}}$.

Proposition 4. *For* $\mathbf{p} \in \Bbbk^{\mathtt{I}}$ *we have:*
(*a*) $\mathrm{supp}(L(\mathbf{p})) = \overline{\mathbf{p}}$;
(*b*) *if* $\mathbf{s} \in \Bbbk^{\mathtt{I}}$, *then* $L(\mathbf{p}) \cong L(\mathbf{s})$ *if and only if* $\mathbf{s} \in \overline{\mathbf{p}}$.

Proof. First we show that the module N/N' constructed in the previous subsection is simple. Let $\mathbf{m} \in \overline{\mathbf{p}}$ and $v \in (N/N')_{\mathbf{m}}$ be a nonzero element. We have to show that v generates N/N'. For this it is enough to check that, given $i \in \mathtt{I}$, we have $X_i v \neq 0$ if $m_i \neq -1$ and we have $Y_i v \neq 0$ if $m_i \neq 0$. However, both these claims follow directly from the definitions.

As $(N/N')_{\mathbf{p}} \neq 0$ by construction, we have $N/N' \cong L(\mathbf{p})$ by Proposition 2(b). Now claim (a) follows by construction and claim (b) follows from claim (a) and Proposition 2. □

The above results can now be summarized as follows: let $\sim$ be the equivalence relation on $\Bbbk^{\mathtt{I}}$ defined as follows: $\mathbf{p} \sim \mathbf{m}$ if and only if $\overline{\mathbf{p}} = \overline{\mathbf{m}}$.

Theorem 5 (Classification of simple weight A-modules)**.** *The map* $\overline{\mathbf{p}} \mapsto L(\mathbf{p})$, $\mathbf{p} \in \Bbbk^{\mathtt{I}}$, *is a bijection between* $\Bbbk^{\mathtt{I}}/\sim$ *and the set of isomorphism classes of simple weight A-modules.*

Example 6. Take $\mathbf{p} = (1, 1, ...)$, i.e. $p_i = 1$ for every i. The simple module $L(\mathbf{p})$ locally finite with respect to each ∂_i, however, there is no $v \in L(\mathbf{p})$ for which $\partial_i v = 0$ for every i. Similar examples were considered in [MZ, Subsection 4.2].

Remark 7. Theorem 5 transfers mutatis mutandis to tensor products of rank one generalized Weyl algebras in the sense of [Ba] associated to $\Bbbk[x]$ and $\sigma : \Bbbk[x] \to \Bbbk[x]$ defined by $\sigma(x) = x + 1$.

3.6. **Twisted polynomial realizations.** For $\mathtt{J} \subset \mathtt{I}$ let θ_J denote the automorphism of A given by

$$\begin{array}{rl} \theta_{\mathtt{J}}(X_j) = Y_j, \theta_{\mathtt{J}}(Y_j) = -X_j, & \text{if} \quad j \in \mathtt{J}; \\ \theta_{\mathtt{J}}(X_i) = X_i, \theta_{\mathtt{J}}(Y_i) = Y_i, & \text{if} \quad i \notin \mathtt{J}. \end{array}$$

For $i \in \mathtt{I}$ we have

$$\theta_{\mathtt{J}}(t_i) = \begin{cases} -t_i - 1, & i \in \mathtt{J} \\ t_i, & i \notin \mathtt{J}. \end{cases} \tag{3.2}$$

For an A-module M, the module obtained by twisting the A-action on M by $\theta_{\mathtt{J}}$ will be denoted by $M^{\theta_{\mathtt{J}}}$. From the above we have that $M^{\theta_{\mathtt{J}}}$

is a weight module if and only if M is. Furthermore, $\mathbf{p} \in \mathrm{supp}(M)$ if and only if $\theta_J(\mathbf{p}) \in \mathrm{supp}(M^{\theta_{\mathtt{J}}})$, where $\theta_J(\mathbf{p})_i = p_i$ if $i \notin \mathtt{J}$ and $\theta_J(\mathbf{p})_i = -p_i - 1$ if $i \in \mathtt{J}$.

