Classification of simple weight Virasoro modules with a finite-dimensional weight space

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Abstract

We show that the support of a simple weight module over the Virasoro algebra, which has an infinite-dimensional weight space, coincides with the weight lattice and that all non-trivial weight spaces of such module are infinite dimensional. As a corollary we obtain that every simple weight module over the Virasoro algebra, having a non-trivial finite-dimensional weight space, is a Harish-Chandra module (and hence is either a simple highest or lowest weight module or a simple module from the intermediate series). This implies positive answers to two conjectures about simple pointed and simple mixed modules over the Virasoro algebra.

1 Description of the results

The *Virasoro algebra* $\mathcal{V}$ over an algebraically closed field, $\Bbbk$, of characteristic zero has a basis, consisting of a central element, c, and elements e_i, $i \in \mathbb{Z}$, with the Lie bracket defined for the basis elements as follows:

$$[e_i, e_j] = (j-i)e_{i+j} + \delta_{i,-j}\frac{i^3-i}{12}c.$$

The linear span of c and e_0 is called the *Cartan subalgebra* $\mathfrak{H}$ of $\mathcal{V}$ and an $\mathfrak{H}$-diagonalizable $\mathcal{V}$-module is usually called a *weight module*. If, additionally, all weight spaces of a weight $\mathcal{V}$-module are finite-dimensional, the module is called a *Harish-Chandra module*, see for example [M]. All simple Harish-Chandra modules were classified in [MP1, MP2, M] and are exhausted by simple highest weight modules, simple lowest weight modules and simple modules from the so-called *intermediate series* (see e.g. [M] for definitions).

If M is a simple weight $\mathcal{V}$-module, then c acts on M by a scalar, called the *central charge* of M. Furthermore, M can be written as a direct sum of its *weight spaces*, $M = \oplus_{\lambda\in\Bbbk} M_\lambda$, where M_λ is the set of all elements of M on which e_0 acts as the multiplication with λ. The set of all λ for which $M_\lambda \neq 0$ is called the *support* of M and is denoted by $\mathrm{supp}(M)$. Obviously, if M is a simple weight $\mathcal{V}$-module, then there exists $\lambda \in \Bbbk$ such that $\mathrm{supp}(M) \subset \lambda + \mathbb{Z}$. A simple weight $\mathcal{V}$-module, M, is called *pointed* provided that there exists $\lambda \in \Bbbk$ such that $\dim M_\lambda = 1$ (for example from the classification of simple Harish-Chandra modules it follows that they all are pointed). The following question was formulated in [Xu, Problem 3.3]:

Question: *Is any simple pointed $\mathcal{V}$-module a Harish-Chandra module?*

A simple weight $\mathcal{V}$-module, M, is called *mixed* provided that there exist $\lambda \in \Bbbk$ and $k \in \mathbb{Z}$ such that $\dim M_\lambda = \infty$ and $\dim M_{\lambda+k} < \infty$. The following conjecture, a positive answer to which implies a positive answer to the question above, was formulated in [Maz, Conjecture 1]:

Conjecture: *There are no simple mixed $\mathcal{V}$-modules.*

In the present paper we prove the following result, which implies positive answers to both the Question and the Conjecture above:

Theorem 1. *Let M be a simple weight $\mathcal{V}$-module. Assume that there exists $\lambda \in \Bbbk$ such that $\dim M_\lambda = \infty$. Then $\mathrm{supp}(M) = \lambda + \mathbb{Z}$ and for every $k \in \mathbb{Z}$ we have $\dim M_{\lambda+k} = \infty$.*

Apart from the positive answers to the Question and the Conjecture above, Theorem 1 also implies the following classification of all simple weight $\mathcal{V}$-modules which admit a non-trivial finite-dimensional weight space:

Corollary 2. *Let M be a simple weight $\mathcal{V}$-module. Assume that there exists $\lambda \in \Bbbk$ such that $0 < \dim M_\lambda < \infty$. Then M is a Harish-Chandra module. Consequently, M is either a simple highest or lowest weight module or a simple module from the intermediate series.*

The paper is organized as follow: Theorem 1 is proved in Section 2 and in Section 3 we discuss the corollaries from this theorem.

