On simple mixed modules over the Virasoro algebra

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Abstract

We describe the support of a simple weight module over the Virasoro Lie algebra and classify all those modules, whose support is one element less than the weight lattice. Using the ideas from the proof of the latter result we derive some properties of the so-called mixed modules, that is the modules, which have both finite-dimensional and infinite-dimensional weight spaces.

1 Introduction and preliminaries

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Let $\mathcal{V}$ denote the Virasoro algebra over the complex field $\mathbb{C}$. Recall that $\mathcal{V}$ is generated by a central element, c, and elements e_i, $i \in \mathbb{Z}$, and has the following Lie bracket:

$$[e_i, e_j] = (j-i)e_{i+j} + \delta_{i,-j}\frac{i^3-i}{12}c.$$

The elements c and e_0 span the Cartan subalgebra $\mathfrak{H}$ of $\mathcal{V}$ and any $\mathfrak{H}$-diagonalizable $\mathcal{V}$-module is usually called a *weight module*. If, additionally, all weight spaces of a weight $\mathcal{V}$-module are finite-dimensional, this module is called a *Harish-Chandra module*, see for example [Mat]. All simple Harish-Chandra modules were classified in [MP1, MP2, Mat] and are exhausted by simple highest weight modules, simple lowest weight modules and simple modules from the *intermediate series*. The last one contains all subquotients of the modules $V(a,b)$, $a,b \in \mathbb{C}$, the latter being defined as follows: $V(a,b)$ has a basis v_i, $i \in \mathbb{Z}$, in which $e_i \cdot v_j = (ai+b+j)v_{i+j}$, $c \cdot v_j = 0$.

It is easy to see that c acts by a scalar (called *central charge*) on every simple Harish-Chandra $\mathcal{V}$-module. Using the above classification, one can also easily answer the following question: what does the support of a simple Harish-Chandra $\mathcal{V}$-module (i.e. the set of all eigenvalues of e_0 on this module) look like? It happens that for a simple Harish-Chandra $\mathcal{V}$-module V only one of the following seven possibilities can occur.

1. $\mathrm{Supp}\, V = \{0\}$. In this case V is the trivial $\mathcal{V}$-module. It is a simple highest weight module, a simple lowest weight module, and a simple module from the intermediate series simultaneously.

2. $\operatorname{Supp} V = \{0, -2, -3, -4, \dots\}$. In this case V is a highest weight module with the highest weight 0 and a non-zero central charge.

3. $\operatorname{Supp} V = \{0, 2, 3, 4, \dots\}$. In this case V is a lowest weight module with the lowest weight 0 and a non-zero central charge.

4. There exists $\lambda \in \mathbb{C}$, $\lambda \neq 0$, such that $\operatorname{Supp} V = \{\lambda - k, k \in \mathbb{Z}_+\}$. Such V is a highest weight module. All $\lambda \neq 0$ really occur.

5. There exists $\lambda \in \mathbb{C}$, $\lambda \neq 0$, such that $\operatorname{Supp} V = \{\lambda + k, k \in \mathbb{Z}_+\}$. Such V is a lowest weight module. All $\lambda \neq 0$ really occur.

6. There exists $\lambda \in \mathbb{C}$ such that $\operatorname{Supp} V = \{\lambda + k, k \in \mathbb{Z}\}$. In this case V is a simple module from the intermediate series. All $\lambda \neq 0$ really occur.

7. $\operatorname{Supp} V = \mathbb{Z} \setminus \{0\}$. Such V is a simple module from the intermediate series.

Using the terminology, proposed in [Ma1], we call a weight $\mathcal{V}$-module, V, *cut* if $\operatorname{Supp} V \subset \lambda + \mathbb{Z}_+$ or $\operatorname{Supp} V \subset \lambda - \mathbb{Z}_+$ for some λ, *dense* if $\operatorname{Supp} V = \lambda + \mathbb{Z}$ for some λ and *pinned* if $\operatorname{Supp} V = \lambda + \mathbb{Z} \setminus \{\lambda\}$ for some λ. Now, the above description of the support of a simple Harish-Chandra $\mathcal{V}$-module can be roughly collected in the following statement: every simple Harish-Chandra $\mathcal{V}$-module is either cut or pinned or dense.

