Partial Horn logic and cartesian categories

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Abstract

A logic is developed in which function symbols are allowed to represent partial functions. It has the usual rules of logic (in the form of a sequent calculus) except that the substitution rule has to be modified. It is developed here in its minimal form, with equality and conjunction, as "partial Horn logic".

Various kinds of logical theory are equivalent:

- partial Horn theories
- "quasi-equational" theories, partial Horn theories without predicate symbols
- cartesian theories
- essentially algebraic theories

The logic is sound and complete with respect to models in **Set**, and sound with respect to models in any cartesian (finite limit) category.

The simplicity of the quasi-equational form allows an easy predicative constructive proof of the free partial model theorem for cartesian theories: that if a theory morphism is given from one cartesian theory to another, then the forgetful (reduct) functor from one model category to the other has a left adjoint.

Various examples of quasi-equational theory are studied, including those of cartesian categories and of other classes of categories. For each quasi-equational theory $\mathbb{T}$ another, $\mathrm{Cart}\varpi\mathbb{T}$, is constructed, whose models are cartesian categories equipped with models of $\mathbb{T}$. Its initial model, the "classifying category" for $\mathbb{T}$, has properties similar to those of the syntactic category, but more precise with respect to strict cartesian functors.

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1   Introduction

One of the most fundamental results in universal algebra is the general ability to construct free algebras. From it spring other techniques, such as the presentation of algebras by generators and relations, and the construction of left adjoints to forgetful functors. The universal characterization of free algebras also encapsulates induction and recursion in a general way.

In logical terms, the result concerns a particular class of logical theories, the *algebraic* theories. They are presented using function symbols (the *operators*) but no predicates; and axioms in the form of equational laws – universally quantified equations. Without introducing any serious difficulties, one can also allow the theory to be many-sorted.

However, the construction of free algebras is not limited to algebraic theories. For instance, there are many examples involving categories – see [Hig71] for instance. One can construct the free category over a graph, the free groupoid over a category, and so on. But the theory of categories is not algebraic.

In fact there is a broader class of theories for which good free algebra results hold. They are known variously as cartesian theories, essentially algebraic theories or finite limit theories. Their notable feature is that the operators may be partial, and the logic provides means (typically using equations) for controlling their domains of definition. Because of these free algebra results, we contend that the class of cartesian theories is a very important one logically. In fact we give some substantial examples of logical constructions (for example, the construction of classifying category) that can be carried out by defining a suitable cartesian theory and then taking an initial model.

The proof of the free algebra theorem for algebraic theories is simple in outline. First form the set of all terms constructed using the operators, and then factor out a congruence generated using the equational laws.

However, with partial operators there is a difficulty with this 2-step process. Factoring out a congruence creates equalities that can bring argument tuples into the domain of definition of an operator, and thereby create new terms. Standard proofs of the free algebra theorem for cartesian theories use sophisticated categorical techniques.

In addition, the descriptions of the theories are a little complicated. For example, in a cartesian theory one can use existential quantification, but it must always be accompanied by side conditions proving that the existential witness is unique. This complexity means that when giving a general account of results such as the free algebra theorem, it difficult to do it directly in terms of the theory structure – typically, one transforms to a categorical representation.

The present work is motivated by the following idea. Since the key difference between algebraic and cartesian is the partiality of operators, can we simplify the presentation of cartesian theories by building the partiality directly into the logic? We describe such a logic. Essentially it is the same as a system [Joh02b] for logic of total terms, but with a modified substitution axiom. It identifies definedness with self-equality.

A minimal *quasi-equational* kind of theory in this partial logic has functions but no predicates (other than equality), and has axioms in a sequent form with conjunction of equations entailing an equation. It turns out that such theories are equivalent to

cartesian theories. Using the quasi-equational presentation, it is very easy to prove the free model theorem in a 2-step process similar to that for algebraic theories. One first forms the set of all *partial* terms constructed from the operators, and then factors out a *partial* congruence generated using the axioms.

A category is the prime example of an algebra with a partial operation (the composition). Further properties of categories can often be described in partial algebraic terms, for instance using the *essentially algebraic theories* of [Fre72] (see also [FS90] and [AR94]), or employing the *limit theories* of [Cos79] appearing as *cartesian theories* in [Joh02b] (see also [Kea75], [Vol79] and the closely related *cartesian logic* [Fre02]). For an extensive general theory of partial algebras, in a non-categorical context, we refer to [ABN81], [Bur86], [Bur02]. In this paper we employ a logic for such algebras which permits axioms to be universal Horn formulae. We prove it sound and complete (Sections 3, 4.2). As one application we show that freely generated partial algebras exist (Theorem 5.4). This theorem is already known from topos theory [BW84]. We show here that it is provable in a predicative meta-theory using a generalisation of Birkhoff's term model construction for total algebras (cf. [MT92]). This turns out be quite straightforward once the appropriate "term logic" is in place. The proof does not use the axiom of choice and is indeed formalisable in a constructive and predicative theory. The question what a minimal categorical theory of this kind might be is still open, but the proofs should in any case be formalisable in the predicative topos of [MP02]. It should be noted that the crucial completeness theorem (Theorem 4.4) we use was obtained already by Andréka and Németi [AN83] but using one-sorted signatures, and thus not directly suitable for categorical applications.

A different approach to the description of categories are using total operations and sort introduction axioms, as in Ehresmann's *sketches* or as in *left exact logic* [McL86] Yet another characterisation of the cartesian categories is as *locally presentable categories* [AR94].

The main contributions of this paper are the quasi-equational logic for partial algebras, called *partial Horn logic* (PHL), which is adapted to a standard presentation of categorical logic (Section 2); its completeness theorem (Section 4.2); its interpretability in any cartesian category, i.e. a category which have all finite limits (Sections 7 and 8); and finally a new characterisation of cartesian categories using the quasi-equational logic and natural construction of syntactic categories (Section 9). This makes it easy to define and reason about categorical structures inside other weak categories. We give PHL axioms for some theories arising in category theory, including cartesian and locally cartesian closed categories (Section 6). As further application of the partial algebraic point of view, we show how to improve the notation of the first order internal language of categories. A logical characterisation of locally cartesian closed categories among Heyting categories is given (Section 6.3).

2 Partial Horn logic

The inference rules of this logic are obtained by a modification of the Horn logic as presented in [Joh02b, D 1]. We refer to this chapter for background. First recall some basic definitions.

Let Σ be a many-sorted first-order signature. There is in addition a binary predicate $=_A$ on each sort A of the signature. Self-equality $t =_A t$ is abbreviated $t\downarrow$ ("t is defined"). A *Horn formula over* Σ is a formula built up from atomic formulae and the truth constant $\top$ using conjunction $\wedge$. We shall take a conjunction $\varphi_1 \wedge \cdots \wedge \varphi_n$ to mean $\top$ if $n = 0$, and $(\cdots ((\varphi_1 \wedge \varphi_2) \wedge \varphi_3) \cdots \wedge \varphi_n)$ if $n > 0$.

The set of free variables $\mathrm{FV}(\varphi)$ of a Horn formula φ is simply the variables of the formula. A *context* is a finite list of distinct typed variables $\vec{x} = x_1, \ldots, x_n$, which may be empty, and then denoted (). A formula φ is *suitable* for the context $\vec{x}$ if each free variable of φ is in the context. A *formula-in-context* is an expression $\vec{x}.\varphi$, where φ is a formula suitable for the context $\vec{x}$. The same terminology is also applied to terms.

A *sequent* over the signature Σ is an expression of the form

$$\varphi \vdash^{\vec{x}} \psi \tag{1}$$

where φ and ψ are formulae over Σ which are suitable for the context $\vec{x}$. The sequent is a *Horn sequent* if both φ and ψ are Horn formulae.

That a term t has sort A is indicated by $t : A$. Each term t over the signature Σ has a unique sort which is denoted $\sigma(t)$. For a sequence of terms $\vec{t} = t_1, \ldots, t_n$ define $\sigma(\vec{t}) = \sigma(t_1), \ldots, \sigma(t_n)$. Two sequences of terms $\vec{s}$ and $\vec{t}$ are *sort compatible* if $\sigma(\vec{s}) = \sigma(\vec{t})$.

We shall introduce some further notation for sequents. A *bisequent* $\varphi \vdash\!\!\dashv^{\vec{x}} \psi$ is understood as the conjunction of the sequents $\varphi \vdash^{\vec{x}} \psi$ and $\psi \vdash^{\vec{x}} \varphi$. If it occurs in a theory, it really means that these two sequents are in the theory. We use

$$\varphi_1, \ldots, \varphi_n \vdash^{\vec{x}} \psi \tag{2}$$

or even $\vec{\varphi} \vdash^{\vec{x}} \psi$ as an abbreviation of $\varphi_1 \wedge \cdots \wedge \varphi_n \vdash^{\vec{x}} \psi$. Thus for $n = 1$ the notation is consistent with the old one. For $n = 0$ then sequent (2) is $\vdash^{\vec{x}} \psi$, and means $\top \vdash^{\vec{x}} \psi$.

Definition 2.1 The basic rules and axioms of *partial Horn logic* (PHL) are the following, grouped into (a) structural rules, (b) equality rules and (c) conjunctive rules.

(a1) *Identity axiom:* for a formula φ suitable for $\vec{x}$:

$$\varphi \vdash^{\vec{x}} \varphi$$

(a2) *Cut rule:* for formulae φ, ψ, θ suitable for $\vec{x}$

$$\frac{\varphi \vdash^{\vec{x}} \psi \qquad \psi \vdash^{\vec{x}} \theta}{\varphi \vdash^{\vec{x}} \theta}$$

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(a3) *Partial term substitution rule:* For $\vec{t} = t_1, \ldots, t_n$ a sequence of terms, sort compatible with the context $\vec{x}$, and whose free variables are among $\vec{y}$, the rule

$$\frac{\varphi \vdash^{\vec{x}} \psi}{\vec{t}\downarrow \wedge \; \varphi(\vec{t}/\vec{x}) \vdash^{\vec{y}} \psi(\vec{t}/\vec{x})}$$

is applicable. Here $\vec{t}\downarrow$ is the conjunction $t_1\downarrow \wedge \cdots \wedge t_n\downarrow$.

The rules for equality are the following

(b1) *Reflexivity axiom:* $\top \vdash^{\vec{x}} x_k\downarrow$, where x_k is some variable in the context $\vec{x}$.

(b2) *Equality axiom:* $\vec{x} = \vec{y} \wedge \varphi \vdash^{\vec{z}} \varphi(\vec{y}/\vec{x})$. Here $\vec{x} = x_1, \ldots, x_n$ are distinct variables, and the variables $\vec{y} = y_1, \ldots, y_n$ are distinct, and sort compatible with $\vec{x}$. The expression $\vec{x} = \vec{y}$ is short for $x_1 = y_1 \wedge \cdots \wedge x_n = y_n$.

(b3) *Strictness of predicates axiom:* $R(t_1, \ldots, t_n) \vdash^{\vec{x}} t_k\downarrow$, for any predicate symbol R and terms $t_1, \ldots, t_n$ with free variables in the context $\vec{x}$.

(b4) *Strictness of equality axiom:* $t_1 = t_2 \vdash^{\vec{x}} t_k\downarrow$, for $k = 1, 2$ and terms t_1, t_2 with free variables in the context $\vec{x}$.

(b5) *Strictness of functions axiom:* $f(t_1, \ldots, t_n)\downarrow \vdash^{\vec{x}} t_k\downarrow$, for any function symbol f and terms $t_1, \ldots, t_n$ with free variables in the context $\vec{x}$.

The conjunctive rules are, for formulae φ, ψ, θ suitable for $\vec{x}$:

(c1-3) $\qquad\qquad \varphi \vdash^{\vec{x}} \top \qquad \varphi \wedge \psi \vdash^{\vec{x}} \varphi \qquad \varphi \wedge \psi \vdash^{\vec{x}} \psi$

(c4)

$$\frac{\varphi \vdash^{\vec{x}} \psi \qquad \varphi \vdash^{\vec{x}} \theta}{\varphi \vdash^{\vec{x}} \psi \wedge \theta}.$$

Remark 2.2 The only difference from Definition D1.1.5(a) in [Joh02b] is the rule (a3). In D1.1.5(b) the reflexivity rule (b1) has been modified to hold in any context. In a usual logic, substitution could be applied to (b1) to infer reflexivity for arbitrary terms. Our (a3) does not allow this, however, and we add explicit strictness axioms (b3) – (b5). The conjunctive rules above are the same as in D1.1.5(c).

Andréka and Németi [AN83, Section 4] present a similar logic, which is however one-sorted and lacks the labelling of sequents with variables, which is necessary in order to deal with empty carriers and give a clean statement of the initial model theorem.

Remark 2.3 We shall below consider the extension of PHL to partial first order logic (PFOL). This consists of the rules of PHL together with the rules for the logical constants $\bot$, $\lor$, $\longrightarrow$, $\forall$ and $\exists$ as in Definition D1.3.1(d)–(g) of [Joh02b]. No term substitution occurs in those quantifier rules. It is straightforward then to derive these rules (with obvious provisos) in PFOL

$$\frac{\varphi \vdash^{\vec{x}} t\downarrow \qquad \varphi \vdash^{\vec{x}} (\forall y \in A)\psi}{\varphi \vdash^{\vec{x}} \psi(t/y)} \qquad \frac{\varphi \vdash^{\vec{x}} t\downarrow \qquad \varphi \vdash^{\vec{x}} \psi(t/y)}{\varphi \vdash^{\vec{x}} (\exists y \in A)\psi}.$$

We refer to [Bee85], [Bee86] and [Sco79] for disscusions of alternative formulations of partial first order logic. The system in [Bee86] is essentially our PFOL, with the minor difference that all constants are assumed to be defined there. An axiom scheme for equality is

$$\text{(BAx)} \qquad s \cong t \land \varphi(s/z) \vdash^{\vec{x}} \varphi(t/z)$$

where $s \cong t$ is $(s\downarrow \longrightarrow s = t) \land (t\downarrow \longrightarrow s = t)$. This scheme with $=$ in place of $\cong$ follows readily in PFOL from (b2), (a3) and (b4). However also the stronger (BAx) can be proved in PFOL by an easy induction on φ. For atomic φ one makes a distinction as to whether z occurs in φ. If so, the strictness axioms (b3-5) can be applied to show that s is defined. Then the case follows by the weaker scheme.

For full first order logic it is always possible to interpret partial operations as relations, and introduce composition of partial operation by a suitable translation. This is a method common when formalising recursion theory in arithmetic and also the one employed in Feferman's systems of explicit mathematics (cf. [Bee85]).

A *Horn theory* $\mathbb{T}$ in a signature Σ is a set of Horn sequents over Σ. In case the signature is devoid of predicate symbols the theory is called *quasi-equational*.

Example 2.4 The quasi-equational theory $\mathbb{T}_{\text{cat}}$ for a category: The sorts are objects and arrows:

$$S = \{\text{obj}, \text{arr}\}.$$

There are three unary operations (identity, domain and codomain)

$$\text{id} : \text{obj} \longrightarrow \text{arr}, \qquad \text{d}, \text{c} : \text{arr} \longrightarrow \text{obj}$$

and a binary composition operation

$$\circ : \text{arr} \times \text{arr} \longrightarrow \text{arr}$$

The domains of definitions are specified by the four (bi)sequents

$$\vdash^{x} \text{id}(x)\downarrow, \qquad \vdash^{f} \text{d}(f)\downarrow, \qquad \vdash^{f} \text{c}(f)\downarrow$$

$$(f \circ g)\downarrow \dashv\vdash^{f,g} \text{d}(f) = \text{c}(g)$$

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Thus the unary operations are total. Further axioms are

$$(f \circ g)\downarrow \vdash\!\!\!\frac{f,g}{}\!\!\!\!\!\! \mathsf{d}(f \circ g) = \mathsf{d}(g)$$

$$(f \circ g)\downarrow \vdash\!\!\!\frac{f,g}{}\!\!\!\!\!\! \mathsf{c}(f \circ g) = \mathsf{c}(f)$$

$$(f \circ (g \circ h))\downarrow \vdash\!\!\!\frac{f,g,h}{}\!\!\!\!\!\! f \circ (g \circ h) = (f \circ g) \circ h$$

$$\vdash\!\!\!\frac{x}{}\!\!\!\!\!\! \mathsf{c}(\mathsf{id}(x)) = x, \qquad \vdash\!\!\!\frac{x}{}\!\!\!\!\!\! \mathsf{d}(\mathsf{id}(x)) = x$$

$$\vdash\!\!\!\frac{f}{}\!\!\!\!\!\! f \circ \mathsf{id}(\mathsf{d}(f)) = f, \qquad \vdash\!\!\!\frac{f}{}\!\!\!\!\!\! \mathsf{id}(\mathsf{c}(f)) \circ f = f \qquad \square$$

Example 2.5 Quasi-equational theory $\mathbb{T}_{\mathrm{gr}}$ for a graph: The sorts are $\{\mathrm{obj}, \mathrm{arr}\}$ and the operations for source and target are denoted $\mathsf{d}, \mathsf{c} : \mathrm{arr} \longrightarrow \mathrm{obj}$. The only axioms state that these are total

$$\vdash\!\!\!\frac{f}{}\!\!\!\!\!\! \mathsf{d}(f)\downarrow, \qquad \vdash\!\!\!\frac{f}{}\!\!\!\!\!\! \mathsf{c}(f)\downarrow \qquad \square$$

A *PHL derivation relative to* $\mathbb{T}$ is a derivation, in which all sequents are Horn sequents over Σ, and which is obtained by following the rules of PHL, and whose axioms are either those of the logic, or belong to $\mathbb{T}$. The last sequent of such a derivation is a *PHL-theorem* of $\mathbb{T}$.