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2 Proof of Theorem 1

Our strategy to prove Theorem 1 is the following: first we show in Lemma 3 that a simple weight $\mathcal{V}$-module with an infinite-dimensional weight space can have at most one weight space of finite dimension in the weight lattice. Then in Lemma 5 we show that this finite-dimensional weight must belong to $\{-1, 0, 1\}$. These three cases are excluded in Lemma 6 and Lemma 7 by a case-by-case analysis. The key point of our proof is Lemma 4, which is an easy technical statement claiming that some special element from the universal enveloping algebra $U(\mathcal{V})$ annihilates certain elements of the module. The statement is very easy to prove by a direct computation, however, the main and perhaps most non-trivial idea of the paper is that there should exist an element in the universal enveloping algebra $U(\mathcal{V})$, which satisfies the conclusion of Lemma 4.

Let M be a simple weight $\mathcal{V}$-module. We start with the following obvious observation:

Principal Observation: *Assume that there exists $\mu \in \Bbbk$ and a non-zero element, $v \in M_\mu$, such that $e_1 v = e_2 v = 0$ or $e_{-1} v = e_{-2} v = 0$. Then M is a Harish-Chandra module.*

Proof. Indeed, under these conditions v is either a highest or a lowest weight vector and hence M is either a highest or a lowest weight module. Hence M is a Harish-Chandra module (see e.g. [M]). □

Assume now that M is a simple weight $\mathcal{V}$-module and that there exists $\lambda \in \Bbbk$ such that $\dim M_\lambda = \infty$.

Lemma 3. *There exists at most one $i \in \mathbb{Z}$ such that $\dim M_{\lambda+i} < \infty$.*

Proof. Assume that $\dim M_{\lambda+i} < \infty$ and $\dim M_{\lambda+j} < \infty$ for different $i, j \in \mathbb{Z}$. Without loss of generality we may assume $i = 1$ and $j > 1$. Let V denote the intersection of the kernels of the linear maps $e_1 : M_\lambda \to M_{\lambda+1}$ and $e_j : M_\lambda \to M_{\lambda+j}$. Since $\dim M_\lambda = \infty$, $\dim M_{\lambda+1} < \infty$ and $\dim M_{\lambda+j} < \infty$, we have $\dim V = \infty$. Since $[e_1, e_k] = (k-1)e_{k+1} \neq 0$ for all $k > 1$, and $j > 1$, inductively we get

$$e_k V = 0 \quad \text{for all} \quad k = 1, j, j+1, j+2, \dots. \tag{2.1}$$

(Here we cannot directly use the well known [M, Lemma 1.5] to deduce that M is a highest weight module). If there would exist $0 \neq v \in V$ such that $e_2 v = 0$, then $e_1 v = e_2 v = 0$ and M would be a Harish-Chandra module by the Principal Observation. A contradiction. Hence $e_2 v \neq 0$ for all $v \in V$.

In particular, $\dim e_2V = \infty$. Since $\dim M_{\lambda+1} < \infty$, there exists $0 \neq w \in e_2V$ such that $e_{-1}w = 0$. Let $w = e_2u$ for some $u \in V$. For all $k \geqslant j$, using (2.1) we have

$$e_k w = e_k e_2 u = e_2 e_k u + (2-k)e_{k+2}u = 0 + 0 = 0.$$

Hence $e_k w = 0$ for all $k = -1, j, j+1, j+2, \dots$. Since $[e_{-1}, e_l] = (l+1)e_{l-1} \neq 0$ for all $l > 1$, inductively we get $e_k w = 0$ for all $k = 1, 2, \dots$. Hence M is a Harish-Chandra module by the Principal Observation. A contradiction. The statement follows. $\square$

Because of Lemma 3 we can now fix the following notation until the end of this section: M is a simple weight $\mathcal{V}$-module, $\mu \in \Bbbk$ is such that $\dim M_\mu < \infty$ and $\dim M_{\mu+i} = \infty$ for every $i \in \mathbb{Z} \setminus \{0\}$.

Lemma 4. *Let $0 \neq v \in M_{\mu-1}$ be such that $e_1 v = 0$. Then*

$$(e_1^3 - 6e_2e_1 + 6e_3)e_2 v = 0.$$

Proof.

$$\begin{aligned}(e_1^3 - 6e_2e_1 + 6e_3)e_2v = (e_1^3e_2 - 6e_2e_1e_2 + 6e_3e_2)v = \\ (e_2e_1^3 + 3e_3e_1^2 + 6e_4e_1 + 6e_5 - 6e_2^2e_1 - 6e_3e_2 - 6e_5 + 6e_3e_2)v = \\ (e_2e_1^3 + 3e_3e_1^2 + 6e_4e_1 - 6e_2^2e_1)v = [\text{using } e_1v = 0] = 0.\end{aligned}$$

$\square$

Lemma 5. $\mu \in \{-1, 0, 1\}$.