It is not difficult to find out that the requirement for V to be a Harish-Chandra module (that is to have finite-dimensional weight spaces) is too restrictive for the last statement. In Section 2 we prove analogous statement for any weight $\mathcal{V}$-module. As soon as we have this result, three natural problems arise:

(I) Classify all simple cut modules.

(II) Classify all simple pinned modules.

(III) Classify all simple dense modules.

The first one is very easy. Indeed, any simple cut module is necessarily a highest weight module or a lowest weight module, thus is a Harish-Chandra module. The third one seems to be very difficult and not much is known here. For instance, there are examples of simple dense $\mathcal{V}$-modules all weight spaces of which are infinite-dimensional, see for example [Zh]. Our motivation for this paper is to determine all simple pinned modules. In fact, we will prove that they are exhausted by the simple pinned modules from the intermediate series. The main result of this paper is the following Theorem.

tmain **Theorem 1.** *Any simple pinned $\mathcal{V}$-module is a Harish-Chandra module and hence is a module from the intermediate series.*

The arguments we present in the proof of Theorem 1 motivate the definition of a new class of modules. We call a simple weight $\mathcal{V}$-module, V, *mixed* provided that there exist a weight, $\lambda \in \mathbb{C}$, and $n \in \mathbb{Z}$ such that $\dim V_\lambda < \infty$ and $\dim V_{\lambda+n} = \infty$. This notion is closely related to the old conjecture, which asserts that any simple pointed $\mathcal{V}$-module (i.e. a weight module having a one-dimensional weight space) is a Harish-Chandra module, see [Xu]. More generally, we conjecture the following:

Conjecture 1. *There are no simple mixed $\mathcal{V}$-modules.*

Theorem 1 in fact implies that there are no pinned mixed modules, and the positive solution of the conjecture from [Xu], mentioned above, would imply that there are no pointed mixed modules.

2 The support of a weight $\mathcal{V}$-module

s2

We start with a description of the support of a weight $\mathcal{V}$-module. Note that this result is easier than the corresponding result for some higher rank Virasoro algebras, obtained in [Ma1, Ma2]. However, we present it since we have never seen it published.

t1 **Theorem 2.** *Any simple weight $\mathcal{V}$-module is either cut or pinned or dense.*

Proof. Let V be a simple weight $\mathcal{V}$-module. Clearly, there exists $\lambda \in \mathbb{C}$ such that $\operatorname{Supp} V \subset \lambda + \mathbb{Z}$. Assume that V is neither dense nor pinned. We will show that in this case V is a highest or a lowest weight module (and hence cut). As V is neither dense nor pinned there exist at least two $k < l \in \mathbb{Z}$ such that $\lambda + k, \lambda + l \notin \operatorname{Supp} V$. Up to the standard involution σ on $\mathcal{V}$, which interchanges the highest and the lowest weight modules, we can assume that $\operatorname{Supp} V$ contains $\lambda + m$ for some $m < k$. Consider the following cases.

Case 1. If we can choose $l = k + 1$, then any non-zero element in $V_{\lambda+m}$, where m is maximal such that $m < k$ and $\lambda + m \in \operatorname{Supp} V$, is annihilated by e_1 and e_2 and thus generates a highest weight submodule of V. This means that V is a highest weight module.

Case 2. Assume that we are outside Case 1 and we can choose $l = k+n$, $n > 2$. Then $\lambda+(k+1) \in \operatorname{Supp} V$ since we ore outside Case 1 and for any $v \in V_{\lambda+(k+1)}$ we have $e_{-1}v = 0$ and $e_{n-1}v = 0$. Commuting e_{-1} with e_{n-1} we obtain that $e_i v = 0$ for any $1 \leqslant i \leqslant n-1$ and hence $e_1 v = e_2 v = 0$ since $n > 2$. As in Case 1 we now have that V is a highest weight module.