Proposition 2.6 *In PHL the following* context weakening rule *is derivable: For $\vec{y}$ a sequence of distinct variables including $\vec{x}$,*

$$\frac{\varphi \vdash\!\!\!\frac{\vec{x}}{}\!\!\!\!\!\! \psi}{\varphi \vdash\!\!\!\frac{\vec{y}}{}\!\!\!\!\!\! \psi}. \tag{3}$$

Proof. Applying the partial term substitution to the premiss of (3) gives

$$\vec{x}\downarrow \wedge \; \varphi(\vec{x}/\vec{x}) \vdash\!\!\!\frac{\vec{y}}{}\!\!\!\!\!\! \psi(\vec{x}/\vec{x}). \tag{4}$$

Repeated use of (b1) and (c4) gives $\top \vdash\!\!\!\frac{\vec{y}}{}\!\!\!\!\!\! \vec{x}\downarrow$. Then using cut and conjunctive rules the conclusion of (3) follows from (4). $\square$

The conjunctive rules and cut yields the following general permutation–weakening rule for formulae as a derived rule of PHL:

$$\frac{\varphi_1, \ldots, \varphi_n \vdash\!\!\!\frac{\vec{x}}{}\!\!\!\!\!\! \psi}{\theta_1, \ldots, \theta_m \vdash\!\!\!\frac{\vec{x}}{}\!\!\!\!\!\! \psi} \tag{5}$$

whenever $\{\varphi_1, \ldots, \varphi_n\} \subseteq \{\top, \theta_1, \ldots, \theta_m\}$ and $\theta_1, \ldots, \theta_m$ suitable for $\vec{x}$.

The Equality Axiom (b2) shows how if $x = y$ then all occurrences of x in φ can be replaced by y. It is often useful to know that we can choose to replace just a selection of the occurrences of x. That is the content of the next lemma.

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Lemma 2.7 *Let $\vec{w}$ be a context that includes x and y with the same sort, and let φ be a formula suitable for $\vec{w}$. Then*

$$x = y \wedge \varphi(x/y) \xvdash{\vec{w}} \varphi.$$

Proof. Let z be a variable not in $\vec{w}$ and of the same sort as x and y. Then

$$z = y \wedge \varphi(z/y) \xvdash{\vec{w},z} \varphi(z/y)(y/z)$$

and the right-hand side is just φ (because z is not free in φ). Substituting x for z, we obtain

$$x\downarrow \wedge\, x = y \wedge \varphi(z/y)(x/z) \xvdash{\vec{w}} \varphi(x/z)$$

which then gives the result. □

Proposition 2.8 *Relative to the empty theory in any signature, PHL proves the following. r, s and t are terms in context $\vec{x}$.*

(i) $r = s \wedge s = t \xvdash{\vec{x}} r = t$.

(ii) $r = s \xvdash{\vec{x}} s = r$.

(iii) Let $\vec{u}$ and $\vec{v}$ be vectors of terms in context $\vec{y}$, with sorts compatible with $\vec{x}$. Then

$$\vec{u} = \vec{v} \wedge t(\vec{u}/\vec{x})\downarrow \xvdash{\vec{y}} t(\vec{u}/\vec{x}) = t(\vec{v}/\vec{x}).$$

Proof. As an illustration we prove (iii). We apply Lemma 2.7 to the formula $t = t(\vec{x'}/\vec{x})$ and deduce

$$\vec{x} = \vec{x'} \wedge t\downarrow \xvdash{\vec{x},\vec{x'}} t = t(\vec{x'}/\vec{x}).$$

The result now follows if we substitute $\vec{u}/\vec{x}$ and $\vec{v}/\vec{x'}$ and use the fact that $\vec{u} = \vec{v}$ implies that both $\vec{u}$ and $\vec{v}$ are defined. □

For a signature Σ, let $\mathrm{Tot}(\Sigma)$ be the theory consisting of all the axioms

$$\top \xvdash{\vec{x}} f(\vec{x})\downarrow$$

where f is a function symbol of Σ. This theory expresses that all functions are total and constants are defined. It is easy to prove by induction that $\top \xvdash{\vec{x}} t\downarrow$ is a PHL-theorem of $\mathrm{Tot}(\Sigma)$, for any Σ-term t. Under the assumption of these totality axioms PHL becomes ordinary Horn logic, since (b3-b5) are trivially true and the proviso of (a3) may removed. More precisely we have:

Theorem 2.9 *Let $\mathbb{T}$ be a Horn theory over the signature Σ. Then a sequent σ is PHL-theorem of $\mathbb{T} \cup \mathrm{Tot}(\Sigma)$ iff σ is a Horn logic theorem of $\mathbb{T}$.* □

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The following two results are usually known as the Theorem on Constants and the Deduction Theorem [Sho67].

Theorem 2.10 *Let $\mathbb{S}$ be a Horn theory over some signature Σ. Let c be a constant not occurring in $\mathbb{S}$. Then following are equivalent*

(i) $\varphi \vdash^{\vec{x}} \psi$ *is a PHL-theorem of* $\mathbb{S} \cup \{\top \vdash c\downarrow\}$

(ii) $\varphi(z/c) \vdash^{\vec{x},z} \psi(z/c)$ *is a PHL-theorem of* $\mathbb{S}$, *for any variable z of the same sort as c, and not in $\vec{x}$.*

In addition, the above equivalence holds for PFOL-theorems and arbitrary first-order theories $\mathbb{S}$.

Proof. That (ii) implies (i) follows by a simple application of partial substitution and cut.

The implication (i) $\Rightarrow$ (ii) is proved by induction on the height of derivations. The logical axioms are checked by noting they are logical axioms also when making the substitution of z for c. The non-logical axioms are verified by observing that the substitution has no effect on them, since c is not in $\mathbb{S}$. The two binary rules are straightforward to verify using the inductive hypothesis. The partial term substitution rule is verified as follows: suppose it was the last rule applied as the instance

$$\frac{\varphi \vdash^{\vec{x}} \psi}{\vec{t}\downarrow \wedge\ \varphi(\vec{t}/\vec{x}) \vdash^{\vec{y}} \psi(\vec{t}/\vec{x})},$$

where the substituted sequence is $\vec{t} = t_1, \ldots, t_n$. Consider any z of the same sort as c and not in $\vec{y}$. Let u be some variable of the same sort as c and not in $\vec{x}, \vec{y}, z$. We may then invoke the inductive hypothesis, that

$$\varphi(u/c) \vdash^{\vec{x},u} \psi(u/c) \tag{6}$$

is PHL-theorem of $\mathbb{S}$. Let $\vec{s} = \vec{t}(z/c)$. Now make the substitution $(\vec{s}, z/\vec{x}, u)$ into (6) and get

$$\vec{s}\downarrow \wedge z\downarrow \wedge \varphi(u/c)(\vec{s}, z/\vec{x}, u) \vdash^{\vec{y},z} \varphi(u/c)(\vec{s}, z/\vec{x}, u)$$

Since z is a variable the conjunct $z\downarrow$ may be eliminated using the reflexivity axiom. We have the syntactic equality $\vec{s}\downarrow \equiv (\vec{t}\downarrow)(z/c)$. Moreover since z is not in $\vec{x}$, we may interchange some substitutions

$$\varphi(u/c)(\vec{s}, z/\vec{x}, u) \equiv \varphi(z/c)(\vec{s}/\vec{x}) \equiv \varphi(\vec{t}/\vec{x})(z/c),$$

and similarly $\psi(u/c)(\vec{s}, z/\vec{x}, u) \equiv \psi(\vec{t}/\vec{x})(z/c)$.

We thus have the desired PHL-theorem

$$(\vec{t}\downarrow)(z/c) \wedge \varphi(\vec{t}/\vec{x})(z/c) \vdash^{\vec{y},z} \psi(\vec{t}/\vec{x})(z/c)$$

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of $\mathbb{S}$. This completes the inductive step.

To extend the result to PFOL is straightforward for the propositional connectives, and for quantifiers one need to apply the variable rule to permute and rename variables. □

Theorem 2.11 *Let $\mathbb{S}$ be a Horn theory over some signature Σ. Let θ be a closed Horn formula over Σ. The following are equivalent:*

(i) $\varphi \vdash\!\!\!\overset{\vec{x}}{\rule{2em}{0pt}}\!\!\! \psi$ is a PHL-theorem of $\mathbb{S} \cup \{\top \vdash\!\!\!\rule{2em}{0pt}\!\!\! \theta\}$,

(ii) $\theta \wedge \varphi \vdash\!\!\!\overset{\vec{x}}{\rule{2em}{0pt}}\!\!\! \psi$ is a PHL-theorem of $\mathbb{S}$.

Moreover, this equivalence holds for PFOL-theorems and arbitrary first-order theories $\mathbb{S}$ and closed formulae θ over Σ.

Proof. (i) $\Rightarrow$ (ii) is proved by a straightforward induction on the height of derivation. (ii) $\Rightarrow$ (i) follows by a simple application of cut and conjunctive axioms. □

For any Horn theory $\mathbb{T}$ over the signature Σ, we get as an immediate corollary that: the PHL-provability of $\varphi \vdash\!\!\!\overset{\vec{z}}{\rule{2em}{0pt}}\!\!\! \psi$ from $\mathbb{T}$ is equivalent to the PHL-provability of $\top \vdash\!\!\!\rule{2em}{0pt}\!\!\! \psi(\vec{c}/\vec{z})$ from $\mathbb{T} \cup \{\top \vdash\!\!\!\rule{2em}{0pt}\!\!\! \vec{c}{\downarrow}\} \cup \{\top \vdash\!\!\!\rule{2em}{0pt}\!\!\! \varphi(\vec{c}/\vec{z})\}$, where $\vec{c}$ is sequence of distinct constants not in $\mathbb{T}$ and sort compatible with $\vec{z}$.

3 Structures, models and homomorphisms in PHL

In this section we show how PHL is interpreted using set-theoretic models.

Let Σ be a many-sorted first-order signature. We write $f : A_1 \cdots A_n \longrightarrow B$ for f being a function symbol with values in B and with n arguments of respective sorts $A_1, \ldots, A_n$. For $n = 0$ it is a constant of sort B. Writing $R \rightarrowtail A_1 \cdots A_n$ indicates that R is a predicate symbol with n arguments of sorts in $A_1, \ldots, A_n$, respectively. In addition to the predicates in Σ, formulae over Σ may be formed using binary equality predicates $(=_A) \rightarrowtail AA$ for each sort A of Σ.

Like the system presented in [Joh02b] (though unlike most classical model theory) we permit sorts to be modelled by empty sets. The contexts labelling the sequent turnstiles ensure that the logic can handle empty carriers correctly, since they mean that elements of the carriers can be explicitly *hypothesized* instead of being presumed always to exist. (This device is due to Mostowski – see [LS86].)

Where we differ from [Joh02b] is that each function symbol will be interpreted by a partial function. This is why we need the partial substitution rule. A *partial function* $f : A \rightharpoonup B$ is a relation $f \subseteq A \times B$ such that

$$(x, y), (x, z) \in f \implies y = z. \tag{7}$$

We employ standard notation for application and take "$f(a)$ *defined*" to mean that there is some $b \in B$ with $(a, b) \in f$. The restriction $f|S$ of f to a subset $S \subseteq A$ is

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defined as the graph $f \cap (S \times B)$. Equality of partial functions means equality of their graphs. Composition of partial functions $A \xrightarrow{f} B \xrightarrow{g} C$ is defined as composition of relations. Thus $(g \odot f)(a)$ is defined iff both $f(a)$ and $g(f(a))$ are defined, and in such case $(g \odot f)(a) = g(f(a))$. Note that partial functions $A \rightharpoonup 1$ are in a 1-1 correspondence with subsets of A.

Remark 3.1 We make a short digression (this may preferably be skipped on first reading) about subsets in constructive systems which will motivate the categorical version of partial map, using monics to represent subobjects, that will be used later in Section 7. In constructive mathematics as in [BB85] subsets are defined in a quite categorical way and suitable to formalisation in type theories. Let X be a set (a type with a prescribed equivalence relation). A subset A of X is a set ∂A together with an injection $\iota_A : \partial A \longrightarrow X$. An element x of X is a member of this subset A if $x = \iota_A(d)$ for some (necessarily) unique $d \in \partial A$. Then define the subset relation between subsets using logic, as for subclasses in set theory. It is easy to see that $A \subseteq B$ is equivalent to there being a function (necessarily unique and injective) $\iota_{A,B} : \partial A \longrightarrow \partial B$ so that $\iota_B \circ \iota_{A,B} = \iota_A$. Now two subsets A and B of X are equal if $\iota_{A,B}$ is a bijection satisfying the same equation.

Next we see how this gives an alternative definition of partial functions in terms of arrows only. A subset R of $A \times B$ is represented by a pair $\iota_R = \langle r_1, r_2 \rangle : \partial R \longrightarrow A \times B$. It is now easy to check that for any $r_1 : \partial R \longrightarrow A$, $r_2 : \partial R \longrightarrow B$, the pair $\langle r_1, r_2 \rangle$ is injective and the corresponding relation R satisfies the univalency condition (7) if, and only if, the function r_1 is injective. Thus a partial function f from A to B can be taken to be a *span* from A to B – a triple (D_f, d_f, m_f) where $d_f : D_f \longrightarrow A$, $m_f : D_f \longrightarrow B$ – whose first leg d_f is injective. The subset $d_f : D_f \longrightarrow A$ is called the *domain of definition* of f. Such a partial function is included in another partial function $g = (D_g, d_g, m_g)$ as a subset if, and only if, there is a function $i : D_f \longrightarrow D_g$ (necessarily unique and an injection) so that $d_g \circ i = d_f$ and $m_g \circ i = m_f$. These partial functions are equal as subsets if this i is an isomorphism. For $f : A \rightharpoonup B$, $g : B \rightharpoonup C$ with $f = (D_f, d_f, m_f)$ and $g = (D_g, d_g, m_g)$ their relational composition $g \odot f$ may be constructed as

$$
\begin{aligned}
D_{g \odot f} &= \{(u, v) \in D_f \times D_g : m_f(u) = d_g(v)\}, \\
d_{g \odot f}(u, v) &= d_f(u), \\
m_{g \odot f}(u, v) &= m_g(v).
\end{aligned}
$$

This construction is in effect a pullback construction, compare to (19) below.

A *partial structure M for Σ* is an assignment M, which to each sort A in Σ assigns a set $M A$, and to each sequence $\vec{A}$ of sorts assigns the set $M(\vec{A}) = M A_1 \times \cdots \times M A_n$. If $n = 0$ this is a canonical 1-element set $\{*\}$ (a fixed terminal object in the category of sets). To each function symbol $f : \vec{A} \longrightarrow B$ in Σ a partial function $M f : M(\vec{A}) \rightharpoonup M B$

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is assigned, and to each predicate symbol $R \rightarrowtail \vec{A}$ in Σ a subset $M R$ of $M(\vec{A})$. Moreover, $M (=_A)$ is the identity relation on $M A$. Let M and N be partial structures for Σ.

For each term $\vec{x}.t$ we define a partial function

$$[\![\vec{x}.t]\!]_M : M \sigma(\vec{x}) \rightharpoonup M \sigma(t). \tag{8}$$

by induction. A variable is interpreted by a projection (which is a total function)

$$[\![\vec{x}.x_k]\!]_M = \pi_k^n : M(\sigma(\vec{x})) \longrightarrow M \sigma(x_k). \tag{9}$$

For a constant c let

$$[\![\vec{x}.c]\!]_M = (M\,c) \odot\, !_{M(\sigma(\vec{x}))} \tag{10}$$

where the second factor indicates the unique map $M(\sigma(\vec{x})) \longrightarrow 1$. For a sequence of partial functions $f_i : A \rightharpoonup B_i$ $(1 \leq i \leq m)$, define the strict tupling

$$\langle \vec{f} \rangle_p : A \rightharpoonup B_1 \times \cdots \times B_m$$

by letting $\langle \vec{f} \rangle_p(a)$ be defined iff each $f_k(a)$ is defined, and if so, then set to $(f_1(a), \ldots, f_m(a))$. Note that $\langle \rangle_p$ may be identified with the unique map $A \longrightarrow 1$. We write

$$[\![\vec{x}.\vec{t}]\!]_M \triangleq \langle [\![\vec{x}.t_1]\!]_M \ldots [\![\vec{x}.t_m]\!]_M \rangle_p$$

and now the interpretation of a function symbol g applied to terms of appropriate sorts is defined as

$$[\![\vec{x}.g(\vec{t})]\!]_M \triangleq (M\,g) \odot [\![\vec{x}.\vec{t}]\!]_M. \tag{11}$$

A formula φ (suitable for $\vec{x}$) will in general be interpreted as $[\![\vec{x}.\varphi]\!]_M \subseteq M(\vec{A})$. For a sequence of terms $\vec{t}$ we now define $[\![\vec{x}.R(\vec{t})]\!]_M$ to be the set of tuples

$$\{\vec{a} \in M(\vec{A}) : \ [\![\vec{x}.\vec{t}]\!]_M(\vec{a}) \text{ defined and belongs to } M\,R\}. \tag{12}$$

Moreover $[\![\vec{x}.t_1 = t_2]\!]_M$ is the set

$$\{\vec{a} \in M(\vec{A}) : \ [\![\vec{x}.t_1]\!]_M(\vec{a}) \text{ and } [\![\vec{x}.t_2]\!]_M(\vec{a}) \text{ are defined and equal }\}. \tag{13}$$

(Note that this is consistent with (12) if $M R = M (=_A)$.) Thus in particular $[\![\vec{x}.t \downarrow]\!]_M$ consists of those $\vec{a}$ for which $[\![\vec{x}.t]\!]_M(\vec{a})$ is defined. Further, we have that $[\![\vec{x}.t_1]\!]_M$ and $[\![\vec{x}.t_2]\!]_M$ are equal as partial functions iff $[\![\vec{x}.t_1 \downarrow]\!]_M = [\![\vec{x}.t_2 \downarrow]\!]_M = [\![\vec{x}.t_1 = t_2]\!]_M$. The interpretation may be extended to all first-order formulae over Σ as in [Joh02b].

We say that a sequent $\varphi \overset{\vec{x}}{\vdash\!\!\!-\!\!\!-} \psi$ over Σ is *valid* in a partial structure M for Σ if

$$[\![\vec{x}.\varphi]\!]_M \subseteq [\![\vec{x}.\psi]\!]_M.$$

Such a structure M *is a model for theory* $\mathbb{T}$ over Σ, if each sequent in $\mathbb{T}$ is valid in M. In this case we write, as usual, $M \models \mathbb{T}$.