Denote by $\Bbbk^{\mathtt{I}}_+$ the set of all $\mathbf{p} \in \Bbbk^{\mathtt{I}}$ such that $p_i \in \mathbb{Z}$ implies $p_i \in \{0, 1, 2, \dots\}$ for all i. For $\mathbf{p} \in \Bbbk^{\mathtt{I}}_+$ denote by $\mathtt{J}_{\mathbf{p}}$ the set of all $i \in \mathtt{I}$ such that $p_i \in \mathbb{Z}$. From Theorem 5 we immediately obtain the following:

Corollary 8. (*a*) *Let L be a simple weight A-module. Then there are unique $\mathbf{p} \in \Bbbk^{\mathtt{I}}_+$ and $\mathtt{J} \subset \mathtt{J}_{\mathbf{p}}$ such that $L \cong L(\mathbf{p})^{\theta_{\mathtt{J}}}$. In fact, if $L = L(\mathbf{p})$, then $\mathtt{J} = \{i \in \mathtt{I} : q_i \in \{-1, -2, ...\}\}$ and $\mathbf{p} = \theta_{\mathtt{J}}(\mathbf{q})$.*

(*b*) *For $\mathbf{p}, \mathbf{q} \in \Bbbk^{\mathtt{I}}_+$, $\mathtt{J} \subset \mathtt{J}_{\mathbf{p}}$ and $\mathtt{J}' \subset \mathtt{J}_{\mathbf{q}}$ we have $L(\mathbf{p})^{\theta_{\mathtt{J}}} \cong L(\mathbf{q})^{\theta_{\mathtt{J}'}}$ if and only if $\mathbf{p} = \mathbf{q}$ and $\mathtt{J} = \mathtt{J}'$.*

Combining this with Proposition 3 we obtain:

Corollary 9. *Let $\mathbf{p} \in \Bbbk^{\mathtt{I}}_+$ and $\mathtt{J} \subset \mathtt{J}_{\mathbf{p}}$. Then $B(\mathbf{p})^{\theta_{\mathtt{J}}}$ and $(B(\mathbf{p})^{\theta_{\mathtt{J}}})^\star$ are the injective envelope and the projective cover of $L(\mathbf{p})^{\theta_{\mathtt{J}}}$, respectively.*

3.7. **Localization realizations.** Let $\mathtt{J} \subset \mathtt{I}$. The adjoint action of X_i on A is locally nilpotent and hence X_i, $i \in \mathtt{J}$, generate a multiplicative Ore subset of A giving rise to the corresponding Ore localization $\mathbf{D}_J A$ of A. Define the functor $\mathrm{F}_J := \mathrm{Res}^{\mathbf{D}_J A}_A \circ \mathrm{Ind}^{\mathbf{D}_J A}_A$ on the category A-mod.

Furthermore, similarly to [Ma, Lemma 4.3], the algebra $\mathbf{D}_J A$ has a family of automorphisms $\varphi_{\mathbf{x}}$, $\mathbf{x} \in \Bbbk^{\mathtt{J}}$, which are polynomial in components of $\mathbf{x}$ and such that for $\mathbf{x} \in \mathbb{Z}^{\mathtt{J}}$ and for any $a \in A$ we have

$$\varphi_{\mathbf{x}}(a) = \prod_{i \in \mathtt{J}} X_i^{-x_i}\, a \prod_{i \in \mathtt{J}} X_i^{x_i}$$

(note that if a is fixed, then it commutes with all but finitely many of the X_i's and hence the expression makes sense by canceling out all other terms). For a $\mathbf{D}_J A$-module M, denote by $M^{\varphi_{\mathbf{x}}}$ the $\mathbf{D}_J A$-module obtained from M after twisting by $\varphi_{\mathbf{x}}$. Note that $\mathrm{supp}(M^{\varphi_{\mathbf{x}}}) = \mathrm{supp}(M) + \mathbf{x}$, where $\mathbf{x}$ is considered as an element of $\Bbbk^{\mathtt{I}}$ by setting all components in $\mathtt{I} \setminus \mathtt{J}$ to be zero.