Case 3. Assume that we are outside Case 1 and Case 2. This means that we can choose $l = k + 2$ and $\lambda + m \in \operatorname{Supp} V$ for any $m \in \mathbb{Z} \setminus \{k, l\}$. Consider a non-zero, $v \in V_{\lambda+(k-1)}$. We have $e_1 v = e_3 v = 0$ and hence $e_i v = 0$ for any $i \geqslant 3$. If $w = e_2 v = 0$, then v generates a highest weight submodule of V and hence V is a highest weight module. Assume that $w \neq 0$. For any $i \geqslant 3$ we still have $e_i w = e_i e_2 v = e_2 e_i v + (2 - i)e_{2+i} v = 0$. Moreover, $e_{-1} w = 0$ and, as in Case 2, we obtain that $e_1 w = e_2 w = 0$ implying that V is a highest weight module. This completes the proof of the theorem. $\square$

3 Proof of Theorem 1

s3

l3.1 **Lemma 1.** *Let V be a simple $\mathcal{V}$-module. Assume that there is $0 \neq v \in V$ and $n \in \mathbb{N}$ such that $e_l v = 0$ for all $l \geqslant n$. Then for any $w \in V$ there exists $N(w) \in \mathbb{N}$ such that $e_l w = 0$*

for all $l \geqslant N(w)$.

Proof. As V is simple, we have $V = U(\mathcal{V})v$. Note that if we have that $e_l w_1 = 0$ for all $l > N_1$ and $e_l w_2 = 0$ for all $l > N_2$, then, by linearity, $e_l(a_1 w_1 + a_2 w_2) = 0$ for all $l > \max\{N_1, N_2\}$ and all $a_1, a_2 \in \mathbb{C}$. Hence, choosing a monomial PBW-basis in $U(\mathcal{V})$, we obtain that it is sufficient to prove the statement for the elements $w = e_{i_1} \dots e_{i_s} v$. Further, by induction, it is sufficient to prove the statement just for $w = e_i v$, $i \in \mathbb{Z}$. In the last case we set $N(w) = \max\{n, n-i, |i|+1, 2\}$ and obtain the following: for $l \geqslant N(w)$ we have $e_l w = e_l e_i v = e_i e_l v + (i-l)e_{i+l} v$. Now $e_l v = 0$ since $l \geqslant N(w) \geqslant n$ and $e_{i+l} v = 0$ since $i + l \geqslant i + N(w) \geqslant i + (n-i) = n$. This gives $e_l w = 0$ completing the proof. □

l3.2 **Lemma 2.** *There does not exist a simple pinned $\mathcal{V}$-module, V, with $\operatorname{Supp} V = \{\lambda+\mathbb{Z}\} \setminus \{\lambda\}$, which satisfies the following condition: there is $0 \neq v \in V$ and $n \in \mathbb{N}$ such that $e_l v = 0$ for all $l \geqslant n$.*

Proof. Assume that such module exists. Take any non-zero $w \in V_{\lambda+1}$. For some $N \in \mathbb{N}$ we have $e_l w = 0$ for all $l \geqslant N$ by Lemma 1. Moreover, we have $e_{-1} w = 0$ since $\lambda \notin \operatorname{Supp} V$. Now let us prove by induction that $e_l w = 0$ for all $l \in \mathbb{N}$. We know this already for all $l > N$. But if $e_l w = 0$ and $l > 1$, we have $e_{l-1} w = \frac{1}{-1-l}[e_l, e_{-1}]w = 0$ as well, which completes the induction. As the result we obtain that V is generated by a highest weight vector and hence is a highest weight module. This contradicts the assumption that V is pinned and completes the proof. □

c3.3 **Corollary 1.** *Let V be a simple pinned $\mathcal{V}$-module such that $\operatorname{Supp} V = \{\lambda+\mathbb{Z}\} \setminus \{\lambda\}$. Then*