Continuing Examples 2.4 and 2.5 we have

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Example 3.2 The models of $\mathbb{T}_{\mathrm{cat}}$ and $\mathbb{T}_{\mathrm{gr}}$ are the categories and directed graphs respectively.

We need the following lemmas for interpreting substitutions:

Lemma 3.3 *Let M be a partial Σ-structure. Suppose we are given Σ-terms $\vec{x}.s$ and $\vec{y}.\vec{t}$, where $\vec{t}$ is sort compatible with $\vec{x}$. Then:*

$$[\![\vec{y}.s(\vec{t}/\vec{x})]\!]|[\![\vec{y}.\vec{t}\downarrow]\!] = [\![\vec{x}.s]\!] \odot [\![\vec{y}.\vec{t}]\!],$$

writing $[\![\cdot]\!]$ for $[\![\cdot]\!]_M$.

Proof. By induction on the structure of the term s, and using the following laws for partial functions

$$
\begin{aligned}
\pi_k^n \odot \langle f_1, \ldots, f_n \rangle_p &= f_k | \mathrm{d}_{f_1} \cap \cdots \cap \mathrm{d}_{f_n} \\
(f \odot g)|S &= f \odot (g|S) \\
\langle f_1, \ldots, f_n \rangle_p | S &= \langle f_1 | S, \ldots, f_n | S \rangle_p | S \\
\langle f_1, \ldots, f_n \rangle_p \odot g &= \langle f_1 \odot g, \ldots, f_n \odot g \rangle_p | \mathrm{d}_g. \qquad \square
\end{aligned}
$$

Lemma 3.4 *Let M be a partial Σ-structure. Consider any first-order formula $\vec{x}.\varphi$ over the signature Σ. For any sequence of Σ-terms $\vec{y}.\vec{t}$, sort compatible with $\vec{x}$, the set $[\![\vec{y}.(\vec{t}\downarrow \wedge \varphi(\vec{t}/\vec{x}))]\!]_M$ is*

$$\{\vec{a} \in M(\sigma(\vec{y})) : \ [\![\vec{y}.\vec{t}]\!]_M(\vec{a}) \ \text{defined and belongs to} \ [\![\vec{x}.\varphi]\!]_M\}.$$

Proof. By induction on the structure of the formula. The inductive steps are straightforward using a little intuitionistic logic for the implicational case. The base case for non-constant atomic fomulae $\varphi \equiv R(\vec{s})$ follows straightforwardly from Lemma 3.3. The base cases for $\top$ and $\bot$ are trivial. $\square$

Theorem 3.5 *(Soundness) PHL and PFOL are sound for the semantics above.*

Proof. The only nonstandard rules to verify are the partial substitution rule (a3) and the strictness rules (b3–5). The rule (a3) is evidently valid since by Lemma 3.4, $[\![\vec{x}.\varphi]\!]_M \subseteq [\![\vec{x}.\psi]\!]_M$ implies

$$[\![\vec{y}.\vec{t}\downarrow \wedge \ \varphi(\vec{t}/\vec{x})]\!]_M \subseteq [\![\vec{y}.\vec{t}\downarrow \wedge \ \psi(\vec{t}/\vec{x})]\!]_M \subseteq [\![\vec{y}.\psi(\vec{t}/\vec{x})]\!]_M.$$

The strictness axioms (b3-5) are easily verified from the definitions above. $\square$

A Horn theory in a signature with no predicate symbols is called *quasi-equational*. The following result shows that this is no great restriction when setting up a theory:

Proposition 3.6 *For any Horn theory $\mathbb{T}$ there is an equivalent quasi-equational theory.*

Proof. First, introduce a new sort U with constant $u : U$ and axioms $\top \vdash\!\!\!\!- u\!\downarrow$ and $\top \vdash\!\!\!\overset{x}{\;}\!\!- x = u$. This guarantees that U is interpreted as a one-element set, so adding it makes no essential difference to $\mathbb{T}$. Now we use the 1-1 correspondence between subsets and partial functions into one-element sets. Each predicate $R \rightarrowtail A_1 \cdots A_n$ may be replaced by a function symbol $f_R : A_1 \cdots A_n \longrightarrow U$, and then instead of $R(t_1, \ldots, t_n)$ use $f_R(t_1, \ldots, t_n)\!\downarrow$. $\square$

3.1 Homomorphisms

If M and N are partial structures for a signature Σ, then a Σ-*homomorphism* $\alpha : M \longrightarrow N$ is a family of *total* functions

$$\alpha_S : M\,S \longrightarrow N\,S \qquad (S \text{ is a sort in } \Sigma)$$

satisfying certain conditions. We can normally omit the subscript S, and indeed for a sequence $\vec{a}$ of variables, terms or elements with sorts $\vec{S}$ we write $\alpha(\vec{a})$ for the sequence

$$\alpha_{S_1}(a_1), \ldots, \alpha_{S_n}(a_n)$$

The first condition on α is that for each function symbol $f : \vec{A} \longrightarrow B$ in Σ, and for all $\vec{a} \in M(\vec{a})$, if $(M\,f)(\vec{a})$ is defined, then so is $(N\,f)(\alpha(\vec{a}))$ and

$$\alpha((M\,f)(\vec{a})) = (N\,f)(\alpha(\vec{a})).$$

(For $m = 0$, we get the condition for constants.) Further, for each predicate symbol $R \rightarrowtail \vec{A}$ the α-functions should satisfy for all $\vec{a} \in M(\vec{A})$ that

$$\vec{a} \in (M\,R) \Longrightarrow \alpha(\vec{a}) \in (N\,R)$$

Example 3.7 For $\mathbb{T}_{\mathrm{cat}}$, where the models are the categories, the homomorphisms are the functors. Note how if $f \circ g$ is defined then so is $F(f \circ g)$ and $F(f \circ g) = F(f) \circ F(g)$. But even if $F(f) \circ F(g)$ is defined, $f \circ g$ need not be.

For $\mathbb{T}_{\mathrm{gr}}$, the homomorphisms are the directed graph morphisms.

Remark 3.8 This notion of homomorphism is also used in connection with partial metric algebras [SHT03].

The class of partial structures for Σ and Σ-homomorphisms form the *category of partial structures for* Σ denoted Σ-PStr. Note that the standard category Σ-Str of structures for Σ [Joh02b] is isomorphic to a full subcategory of Σ-Pstr. Let $\mathbb{T}$ be a theory over Σ. The full subcategory of Σ-PStr consisting of those partial structures that are models of $\mathbb{T}$ is denoted $\mathbb{T}$-PMod.

We have already seen how a partial structure can be extended from the function and predicate symbols to all terms and formulae. The same goes for a homomorphism, though not for all formulae. A formula is *coherent* if it is constructed from atomic formulae (including equations) using conjunction, disjunction and existential quantification.

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Proposition 3.9 *Let $\mathbb{T}$ be a quasi-equational theory and let M and N be partial structures for it. Let $\alpha : M \longrightarrow N$ be a homomorphism. For every sequence $\vec{S}$ of sorts, we get a product function*

$$\alpha_{\vec{S}} : M(\vec{S}) \longrightarrow N(\vec{S}).$$

Then

1. *For each term $\vec{x}.t$ we have commutative squares*

$$
\begin{array}{ccccc}
M(\sigma(\vec{x})) & \longleftarrow & [\![\vec{x}.t{\downarrow}]\!]_M & \xrightarrow{\;[\![\vec{x}.t]\!]_M\;} & M(\sigma(t)) \\
\Big\downarrow{\scriptstyle\alpha_{\sigma(\vec{x})}} & & \Big\downarrow{\scriptstyle[\![\vec{x}.t{\downarrow}]\!]_\alpha} & & \Big\downarrow{\scriptstyle\alpha_{\sigma(\vec{x})}} \\
N(\sigma(\vec{x})) & \longleftarrow & [\![\vec{x}.t{\downarrow}]\!]_N & \xrightarrow{\;[\![\vec{x}.t]\!]_N\;} & N(\sigma(t))
\end{array}
$$

   where the row spans are the interpretations of the term t in the two models.

2. *For each coherent formula $\vec{x}.\varphi$ we have a commutative square*

$$
\begin{array}{ccc}
M(\sigma(\vec{x})) & \longleftarrow & [\![\vec{x}.\varphi]\!]_M \\
\Big\downarrow{\scriptstyle\alpha_{\sigma(\vec{x})}} & & \Big\downarrow{\scriptstyle[\![\vec{x}.\varphi]\!]_\alpha} \\
N(\sigma(\vec{x})) & \longleftarrow & [\![\vec{x}.\varphi]\!]_N
\end{array}
$$

Proof. Use induction on the structure of the terms and formulae. $\square$

   With homomorphisms as we have defined them the result fails to work for negation, implication and universal quantification. (For a simple example of the last, consider the fact that a homomorphism of groups does not in general map centre to centre.)

4 The term model and completeness

We now turn to the completeness theorem, using a construction analogous to the term model of equational logic as set out in [MT92]. Note, however, that they assume sufficient constants to ensure the carriers are non-empty. This is because their equational logic is not properly adapted to empty carriers. We avoid this problem by our use of contexts of variables labelling the turnstiles as discussed in the previous section. The first completeness theorem for partial models is due to Andréka and Németi [AN83]. Their construction avoided the labelling of the sequents, but the theorem was restricted to single-sorted theories.

4.1 The term model

In Section 9 we shall show that quasi-equational theories are equivalent to cartesian theories. This implies that there should be various free model constructions. These are best known for algebraic theories, but work also for cartesian theories (even with infinitary operations) and are discussed in [BW84, Chapter 4]. The most fundamental of these is the Initial Model Theorem – every cartesian theory has an initial model –, and the others can be proved from it. That is the term model which we now construct. Its initiality encapsulates important uses of induction on the structure of terms and proofs.

The proof, by taking all terms and factoring out a partial congruence of provable equality, is hardly any more complicated in our partial Horn logic than the usual proof for algebraic theories. Nonetheless, it is constructive and predicative – unlike the discussion in [BW84], which uses the Adjoint Functor Theorem.

Let $\mathbb{T}$ be a quasi-equational theory in the signature Σ. Define the closed term model $\mathrm{Ter}(\mathbb{T}) = M$ as follows. For each sort S in Σ, let $M_0\,S$ be the set of its closed terms. (Note that this set may well be empty.) M_0 is a (total) structure for Σ.

Now we define a partial equivalence relation $=_S$ on each $M_0\,S$ by provable equality: $t_1 =_S t_2$ if $\vdash\!\!\!\!-\!\!\!- t_1 = t_2$ is a PHL-theorem of $\mathbb{T}$.

In fact, the family of partial equivalence relations $=_S$ is a partial congruence on M_0. If $\vec{t_1}$ and $\vec{t_2}$ are sequences of closed terms sort compatible with the arguments of a function symbol f, and if $\vdash\!\!\!\!-\!\!\!- \vec{t_1} = \vec{t_2}$ is a PHL-theorem of $\mathbb{T}$, then by Proposition 2.8.(iii) so is $\vdash\!\!\!\!-\!\!\!- f(\vec{t_1}) = f(\vec{t_2})$.

We deduce that factoring out the partial congruence on the structure M_0 gives a partial structure M. Its carriers $M\,S$ are

$$\{t \text{ is a closed } \Sigma\text{-term} \ : \ \vdash\!\!\!\!-\!\!\!- t\!\downarrow \text{ is a PHL-theorem of } \mathbb{T}\}.$$

with equality

$$M(=_S) = \{(t_1, t_2) \in M\,S \times M\,S : \vdash\!\!\!\!-\!\!\!- t_1 = t_2 \text{ is a PHL-theorem of } \mathbb{T}\}.$$

This is an equivalence relation on $M\,S$. It defines the partial Σ-structure $\mathrm{Ter}(\mathbb{T})$. We show by structural induction that for closed Σ-terms t of sort S:

$$t \in M(S) \implies [\![().t]\!](*) \text{ defined and } =_{M\,S} t. \tag{14}$$

Here we are writing $[\![\cdot]\!]$ for $[\![\cdot]\!]_M$. For a constant $t = c$

$$[\![().c]\!] = (M\,c) \odot\, !_{M()}.$$

and hence $[\![().c]\!](*) = c$, when $c \in M(S)$. More generally, for $f(\vec{t})$ with $f : \sigma(\vec{t}) \longrightarrow S$,

$$[\![().f(\vec{t})]\!] = (Mf) \odot [\![().\vec{t}]\!].$$

Now if $f(\vec{t}) \in M(S)$, then by the strictness axiom (b5) we also have $t_k \in M(\sigma(t_k))$ for each component t_k of $\vec{t}$. By inductive hypothesis $[\![().t_k]\!](*)$ is defined and equal to t_k. Thus $[\![().\vec{t}\,]\!](*)$ is defined and equals $\vec{t}$, and since $f(\vec{t}) \in M(S)$ also $[\![().f(\vec{t})]\!](*)$ is indeed defined and equals $f(\vec{t})$.

The crucial lemma for the term model is the following

Lemma 4.1 *Let* $M = \mathrm{Ter}(\mathbb{T})$*, where* $\mathbb{T}$ *is a quasi-equational theory over* Σ*. For any* Σ*-equation* $\vec{x}.\varphi$*,*

$$[\![\vec{x}.\varphi]\!]_M = \{\vec{t} \in M(\sigma(\vec{x})) : \top \longmapsto \varphi(\vec{t}/\vec{x}) \text{ is a PHL-theorem of } \mathbb{T}\}. \qquad (15)$$

Proof. Let $\varphi = (r_1 =_A r_2)$ be an equation in context $\vec{x}$. Consider $\vec{t}$ in $M(\sigma(\vec{x}))$. We write just $[\![\cdot]\!]$ for $[\![\cdot]\!]_M$ below. >From Lemma 3.3 follows for $j = 1, 2$ that

$$[\![().r_j(\vec{t}/\vec{x})]\!] | [\![().\vec{t} \downarrow]\!] = [\![\vec{x}.r_j]\!] \odot [\![().\vec{t}]\!]. \qquad (16)$$

Note that any closed term is suitable for the empty context. We check (15): If $\vec{t} \in [\![\vec{x}.\varphi]\!]$, then by (13) $[\![\vec{x}.r_1]\!](\vec{t})$ and $[\![\vec{x}.r_2]\!](\vec{t})$ are both defined and equal. Now $[\![().t_k]\!](*) = t_k$ by (14). By (16) it follows that $[\![().r_1(\vec{t}/\vec{x})]\!](*)$ and $[\![().r_2(\vec{t}/\vec{x})]\!](*)$ are defined and equal, and hence by (14)

$$r_1(\vec{t}/\vec{x}) =_{MA} r_2(\vec{t}/\vec{x}),$$

i.e. $\top \longmapsto \varphi(\vec{t}/\vec{x})$ is a PHL-theorem of $\mathbb{T}$. Conversely, assume this to be the case. Thus we get by strictness of equality that $r_1(\vec{t}/\vec{x}), r_2(\vec{t}/\vec{x}) \in MA$ and by assumption they are equal elements. Moreover (14) gives $[\![().r_1(\vec{t}/\vec{x})]\!](*)$ and $[\![().r_2(\vec{t}/\vec{x})]\!](*)$ are defined and equal. Now (16) yields that $[\![\vec{x}.r]\!](\vec{t})$ and $[\![\vec{x}.s]\!](\vec{t})$ are both defined and equal, and we have $\vec{t} \in [\![\vec{x}.\varphi]\!]$. $\square$

Theorem 4.2 *Let* $\mathbb{T}$ *be a quasi-equational theory. Then* $\mathrm{Ter}(\mathbb{T})$ *is an initial model of* $\mathbb{T}$*.*

Proof. First we must show it is indeed a model. Consider an axiom of $\mathbb{T}$

$$\varphi \longmapsto^{\vec{x}} \psi \qquad (17)$$

We wish to show that it is valid in $M = \mathrm{Ter}(\mathbb{T})$, which means that for any $\vec{t} \in M(\sigma(\vec{x}))$,

$$\vec{t} \in [\![\vec{x}.\varphi]\!]_M \Longrightarrow \vec{t} \in [\![\vec{x}.\psi]\!]_M.$$

From (17) we get by the partial substitution rule with $\vec{t}$, the PHL-theorem of $\mathbb{T}$:

$$\vec{t}\downarrow \wedge \ \varphi(\vec{t}/\vec{x}) \longmapsto \psi(\vec{t}/\vec{x}). \qquad (18)$$

By Lemma 4.1, $\vec{t} \in [\![\vec{x}.\varphi]\!]_M$ implies that $\top \longmapsto \varphi(\vec{t}/\vec{x})$ is a PHL-theorem of $\mathbb{T}$ and hence so is $\top \longmapsto \vec{t}\downarrow \wedge \ \varphi(\vec{t}/\vec{x})$, since $\vec{t}$ are all provably defined. Then the cut rule and (18) gives that $\top \longmapsto \psi(\vec{t}/\vec{x})$ is a PHL-theorem of $\mathbb{T}$, which by the lemma, is nothing but $\vec{t} \in [\![\vec{x}.\psi]\!]_M$.

For initiality, let N be another model. If $\alpha : M \longrightarrow N$ is to be a homomorphism, then by induction on the structure of terms t we see that $\alpha(t)$ can only be $[\![().t]\!]_N(*)$. This is well defined if $t \in MS$. For then $\mathbb{T}$ proves $\top \longmapsto t\downarrow$. This is therefore also valid in N, so $[\![().t]\!]_N$ is total and $[\![().t]\!]_N(*)$ exists. Similarly if $t_1 = t_2$ in M then $[\![().t_1]\!]_N(*) = [\![().t_2]\!]_N(*)$. $\square$

4.2 Completeness

We can now use the term model to prove completeness results.