Proof. Let V denote the kernel of $e_1 : M_{\mu-1} \to M_\mu$. Since $\dim M_{\mu-1} = \infty$ and $\dim M_\mu < \infty$ we have $\dim V = \infty$. For any $v \in V$ consider the element e_2v. By the Principal Observation, $e_2v = 0$ would imply that M is a Harish-Chandra module, a contradiction. Hence $e_2v \neq 0$, in particular, $\dim e_2V = \infty$. This implies that there exists $w \in e_2V$ such that $w \neq 0$ and $e_{-1}w = 0$. From Lemma 4 we have $(e_1^3 - 6e_2e_1 + 6e_3)w = 0$, in particular, $e_{-1}^3(e_1^3 - 6e_2e_1 + 6e_3)w = 0$. However, by a direct calculation we obtain

$$e_{-1}^3(e_1^3 - 6e_2e_1 + 6e_3) = 48e_0^3 - 144e_0^2 + 96e_0 \mod U(\mathcal{V})e_{-1}.$$

This implies $(48e_0^3 - 144e_0^2 + 96e_0)w = 0$. But $w \in M_{\mu+1}$, which implies $e_0w = (\mu+1)w$, and hence $(\mu+1)^3 - 3(\mu+1)^2 + 2(\mu+1) = 0$, that is $\mu \in \{-1, 0, 1\}$. $\square$

Lemma 6. $\mu \in \{-1, 1\}$ *is not possible.*

Proof. We show that $\mu = 1$ is not possible and for $\mu = -1$ the statement will follow by applying the canonical involution on $\mathcal{V}$. Assume $\mu = 1$ and denote by V the infinite-dimensional kernel of the linear map $e_1 : M_0 \to M_1$. For $v \in V$, using $e_1 v = e_0 v = 0$ we have

$$e_1 e_{-1} v = e_{-1} e_1 v - 2e_0 v = 0 + 0 = 0. \tag{2.2}$$

Hence if $e_{-1}V$ would be infinite-dimensional, there would exist $0 \neq w \in e_{-1}V$ such that $e_1 w = 0$ (by (2.2)) and $e_2 w = 0$ (since $\dim V_1 < \infty$). The Principal Observation then would imply that M is a Harish-Chandra module, a contradiction. Hence $\dim e_{-1}V < \infty$. This means that the kernel W of the linear map $e_{-1} : V \to M_{-1}$ is infinite-dimensional. For every $x \in W$ we have

$$e_1 e_{-2} x = e_{-2} e_1 x - 3e_{-1} x = [\text{using } e_{-1}x = e_1 x = 0] = 0 + 0 = 0. \tag{2.3}$$

If there would exist $0 \neq x \in W$ such that $e_{-2}x = 0$, then we would have $e_{-2}x = e_{-1}x = 0$ and the Principal Observation would imply that M is a Harish-Chandra module, a contradiction. Thus $\dim e_{-2}W = \infty$. Let H denote the kernel of the linear map $e_3 : e_{-2}W \to M_1$. Since $\dim e_{-2}W = \infty$ and $\dim M_1 < \infty$, we have $\dim H = \infty$. For every $y \in H$ we also have $e_1 y = 0$ by (2.3), implying by induction that $e_k H = 0$ for all $k = 1, 3, 4, \dots$.

If $e_2 h = 0$ for some $0 \neq h \in H$ then the Principal Observation implies that M is a Harish-Chandra module, a contradiction. Hence $\dim e_2 H = \infty$. For every $h \in H$ and $k \geqslant 3$ we have

$$e_k e_2 h = e_2 e_k h + (2-k) e_{k+2} h = [\text{using } e_i h = 0 \text{ for } i \geqslant 3] = 0 + 0 = 0.$$

Hence $e_k e_2 H = 0$ for all $k \geqslant 3$. Let, finally, K denote the infinite-dimensional kernel of the linear map $e_1 : e_2 H \to M_1$. If $e_2 z = 0$ for some $0 \neq z \in K$ then the Principal Observation implies that M is a Harish-Chandra module, a contradiction. Hence $\dim e_2 z \neq 0$ for all $z \in K$. For every $z \in K$ and $k \geqslant 3$ we have

$$e_k e_2 z = e_2 e_k z + (2-k) e_{k+2} z = [\text{using } e_i z = 0 \text{ for } i \geqslant 3] = 0 + 0 = 0.$$