1. *for any $k \in \mathbb{Z}$, $k \neq 0, -1$, the action of the linear operator $e_1 : V_{\lambda+k} \to V_{\lambda+k+1}$ is injective;*
2. *for any $k \in \mathbb{Z}$, $k \neq 0, 1$, the action of the linear operator $e_{-1} : V_{\lambda+k} \to V_{\lambda+k-1}$ is injective.*

Proof. Applying σ it is easy to see that it is enough to prove only the first statement. Assume that $e_1 v = 0$ for some $k \in \mathbb{Z}$, $k \neq 0, -1$, and some $v \in V_{\lambda+k}$, $v \neq 0$. Remark that $e_{-k} v = 0$ since $e_{-k} v \in V_\lambda = 0$. We consider the following cases.

Case 1. $k < 0$. Commuting e_1 with e_{-k} we inductively get that $e_l v = 0$ for all $l > -k$. From Lemma 2 it follows that this case is not possible.

Case 2. $k > 1$. Commuting e_1 with e_{-k} we inductively get that $e_l v = 0$ for all $-k \leqslant l \leqslant -1$, in particular, both e_{-2} and e_{-1} annihilate v. Thus $e_l v = 0$ for all $l < 0$ and it follows that V is a lowest weight module. This contradicts the condition that V is pinned and hence this case is not possible either.

Case 3. $k = 1$. In this case $e_{-1} v \in V_\lambda = 0$. Consider the element $w = e_2 v$. If $w = 0$, then both $e_1 v = 0$ and $e_2 v = 0$, which implies that $e_l v = 0$ for all $l > 0$, which is impossible by Lemma 2. If $w \neq 0$ we have

$$e_{-1} w = e_{-1} e_2 v = e_2 e_{-1} v + 3 e_1 v = 0.$$

Moreover, $e_{-3}w \in V_\lambda = 0$. Using induction we get $e_l w = 0$ for all $l < e_{-3}$ and from (the σ-opposite version of) Lemma 2 it follows that this situation is not possible either. This completes the proof. $\square$

c3.4 **Corollary 2.** *Let V be a simple, mixed, and pinned $\mathcal{V}$-module such that* $\operatorname{Supp} V = \{\lambda + \mathbb{Z}\} \setminus \{\lambda\}$. *Then* $\dim V_{\lambda+k} = \infty$ *for all* $k \in \mathbb{Z} \setminus \{0\}$.

Proof. Since V is mixed, we have that $\dim V_{\lambda+k} = \infty$ for some $k \in \mathbb{Z}$ and, without loss of generality, we can assume $k < 0$ (otherwise we can apply σ and come to this situation). From Corollary 1 it follows immediately that $\dim V_{\lambda+k} = \infty$ for all $k < 0$. Assume now that $\dim V_{\lambda+k} = N < \infty$ for some $k > 0$. Then the linear operator $e_{k+1} : V_{\lambda-1} \to V_{\lambda+k}$ starts in an infinite-dimensional space and has finite rank. Hence there should exist $0 \neq v \in V_{\lambda-1}$ such that $e_{k+1}v = 0$. But $e_1 v \in V_\lambda = 0$ and hence, by induction, we obtain $e_l v = 0$ for all $l > k+1$. From Lemma 2 it follows that this contradicts our assumption that V is pinned. This completes the proof. $\square$

Now we are ready to prove Theorem 1. We denote by $\mathfrak{a}$ the $\mathfrak{sl}_2$-subalgebra of $\mathcal{V}$, generated by $e_{\pm1}$. Let V be a simple, mixed, and pinned $\mathcal{V}$-module such that $\operatorname{Supp} V = \{\lambda+\mathbb{Z}\}\setminus\{\lambda\}$. Fix $v = v_{-1} \in V_{\lambda-1}$, $v \neq 0$, and denote $v_{-k} = e_{-1}^{k-1}v$ for $k \in \mathbb{Z}$, $k > 1$. Denote further $v_1 = e_2 v$ and $v_k = e_1^{k-1}v_1$ for $k \in \mathbb{Z}$, $k > 1$. Then Theorem 1 follows directly from the following statement.