Theorem 4.3 *(Strong completeness for equations) Let $\mathbb{T}$ be a quasi-equational theory over Σ. For each closed equation $r = s$ over Σ:*

$$\mathrm{Ter}(\mathbb{T}) \models r = s \ \text{ iff } \ \top \vdash\!\!\!\!-\!\!\!- r = s \text{ is a PHL-theorem of } \mathbb{T}.$$

Proof. This is the special case of Lemma 4.1 when $\varphi \equiv (r = s)$ is closed. $\square$

Theorem 4.4 *(Completeness) Let $\mathbb{T}$ be a quasi-equational theory over Σ. For each formula in context $\vec{x}.\varphi$ over Σ there is a partial Σ-structure M which is a model of $\mathbb{T}$ and such that for every formula $\vec{x}.\psi$,*

$$[\![\vec{x}.\varphi]\!]_M \subseteq [\![\vec{x}.\psi]\!]_M \ \text{ iff } \ \varphi \vdash\!\!\overset{\vec{x}}{\!-\!\!\!-} \psi \text{ is a PHL-theorem of } \mathbb{T}$$

In particular, a sequent $\varphi \vdash\!\!\overset{\vec{x}}{\!-\!\!\!-} \psi$ holds in all models of $\mathbb{T}$ iff $\varphi \vdash\!\!\overset{\vec{x}}{\!-\!\!\!-} \psi$ is a PHL-theorem of $\mathbb{T}$.

Proof. Let
$$\mathbb{T}' = \mathbb{T} \cup \{\top \vdash\!\!\!\!-\!\!\!- \vec{c}{\downarrow}\} \cup \{\top \vdash\!\!\!\!-\!\!\!- \varphi(\vec{c}/\vec{x})\},$$

where $\vec{c}$ are some distinct constants, sort compatible with $\vec{x}$, and not in Σ. Let Σ' denote Σ extended by these constants. Form the term model $M' = \mathrm{Ter}(\mathbb{T}')$ over Σ' and the restriction $M = M'|\Sigma$ to the smaller signature. For equations θ over Σ suitable for $\vec{x}$ we thus have $[\![\vec{x}.\theta]\!]_M = [\![\vec{x}.\theta]\!]_{M'}$. Suppose now $[\![\vec{x}.\varphi]\!]_M \subseteq [\![\vec{x}.\psi]\!]_M$. Then this holds also for M' in place of M. Next, using this inclusion for $\vec{c}$, we get by Lemma 4.1 the implication

$$\top \vdash\!\!\!\!-\!\!\!- \varphi(\vec{c}/\vec{x}) \text{ is a PHL-theorem of } \mathbb{T}' \implies \top \vdash\!\!\!\!-\!\!\!- \psi(\vec{c}/\vec{x}) \text{ is a PHL-theorem of } \mathbb{T}'.$$

The lefthand side is immediate by the extra axiom of $\mathbb{T}'$. Thus $\top \vdash\!\!\!\!-\!\!\!- \psi(\vec{c}/\vec{x})$ is a PHL-theorem of $\mathbb{T}'$. The Deduction Theorem 2.11 then implies that $\varphi(\vec{c}/\vec{x}) \vdash\!\!\!\!-\!\!\!- \psi(\vec{c}/\vec{x})$ is a PHL-theorem of $\mathbb{T}$. Theorem 2.10 on Constants finally gives that $\varphi \vdash\!\!\overset{\vec{x}}{\!-\!\!\!-} \psi$ is PHL-theorem of $\mathbb{T}$.

The implication ($\Rightarrow$) of the final statement of the theorem follows now by specialisation to M. The reverse implication ($\Leftarrow$) is the soundness theorem. $\square$

The corollary below shows that PFOL is "conservative" over PHL for quasi-equational sequents.

Corollary 4.5 *Consider a quasi-equational theory $\mathbb{T}$ over Σ. If a quasi-equational sequent $\varphi \vdash\!\!\overset{\vec{x}}{\!-\!\!\!-} \psi$ over Σ is a PFOL-theorem of $\mathbb{T}$, then it is also a PHL-theorem of $\mathbb{T}$.*

Proof. By soundness of PFOL this follows from Theorem 4.4. $\square$

5 Free partial models

The freeness theorem for partial models will be formulated as the existence of left adjoints to very general forgetful functors. Consider two signatures Σ and Σ', without relation symbols, since we do not need them here. A *signature morphism* $\rho : \Sigma \longrightarrow \Sigma'$ consists of an assignment: to each sort A of Σ, a sort A^ρ of Σ' and to each function symbol $f : A_1 \cdots A_n \longrightarrow A$ of Σ, a function symbol $f^\rho : A_1^\rho \cdots A_n^\rho \longrightarrow A^\rho$ of Σ'. This gives a translation of a formula φ over Σ to a formula φ^ρ over Σ' by applying ρ to its symbols for sorts and functions. The signature morphism ρ induces a *restriction* (or *reduct*) functor $(\cdot)|\rho : \Sigma'\text{-PStr} \longrightarrow \Sigma\text{-PStr}$ as follows: for a partial Σ'-structure N define $N|\rho$ to be the partial Σ-structure M given by

$$
\begin{aligned}
M(A) &= N(A^\rho) \\
(M(f) : M(A_1, \ldots A_n) \rightharpoonup M(A)) &= (N(f^\rho) : N(A_1^\rho, \ldots, A_n^\rho) \rightharpoonup N(A^\rho))
\end{aligned}
$$

For a morphism $\alpha : N \longrightarrow N'$ in $\Sigma'\text{-PStr}$ define a morphism $\alpha|\rho : N|\rho \longrightarrow N'|\rho$ in $\Sigma\text{-PStr}$ by letting

$$
(\alpha|\rho)_A = \alpha_{(A^\rho)} \qquad (A \text{ sort in } \Sigma).
$$

Naturally, any inclusion $\Sigma \subseteq \Sigma'$ of sort and function symbols will be an example of a signature morphism. In this case we write (as usual) $N|\Sigma$ for the restriction.

Example 5.1 A different example is that which maps the signature of modules Σ_{Mod} to that of rings Σ_{Rng}. The signature Σ_{Mod} has a sort K for scalars and a sort V for vectors, and appropriate function symbols, in particular multiplication of scalars with vectors $g : K, V \longrightarrow V$. The ring signature has a single sort R and a multiplication $m : R, R \longrightarrow R$. Now map $K^\rho = V^\rho = R$ and $g^\rho = m$. The restriction functor treats any ring as a module over itself.

Lemma 5.2 *Let $\rho : \Sigma \longrightarrow \Sigma'$ be a signature morphism. For $N \in \Sigma'\text{-PStr}$, and term t and formula φ over Σ suitable for $\vec{x}$,*

$$
\begin{aligned}
[\![\vec{x}.t^\rho]\!]_N &= [\![\vec{x}.t]\!]_{(N|\rho)}, \\
[\![\vec{x}.\varphi^\rho]\!]_N &= [\![\vec{x}.\varphi]\!]_{(N|\rho)}.
\end{aligned}
$$

If $\rho : \Sigma \subseteq \Sigma'$ is an inclusion, then $[\![\vec{x}.t]\!]_N = [\![\vec{x}.t]\!]_{(N|\Sigma)}$ and $[\![\vec{x}.\varphi]\!]_N = [\![\vec{x}.\varphi]\!]_{(N|\Sigma)}$.

Proof. The first equation follows by a straightforward induction on t. The base case of the second equation then follows from this. The inductive steps for formulae are trivial. $\square$

Suppose that $\mathbb{T}$ and $\mathbb{T}'$ are two theories of signature Σ and Σ', respectively. A signature morphism $\rho : \Sigma \longrightarrow \Sigma'$ is a *theory morphism* $\mathbb{T} \longrightarrow \mathbb{T}'$, if for each axiom $\varphi \vdash^{\vec{x}:\vec{A}} \psi$ of $\mathbb{T}$, the ρ-translation $\varphi^\rho \vdash^{\vec{x}:\vec{A}^\rho} \psi^\rho$ is an axiom of $\mathbb{T}'$. Note that for such a theory

morphism, the restriction functor maps $\mathbb{T}'$-PMod into $\mathbb{T}$-PMod. Suppose $N \models \mathbb{T}'$. For an axiom $\varphi \xmapsto{\;\vec{x}\;} \psi$ of $\mathbb{T}$, we have that its translation $\varphi^\rho \xmapsto{\;\vec{x}\;} \psi^\rho$ is an axiom of $\mathbb{T}'$, so by assumption $[\![\vec{x}.\varphi^\rho]\!]_N \subseteq [\![\vec{x}.\psi^\rho]\!]_N$. By Lemma 5.2 this is the same as $[\![\vec{x}.\varphi]\!]_{(N|\rho)} \subseteq [\![\vec{x}.\psi]\!]_{(N|\rho)}$. This proves $(N|\rho) \models \mathbb{T}$. We may now conclude the following result.

Proposition 5.3 *Each theory morphism $\rho : \mathbb{T} \longrightarrow \mathbb{T}'$ induces a functor — the forgetful functor —*

$$U_\rho : \mathbb{T}'\text{-}PMod \longrightarrow \mathbb{T}\text{-}PMod$$

given by $U_\rho = (\cdot)|\rho$. $\square$

Our aim now is to prove that U_ρ has a left adjoint, a *free model functor* F_ρ. We shall construct $F_\rho(M)$ as the initial partial model for a theory in which $\mathbb{T}'$ is augmented with constants and axioms for the elements and equalities in M. This uses an auxiliary construction of the kind familiar from model theory of total structures. For a partial Σ-structure M, and a signature morphism $\rho : \Sigma \longrightarrow \Sigma'$, let the *diagram (theory)* $\mathbb{D}_\rho(M)$ of M be the theory whose signature $\Sigma_{\mathbb{D}_\rho(M)}$ is Σ', extended with a fresh constant $c_{A,s} : A^\rho$, for each sort A of Σ and each $s \in M(A)$. (Note carefully the "typing" of the constants.) Thus $\rho : \Sigma \longrightarrow \Sigma' \subseteq \Sigma_{\mathbb{D}_\rho(M)}$. Its axioms are totality axioms for the new constants

$$\vphantom{x}\xmapsto{} c_{A,s}\downarrow \qquad (s \in M(A), A \in \Sigma\text{-Sort})$$

and axioms for graphs of partial functions

$$\vphantom{x}\xmapsto{} f^\rho(c_{A_1,s_1}, \ldots, c_{A_n,s_n}) = c_{A,s}$$

for all $s_1 \in M(A_1), \ldots, s_n \in M(A_n), s \in M(A)$ and $f : A_1 \cdots A_n \longrightarrow A$ in Σ with $(M\,f)(s_1, \ldots, s_n) = s$. (Again, note the typing.)

Theorem 5.4 *(Free Partial Model Theorem) Let $\mathbb{T}$ and $\mathbb{T}'$ be quasi-equational theories with respective signatures Σ and Σ' and let $\rho : \mathbb{T} \longrightarrow \mathbb{T}'$ be a theory morphism. Then the forgetful functor $U_\rho : \mathbb{T}'\text{-}PMod \longrightarrow \mathbb{T}\text{-}PMod$ has a left adjoint.*

Proof. For a Σ-structure M in $\mathbb{T}$-PMod extend the theory $\mathbb{T}'$ as follows: $\mathbb{T}'' = \mathbb{T}' \cup \mathbb{D}_\rho(M)$ whose signature is $\Sigma'' = \Sigma_{\mathbb{D}_\rho(M)}$.
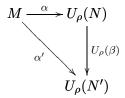
If L is a model of $\mathbb{T}''$, then it has a reduct N to $\mathbb{T}'$. The only information in L that is not already contained in N is the interpretation of the constants $c_{A,s}$, and this is determined by carrier maps

$$\alpha_A : M\,A \longrightarrow N\,A^\rho = U_\rho(N)\,A.$$

The axioms of $\mathbb{D}_\rho(M)$ are equivalent to requiring these to form a homomorphism $\alpha : A \longrightarrow U_\rho(N)$.

Hence, the models of $\mathbb{T}''$ are equivalent to pairs (N, α) where N is a model of $\mathbb{T}'$ and $\alpha : M \longrightarrow U_\rho(N)$ is a homomorphism.

A $\mathbb{T}''$-homomorphism γ from (N, α) to (N', α') reduces to a $\mathbb{T}'$ homomorphism $\beta : N \longrightarrow N'$. β already contains all the structure (the carrier maps) needed for γ. The only additional information needed for it to be a $\mathbb{T}''$-homomorphism are that it should preserve the constants $c_(A, s)$, and this is expressed by commutativity of the following diagram.

$$
\begin{array}{ccc}
M & \xrightarrow{\ \alpha\ } & U_\rho(N) \\
& {\scriptstyle \alpha'} \searrow & \downarrow {\scriptstyle U_\rho(\beta)} \\
& & U_\rho(N')
\end{array}
$$

Hence, $\mathbb{T}''$-homomorphisms from (N, α) to (N', α') are equivalent to $\mathbb{T}'$ homomorphisms $\beta : N \longrightarrow N'$ making the above diagram commute.

Now the term model $\mathrm{Ter}(\mathbb{T}'')$ can be expressed as $(F_\rho(M), \eta_M)$ where

$$
F_\rho(M) = \mathrm{Ter}(\mathbb{T}'')|_{\Sigma'}
$$

is a model of $\mathbb{T}'$, and $\eta_M : M \longrightarrow U_\rho(F_\rho(M))$ is a Σ-homomorphism.

For any model N of $\mathbb{T}'$, we now see that the following are equivalent.

- a homomorphism $\alpha : M \longrightarrow U_\rho(N)$

- a models (N, α) for $\mathbb{T}''$ whose Σ'-reduct is N

- a $\mathbb{T}''$-homomorphism $(F_\rho(M), \eta_M) \longrightarrow (N, \alpha)$ whose codomain has N as Σ'-reduct (here we use initiality of the term model)

- a $\mathbb{T}'$-homomorphism $F_\rho(M) \longrightarrow N$ (since α is determined as $U_\rho(\beta) \circ \eta_M$)

This shows that η_M is a universal arrow from M to U_ρ and hence [ML71] that F_ρ provides a left adjoint to U_ρ. $\square$

As an application of the theorem we see that the left adjoint to $U : \mathbb{T}_{\mathrm{cat}}\text{-PMod} \longrightarrow \mathbb{T}_{\mathrm{gr}}\text{-PMod}$ generates categories from graphs. Further examples of interesting left adjoints follows from the examples given in the next section.

6 Examples from category theory

The sequents may also be useful for expressing uniqueness conditions. The theory $\mathbb{T}_{\mathrm{ter}}$ of a category with a terminal object is $\mathbb{T}_{\mathrm{cat}}$ extended with a constant $\top : \mathrm{obj}$ and a function symbol $! : \mathrm{obj} \longrightarrow \mathrm{arr}$ satisfying the following axioms. (x and f are of sorts obj and arr.)

(i) $\longmapsto \top \downarrow \qquad \xmapsto{\ x\ } !_x \downarrow$,

(ii) $\xmapsto{\ x\ } \mathsf{d}(!_x) = x \qquad \xmapsto{\ x\ } \mathsf{c}(!_x) = \top$,

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(iii) $\mathsf{d}(f) = x, \mathsf{c}(f) = \top \vdash^{xf} f = !_x$.

Let $\mathcal{C}$ be a category with finite limits, i.e. with pullbacks and a terminal object. Such categories are also known as *cartesian* categories, or *lex* categories for short. But note that we are assuming here that they have *canonical* finite limits. More precisely, they have canonical terminal objects and pullbacks, and other finite limits are canonically constructed from them. We use the following notation for canonical pullbacks. For $\alpha : A \longrightarrow X$ and $\beta : B \longrightarrow X$ let $\mathsf{P}(\alpha, \beta)$ be the pullback object, and let $\mathsf{p}^1_{\alpha,\beta}$ and $\mathsf{p}^2_{\alpha,\beta}$ be the projections into A and B respectively, which satisfy

$$\alpha \mathsf{p}^1_{\alpha,\beta} = \beta \mathsf{p}^2_{\alpha,\beta}.$$

The common value of these compositions is denoted $\mathsf{p}(\alpha, \beta)$. For $f : C \longrightarrow A$ and $g : C \longrightarrow B$ with $\alpha f = \beta g$, let $\langle f, g \rangle_{\alpha,\beta}$ be the unique arrow h such that the following commutes

For the case when $X = \top$ is the terminal object, $\mathsf{P}(!_A, !_B)$ is an ordinary product $A \times B$ and $\mathsf{p}^1_{A,B} = \mathsf{p}^1_{!_A,!_B}, \mathsf{p}^2_{A,B} = \mathsf{p}^2_{!_A,!_B}$ are its projections, and $\langle \cdot, \cdot \rangle_{A,B} = \langle \cdot, \cdot \rangle_{!_A,!_B}$ is the ordinary pairing operation.

The canonical n-ary pullback of $\alpha_k : A_k \longrightarrow X$, $k = 1, \ldots, n$, has projections $\mathsf{p}^k_{\alpha_1,\ldots,\alpha_n} : \mathsf{P}(\alpha_1, \ldots, \alpha_n) \longrightarrow A_k$, and tupling operation $\langle \cdot, \ldots, \cdot \rangle_{\alpha_1,\ldots,\alpha_n}$. Write

$$\mathsf{p}(\alpha_1, \ldots, \alpha_n) = \alpha_k \mathsf{p}^k_{\alpha_1,\ldots,\alpha_n}.$$

6.1 Cartesian categories

To facilitate the formulation of certain theories, introduce the following notion of a *partial equality*: for terms t and t' which together contains *exactly* the free variables $\vec{x}$, and let $t \rightleftharpoons t'$ denote the sequent

$$t \downarrow, t' \downarrow \vdash^{\vec{x}} t = t'.$$

Thus the terms are considered equal whenever they are both defined.

The following is a quasi-equational theory for cartesian categories. The theory $\mathbb{T}_{\mathrm{cart}}$ consists of $\mathbb{T}_{\mathrm{ter}}$ extended with function symbols $\mathsf{p}^1, \mathsf{p}^2 : \mathrm{arr} \times \mathrm{arr} \longrightarrow \mathrm{arr}$ and $\langle \cdot, \cdot \rangle_{\cdot,\cdot} : \mathrm{arr} \times \mathrm{arr} \times \mathrm{arr} \times \mathrm{arr} \longrightarrow \mathrm{arr}$ satisfying the axioms

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(i) $\mathsf{p}^k_{f_1,f_2} \downarrow \xrightarrow{f_1,f_2} \mathsf{c}(f_1) = \mathsf{c}(f_2)$, for $k = 1, 2$.