Proposition 10. *Let $\mathbf{p} \in \Bbbk^{\mathtt{I}}_+$.*

(*a*) *$L(\mathbf{p}) \simeq \mathrm{F}^{\mathbf{p}'}_{\mathtt{I}\setminus\mathtt{J}_{\mathbf{p}}} L(\mathbf{0})$, where $\mathbf{p}' \in \Bbbk^{\mathtt{I}\setminus\mathtt{J}_{\mathbf{p}}}$ with $p'_i = p_i$.*

(*b*) *$B(\mathbf{p}) \simeq \mathrm{F}_{\mathtt{J}_{\mathbf{p}}} L(\mathbf{p})$.*

Proof. By Theorem 5, $L \mapsto \mathrm{supp}(L)$ defines a bijection between the set of simple weight A-modules and $\Bbbk^{\mathtt{I}}/\sim$. Since every simple module has weight multiplicities at most 1, for a weight A-module M we have that $M \simeq L(\mathbf{p})$ if and only if $\mathrm{supp}(M) = \mathrm{supp}(L(\mathbf{p}))$. On the other hand, $\mathrm{supp}(\mathrm{F}^{\mathbf{p}}_{\mathtt{I}\setminus\mathtt{J}_{\mathbf{p}}} L(\mathbf{0})) = \overline{\mathbf{p}} = \mathrm{supp}(L(\mathbf{p}))$, which implies claim (a). For claim (b) we use the polynomial description of $B(\mathbf{p})$ from Subsection 3.4. A basis vector of $\mathrm{F}_{\mathtt{J}_{\mathbf{p}}} L(\mathbf{p})$ can be written uniquely in the form $\mathbf{x}^{\mathbf{m}}\mathbf{x}^{\mathbf{q}}$, where $m_i \in \{0, -1, -2, ...\}$, $i \in \mathtt{J}_{\mathbf{p}}$, $m_j = 0$, $j \notin \mathtt{J}_{\mathbf{p}}$, and $\mathbf{q} \in \overline{\mathbf{p}}$. Then $\mathbf{x}^{\mathbf{m}}\mathbf{x}^{\mathbf{q}} \mapsto \mathbf{x}^{\mathbf{m}+\mathbf{q}}$ defines an isomorphism $\mathrm{F}_{\mathtt{J}_{\mathbf{p}}} L(\mathbf{p}) \simeq B(\mathbf{p})$. $\square$

Note that due to Corollary 8, the injective envelope of every simple weight module L is isomorphic to the localization of L relative to an appropriate Ore's multiplicative subset of A. Presenting the indecomposable injectives as localizations of their simple submodules is an idea explored for categories of weight modules of the symplectic Lie algebras in [GS].

4. Quiver of $\mathfrak{W}$

4.1. **Explicit description of the quiver.** For a nonempty set E define the quiver $Q = Q_E$ as follows: The vertices of Q are all finite subsets of E. For $U, W \in Q$ there is one arrow from U to W if and only if the symmetric difference $U \triangle W$ is a singleton. Impose in Q the following relations: if U and W are such that there are arrows $\alpha : U \to W$ and $\beta : W \to U$, then $\alpha\beta = \beta\alpha = 0$; if U, W, U', W' are different subsets such that there are arrows $\alpha : U \to W$, $\beta : W \to W'$, $\alpha' : U \to U'$ and $\beta' : U' \to W'$, then $\beta\alpha = \beta'\alpha'$. For example, if $E = \{1, 2\}$, then the corresponding quiver Q looks as follows:

$$\begin{tikzcd}
\varnothing \arrow[r, bend left, "\eta"] \arrow[d, bend right, "\xi"'] & \{1\} \arrow[l, bend left, "\eta"] \arrow[d, bend left, "\xi"] \\
\{2\} \arrow[u, bend right, "\xi"'] \arrow[r, bend right, "\eta"'] & \{1,2\} \arrow[l, bend right, "\eta"'] \arrow[u, bend left, "\xi"]
\end{tikzcd}$$

The relations are: $\eta^2 = \xi^2 = 0$ and $\eta\xi = \xi\eta$. The path category of Q with the above relations will be denoted C_E.

The category C_E is canonically isomorphic to the tensor product $\bigotimes_{i\in E} C_{\{i\}}$, where in the case of infinite E the tensor product is understood as the direct limit of the directed system formed by all tensor products with respect to finite subsets of E (see e.g. [Bl]). Note that the algebra $C_{\{1\}}$ is the path category of the following quiver with relations:

$$\begin{tikzcd}
\varnothing \arrow[r, bend left, "\alpha"] & \{1\} \arrow[l, bend left, "\alpha"]
\end{tikzcd}, \qquad \alpha^2 = 0. \tag{4.1}$$

It is easy to see that in the case of finite E all projective C_E-modules are injective and have the same Loewy length.