p3.5 **Proposition 1.** *The linear span $N = \langle v_i \,:\, i \in \mathbb{Z} \setminus \{0\}\rangle$ is a $\mathcal{V}$-submodule of V.*

Proof. Let N_+ and N_- denote the linear spans $\langle v_i \,:\, i > 0\rangle$ and$\langle v_i \,:\, i < 0\rangle$ respectively. Since $\mathcal{V}$ is generated, as a Lie algebra, by $e_{\pm1}$ and $e_{\pm2}$, it is enough to show that these four elements preserve N.

Step 1. $e_{\pm1}N \subset N$. We have $e_1 N_+ \subset N_+$ by definition of N_+. Since $e_1 v_{-1} \in V_\lambda = 0$, the module $U(\mathfrak{a})v_{-1}$ is a Verma module over $\mathfrak{a}$, in particular, it has 1-dimensional weight spaces since $\mathfrak{a} \cong \mathfrak{sl}_2$. This implies that N_- is invariant under e_1 as well. The arguments for e_+ are analogous.

Step 2. $e_2 N_- \subset N$. We have $e_2 v_{-1} = v_1 \in N$, $e_2 v_{-2} \in V_\lambda = 0$. Assume that we have already shown that $e_2 v_l \in N$ for all $n < l \leqslant -1$, where $n < -2$. We have

$$e_2 v_n = e_2 e_{-1} v_{n-1} = e_{-1} e_2 v_{n-1} - 3 e_1 v_{n-1}.$$

We have $e_2 v_{n-1} \in N$ by induction, hence $e_{-1}e_2 v_{n-1} \in N$ by Step 1. Further, $e_1 v_{n-1} \in N$ again by Step 1. It follows that $e_2 v_n \in N$ and this step is completed by induction.

Step 3. $e_2 N_+ \subset N$. For every $n \in \mathbb{N}$ we have

$$e_{-1} e_2 v_n = e_2 e_{-1} v_n + 3 e_1 v_n.$$

Here $e_{\pm1} v_n \in N$ by Step 1 and thus $e_2 e_{-1} v_n \in N$ by induction. It follows that $e_{-1}e_2 v_n \in N$ and thus $e_{-1}e_2 v_n = a v_{n+1}$ for some $a \in \mathbb{C}$, $a \neq 0$. But, by Corollary 1, the action of e_{-1} on $V_{\lambda+n+2}$ is injective. Since $e_{-1}v_{n+2} \in N$ and is non-zero (as e_{-1} acts injectively), we have

$e_{-1}v_{n+2} = bv_{n+1}$ for some $\mathbb{C} \ni b \neq 0$. This implies that $e_2 v_n = b' v_{n+2}$ for some $b' \in \mathbb{C}$ and completes this step.

Step 4. $e_{-2}N_- \subset N$. For every $n \in -\mathbb{N}$ we have

$$e_1 e_{-2} v_n = e_{-2} e_1 v_n - 3e_{-1} v_n.$$

Here $e_{\pm 1} v_n \in N$ by Step 1 and thus $e_{-2}e_1 v_n \in N$ by induction. It follows that $e_1 e_{-2} v_n \in N$ and thus $e_1 e_{-2} v_n = a v_{n-1}$ for some $a \in \mathbb{C}$, $a \neq 0$. But, by Corollary 1, the action of e_1 on $V_{\lambda+n-2}$ is injective. Since $e_1 v_{n+2} \in N$ and is non-zero (as e_1 acts injectively), we have $e_1 v_{n-2} = b v_{n-1}$ for some $\mathbb{C} \ni b \neq 0$. This implies that $e_{-2} v_n = b' v_{n-2}$ for some $b' \in \mathbb{C}$ and completes this step.