(ii) $\mathsf{c}(\mathsf{p}^k_{f_1,f_2}) \leftrightharpoons \mathsf{d}(f_k)$, for $k = 1, 2$.

(iii) $f_1 \circ \mathsf{p}^1_{f_1,f_2} \leftrightharpoons f_2 \circ \mathsf{p}^2_{f_1,f_2}$.

(iv) $\langle h_1, h_2 \rangle_{f_1,f_2} \downarrow \xrightarrow{h_1 h_2 f_1 f_2} f_1 \circ h_1 = f_2 \circ h_2$

(v) $\mathsf{d}(\langle h_1, h_2 \rangle_{f_1,f_2}) \leftrightharpoons \mathsf{d}(h_k)$, for $k = 1, 2$.

(vi) $\mathsf{c}(\langle h_1, h_2 \rangle_{f_1,f_2}) \leftrightharpoons \mathsf{d}(\mathsf{p}^k_{f_1,f_2})$, for $k = 1, 2$.

(vii) $\mathsf{p}^k_{f_1,f_2} \circ \langle h_1, h_2 \rangle_{f_1,f_2} \leftrightharpoons h_k$, for $k = 1, 2$.

(viii) $\langle \mathsf{p}^1_{f_1,f_2} \circ h, \mathsf{p}^2_{f_1,f_2} \circ h \rangle_{f_1,f_2} \leftrightharpoons h$

We use the abbreviation $\mathsf{p}(f_1, f_2)$ for $f_1 \circ \mathsf{p}^1_{f_1,f_2}$, and $\mathsf{P}(f_1, f_2)$ for $\mathsf{d}(\mathsf{p}(f_1, f_2))$. Note that

$$\mathsf{c}(f_1) = \mathsf{c}(f_2) \xrightarrow{f_1 f_2} \mathsf{p}(f_1, f_2) = f_2 \circ \mathsf{p}^2_{f_1,f_2}.$$

A homomorphism between cartesian categories $\mathcal{C}$ and $\mathcal{C}'$ is thus a strict cartesian functor, i.e. a functor F satisfying

$$
\begin{aligned}
F(\top) &= \top' \\
F(!_x) &= (!')_x \\
F(\mathsf{p}^k_{f_1,f_2}) &= \mathsf{p}'^k_{F(f_1),F(f_2)} \\
F(\langle h_1, h_2 \rangle_{f_1,f_2}) &= \langle F(h_1), F(h_2) \rangle'_{F(f_1),F(f_2)}
\end{aligned}
$$

whenever the left hand side expressions are defined. We have also $F(\mathsf{p}(f_1, f_2)) = \mathsf{p}'(F(f_1), F(f_2))$.

6.2 Locally cartesian closed categories

First we indicate how the theory of cartesian categories yields that products exist in every slice. Denote by $f : a \curvearrowright b$ the equation $b \circ f = a$. This means that a and b have common codomain, and f is a morphism in the slice category.

Proposition 6.1 *In $\mathbb{T}_{\mathrm{cart}}$ we have*

(i) $\mathsf{c}(f_1) = \mathsf{c}(f_2) \xrightarrow{f_1 f_2} \mathsf{p}^k_{f_1,f_2} : \mathsf{p}(f_1, f_2) \curvearrowright f_k,$ *for $k = 1, 2$*

(ii) $q_1 : g \curvearrowright f_1, q_2 : g \curvearrowright f_2 \xrightarrow{q_1 q_2 f_1 f_2} \langle q_1, q_2 \rangle_{f_1,f_2} : g \curvearrowright \mathsf{p}(f_1, f_2)$

(iii) $q_1 : g \curvearrowright f_1, q_2 : g \curvearrowright f_2 \xrightarrow{q_1 q_2 f_1 f_2} \mathsf{p}^k_{f_1,f_2} \circ \langle q_1, q_2 \rangle_{f_1,f_2} = q_k,$ *for $k = 1, 2$*

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(iv) $q : g \frown \mathsf{p}(f_1, f_2) \xmapsto{\quad f_1 f_2 qg \quad} \langle \mathsf{p}^1_{f_1,f_2} \circ q, \mathsf{p}^2_{f_1,f_2} \circ q \rangle_{f_1,f_2} = q.$ $\square$

The theory $\mathbb{T}_{\mathrm{lccc}}$ of a locally cartesian closed category is $\mathbb{T}_{\mathrm{cart}}$ extended with the function symbols $(\Leftarrow) : \mathrm{arr} \times \mathrm{arr} \longrightarrow \mathrm{arr}$, $\varepsilon : \mathrm{arr} \times \mathrm{arr} \longrightarrow \mathrm{arr}$, $\Lambda : \mathrm{arr} \times \mathrm{arr} \times \mathrm{arr} \times \mathrm{arr} \longrightarrow \mathrm{arr}$. Write $\Lambda^k_{f,g}(h)$ for $\Lambda(k, f, g, h)$. Furthermore these axioms are included:

(i) $(f \Leftarrow g){\downarrow} \xmapsto{\quad fg \quad} \mathsf{c}(f) = \mathsf{c}(g)$,

(ii) $\mathsf{c}(f \Leftarrow g) \leftmapsto \mathsf{c}(f)$,

(iii) $\varepsilon_{f,g}{\downarrow} \xmapsto{\quad fg \quad} \mathsf{c}(f) = \mathsf{c}(g)$,

(iv) $\mathsf{c}(f) = \mathsf{c}(g) \xmapsto{\quad fg \quad} \varepsilon_{f,g} : \mathsf{p}(f \Leftarrow g, g) \frown f$,

(v) $\Lambda^k_{f,g}(h){\downarrow} \xmapsto{\quad kfgh \quad} h : \mathsf{p}(k, g) \frown f$

(vi) $h : \mathsf{p}(k, g) \frown f \xmapsto{\quad kfgh \quad} \Lambda^k_{f,g}(h) : k \frown (f \Leftarrow g)$,

(vii) $\varepsilon_{f,g} \circ \langle \Lambda^k_{f,g}(h) \circ \mathsf{p}^1_{k,g}, \mathsf{p}^2_{k,g} \rangle_{f \Leftarrow g, g} \leftmapsto h$,

(viii) $\Lambda^k_{f,g}(\varepsilon_{f,g} \circ \langle m \circ \mathsf{p}^1_{k,g}, \mathsf{p}^2_{k,g} \rangle_{f \Leftarrow g, g}) \leftmapsto m$.

6.3   A logical characterisation of locally cartesian closed categories

Next we give an application of a stronger logic, partial first order logic.

Let $\mathcal{C}$ be a Heyting category [Joh02a], thus admitting first-order intuitionistic logic as internal logic — see Section 8. The following notation for fibers of arrows is used in the internal language. If $\alpha : A \longrightarrow X$ is an arrow in $\mathcal{C}$, then $a \in A_x$ is short for $\alpha(a) = x$, and $(\forall a \in A_x) P(x, a)$ is an abbreviation of $(\forall a \in A)(\alpha(a) = x \Rightarrow P(x, a))$ whereas $(\exists a \in A_x) P(x, a)$ is short for $(\exists a \in A)(\alpha(a) = x \land P(x, a))$.

Then there is this "logical" characterisation: $\mathcal{C}$ is a locally cartesian closed category iff for all $a : A \longrightarrow X$ and $b : B \longrightarrow X$ in $\mathcal{C}$ there are $\varphi_{a,b} : \Phi(a, b) \longrightarrow X$ and

$$\mathrm{ev}_{a,b} : \Phi(a, b) \times A \rightharpoonup B$$

in $\mathcal{C}$ such that for all $P \rightarrowtail A \times B \times C$ the following sequents are valid in the internal language of $\mathcal{C}$.

(i) $\mathrm{ev}_{a,b}(f, y){\downarrow} \xmapsto{\quad fy \quad} \varphi_{a,b}(f) = a(y)$

(ii) $f \in \Phi(a, b)_x, y \in A_x \xmapsto{\quad xyf \quad} \mathrm{ev}_{a,b}(f, y) \in B_x$

(iii) $(\forall x \in X)(\forall f, g \in \Phi(a, b)_x)[(\forall y \in A_x)(\mathrm{ev}_{a,b}(f, y) = \mathrm{ev}_{a,b}(g, y)) \longrightarrow f = g]$

(iv) $(\forall y \in A_x)(\exists! u \in B_x) P(y, u; t) \xmapsto{\quad xt \quad} (\exists f \in \Phi(a, b)_x)(\forall y \in A_x) P(y, \mathrm{ev}_{a,b}(f, y); t)$.

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6.4 Finitary sheaves

Definition 6.2 Let L be a distributive lattice, with operations 0, 1, $\vee$ and $\wedge$ (not to be confused with logical conjunction and disjunction). Then a *finitary sheaf* over L is a presheaf $F : L^{op} \longrightarrow \mathbf{Sets}$ that satisfies the following *pasting condition*. Let $a_i \in L$ ($1 \leq i \leq n$), and let $x_i \in F(a_i)$ be such that $x_i|(a_i \wedge a_j) = x_j|(a_i \wedge a_j)$ for all i, j in the range 1 to n. (We write "$|$" for the restriction operations.) Then there is a unique $x \in F(\bigvee_i a_i)$ such that $x|a_i = x_i$ for all i.

With a little work one can show that it suffices to have the pasting condition in the cases $n = 0$ and $n = 2$. The nullary pasting just says that $F(0)$ is a singleton. The finitary sheaves are then the models of a quasi-equational theory described as follows.

First, for each $a \in L$ there is a sort X_a.

For each $a \leq b$ in L, there is a total *restriction* function $\rho_{ab} : X_b \longrightarrow X_a$. Totality is axiomatized by

$$\top \overset{x:X_b}{\longmapsto\!\!\!\!\longmapsto} \rho_{ab}(x)\!\downarrow.$$

Functoriality is axiomatized by

$$\top \overset{x:X_a}{\longmapsto\!\!\!\!\longmapsto} \rho_{aa}(x) = x$$

$$\top \overset{x:X_c}{\longmapsto\!\!\!\!\longmapsto} \rho_{ac}(x) = \rho_{ab}(\rho_{bc}(x))$$

(Note: what we have introduced so far is the theory of *presheaves* over L.)

X_0 is forced to be a singleton by a constant $* : X_0$ axiomatized to be total and unique.

$$\top \longmapsto\!\!\!\!\longmapsto *\!\downarrow$$

$$\top \overset{x:X_0}{\longmapsto\!\!\!\!\longmapsto} x = *$$

For each $a, b \in L$, there is a partial *pasting* operation $\pi_{ab} : X_a \times X_b \longrightarrow X_{a \vee b}$. Its domain of definition is axiomatized by

$$\pi_{ab}(x, y)\!\downarrow \overset{xy}{\longmapsto\!\!\!\!\longmapsto} \rho_{a \wedge b, a}(x) = \rho_{a \wedge b, b}(y)$$

(though the $\vdash$ direction is unnecessary, since it is implied by the other axioms). Its characterizing condition is axiomatized by

$$\pi_{ab}(x, y)\!\downarrow \overset{xy}{\longmapsto\!\!\!\!\longmapsto} \rho_{a, a \vee b}(\pi_{ab}(x, y)) = x \wedge \rho_{b, a \vee b}(\pi_{ab}(x, y)) = y.$$

Finally, uniqueness is axiomatized by

$$\top \overset{z:X_{a \vee b}}{\longmapsto\!\!\!\!\longmapsto} z = \pi_{ab}(\rho_{a, a \vee b}(z), \rho_{b, a \vee b}(z)).$$

Because the theory of finitary sheaves is cartesian and extends the theory of presheaves, the free partial model theorem shows us that each presheaf can be *sheavified*, at least

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finitarily. The usual sheavification, for the case where L is a frame (complete, with meet distributing over arbitrary joins), requires infinitary pasting. The consequent need for partial operators that are infinitary, and more seriously of unbounded arity, means that the freeness theorem cannot be applied directly. Nonetheless, it encourages us to expect general sheavification to exist, as indeed it does.

The lattice L can also be internalized: we can define a theory whose models are pairs (L, F) with F a finitary sheaf over L. For this we have a sort L with operators and laws to make it a distributive lattice. A sort X represents the disjoint union of the X_as, and is equipped with a total function $p : X \longrightarrow L$. Restriction is most conveniently given by a total operation $\rho : X \times L \longrightarrow X$, with $\rho(x, a)$ representing $x|(p(x) \wedge a)$, with axioms

$$\top \vdash^{x:X} \rho(x, p(x)) = x$$

$$\top \vdash^{x:X} \rho(x, a \wedge b) = \rho(\rho(x, a), b).$$

Then pasting $\pi : X \times X \longrightarrow X$ has

$$\pi(x, y) \downarrow \vdash^{xy} \rho(x, p(y)) = \rho(y, p(x)).$$

It is axiomatized by

$$p(\pi(x, y)) \mathrel{\leftdashv\!\vdash} (p(x) \vee p(y))$$

$$\rho(\pi(x, y), p(x)) \mathrel{\leftdashv\!\vdash} x$$

$$\rho(\pi(x, y), p(y)) \mathrel{\leftdashv\!\vdash} y$$

$$p(z) = (a \vee b) \vdash^{z:X, a, b:L} z = \pi(\rho(z, a), \rho(z, b)).$$

The cartesian nature of the theory of finitary sheaves is exploited in [Vig04] to investigate sheaves over stably locally compact locales.
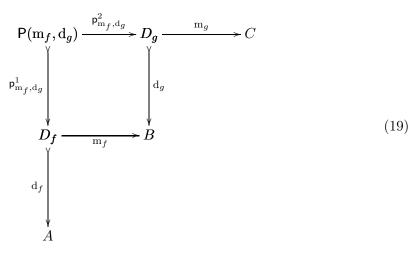
7 Partial morphisms

We refer to [Joh02b] for basic results about partial morphisms in categories. In this section we let $\mathcal{C}$ be a cartesian category. A *partial morphism* f from the object A to the object B in $\mathcal{C}$, in symbols $f : A \rightharpoonup B$, is a pair $(\mathrm{d}_f, \mathrm{m}_f)$ where $\mathrm{d}_f : \mathrm{D}_f \longrightarrow A$ is a monomorphism in $\mathcal{C}$, the *domain of definition*, and $\mathrm{m}_f : \mathrm{D}_f \longrightarrow B$ is an arbitrary morphism in $\mathcal{C}$. It *extends* another $f' : A \rightharpoonup B$, in symbols $f' \subseteq f$, if there is a $\varphi : \mathrm{D}_{f'} \longrightarrow \mathrm{D}_f$ with

$$\mathrm{d}_f \circ \varphi = \mathrm{d}_{f'} \qquad \mathrm{m}_f \circ \varphi = \mathrm{m}_{f'}.$$

Note that φ is necessarily a unique monomorphism. The relation $\subseteq$ is clearly a preorder, and in fact we make it a partial order on partial morphisms by *defining* equality of partial morphisms to be the corresponding equivalence relation: $f = f'$ iff $f' \subseteq f$ and $f \subseteq f'$. Equivalently, $f = f'$ iff $f' \subseteq f$ with the corresponding $\varphi : \mathrm{D}_{f'} \longrightarrow \mathrm{D}_f$ an *isomorphism*.

Now let $g = (\mathrm{d}_g, \mathrm{m}_g)$ be partial morphism from B to C. The (partial) composition of g and f, denoted $g \odot f$, is by definition $(\mathrm{d}_f \mathsf{p}^1_{\mathrm{m}_f, \mathrm{d}_g}, \mathrm{m}_g \mathsf{p}^2_{\mathrm{m}_f, \mathrm{d}_g})$.

$$
\begin{array}{ccccc}
\mathsf{P}(\mathrm{m}_f, \mathrm{d}_g) & \xrightarrow{\ \mathsf{p}^2_{\mathrm{m}_f, \mathrm{d}_g}\ } & D_g & \xrightarrow{\ \mathrm{m}_g\ } & C \\[2mm]
\Big\downarrow {\scriptstyle \mathsf{p}^1_{\mathrm{m}_f, \mathrm{d}_g}} & & \Big\downarrow {\scriptstyle \mathrm{d}_g} & & \\[6mm]
D_f & \xrightarrow[\ \mathrm{m}_f\ ]{} & B & & \\[2mm]
\Big\downarrow {\scriptstyle \mathrm{d}_f} & & & & \\[6mm]
A & & & &
\end{array}
\tag{19}
$$

Then it is clear that as subobjects of A

$$
\mathrm{d}_{g \odot f} \le \mathrm{d}_f.
\tag{20}
$$

Lemma 7.1 *For $f_1, g_1 : X \to Y$ and $f_2, g_2 : Y \to Z$ with $f_1 \subseteq g_1$, $f_2 \subseteq g_2$,*

$$
f_2 \odot f_1 \subseteq g_2 \odot g_1.
$$

Hence partial composition respects the defined equality. It is easily shown to be associative. Note that any (total) morphism $f : A \longrightarrow B$ can be regarded as a partial morphism $(1_A, f) : A \to B$. The total identities for objects will also serve as identities for partial composition.

