Lie Algebras VT 2025

ASSIGNMENT 3

1. (10 credits) Let Φ be a root systs in $(\mathbb{E}, (-, -))$. Define $\Phi^* := \{\frac{2\alpha}{(\alpha, \alpha)} \mid \alpha \in \Phi\}$. Show that Φ^* is a root system in $(\mathbb{E}, (-, -))$ with the same Weyl group as Φ .

2. (10 credits) Let Φ be a root system and W the corresponding Weyl group. Let π be a basis in Φ . Show that there is a unique element, $w_0 \in W$, which sends Φ_+ to Φ_- . Moreover, show that $|\Phi_+|$ coincides with the length of the shortest decomposition of w_0 into a product of simple reflections. Is it true that w_0 must be central in W?

3. (10 credits) What is the eigenvalue of the Casimir element for a semi-simple complex Lie algebra on the adjont module?

4. (10 credits) Let \mathfrak{g} be the 2-dimensional Lie algebra with basis h, e such that [h, e] = 2e. Prove that the center of $U(\mathfrak{g})$ is trivial, that is consists only of scalars.

Submission deadline: Friday, the 23-d of May at 12.00 (noon).