Hence $e_k e_2 K = 0$ for all $k \geqslant 3$. At the same time, since $\dim e_2 K = \infty$ and $\dim M_1 < \infty$, we can find some $0 \neq t \in e_2 K$ such that $e_{-1} t = 0$. By induction we get $e_i t = 0$ for all $i > 0$ and thus M is a Harish-Chandra module by the Principal Observation. This last contradiction completes the proof. $\square$

Now the proof of Theorem 1 follows from the following lemma:

Lemma 7. *$\mu = 0$ is not possible.*

Proof. Define

$$V = \operatorname{Ker}(e_1 : M_{-1} \to M_0) \cap \operatorname{Ker}(e_{-1}e_2 : M_{-1} \to M_0) \cap \\ \cap \operatorname{Ker}(e_1 e_{-2} e_2 : M_{-1} \to M_0),$$

$$W = \operatorname{Ker}(e_{-1} : M_1 \to M_0) \cap \operatorname{Ker}(e_1 e_{-2} : M_1 \to M_0) \cap \\ \cap \operatorname{Ker}(e_{-1} e_2 e_{-2} : M_1 \to M_0).$$

Since $\dim M_{-1} = \infty$ and $\dim M_0 < \infty$, V is a vector subspace of finite codimension in M_{-1}. Since $\dim M_1 = \infty$ and $\dim M_0 < \infty$, W is a vector subspace of finite codimension in M_1. In order not to get a direct contradiction using the Principal Observation, we assume $e_2 v \neq 0$ for all $0 \neq v \in V$ and $e_{-2} w \neq 0$ for all $0 \neq w \in W$. Then $\dim e_2 V = \infty$ and, by Lemma 4, for every $0 \neq v \in V$ we have $(e_1^3 - 6e_2e_1 + 6e_3)e_2 v = 0$.

Since the codimension of W in M_1 is finite, the intersection $W' = e_2 V \cap W$ is infinite-dimensional. Note that

$$e_{-1}(e_1^3 - 6e_2e_1 + 6e_3) = 6e_1^2 e_0 + 6e_1^2 - 12e_2e_0 - 18e_1^2 + 24e_2 \mod U(\mathcal{V})e_{-1}.$$

Choose $v \in V$ such that $w_v := e_2 v \in W' \setminus \{0\}$. The equality $e_{-1}(e_1^3 - 6e_2e_1 + 6e_3)e_2 v = 0$ implies that $(2e_2 - e_1^2)w_v = 0$. In particular, for this v we have $e_{-2}(2e_2 - e_1^2)w_v = 0$. However,

$$e_{-2}(2e_2 - e_1^2) = 2e_2e_{-2} + 2e_0 - c - e_1^2 e_{-2} - 6e_1e_{-1},$$

and since $e_1 e_{-2} w_v = e_{-1} w_v = 0$ by assumptions, we get $e_2 e_{-2} w_v = \tau w_v$ for some $\tau \in \Bbbk$. In order not to get a direct contradiction using the Principal Observation, we must assume $e_{-2} w_v \neq 0$. Since $e_1 e_{-2} w_v = 0$, we also must assume $e_2 e_{-2} w_v \neq 0$, that is $\tau \neq 0$.

Denote $y = w_v$ and $x = e_{-2} y$. Let us sum up, what we know about x and y:

$$e_1 x = 0, \quad e_{-1} y = 0, \quad x = e_{-2} y, \quad \tau y = e_2 x. \tag{2.4}$$

Let U_+ and U_- denote the subalgebras of $U(\mathcal{V})$, generated by e_1, e_2 and e_{-1}, e_{-2} respectively. Consider the vector space

$$N = U_- x \oplus U_+ y \subset M.$$

From the definition it follows that both U_+ and U_- are stable under the adjoint action of e_0. Since both x and y are eigenvectors for e_0, we derive that N decomposes into a direct sum of weight spaces which are obviously finite-dimensional. Hence, to complete the proof we have just to show that

N is stable under the action of the whole $\mathcal{V}$. Since $\mathcal{V}$ is generated by e_1, e_{-1}, e_2, e_{-2}, it is enough to show that N is stable under the action of these four operators. Because of the symmetry of our situation, it is even enough to show that N is stable under the action of, say e_1 and e_2.