If $f_1 : A \to B_1, \ldots, f_n : A \to B_n$ are partial morphisms, let $f = \langle f_1, f_2, \ldots, f_n \rangle_p : A \to B_1 \times \cdots \times B_n$ be

$$
(\mathsf{p}(\mathrm{d}_{f_1}, \ldots, \mathrm{d}_{f_n}), \langle m_{f_1} \mathsf{p}^1_{\mathrm{d}_{f_1}, \ldots, \mathrm{d}_{f_n}}, \ldots, \mathrm{m}_{f_n} \mathsf{p}^n_{\mathrm{d}_{f_1}, \ldots, \mathrm{d}_{f_n}} \rangle_{B_1, \ldots, B_n}).
$$

The notation for binary pullbacks is extended here in an obvious way. For $n > 2$ we make an arbitrary choice of the order in which binary pullbacks are iterated to construct n-ary pullbacks. Then

$$
\mathrm{d}_f = \mathrm{d}_{f_1} \wedge \cdots \wedge \mathrm{d}_{f_n}.
\tag{21}
$$

Lemma 7.2 *For $f_k, g_k : X \to Y_k$, with $f_k \subseteq g_k$ for all $k = 1, \ldots, n$*

$$
\langle f_1, \ldots, f_n \rangle_p \subseteq \langle g_1, \ldots, g_n \rangle_p.
$$

As a consequence of this lemma, the tupling operation respects equality of partial functions. The restriction of $f : X \to Y$ to a subobject $a : A \rightarrowtail X$, is the partial morphism $f|a = f \odot (a, a) : X \to Y$. Thus for $g : Y \to Z$ we have by associativity

$$
(g \odot f)|a = g \odot (f|a).
$$

Obviously $(f|a) \subseteq f$ and $\mathrm{d}_{(f|a)} = a \wedge \mathrm{d}_f$. The next lemma is also straightforward.

Lemma 7.3 *Suppose that* $f_1 : Y \rightharpoonup Z_1, \ldots, f_n : Y \rightharpoonup Z_n$ *and* $g : X \rightharpoonup Y$. *Then*

(i) $\langle f_1, \ldots, f_n \rangle_p \odot g = \langle f_1 \odot g, \ldots, f_n \odot g \rangle_p | \mathrm{d}_g$.

(ii) $\mathsf{p}_k \odot \langle f_1, \ldots, f_n \rangle_p = f_k | \mathrm{d}_{\langle f_1, \ldots, f_n \rangle_p}$

As a corollary of part (i) of the lemma, if a is a monomorphism with codomain Y then we can take g to be the partial morphism (a, a) and obtain
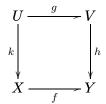
$$\langle f_1, \ldots, f_n \rangle_p | a = \langle f_1 | a, \ldots, f_n | a \rangle_p | a.$$

The following notion will be useful. Consider partial morphisms $f = (D, d, m) : X \rightharpoonup Y$ and $g = (E, e, n) : U \rightharpoonup V$, and morphisms $k : U \longrightarrow X$ and $h : V \longrightarrow Y$. If there is a morphism $j : E \longrightarrow D$ so that both squares of

$$
\begin{array}{ccccc}
U & \xleftarrow{\;\;e\;\;} & E & \xrightarrow{\;\;n\;\;} & V \\
{\scriptstyle k}\downarrow & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle h} \\
X & \xleftarrow{\;\;d\;\;} & D & \xrightarrow{\;\;m\;\;} & Y
\end{array}
\tag{22}
$$

are pullback squares, then we call

$$
\begin{array}{ccc}
U & \xrightarrow{\;\;g\;\;} & V \\
{\scriptstyle k}\downarrow & & \downarrow{\scriptstyle h} \\
X & \xrightarrow{\;\;f\;\;} & Y
\end{array}
$$

a *partial pullback square*. It is clear that the partial morphisms f and g may be replaced by equivalent partial morphisms, and it will remain a partial pullback square.

Example 7.4 An example of a partial pullback square is given by in diagram (22) letting $U = X \times Z$, $E = D \times Z$, $V = Y \times Z$, $e = d \times 1$, $n = m \times 1$ and taking the vertical arrows to be projections to the first coordinate. Denote the partial morphism g by $f \times_p 1_Z$.

8   Categorical semantics of partial Horn logic

The semantics of PHL is now generalised from the category of sets to an arbitrary cartesian category $\mathcal{C}$. We assume throughout that cartesian categories have *canonical* terminal objects and pullbacks, and hence are models for the theory Cart. However, a cartesian *functor* (a functor that preserves finite limits) does not have to preserve canonicity. A cartesian functor is *strict* if it preserves canonical limits (i.e. if it is a homomorphism for Cart).

A partial structure M for a signature Σ in $\mathcal{C}$, assigns to each sort S, an object MS in $\mathcal{C}$, to each function symbol $f : S_1 \cdots S_n \longrightarrow S$ a partial morphism $M(f) :$

$M(S_1, \ldots, S_n) \twoheadrightarrow M(S)$ in $\mathcal{C}$. Here $M(S_1, \ldots, S_n)$ is some canonical choice of product of the objects $M(S_1), \ldots, M(S_n)$. The interpretations of terms are now built up just as in Section 3, see (8 — 11). Furthermore, to each relation symbol $R \rightarrowtail S_1 \cdots S_n$ a subobject $M(R)$ of $M(S_1, \ldots, S_n)$ is assigned. The identity relation symbol $=_S$ is assigned the diagonal subobject $\Delta : M(S) \rightarrowtail M(S, S)$.
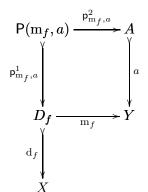
Depending on further properties of the category, it may be possible extend this semantics to first order formulas over the signature Σ. To each formula φ with free variables $\vec{x} = x_1, \ldots, x_n$, of sorts $\vec{S} = S_1, \ldots, S_n$ we assign a subobject $[\![\vec{x}.\varphi]\!]_M$ of $M(\vec{S})$. The sequent

$$\varphi_1, \ldots, \varphi_m \overset{\vec{x}}{\vdash\!\!\!-\!\!\!-} \psi \tag{23}$$

is valid in M, if the following inclusion of subobjects holds

$$[\![\vec{x}.\varphi_1]\!]_M \wedge \cdots \wedge [\![\vec{x}.\varphi_m]\!]_M \leq [\![\vec{x}.\psi]\!]_M. \tag{24}$$

From the algebraic properties of the semilattice of subobjects of a given object, the identity rule (a1), the cut rule (a2) and the conjunctive rules (c1-4) are easily verified. This is exactly as for standard Horn logic. The crucial difference is in the interpretation of a predicate applied to a sequence of partial terms. Say that a subobject of Y is represented by a monomorphism $a : A \rightarrowtail Y$. The resulting subobject when substituting a partial morphism $f : X \rightarrow Y$ into the subobject is represented by the monomorphism $\mathrm{d}_f \mathsf{p}^1_{\mathrm{m}_f, a}$.

Write $f^{\sim 1}(a)$, with a wavy minus sign, for this subobject. Note that

$$f^{\sim 1}(a) \leq \mathrm{d}_f \tag{25}$$

as subobjects of X, and that if $a = 1_Y : Y \longrightarrow Y$, then $f^{\sim 1}(a) = \mathrm{d}_f$. If f is *total* (that is, d_f is an isomorphism), then $f^{\sim 1}(a)$ coincides with the usual $f^{-1}(a)$ as subobjects of X. Note that the $f^{\sim 1}(a)$ is also the domain of definition of the composition

$$\hat{a} \odot f.$$

Here $\hat{a} = (a, !_A)$ is the partial morphism $Y \rightarrow 1$ corresponding to the mono a. The following results are now easily checked directly.

Lemma 8.1 *For $f, g : X \to Y$ and monomorphisms $a : A \rightarrowtail Y$, $b : B \rightarrowtail Y$*

(i) $a \leq b$, $f \subseteq g \implies f^{\sim 1}(a) \leq g^{\sim 1}(b)$,

(ii) $f^{\sim 1}(a \wedge b) = f^{\sim 1}(a) \wedge f^{\sim 1}(b)$.

Lemma 8.2 *For $g : X \to Y$, $f : Y \to Z$ and monomorphisms $a : A \rightarrowtail Z$, $b : B \longrightarrow Y$*

(i) $(f \odot g)^{\sim 1}(a) = g^{\sim 1}(f^{\sim 1}(a))$,

(ii) $(f|b)^{\sim 1}(a) = b \wedge f^{\sim 1}(a)$.

Lemma 8.3 *For $f : X \to Y$,*

$$\langle f, f \rangle_p^{\sim 1}(\Delta_Y) = \mathrm{d}_f = f^{\sim 1}(1_Y),$$

as subobjects of X.

We now extend the usual interpretation of atomic formulae to allow also partial terms. We fix a partial structure M for a signature Σ in a cartesian category. Write simply $[\![\cdot]\!]$ for $[\![\cdot]\!]_M$ in the remainder of this subsection.

Consider a predicate symbol $R \rightarrowtail S_1 \cdots S_n$ and partial terms $\vec{t} = t_1, \ldots, t_n$ suitable for $\vec{x}$ and where $\sigma(t_k) = S_k$. Then define

$$[\![\vec{x}.R(\vec{t})]\!] \triangleq [\![\vec{x}.\vec{t}]\!]^{\sim 1}(M(R)) = \langle [\![\vec{x}.t_1]\!], \ldots, [\![\vec{x}.t_n]\!] \rangle_p^{\sim 1}(M(R)),$$

and for $t_1, t_2 : S$

$$[\![\vec{x}.t_1 = t_2]\!] \triangleq \langle [\![\vec{x}.t_1]\!], [\![\vec{x}.t_2]\!] \rangle_p^{\sim 1}(M(=_S)).$$

Thus by Lemma 8.3

$$[\![\vec{x}.t \downarrow]\!] = [\![\vec{x}.t = t]\!] = \mathrm{d}_{[\![\vec{x}.t]\!]}, \tag{26}$$

and hence for $t = x_k$ this is isomorphic to $1_{M(\sigma(\vec{x}))}$. The reflexivity axiom (b1) is thereby evident. The equality axiom (b2) follows by the validity of the rule for total terms, since $[\![\vec{z}.x_k]\!]$ and $[\![\vec{z}.y_k]\!]$ are indeed total.

To prove the validity of the partial term substitution rule (a3) we first prove

Lemma 8.4 *Let $(\vec{x}.\vec{s})$ and $(\vec{y}.\vec{t})$ be term tuples in context over Σ, where $\vec{t}$ is sort compatible with $\vec{x}$. Then*

$$[\![\vec{y}.\vec{s}(\vec{t}/\vec{x})]\!] | \mathrm{d}_{[\![\vec{y}.\vec{t}]\!]} = [\![\vec{x}.\vec{s}]\!] \odot [\![\vec{y}.\vec{t}]\!]. \tag{27}$$

Proof. Suppose for each component s_i of $\vec{s}$, the result holds when $\vec{s}$ is replaced by s_i. Then

$$
\begin{aligned}
[\![\vec{x}.\vec{s}]\!] \odot [\![\vec{y}.\vec{t}]\!] &= \langle \ldots [\![\vec{x}.s_i]\!] \ldots \rangle_p \odot [\![\vec{y}.\vec{t}]\!] \\
&= \langle \ldots [\![\vec{x}.s_i]\!] \odot [\![\vec{y}.\vec{t}]\!] \ldots \rangle_p | \mathrm{d}_{[\![\vec{y}.\vec{t}]\!]} \\
&= \langle \ldots [\![\vec{x}.s_i(\vec{t}/\vec{x})]\!] | \mathrm{d}_{[\![\vec{y}.\vec{t}]\!]} \ldots \rangle_p | \mathrm{d}_{[\![\vec{y}.\vec{t}]\!]} \\
&= \langle \ldots [\![\vec{x}.s_i(\vec{t}/\vec{x})]\!] \ldots \rangle_p | \mathrm{d}_{[\![\vec{y}.\vec{t}]\!]} \\
&= [\![\vec{y}.\vec{s}(\vec{t}/\vec{x})]\!] | \mathrm{d}_{[\![\vec{y}.\vec{t}]\!]}
\end{aligned}
$$

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It remains only to use induction to show the case where $\vec{s}$ is of length 1, a single term s. For $s = x_i$, a variable, the righthand side of (27) is

$$
\begin{aligned}
[\![\vec{x}.x_i]\!] \odot [\![\vec{y}.\vec{t}]\!] &= \pi_i \odot \langle [\![\vec{y}.t_1]\!], \ldots, [\![\vec{y}.t_n]\!] \rangle_p \\
&= [\![\vec{y}.t_i]\!] | \mathrm{d}_{\langle [\![\vec{y}.t_1]\!], \ldots, [\![\vec{y}.t_n]\!] \rangle_p} \\
&= [\![\vec{y}.t_i]\!] | \mathrm{d}_{[\![\vec{y}.\vec{t}]\!]} \\
&= [\![\vec{y}.x_i(\vec{t}/\vec{x})]\!] | \mathrm{d}_{[\![\vec{y}.\vec{t}]\!]}
\end{aligned}
$$

Here Lemma 7.3 is used in the second step. For $s = f(\vec{r})$, the righthand side of (27) is

$$
\begin{aligned}
[\![\vec{x}.f(\vec{r})]\!] \odot [\![\vec{y}.\vec{t}]\!] &= (M\,f) \odot [\![\vec{x}.\vec{r}]\!] \odot [\![\vec{y}.\vec{t}]\!] \\
&= (M\,f) \odot [\![\vec{y}.\vec{r}(\vec{t}/\vec{x})]\!] | \mathrm{d}_{[\![\vec{y}.\vec{t}]\!]} \\
&= [\![\vec{y}.f(\vec{r})(\vec{t}/\vec{x})]\!] | \mathrm{d}_{[\![\vec{y}.\vec{t}]\!]}
\end{aligned}
$$

In this calculation, Lemma 7.3 and properties of the restriction operation have been used. $\square$

Then

Lemma 8.5 *Let φ be a conjunction of atomic Σ-formulae suitable for the context $\vec{x}$, and let $\vec{t}$ be terms over Σ suitable for $\vec{y}$, and sort compatible with $\vec{x}$. Then*

$$
[\![\vec{y}.(\vec{t}\downarrow) \wedge \varphi(\vec{t}/\vec{x})]\!] = [\![\vec{y}.\vec{t}]\!]^{\sim 1}([\![\vec{x}.\varphi]\!]).
$$

Proof. First note that

$$
\begin{aligned}
[\![\vec{y}.(\vec{t}\downarrow)]\!] &= [\![\vec{y}.t_1 \downarrow]\!] \wedge \ldots \wedge [\![\vec{y}.t_n \downarrow]\!] \\
&= \mathrm{d}_{[\![\vec{y}.t_1]\!]} \wedge \ldots \wedge \mathrm{d}_{[\![\vec{y}.t_n]\!]} \\
&= \mathrm{d}_{\langle [\![\vec{y}.t_1]\!], \ldots, [\![\vec{y}.t_n]\!] \rangle_p} \\
&= \mathrm{d}_{[\![\vec{y}.\vec{t}]\!]}
\end{aligned}
$$

Now the proof is done by induction on φ.

Case $\varphi = R(\vec{s})$: By an earlier remark, $[\![\vec{y}.R(\vec{s})(\vec{t}/\vec{x})]\!]$ is the domain of definition of $\widehat{M(R)} \odot [\![\vec{y}.\vec{s}(\vec{t}/\vec{x})]\!]$, and hence

$$
[\![\vec{y}.(\vec{t}\downarrow) \wedge \varphi(\vec{t}/\vec{x})]\!] = \mathrm{d}_{[\![\vec{y}.\vec{t}]\!]} \wedge [\![\vec{y}.R(\vec{s})(\vec{t}/\vec{x})]\!]
$$

is the domain of definition of

$$
\widehat{M(R)} \odot [\![\vec{y}.\vec{s}(\vec{t}/\vec{x})]\!] | \mathrm{d}_{[\![\vec{y}.\vec{t}]\!]} = \widehat{M(R)} \odot [\![\vec{y}.\vec{s}]\!] \odot [\![\vec{y}.\vec{t}]\!].
$$

But this domain of definition is $[\![\vec{y}.\vec{t}]\!]^{\sim 1}([\![\vec{x}.R(\vec{s})]\!])$.

Case $\varphi = \top$: Employing properties of $\wedge$, $\top$ and the equation (26) and Lemma 8.3 we get

$$
\begin{aligned}
[\![\vec{y}.(\vec{t}\downarrow) \wedge \top]\!] &= \mathrm{d}_{[\![\vec{y}.\vec{t}]\!]} \wedge [\![\top]\!] \\
&= [\![\vec{y}.\vec{t}]\!]^{\sim 1}([\![\vec{x}.\top]\!]).
\end{aligned}
$$

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Case $\varphi = \varphi_1 \wedge \varphi_2$: Using the properties of $\wedge$, the inductive hypotheses and Lemma 8.1.(ii) we compute as follows

$$
\begin{aligned}
[\![\vec{y}.(\vec{t}\!\downarrow) \wedge \varphi(\vec{t}/\vec{x})]\!] &= [\![\vec{y}.(\vec{t}\!\downarrow) \wedge \varphi_1(\vec{t}/\vec{x}) \wedge \varphi_2(\vec{t}/\vec{x})]\!] \\
&= [\![\vec{y}.(\vec{t}\!\downarrow) \wedge \varphi_1(\vec{t}/\vec{x})]\!] \wedge [\![\vec{y}.(\vec{t}\!\downarrow) \wedge \varphi_2(\vec{t}/\vec{x})]\!] \\
&= [\![\vec{y}.\vec{t}]\!]^{\sim 1}([\![\vec{x}.\varphi_1]\!]) \wedge [\![\vec{y}.\vec{t}]\!]^{\sim 1}([\![\vec{x}.\varphi_2]\!]) \\
&= [\![\vec{y}.\vec{t}]\!]^{\sim 1}([\![\vec{x}.\varphi_1]\!] \wedge [\![\vec{x}.\varphi_2]\!]) \\
&= [\![\vec{y}.\vec{t}]\!]^{\sim 1}([\![\vec{x}.\varphi_1 \wedge \varphi_2]\!]).
\end{aligned}
$$

$\square$

Using Lemma 8.5 the rule (a3) is now straightforward to check: if

$$[\![\vec{x}.\varphi]\!] \leq [\![\vec{x}.\psi]\!],$$

then we get by Lemma 8.1.(i)

$$[\![\vec{y}.\vec{t}]\!]^{\sim 1}([\![\vec{x}.\varphi]\!]) \leq [\![\vec{y}.\vec{t}]\!]^{\sim 1}([\![\vec{x}.\psi]\!]).$$

Applying Lemma 8.5 on both sides we obtain the first member of

$$[\![\vec{y}.(\vec{t}\!\downarrow) \wedge \varphi(\vec{t}/\vec{x})]\!] \leq [\![\vec{y}.(\vec{t}\!\downarrow) \wedge \psi(\vec{t}/\vec{x})]\!] \leq [\![\vec{y}.\psi(\vec{t}/\vec{x})]\!]$$

The axiom for strictness of predicates (b3) is verified thus:

$$
\begin{aligned}
[\![\vec{x}.R(\vec{t})]\!] &= [\![\vec{x}.\vec{t}]\!]_p^{\sim 1}(M(R)) \\
&\leq d_{[\![\vec{x}.\vec{t}]\!]} \\
&= d_{[\![\vec{x}.t_1]\!]} \wedge \cdots \wedge d_{[\![\vec{x}.t_n]\!]} \\
&\leq d_{[\![\vec{x}.t_k]\!]} \\
&= [\![\vec{x}.t_k\!\downarrow]\!].
\end{aligned}
$$

The first inequality is (25), the following equality is (21) and the last equality is (26). The verification of (b4) is similar, and could in fact be regarded a special case of the above setting R to $=_S$. The validity of strictness axiom for functions (b5) follows by using (26) and then (20):

$$
\begin{aligned}
[\![\vec{x}.f(t_1,\ldots,t_n)\!\downarrow]\!] &= d_{[\![\vec{x}.f(t_1,\ldots,t_n)]\!]} \\
&= d_{M(f) \odot \langle [\![\vec{x}.t_1]\!],\ldots,[\![\vec{x}.t_n]\!] \rangle_p} \\
&\leq d_{\langle [\![\vec{x}.t_1]\!],\ldots,[\![\vec{x}.t_n]\!] \rangle_p} \\
&\leq [\![\vec{x}.t_k\!\downarrow]\!].
\end{aligned}
$$

From the above verifications we conclude

Theorem 8.6 *PHL is sound for the categorical semantics in any cartesian category.* $\square$

We shall prove that also partial intuitionistic first-order logic PFOL is sound for the given semantics but in Heyting categories. It suffices to extend Lemma 8.5 to all first order formulae in order to verify the partial term substitution rule. In fact, we can prove soundness of subsystems of PFOL, corresponding to regular logic, and coherent logic as well. Therefore we look at the different classes of categories: regular categories (cartesian categories with images, which are stable under pullback), coherent categories (a regular category with finite sums, such that all substitution functions f^{-1} preserve finite joins) and Heyting categories (coherent categories in which each f^{-1} has a right adjoint). We refer to [Joh02a] for further background, including the Frobenius and Beck-Chevalley conditions.