Step 5. $e_{-2} v_l \in N$ for $l > 1$. Obviously we have $e_{-2} v_2 \in V_\lambda = 0$ and now we can now proceed by induction. For $n > 2$ we have

$$e_1 e_{-2} v_{n-1} = e_{-2} e_1 v_{n-1} - 3 e_{-1} v_{n-1}.$$

Here $e_{-2} v_{n-1} \in N$ by induction and hence $e_1 e_{-2} v_{n-1} =\in N$ by Step 1. Further, $e_{-1} v_{n-1} \in N$ by Step 1. This implies $e_{-2} e_1 v_{n-1} = e_{-2} v_n \in N$.

Step 6. $e_{-2} v_1 \in N$. We have

$$e_{-2} e_3 v_{-2} = e_3 e_{-2} v_{-2} + 5 e_1 v_{-2}.$$

Here $e_1 v_{-2} \in N$ by Step 1. Moreover, $e_{-2} v_{-2} \in N$ by Step 4, in particular, $e_{-2} v_{-2} = a v_{-4}$ for some $a \in \mathbb{C}$. Further

$$e_3 e_{-2} v_{-2} = e_3 a v_{-4} = a(e_1 e_2 - e_2 e_1) v_{-4} = a e_1 e_2 v_{-4} - a e_2 e_1 v_{-4}.$$

We have $e_2 v_{-4} \in N$ by Step 2 and thus $e_1 e_2 v_{-4} \in N$ by Step 1. Moreover, we have $e_1 v_{-4} \in N$ by Step 1 and thus $e_2 e_1 v_{-4} \in N$ by Step 2. Hence $e_3 e_{-2} v_{-2} \in N$, which implies $e_{-2} e_3 v_{-2} \in N$. But

$$e_3 v_{-2} = (e_1 e_2 - e_2 e_1) v_{-2} = e_1 e_2 v_{-2} - e_2 e_1 v_{-2} = -e_2 e_1 v_{-2}.$$

We have $e_1 v_{-2} \in N$ by Step 1, moreover, $e_1 v_{-2} \neq 0$ by Corollary 1, in particular, $e_1 v_{-2} = b v_{-1}$ for some $b \in \mathbb{C}$, $b \neq 0$. Thus $-e_2 e_1 v_{-2} = -b v_1 \neq 0$ and we obtain $e_{-2}(-b v_1) \in N$ with $b \neq 0$. This completes the proof. □

4 Deriving some properties for mixed modules

s4

In this chapter we assume that V is a simple mixed $\mathcal{V}$-module and $\operatorname{Supp} V \subset \lambda + \mathbb{Z}$ for some $\lambda \in \mathbb{C}$.

l4.1 **Lemma 3.** $|\{k \in \mathbb{Z} : \dim V_{\lambda+k} = \infty\}| = \infty$.

Proof. If this is not the case, we can find $k \in \mathbb{Z}$ such that $\dim V_{\lambda+k} = \infty$ and $\dim V_{\lambda+l} < \infty$ for all $l > k$. But then the linear operators $e_1 : V_{\lambda+k} \to V_{\lambda+k+1}$ and $e_2 : V_{\lambda+k} \to V_{\lambda+k+2}$ both start from an infinite-dimensional space and end up in a finite-dimensional space. It means that there exists $0 \neq v \in V_{\lambda+k}$ such that $e_1 v = e_2 v = 0$. Hence v is a highest weight vector and thus V is a highest weight module, which contradicts the assumption that V is mixed. This completes the proof. $\square$

p4.2 **Proposition 2.** $|\{k \in \mathbb{Z} : \dim V_{\lambda+k} < \infty\}| = 1$.