4.2. **Koszulity.** Recall that a $\mathbb{Z}$-graded associative $\Bbbk$-algebra $C = \bigoplus_{i\in\mathbb{Z}} C_i$ is called *Koszul* if C_0 is semisimple, $C_i = 0$ for $i < 0$, and the i-th component of the minimal graded projective resolution of C_0 is generated in degree i (such resolution is called *linear*). Similarly one defines Koszulity for $\Bbbk$-linear categories.

Proposition 11. *Let E be as in the previous subsection. Then the category C_E is Koszul.*

Proof. For $t \in E$ the algebra $C_{\{t\}}$ is given by (4.1). It is quadratic and monomial, hence Koszul (it is straightforward to write down linear projective resolutions of simple $C_{\{t\}}$-modules). The claim now follows from the standard observation that any tensor product of Koszul algebras is Koszul. To see the latter, fix linear resolutions for each simple module over every tensor factor of the product. Tensoring these resolutions together (one resolution per tensor factor) and taking the total complex in the usual way we obtain linear projective resolutions for simple C_E-modules. The case with finitely many factors can be found for example in [BF]. The infinite case follows by taking the direct limit with respect to the directed system given by finite subsets of E. $\square$

4.3. **Description of the blocks.** Finally, we would like to show that blocks of $\mathfrak{W}$ are described by C_E for appropriate E. Recall that for $\mathbf{p} \in \Bbbk^{\mathtt{I}}$ we denote by $\mathtt{J}_{\mathbf{p}}$ the set of all $i \in \mathtt{I}$ such that $p_i \in \mathbb{Z}$. If $\mathtt{J}_{\mathbf{p}} = \varnothing$, the projective module $P(\mathbf{p})$ is simple which means that $\mathfrak{W}_{\mathbf{p}}$ is semisimple and hence isomorphic to $\Bbbk$-mod.

Theorem 12. *If $\mathtt{J}_{\mathbf{p}} \neq \varnothing$, then the category $\mathfrak{W}_{\mathbf{p}}$ is equivalent to the category of modules over $C_{\mathtt{J}_{\mathbf{p}}}$.*

Proof. Fix some representative in each isomorphism class of indecomposable projectives in $\mathfrak{W}_{\mathbf{p}}$ and let $\mathcal{X}$ be the full subcategory of $\mathfrak{W}_{\mathbf{p}}$ which these fixed representatives generate. To prove our theorem we apply the classical Morita theory for rings with local units, see [Ab, Theorem 4.2]. The only nontrivial thing we have to check is that $\mathcal{X}$ is isomorphic to $C_{\mathtt{J}_{\mathbf{p}}}$.

For $\mathbf{m} \in \mathbf{p} + \mathbb{Z}^{\mathtt{I}}_f$ let $U(\mathbf{m})$ be the set of all $i \in \mathtt{J}_{\mathbf{p}}$ satisfying either $p_i \in \{-1, -2, \dots\}$ while $m_i \in \{0, 1, 2, \dots\}$ or $p_i \in \{0, 1, 2, \dots\}$ while $m_i \in \{-1, -2, \dots\}$. Then the map $P(\mathbf{m}) \mapsto U(\mathbf{m})$ induces a bijection from objects of $\mathcal{X}$ to objects of $C_{\mathtt{J}_{\mathbf{p}}}$. We realize the inverse of this map in the following way: for a finite $U \subset \mathtt{J}_{\mathbf{p}}$ define $\mathbf{p}^{(U)}$ as follows:

$$l_i^{(U)} := \begin{cases} p_i, & p_i \notin \mathbb{Z} \text{ or } i \notin U; \\ 0, & i \in U \text{ and } p_i < 0; \\ -1, & i \in U \text{ and } p_i \geq 0. \end{cases}$$

For each U fix some nonzero $v_U \in P(\mathbf{p}^{(U)})_{\mathbf{p}^{(U)}}$.