Let $\mathcal{C}$ be a regular category. For a partial map $f = (D, d, m) : X \rightharpoonup Y$, the partial substitution operation $f^{\sim 1} : \mathrm{Sub}(Y) \longrightarrow \mathrm{Sub}(X)$ may be decomposed into an ordinary substitution and extential quantifier

$$f^{\sim 1} = \exists_d \circ m^{-1},$$

where $m^{-1} : \mathrm{Sub}(Y) \longrightarrow \mathrm{Sub}(D)$ and $\exists_d : \mathrm{Sub}(D) \longrightarrow \mathrm{Sub}(X)$. The existential quantifier $\exists_d$ quantifies over domains that have at most one element, since d is mono, and indeed in this case

$$\exists_d([h]) = [d \circ h] \tag{28}$$

for any mono h.

Ordinary substitution commutes with all logical operations associated with regular, coherent and Heyting categories respectively. This is naturally not the case with existential quantification, even of the special form above. However we have the following useful results: Lemmas 8.7, 8.8 and 8.10.

Lemma 8.7 *Let $\mathcal{C}$ be a regular category. For any monomorphism $f : X \longrightarrow Y$ and subobjects A and B of X:*

(i) $\exists_f(\top_X) = [f]$

(ii) $\exists_f(A \wedge B) = \exists_f(A) \wedge \exists_f(B)$

(iii) $\exists_f(A) \leq [f]$

(iv) $f^{-1}(\exists_f(A)) = A$

(v) *if $\exists_f(A) \leq \exists_f(B)$, then $A \leq B$*

Proof. Item (i) and (iii) follows directly from (28). The direction $\leq$ of (ii) is monotonicity of $\exists_f$. The reverse direction follows since if P is a pullback of $f\alpha$ and $g\alpha$ it is also a pullback of α and β, whenever f is mono. The equation (iv) is true since for monomorphisms $\alpha : A \longrightarrow X$, the pullback of $f\alpha$ along f is α. The implication (vi) follows using (28) and that f is mono. $\square$

Lemma 8.8 *In a regular category we have for morphisms with $f \circ g = h \circ k$ that*

$$\exists_f \circ \exists_g = \exists_h \circ \exists_k.$$

Proof. Using $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ and adjointness gives $\exists_{f \circ g} = \exists_f \circ \exists_g$. From this the result is immediate. $\square$

Lemma 8.9 *Let $\mathcal{C}$ be a coherent category. For any partial $f : X \to Y$ and subobjects $A \rightarrowtail Y$ and $B \rightarrowtail Y$:*

(i) $f^{\sim 1}(\bot_Y) = \bot_X$

(ii) $f^{\sim 1}(A \vee B) = f^{\sim 1}(A) \vee f^{\sim 1}(B)$.

Proof. Write $f = (D, d, m)$, so that $f^{\sim 1} = \exists_d \circ m^{-1}$. The following holds for *any* $d : D \longrightarrow X$, and subobjects $S \rightarrowtail D$ and $T \rightarrowtail D$, in a coherent category:

(i) $\exists_d(\bot_D) = \bot_X$

(ii) $\exists_d(S \vee T) = \exists_d(S) \vee \exists_d(T)$. $\square$
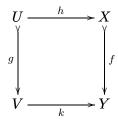
Combining this with the fact that m^{-1} commutes with finite joins in a coherent category we obtain the result. $\square$

Lemma 8.10 *Let $\mathcal{C}$ be a Heyting category.*

(i) *For any monomorphism $f : X \longrightarrow Y$ and subobjects $A \rightarrowtail X$ and $B \rightarrowtail X$:*

$$[f] \wedge (\exists_f(A) \Rightarrow \exists_f(B)) = \exists_f(A \Rightarrow B).$$

(ii) *For any pullback square, where f is mono,*

and for any subobject $A \rightarrowtail U$,

$$[f] \wedge \forall_k(\exists_g(A)) = \exists_f(\forall_h(A)).$$

(iii) *For any partial morphism $f : X \to Y$ and subobjects A B of X:*

$$d_f \wedge (f^{\sim 1}(A) \Rightarrow f^{\sim 1}(B)) = f^{\sim 1}(A \Rightarrow B).$$

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Proof. We subdivide the proof of (i) in two parts

(ia) $\exists_f(A \Rightarrow B) \leq \exists_f(A) \Rightarrow \exists_f(B)$

(ib) $[f] \wedge (\exists_f(A) \Rightarrow \exists_f(B)) \leq \exists_f(A \Rightarrow B)$.

Since $\exists_f(A \Rightarrow B) \leq [f]$ by Lemma 8.7.(iii) this will suffice. The inequality (ia) follows by applying $\exists_f$ to $(A \Rightarrow B) \wedge A \leq B$ and then using Lemma 8.7.(ii) and the adjunction property for implication. Using Lemma 8.7.(i) and the Frobenius reciprocity we may rewrite (ib) as

$$\exists_f(\top_X \wedge f^{-1}(\exists_f(A) \Rightarrow \exists_f(B))) \leq \exists_f(A \Rightarrow B)$$

Using Lemma 8.7.(v) we see that this is equivalent to

$$f^{-1}(\exists_f(A) \Rightarrow \exists_f(B)) \leq A \Rightarrow B$$

Now distributing f^{-1} over the implication and using Lemma 8.7.(iv) we see that it is equivalent to a tautology.

To prove (ii) we start from $g^{-1}(\exists_g(A)) = A$ in Lemma 8.7.(i) and apply $\forall_h$ to get

$$\forall_h(g^{-1}(\exists_g(A))) = \forall_h(A).$$

The Beck-Chevalley property for the pullback square and intersection with $\top$ give

$$\top \wedge f^{-1}(\forall_k(\exists_g(A))) = \forall_h(A).$$

Thus applying $\exists_f$:
$$\exists_f(\top \wedge f^{-1}(\forall_k(\exists_g(A)))) = \exists_f(\forall_h(A)).$$

Using Frobenius reciprocity and Lemma 8.7.(i) the lefthand side may be rewritten as $[f] \wedge \forall_k(\exists_g(A))$ and the equality is proved.

To prove (iii), substitute d_f for f in (i) and use the fact that

$$f^{\sim 1}(A) = \exists_{d_f}(m_f^*(A)).$$

m_f^*, which pulls back subobjects along m_f, preserves $\Rightarrow$. $\square$

Lemma 8.11 (Partial Beck-Chevalley) *Let $\mathcal{C}$ be a regular category. Consider partial morphisms $f = (D, d, m) : X \to Y$ and $g = (E, e, n) : U \to V$ in a partial pullback square*

Then

(i) $f^{\sim1} \circ \exists_h = \exists_k \circ g^{\sim1}$

(ii) if $\mathcal{C}$ is Heyting, then for any subobject $A \rightarrowtail V$:

$$f^{\sim1}(\forall_h(A)) = [d] \wedge \forall_k(g^{\sim1}(A)).$$

Proof. Write the partial pullback square explicitly as

To prove (i) we calculate, applying first the usual Beck-Chevalley and then Lemma 8.8,

$$
\begin{aligned}
f^{\sim1} \circ \exists_h &= \exists_d \circ m^{-1} \circ \exists_h \\
&= \exists_d \circ \exists_j \circ n^{-1} \\
&= \exists_k \circ \exists_e \circ n^{-1} = \exists_k \circ g^{\sim1}.
\end{aligned}
$$

As for (ii), assume $\mathcal{C}$ is a Heyting category and that $A \rightarrowtail V$ is any subobject. Lemma 8.10.(ii) and the ordinary Beck-Chevalley property now yield

$$
\begin{aligned}
[d] \wedge \forall_k(g^{\sim1}(A)) &= [d] \wedge \forall_k(\exists_e(n^{-1}(A))) \\
&= \exists_d(\forall_j(n^{-1}(A))) \\
&= \exists_d(m^{-1}(\forall_h(A))) \\
&= f^{\sim1}(\forall_h(A))
\end{aligned}
$$

$\square$

Lemma 8.12 *Let M be a partial Σ-structure in a cartesian category. Consider a vector $\vec{t}$ of terms over Σ suitable for the context $\vec{y}$ and let $\vec{x}$ be a context sort compatible with $\vec{t}$. Let z be a variable neither in $\vec{x}$ nor in $\vec{y}$. Then these term interpretations fits into a partial pullback square:*

Here p and q are the canonical projections.

Proof. According to Example 7.4

$$\begin{array}{ccc}
M(\sigma(\vec{y})) \times M(\sigma(z)) & \xrightarrow{[\![\vec{y}.\vec{t}]\!]\times_p 1} & M(\sigma(\vec{x})) \times M(\sigma(z)) \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} \\
M(\sigma(\vec{y})) & \xrightarrow[{[\![\vec{y}.\vec{t}]\!]}]{} & M(\sigma(\vec{x}))
\end{array}$$

is a partial pullback square. Composing the upper partial arrow with the canonical isomorphisms $M(\sigma(\vec{y}z)) \cong M(\sigma(\vec{y})) \times M(\sigma(z))$ and $M(\sigma(\vec{x})) \times M(\sigma(z)) \cong M(\sigma(\vec{y}z))$ we get $[\![\vec{y}z.\vec{t}z]\!]$ (using Lemma 8.4) and the desired partial pullback square. $\square$

Lemma 8.13 *Let $\mathcal{C}$ be a cartesian category. Let M be a partial Σ-structure in $\mathcal{C}$, and write $[\![\cdot]\!]$ for $[\![\cdot]\!]_M$. Consider a first-order Σ-formulae φ suitable for the context $\vec{x}$, and $\vec{t}$ terms over Σ suitable for the context $\vec{y}$, and sort compatible with $\vec{x}$. Then*

$$[\![\vec{y}.(\vec{t}\downarrow) \wedge \varphi(\vec{t}/\vec{x})]\!] = [\![\vec{y}.\vec{t}]\!]^{\sim 1}([\![\vec{x}.\varphi]\!]), \tag{29}$$

if

(i) φ *is a regular formula and $\mathcal{C}$ regular, or*

(ii) φ *is a coherent formula and $\mathcal{C}$ coherent, or*

(iii) φ *is a first-order formula and $\mathcal{C}$ Heyting.*

Proof. The proof goes by induction on the formula φ. In Lemma 8.5 we already dealt with the atomic cases and the conjunctive case.

Part (i). For a regular φ and regular category $\mathcal{C}$ we need to check the $\exists$-case.

Case $\varphi = (\exists z \in V)\psi$: We may assume that z is not in $\vec{x}$ or $\vec{y}$. Let p be the projection of $M(\sigma(\vec{y}z))$ on $M(\sigma(\vec{y}))$, q the projection of $M(\sigma(\vec{x}z))$ on $M(\sigma(\vec{x}))$. We write $\varphi(\vec{t}/\vec{x}) = (\exists z \in V)\psi(\vec{t}/\vec{x}) = (\exists z \in V)\psi(\vec{t}, z/\vec{x}, z)$ and calculate

$$\begin{aligned}
[\![\vec{y}.(\vec{t}\downarrow) \wedge (\exists z \in V)\psi(\vec{t}, z/\vec{x}, z)]\!] &= [\![\vec{y}.\vec{t}\downarrow]\!] \wedge [\![\vec{y}.(\exists z \in V)\psi(\vec{t}, z/\vec{x}, z)]\!] \\
&= [\![\vec{y}.\vec{t}\downarrow]\!] \wedge \exists_p([\![\vec{y}z.\psi(\vec{t}, z/\vec{x}, z)]\!]) \\
&= \exists_p(p^{-1}([\![\vec{y}.\vec{t}\downarrow]\!]) \wedge [\![\vec{y}z.\psi(\vec{t}, z/\vec{x}, z)]\!]) \\
&= \exists_p([\![\vec{y}z.\vec{t}\downarrow \wedge z\downarrow]\!] \wedge [\![\vec{y}z.\psi(\vec{t}, z/\vec{x}, z)]\!]) \\
&= \exists_p([\![\vec{y}z.\vec{t}\downarrow \wedge z\downarrow \wedge \psi(\vec{t}, z/\vec{x}, z)]\!]) \\
&= \exists_p([\![\vec{y}z.\vec{t}, z]\!]^{\sim 1}([\![\vec{x}z.\psi]\!])) \\
&= [\![\vec{y}.\vec{t}]\!]^{\sim 1}(\exists_q([\![\vec{x}z.\psi]\!])) \\
&= [\![\vec{y}.\vec{t}]\!]^{\sim 1}[\![\vec{x}.(\exists z \in V)\psi]\!])
\end{aligned}$$

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The third member is the Frobenius reciprocity. The sixth is the inductive hypothesis. The seventh member is Lemma 8.11.(i) using the partial pullback square of Lemma 8.12.

Part (ii). For coherent φ and coherent $\mathcal{C}$ there are in addition cases for $\bot$, $\lor$:

Case $\varphi = \bot$: $[\![\vec{y}.(\vec{t}\downarrow)]\!] \land \bot]\!] = \bot = [\![\vec{y}.\vec{t}\,]\!]^{\sim 1}(\bot) = [\![\vec{y}.\vec{t}\,]\!]^{\sim 1}([\![\vec{x}.\bot]\!])$. Here the second member follows from Lemma 8.9.(i).

Case $\varphi = \varphi_1 \lor \varphi_2$: This case follows using Lemma 8.9.(ii), the inductive hypothesis, and distributivity of the lattice of subobjects.

$$
\begin{aligned}
[\![\vec{y}.(\vec{t}\downarrow) \land (\varphi_1 \lor \varphi_2)(\vec{t}/\vec{x})]\!] &= [\![\vec{y}.\vec{t}\downarrow]\!] \land ([\![\vec{y}.\varphi_1(\vec{t}/\vec{x}]\!] \lor [\![\vec{y}.\varphi_2(\vec{t}/\vec{x})]\!]) \\
&= ([\![\vec{y}.\vec{t}\downarrow]\!] \land [\![\vec{y}.\varphi_1(\vec{t}/\vec{x})]\!]) \lor ([\![\vec{y}.\vec{t}\downarrow]\!] \land [\![\vec{y}.\varphi_2(\vec{t}/\vec{x})]\!]) \\
&= ([\![\vec{y}.(\vec{t}\downarrow) \land \varphi_1(\vec{t}/\vec{x})]\!]) \lor ([\![\vec{y}.(\vec{t}\downarrow) \land \varphi_2(\vec{t}/\vec{x})]\!]) \\
&= [\![\vec{y}.\vec{t}\,]\!]^{\sim 1}([\![\vec{x}.\varphi_1]\!]) \lor [\![\vec{y}.\vec{t}\,]\!]^{\sim 1}([\![\vec{x}.\varphi_2]\!]) \\
&= [\![\vec{y}.\vec{t}\,]\!]^{\sim 1}([\![\vec{x}.\varphi_1]\!] \lor [\![\vec{x}.\varphi_2]\!]) \\
&= [\![\vec{y}.\vec{t}\,]\!]^{\sim 1}([\![\vec{x}.\varphi_1 \lor \varphi_2]\!]).
\end{aligned}
$$

Part (iii). Finally, for a general φ and Heyting category $\mathcal{C}$ there are the additional cases for $\forall$ and $\Rightarrow$.