That $e_1U_+y \subset U_+y$ and $e_2U_+y \subset U_+y$ is clear. Let us show that $e_1U_-x \subset U_-x$. For any $a \in U_-$ we have $e_1ax = ae_1x + [e_1, a]x$. By (2.4), $ae_1x = 0$. Further, $[e_1, a] = \sum_{i,j} a_{i,j}e_0^ic^j$ for some $a_{i,j} \in U_-$. Since $x \in M_{-1}$, we have $e_0^ic^jx = \xi x$ for some $\xi \in \Bbbk$. Therefore $[e_1, a]x \in U_-x$, which means that e_1 preserves U_-x and hence N.

Finally, let us show that $e_2U_-x \subset N$. For any $a \in U_-$ we have $e_2ax = ae_2x + [e_2, a]x$. By (2.4), $e_2x = \tau y \neq 0$. Let $A = e_{i_1} \dots e_{i_l}$ be a monomial, where $i_s \in \{-1, -2\}$ for all $s = 1, \dots, l$. If $i_l = -1$ we have $Ae_2x = 0$ since $e_{-1}y = 0$. If $i_l = -2$ we have $Ae_2x = \zeta e_{i_1} \dots e_{i_{l-1}}x \in U_-x$ for some $\zeta \in \Bbbk$ by (2.4). This implies that $ae_2x \in N$. Let us write the element $[e_2, a]$ in the PBW basis corresponding to the order $\dots, e_{-2}, e_{-1}, e_0, c, e_1$. By (2.4), $e_1x = 0$, and hence all terms, which end on e_1 will vanish. This means that $[e_2, a]x = \sum_{i,j} a_{i,j}e_0^ic^jx$ for some $a_{i,j} \in U_-$. In the previous paragraph it was shown that in this case $[e_2, a]x \in U_-x$. This completes the proof. □

3 Corollaries from Theorem 1

As an immediate corollary from Theorem 1 we have:

Corollary 8. *Let M be a simple weight $\mathcal{V}$-module. Assume that there exists $\lambda \in \Bbbk$ such that $0 < \dim M_\lambda < \infty$. Then M is a Harish-Chandra module. Consequently, M is either a simple highest or lowest weight module or a simple module from the intermediate series.*

Proof. Assume that this M is not a Harish-Chandra module. Then there should exists $i \in \mathbb{Z}$ such that $\dim M_{\lambda+i} = \infty$. In this case Theorem 1 implies $\dim M_\lambda = \infty$, a contradiction. Hence M is a Harish-Chandra module, and the rest of the statement follows from [M, Theorem 1]. □

The following corollary gives a positive answer to [Xu, Problem 3.3]:

Corollary 9. *Every pointed $\mathcal{V}$-module is a Harish-Chandra module.*

Proof. Every pointed module satisfies the conditions of Corollary 8 by definition. Hence the statement follows from Corollary 8. □

The following corollary gives a positive answer to [Maz, Conjecture 1]:

Corollary 10. *There are no simple mixed $\mathcal{V}$-modules.*

Proof. Let M be a simple mixed $\mathcal{V}$-module. Then, by the definition, there exists $\lambda \in \Bbbk$ and $i \in \mathbb{Z}$ such that $\dim M_\lambda = \infty$ and $\dim M_{\lambda+i} < \infty$. However, Theorem 1 implies $\dim M_{\lambda+i} = \infty$. A contradiction. $\square$

References

[MP1] *C.Martin, A.Piard*, Indecomposable modules over the Virasoro Lie algebra and a conjecture of V. Kac. Comm. Math. Phys. 137 (1991), no. 1, 109–132.

[MP2] *C.Martin, A.Piard*, Classification of the indecomposable bounded admissible modules over the Virasoro Lie algebra with weight spaces of dimension not exceeding two. Comm. Math. Phys. 150 (1992), no. 3, 465–493.

[M] *O.Mathieu*, Classification of Harish-Chandra modules over the Virasoro Lie algebra. Invent. Math. 107 (1992), no. 2, 225–234.

[Maz] *V.Mazorchuk*, On simple mixed modules over the Virasoro algebra, Matematychni Studii, v. 22, No. 2.

[Xu] *X.Xu*, Pointed representations of Virasoro algebra. A Chinese summary appears in Acta Math. Sinica 40 (1997), no. 3, 479. Acta Math. Sinica (N.S.) 13 (1997), no. 2, 161–168.

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