Proof. Assume that this is not the case. Then from Lemma 3 it follows that we can assume that $\dim V_\lambda = \infty$, $\dim V_{\lambda+1} < \infty$ and $\dim V_{\lambda+k} < \infty$ for some $k \in \{2, 3, \dots\}$. It follows that there exists $0 \neq v \in V_\lambda$ such that $e_1 v = e_k v = 0$, and hence $e_l v = 0$ for all $l \geqslant k$. If we can choose $k = 2$, then v is a highest weight vector and thus V is a highest weight module, which contradicts the assumption that V is mixed. Otherwise we have that for every $w \in V$ there exists $N(w)$ such that $e_l w = 0$ for all $l \geqslant N(w)$ by Lemma 1. Since $\dim V_{\lambda+1} < \infty$ and $\dim V_{\lambda+2} = \infty$, we can choose $w \in V_{\lambda+2}$ such that $e_{-1} w = 0$. Using $e_l w = 0$ for all $l \geqslant N(w)$ and $e_{-1} w = 0$ one obtains $e_l w = 0$ for all $l > 0$ by induction. Hence w is a highest weight vector and V is a highest weight module, which again contradicts the assumption that V is mixed. This completes the proof. $\square$

After Proposition 2 we can assume $\dim V_\lambda < \infty$ and $\dim V_{\lambda+k} = \infty$ for all $k \in \mathbb{Z} \setminus \{0\}$.

c4.3 **Corollary 3.** *There does not exist a simple mixed $\mathcal{V}$-module in which there would exists $v \neq 0$ with the property $e_l v = 0$ for all $l > N$ (or all $l < N$) and some $N \in \mathbb{N}$ (resp. $N \in -\mathbb{N}$).*

Proof. Assume that such module V exists. Then, by Lemma 1, we have that for every $w \in V$ there exists $N(w)$ such that $e_l w = 0$ for all $l \geqslant N(w)$. Since $\dim V_{\lambda+1} = \infty$ and $\dim V_\lambda < \infty$, we can find $0 \neq w \in V_{\lambda+1}$ such that $e_{-1} w = 0$. Using $e_l w = 0$ for all $l \geqslant N(w)$ and $e_{-1} w = 0$ one obtains $e_l w = 0$ for all $l > 0$ by induction. Hence w is a highest weight vector and V is a highest weight module, which contradicts the assumption that V is mixed. This completes the proof. $\square$

p4.4 **Proposition 3.** *Let V be a simple mixed $\mathcal{V}$-module such that $\dim V_\lambda < \infty$ and $\dim V_{\lambda+k} = \infty$ for all $k \in \mathbb{Z} \setminus \{0\}$. Then $\lambda = 0, 1/2, -1/2$ or $\lambda \notin (\mathbb{Z} \cup 1/2 + \mathbb{Z})$.*

Proof. Because of σ it is of course enough to show that $\lambda \neq 1, 3/2, 2, 5/2, 3, \dots$.

Case 1. $\lambda \neq 3/2, 2, 5/2, 3, \dots$. Assume that $\lambda \in \{3/2, 2, 5/2, 3, \dots\}$. Then the kernel K of $e_1 : V_{\lambda-1} \to V_\lambda$ has finite codimension in the infinite-dimensional space $V_{\lambda-1}$. Each element $v \in K$ is a highest weight vector for $\mathfrak{a}$ of highest weight $\lambda - 1$. The simple highest weight $\mathfrak{a}$-module of highest weight $\lambda - 1$ is finite-dimensional and has dimension $k = 2\lambda - 1 > 1$. Since K is infinite-dimensional, we have that either $\dim e_{-1}^k K = \infty$ or e_{-1}^k has infinite-dimensional kernel on K (note that e_{-1}^{k-1} acts injectively on K).

Subcase 1. $\dim e_{-1}^k K = \infty$. In this case we have $e_1 e_{-1}^k K = 0$ since all these elements correspond to highest weight vector in $M(\lambda - 1)$. This implies the existence of $0 \neq w \in$

$e_{-1}^{k}K$ such that $e_1 w = 0$ and $e_{k+1}w = 0$. From Corollary 3 we obtain that this situation is not possible.