Take now any finite $U \subset \mathtt{J}_{\mathbf{p}}$ and $i \in \mathtt{I}_{\mathbf{p}} \setminus U$. If $p_i < 0$, then define the homomorphism $\alpha_{U,i} : P(\mathbf{p}^{(U)}) \to P(\mathbf{p}^{(U \cup \{i\})})$ by sending v_U to $Y^{-p_i} v_{U \cup \{i\}}$, and also define the homomorphism $\beta_{U,i} : P(\mathbf{p}^{(U \cup \{i\})}) \to P(\mathbf{p}^{(U)})$ by sending $v_{U \cup \{i\}}$ to $X^{-p_i} v_U$, If $p_i > 0$, then define the homomorphism $\alpha_{U,i} : P(\mathbf{p}^{(U)}) \to P(\mathbf{p}^{(U \cup \{i\})})$ by sending v_U to $X^{p_i+1} v_{U \cup \{i\}}$, and also define the homomorphism $\beta_{U,i} : P(\mathbf{p}^{(U \cup \{i\})}) \to P(\mathbf{p}^{(U)})$ by sending $v_{U\cup\{i\}}$ to $Y^{p_i+1}v_U$. It is straightforward to check that these homomorphisms satisfy the defining relations of $C_{\mathtt{J}_{\mathbf{p}}}$. Comparing the characters of indecomposable projective modules in $\mathfrak{W}_{\mathbf{p}}$ and over $C_{\mathtt{J}_{\mathbf{p}}}$ we conclude that these are all defining relations. The claim follows. $\square$

References

[Ab] G. Abrams; Morita equivalence for rings with local units. Comm. Algebra **11** (1983), no. 8, 801–837.

[BF] J. Backelin, R. Fröberg; Koszul algebras, Veronese subrings and rings with linear resolutions. Rev. Roumaine Math. Pures Appl. **30** (1985), no. 2, 85–97.

[Ba] V. Bavula; Generalized Weyl algebras and their representations. Algebra i Analiz **4** (1992), no. 1, 75–97.

[BB] V. Bavula and V. Bekkert; Indecomposable representations of generalized Weyl algebras. Comm. Algebra **28** (2000), 5067–5100.

[BBF] V. Bekkert, G. Benkart and V. Futorny; Weight modules for Weyl algebras. In *Kac-Moody Lie algebras and related topics*. Contemp. Math. **343** Amer. Math. Soc., Providence, RI, 2004, 17–42.

[Bl] B. Blackadar; Infinite tensor products of C^*-algebras, Pac. J. Math. **77** (1977), 313–334.

[Co] B.Cox; Fock Space Realizations of Imaginary Verma Modules. Algebras and Representation Theory **8** (2005), no. 2, 173–206.

[CF] B. Cox and V. Futorny; Intermediate Wakimoto modules for affine $sl(n+1)$, J. Physics A: Math. and General **37** (2004), 5589–5603.

[FF1] B. Feigin and E. Frenkel; Affine Kac-Moody algebras and semi-infinite flag manifolds. Comm. Math. Phys. **128** (1990), no. 1, 161–189.

[FF2] B. Feigin and E. Frenkel; Representations of affine Kac-Moody algebras and bosonization. In *Physics and Mathematics of strings*. World Sci. Publishing, Teaneck, NJ, 1990, 271–316

[GS] D. Grantcharov, V. Serganova, Category of $\mathfrak{sp}(2n)$-modules with bounded weight multiplicities. Mosc. Math. J. **6** (2006), no. 1, 119–134.

[JK] H. Jakobsen and V. Kac, A new class of unitarizable highest weight representations of infinite-dimensional Lie algebras. In *Nonlinear equations in classical and quantum field theory (Meudon/Paris, 1983/1984)*. Springer, Berlin, 1985, 1–20.

[Ma] O. Mathieu; Classification of irreducible weight modules. Ann. Inst. Fourier (Grenoble) **50** (2000), no. 2, 537–592.

[MZ] V. Mazorchuk, K. Zhao; Characterization of simple highest weight modules. preprint arXiv:1105.1123, to appear in Canad. Math. Bull.

[Vo] A. Voronov; Semi-infinite induction and Wakimoto modules. Amer. J. Math. **121** (1999), 1079–1094.

[Wa] M. Wakimoto; Fock representations of the affine Lie algebra $A_1^{(1)}$. Comm. Math. Phys. **104** (1986), no. 4, 605–609.

V.F.: Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo SP, Brasil; e-mail: futorny@ime.usp.br

D.G.: Department of Mathematics, University of Texas at Arlington, Arlington, TX 76019, USA; e-mail: grandim@uta.edu

V.M.: Department of Mathematics, Uppsala University, Box 480, SE-751 06, Uppsala, Sweden; e-mail: `mazor@math.uu.se`