Subcase 2. $\dim\{w \in e_{-1}^{k-1}K : e_{-1}w = 0\} = \infty$. In this case we obtain some $0 \neq w \in \{w \in e_{-1}^{k-1}K : e_{-1}w = 0\}$, such that $e_{-1}w = 0$ and $e_k w = 0$. Since $k > 2$, it follows that w is a highest weight vector, and thus this situation is not possible either.

Case 2 $\lambda \neq 1$. If $\dim e_{-1}^{2}V_{\lambda-1} = \infty$ we get the same contradiction as in Subcase 1 above. So, we can assume that the codimension of the kernel X of e_{-1} on $V_{\lambda-1}$ is finite. But this then means that there exists $w \in V_{\lambda+1}$ such that $e_{-2}w \in X$ and $e_{-1}w = 0$ (since the dimension of the kernel of e_{-1} on $V_{\lambda+1}$ is infinite). We have $e_{-1}w = 0$ and $e_{-1}e_{-2}w = 0$, which implies $e_{-3}w = 0$. Now $e_{-1}w = 0$ and $e_{-3}w = 0$ imply $e_l w = 0$ for all $l < -3$, and hence this situation is impossible by Corollary 3. □

t4.5 **Theorem 3.** *There does not exist a simple mixed $\mathcal{V}$-module V such that* $\dim V_\lambda < \infty$ *and* $\dim V_{\lambda+k} = \infty$ *for all $k \in \mathbb{Z} \setminus \{0\}$, which would satisfy the following conditions:*

$$e_{-1} \text{ acts injectively on } V_{\lambda+k} \text{ for all } k \geqslant 3 \tag{1}$$ eq4.7

$$e_{1} \text{ acts injectively on } V_{\lambda+k} \text{ for all } k \leqslant -3. \tag{2}$$ eq4.8

Proof. Assume that such V exists. Since $V_{\lambda-1}$ is infinite-dimensional and V_λ is finite-dimensional, we can choose $0 \neq v \in V_{\lambda-1}$ such that $e_1 v = e_{-1}e_2 v = 0$. Set $v_{-1} = v$, $v_{-l} = e_{-1}^{l-1}v$ for $l > 1$, $v_1 = e_2 v$, and $v_l = e_1^{l-1}v_1$ for $l > 1$. Using (1) and (2) one proves that $N = \langle v_i : i \in \mathbb{Z} \setminus \{0\}\rangle$ is a $\mathcal{V}$-submodule of V using the same arguments as in the proof of Proposition 1. This implies that V must be a Harish-Chandra module, which contradicts the assumption that V is mixed. The obtained contradiction completes the proof. □

References

MP1 [MP1] *C.Martin, A.Piard*, Indecomposable modules over the Virasoro Lie algebra and a conjecture of V. Kac. Comm. Math. Phys. 137 (1991), no. 1, 109–132.

MP2 [MP2] *C.Martin, A.Piard*, Classification of the indecomposable bounded admissible modules over the Virasoro Lie algebra with weight spaces of dimension not exceeding two. Comm. Math. Phys. 150 (1992), no. 3, 465–493.

M [Mat] *O.Mathieu*, Classification of Harish-Chandra modules over the Virasoro Lie algebra. Invent. Math. 107 (1992), no. 2, 225–234.

Ma [Ma1] *V.Mazorchuk*, Futorny theorem for generalized Witt algebras of rank 2. Comm. Algebra 25 (1997), no. 2, 533–541.

Ma2 [Ma2] *V.Mazorchuk*, Classification of simple Harish-Chandra modules over $\mathbb{Q}$-Virasoro algebra. Math. Nachr. 209 (2000), 171–177.

$\boxed{\texttt{X}}$ [Xu] *X.Xu*, Pointed representations of Virasoro algebra. A Chinese summary appears in Acta Math. Sinica 40 (1997), no. 3, 479. Acta Math. Sinica (N.S.) 13 (1997), no. 2, 161–168.

$\boxed{\texttt{Z}}$ [Zh] *H.Zhang*, A class of representations over the Virasoro algebra. J. Algebra 190 (1997), no. 1, 1–10.

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