Case $\varphi = (\forall z \in V)\psi$: Let p and q be the same projections as in the existential case.

$$
\begin{aligned}
[\![\vec{y}.\vec{t}\downarrow \land (\forall z \in V)\psi(\vec{t}, z/\vec{x}, z)]\!] &= [\![\vec{y}.\vec{t}\downarrow]\!] \land [\![\vec{y}.(\forall z \in V)\psi(\vec{t}, z/\vec{x}, z)]\!] \\
&= [\![\vec{y}.\vec{t}\downarrow]\!] \land \forall_p([\![\vec{y}z.\psi(\vec{t}, z/\vec{x}, z)]\!]) \\
&= [\![\vec{y}.\vec{t}\downarrow]\!] \land ([\![\vec{y}.\vec{t}\downarrow]\!] \land \forall_p([\![\vec{y}z.\psi(\vec{t}, z/\vec{x}, z)]\!])) \\
&= [\![\vec{y}.\vec{t}\downarrow]\!] \land \forall_p(p^{-1}([\![\vec{y}.\vec{t}\downarrow]\!]) \land [\![\vec{y}z.\psi(\vec{t}, z/\vec{x}, z)]\!]) \\
&= d_{[\![\vec{y}.\vec{t}]\!]} \land \forall_p([\![\vec{y}z.(\vec{t}\downarrow \land z\downarrow) \land \psi(\vec{t}, z/\vec{x}, z)]\!]) \\
&= d_{[\![\vec{y}.\vec{t}]\!]} \land \forall_p([\![\vec{y}z.\vec{t}z]\!]^{\sim 1}([\![\vec{x}z.\psi]\!])) \\
&= [\![\vec{y}.\vec{t}\,]\!]^{\sim 1}(\forall_q([\![\vec{x}z.\psi]\!])) \\
&= [\![\vec{y}.\vec{t}\,]\!]^{\sim 1}([\![\vec{x}.(\forall z \in V)\psi]\!])
\end{aligned}
$$

In the fourth member, $\leq$ is the law $A \land \forall_p(B) \leq \forall_p(p^{-1}(A) \land B)$, and $\geq$ is monotonicty of $\forall_p$. The sixth member is the inductive hypothesis. The seventh member is Lemma 8.11.(ii) using the partial pullback square of Lemma 8.12.

Case $\varphi = \varphi_1 \Rightarrow \varphi_2$: This case follows using Lemma 8.9.(ii), the inductive hypothesis, and distributivity of the lattice of subobjects.

$$
\begin{aligned}
[\![\vec{y}.(\vec{t}\downarrow) \land (\varphi_1 \Rightarrow \varphi_2)(\vec{t}/\vec{x})]\!] &= [\![\vec{y}.\vec{t}\downarrow]\!] \land ([\![\vec{y}.\varphi_1(\vec{t}/\vec{x})]\!] \Rightarrow [\![\vec{y}.\varphi_2(\vec{t}/\vec{x})]\!]) \\
&= [\![\vec{y}.\vec{t}\downarrow]\!] \land (([\![\vec{y}.\vec{t}\downarrow]\!] \land [\![\vec{y}.\varphi_1(\vec{t}/\vec{x})]\!]) \Rightarrow ([\![\vec{y}.\vec{t}\downarrow]\!] \land [\![\vec{y}.\varphi_2(\vec{t}/\vec{x})]\!])) \\
&= d_{[\![\vec{y}.\vec{t}]\!]} \land ([\![\vec{y}.\vec{t}\,]\!]^{\sim 1}[\![\vec{x}.\varphi_1]\!] \Rightarrow [\![\vec{y}.\vec{t}\,]\!]^{\sim 1}[\![\vec{x}.\varphi_2]\!]) \\
&= [\![\vec{y}.\vec{t}\,]\!]^{\sim 1}([\![\vec{x}.\varphi_1]\!] \Rightarrow [\![\vec{x}.\varphi_2]\!]) \\
&= [\![\vec{y}.\vec{t}\,]\!]^{\sim 1}([\![\vec{x}.\varphi_1 \Rightarrow \varphi_2]\!]).
\end{aligned}
$$

The fourth step of this uses part (iii) of Lemma 8.10. $\square$

8.1 Homomorphisms

Suppose $\mathbb{T}$ is a quasi-equational theory, $\mathcal{C}$ a cartesian category, and M and N models of $\mathbb{T}$ in $\mathcal{C}$. Then the notion of homomorphism between M and N will be defined in the obvious way.

Some simple category theory helps the discussion of homomorphisms.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, with $\mathcal{C}$ cartesian. Then the functor category $[\mathcal{D}, \mathcal{C}]$ is cartesian. (See [Joh02a, A 1.2.1].) We shall use this in the particular case where $\mathcal{D}$ is the category $\vec{2}$, with two objects and one non-identity morphism between them. Then $[\vec{2}, \mathcal{C}]$ is the arrow category for $\mathcal{C}$, whose objects are morphisms in $\mathcal{C}$ and whose morphisms are commutative squares.

Proposition 8.14 *There is a bijection between –*

- *triples (M, α, N) where M and N are models of $\mathbb{T}$ in $\mathcal{C}$ and $\alpha : M \longrightarrow N$ is a homomorphism, and*

- *models of $\mathbb{T}$ in $[\vec{2}, \mathcal{C}]$.*

Proof. The bijection is obvious for partial structures. Next, Proposition 3.9 generalizes to this categorical setting and this enables us to conclude that the bijection between partial structures restricts to models. □
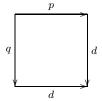
We shall also use later the fact that if $\mathcal{D}$ is another cartesian category, then a cartesian functor $\mathcal{D} \longrightarrow [\vec{2}, \mathcal{C}]$ is equivalent to a pair of cartesian functors $\mathcal{D} \longrightarrow \mathcal{C}$ with a natural transformation between them.

8.2 Cartesian categories equipped with models

In this subsection we show how the categorical semantics is itself quasi-equational. More precisely, if $\mathbb{T}$ is a quasi-equational theory then we can form another quasi-equational theory whose models are cartesian categories equipped with models of $\mathbb{T}$. For this, we introduce the notation $\mathrm{Cart}\varpi\mathbb{T}$, where ϖ is intended to be pronounced "with".

We first discuss some structure of the theory Cart.

An arrow d is monic iff, in its kernel pair

the two projections p and q are equal. Hence monicity can be described by a formula in context d.Mon, where d is of sort arr. in Cart,

$$\mathrm{Mon} \triangleq \mathsf{p}^1_{d,d} = \mathsf{p}^2_{d,d}$$

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Then partial morphisms can be described by a formula $d, m.\mathrm{pArr}$ (m also of sort arr),

$$\mathrm{pArr} \triangleq \mathrm{Mon} \wedge \mathsf{d}(d) = \mathsf{d}(m)$$

We shall use such formulae in a shorthand for describing functions and axioms. For instance, a "constant a of type pArr" is a pair of constants a_d and a_m, with an axiom

$$\vdash\!\!\!\!\!-\ \mathrm{pArr}(a_d, a_m/d, m)$$

We can talk about partial morphisms in the obvious way. For instance, each partial morphism (d, m) has a domain (the codomain of d), a codomain (the codomain of m) and a domain of definition (the common domain of d and m).

Let $\mathbb{T}$ have sets Sort, Fun and Ax of sorts, function symbols and axioms. These come with additional structure. The domains and codomains of the function symbols are expressed by functions

$$\mathrm{dom} : \mathrm{Fun} \longrightarrow \mathrm{Sort}^*$$

$$\mathrm{cod} : \mathrm{Fun} \longrightarrow \mathrm{Sort}$$

where Sort^* is the set of finite lists over Sort.

From these can be constructed the set TT of *term tuples in context*, modulo renaming of variables. (It is clear that the variables in a context are just a device for labelling components in a product of sorts, so their names are inessential.) We have domain and codomain functions

$$\mathrm{dom}, \mathrm{cod} : \mathrm{TT} \longrightarrow \mathrm{Sort}^*.$$

Finally we have a set Form of equation sequences – in other words, the formulae. These are the pairs of term tuples $(\vec{t_1}, \vec{t_2})$ with common domain (for the context) and codomain (for the number and sorts of the component equations). Again, each equation sequence has a domain and codomain.

Now the structure of the axiom set Ax is given by two functions for premiss and conclusion,

$$\mathrm{prem}, \mathrm{conc} : \mathrm{Ax} \longrightarrow \mathrm{Form}$$

such that for each axiom a in Ax,

$$\mathrm{dom}(\mathrm{prem}(a)) = \mathrm{dom}(\mathrm{conc}(a))$$

(for the context of the axiom).

Remark 8.15 It is possible to formalize a type theory whose models are quasi-equational theories. It will not be a theory in finitary first-order logic because of its use of list types such as Sort^*. However, it is possible to formalize it as a *geometric* theory, using infinitary disjunctions [Joh02b], because the type constructions required can be characterized using geometric structure and axioms [Vic99].

We now turn to our description of $\mathrm{Cart}\varpi\mathbb{T}$.

First, its only sorts are obj and arr. In addition, it has all the functions and axioms of Cart.

Next, for each $S \in \mathrm{Sort}$ there is a constant γ^S of sort obj. Since products of objects can be constructed with Cart, we can now find for each sequence $\vec{S} \in \mathrm{Sort}^*$ a term $().\gamma^{\vec{S}}$ representing the product of the components γ^{S_i}.

Next, for each $f \in \mathrm{Fun}$ there is a constant γ^f of type pArr – that is to say, as described above, constants γ_d^f and γ_m^f of sort arr with an axiom

$$\vdash\!\!\!-\!\!-\ \mathrm{pArr}(\gamma_d^f, \gamma_m^f/d, m).$$

If $\mathrm{dom}(f) = \vec{S}$ and $\mathrm{cod}(f) = T$ then we also have axioms

$$\vdash\!\!\!-\!\!-\ \mathsf{c}(\gamma_d^f) = \gamma^{\vec{S}},$$
$$\vdash\!\!\!-\!\!-\ \mathsf{c}(\gamma_m^f) = \gamma^T.$$

Now for each $\vec{t} \in \mathrm{TT}$, the categorical semantics provides corresponding terms in context $().\gamma_d^{\vec{t}}$ and $().\gamma_m^{\vec{t}}$ for the partial morphism corresponding to $\vec{t}$. Also, for each formula in context $\varphi \in \mathrm{Form}$ it provides a term in context $().\gamma^\varphi$ for the monic corresponding to φ.

Next, for each $a \in \mathrm{Ax}$ there is a constant γ^a to represent the monic showing that the premiss φ is a subset of the conclusion ψ. It has a single axiom

$$\vdash\!\!\!-\!\!-\ \gamma^\varphi = \gamma^\psi \circ \gamma^a.$$

That completes the description of $\mathrm{Cart}\varpi\mathbb{T}$. Since it includes Cart, every model certainly reduces to a cartesian category $\mathcal{C}$. But the extra constants and axioms just specify a model of $\mathbb{T}$ in $\mathcal{C}$.

9 Quasi-equational theories are equivalent to cartesian theories

It is well known that total function symbols can be eliminated in favour of predicates. It is done by introducing, for each function symbol $f : \vec{A} \longrightarrow B$, a predicate $\Gamma_f \rightarrowtail \vec{A}, B$. This is constrained by an axiom $\Gamma_f(\vec{x}, y) \overset{\vec{x},y}{\vdash\!\!\!-\!\!-} y = f(\vec{x})$ to represent the graph of f. Then a graph formula $\vec{z}, w.\Gamma_t$ can be defined for every term in context $(\vec{z}.t)$, with provably $\Gamma_t \overset{\vec{z},w}{\vdash\!\!\!-\!\!-} w = t$. In particular, suppose t is $f(\vec{s})$ with $\vec{s}$ compatible with $\vec{x}$. Then Γ_t is

$$(\exists \vec{x})(\Gamma_f(\vec{x}, w) \wedge \bigwedge_i \Gamma_{s_i}(x_i/w)).$$

Then each formula in context $(\vec{x}.\varphi)$ has a provably equivalent version $(\vec{x}.\Gamma_\varphi)$ using graphs instead of terms.

We now find that the original theory is equivalent to one with –

- the original sorts and predicates,

- the graph predicates Γ_f,

- functionhood axioms for each Γ_f (single-valuedness and totality),

- axioms $\Gamma_\varphi \xmapsto{\ \vec{x}\ } \Gamma_\psi$ for each original axiom $\varphi \xmapsto{\ \vec{x}\ } \psi$.

The same can be done with partial function symbols, the only difference being that we do not introduce totality axioms for the graph predicates. It follows that our partial interpretation of function symbols brings no change in expressive power of theories when predicate symbols are available.

It is not hard to see that the above argument works in the presence of *regular* logic, which has $\top$, $\wedge$, $=$ and $\exists$ with their usual rules, together with the Frobenius rule

$$\varphi \wedge (\exists y)\psi \xmapsto{\ \vec{x}\ } (\exists y)(\varphi \wedge \psi).$$

However, it also works for *cartesian* logic (see [Joh02b, D 1.3.4]), in which witnesses for existential quantifications are required to be provably unique.

Definition 9.1 *Let $\mathbb{T}$ be a first-order theory. The formulae in context that are* cartesian *relative to $\mathbb{T}$ are those that can be constructed using atomic formulae $R(\vec{s})$, equations $s = t$, conjunctions $\top$ and $\wedge$, and provably unique existential quantification in the following way. Let $(\vec{x}, y.\varphi)$ be cartesian relative to $\mathbb{T}$, and suppose from $\mathbb{T}$ it can be proved that*

$$\varphi \wedge \varphi(y'/y) \xmapsto{\ \vec{x},y,y'\ } y = y'.$$

Then $(\vec{x}.(\exists y)\varphi)$ is cartesian relative to $\mathbb{T}$.

Definition 9.2 *A regular theory $\mathbb{T}$ is* cartesian *if there is some well-founded partial ordering of its axioms such that in each axiom $\varphi \xmapsto{\ \vec{x}\ } \psi$, the formulae φ and ψ are cartesian relative to the axioms that precede it in the ordering.*

It is easy to check that when you eliminate partial function symbols in favour of predicates, the formulae introduced are all cartesian. Hence if the original theory was cartesian, so is the new one.

The importance of cartesian theories lies in the fact that their logic is embodied in the categorical structure of cartesian categories. This is obscured by the way they are defined using $\exists$, which is *not* interpreted directly in all cartesian categories – in general it requires image factorization. Nonetheless, a simple lemma shows that cartesian formulae are interpretable.

This excursion via regular logic makes cartesian theories slightly difficult to deal with. One of the advantages of our quasi-equational theories is that they provide a more direct way to handle cartesian theories, with a logic whose notation can be interpreted in full in cartesian categories.

In fact, quasi-equational theories are equivalent to cartesian theories. We already know how to eliminate the partial function symbols in favour of predicates and cartesian axioms. The reverse can also be done, with predicates eliminated in favour of partial function symbols, and in Section 9.4 we shall show how to do this explicitly. Meanwhile, we give a more theoretical treatment.

[Joh02b, D 1.4.8] shows that, up to equivalence of model categories, cartesian theories are 'the same thing as' small cartesian categories. What this really means is that cartesian theories are a scheme for presenting cartesian categories – of course, quite different theories may present equivalent categories. The basis for the assertion essentially comprises two parts. First, Theorem D 1.4.7 shows that any cartesian theory $\mathbb{T}$ has a *syntactic category* $\mathcal{C}_{\mathbb{T}}$, a cartesian category characterized by

$$\mathrm{Cart}(\mathcal{C}_{\mathbb{T}}, \mathcal{D}) \simeq \mathbb{T}\text{-}\mathrm{Mod}(\mathcal{D})$$

for any cartesian category $\mathcal{D}$. This is the cartesian category "presented by" $\mathbb{T}$, which appears in $\mathcal{C}_{\mathbb{T}}$ as a generic model. Next, Example D 1.4.8 shows that for any small cartesian category $\mathcal{C}$ there is a cartesian theory $\mathrm{Th}(\mathcal{C})$ such that

$$\mathrm{Cart}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Th}(\mathcal{C})\text{-}\mathrm{Mod}(\mathcal{D})$$

for any cartesian category $\mathcal{D}$. It follows that $\mathcal{C} \simeq \mathcal{C}_{\mathrm{Th}(\mathcal{C})}$. Two theories $\mathbb{T}$ and $\mathbb{T}'$ are said to be *Morita equivalent* if their syntactic categories are equivalent.

As ways of discussing the theories, both forms have their drawbacks. The definition of cartesian theory (using existential quantification but only where it is provably unique) is slightly awkward, and in practice the entire syntactic category is too complex to deal with. Our aim now is to show that quasi-equational theories also are 'the same thing as' small cartesian categories. It follows that cartesian theories can always be replaced by quasi-equational theories. For many purposes the quasi-equational theories are a simple and convenient mode of presentation, and this has already been demonstrated in the free construction, Section 5.

Our proof is in two parts, just as in [Joh02b]. First we show how to construct a syntactic category $\mathcal{C}_{\mathbb{T}}$ for each quasi-equational theory $\mathbb{T}$, and then we show how to construct a quasi-equational theory $\mathrm{Th}(\mathcal{C})$ for each cartesian category $\mathcal{C}$.

9.1 Classifying categories for quasi-equational theories

Our purpose in this section is to show that for each quasi-equational theory $\mathbb{T}$ there is a *classifying category* $\mathcal{C}_{\mathbb{T}}$. This is a cartesian category that is freely generated by a *generic model* $M_{\mathbb{T}}$ of $\mathbb{T}$. What this means technically is that for any cartesian category $\mathcal{D}$, there is a bijection between models of $\mathbb{T}$ in $\mathcal{D}$ and strict cartesian functors from $\mathcal{C}_{\mathbb{T}}$ to $\mathcal{D}$.

For the rest of this subsection, we fix a quasi-equational theory $\mathbb{T}$.

Definition 9.3 We write $(\mathcal{C}_{\mathbb{T}}, M_{\mathbb{T}})$ for the term model for $\mathrm{Cart}\varpi\mathbb{T}$.

The cartesian category $\mathcal{C}_{\mathbb{T}}$, its reduct to Cart, is called the *classifying category* for $\mathbb{T}$, and $M_{\mathbb{T}}$, the model of $\mathbb{T}$ with which $\mathcal{C}_{\mathbb{T}}$ is equipped, is the *generic model* of $\mathbb{T}$.

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Theorem 9.4 *For any cartesian category $\mathcal{D}$, there is an isomorphism of categories between $\mathbb{T}$-PMod$(\mathcal{D})$ and* StrCart$(\mathcal{C}_\mathbb{T}, \mathcal{D})$ *(the category of strict cartesian functors).*

Proof. Suppose M is a model of $\mathbb{T}$ in $\mathcal{D}$. Then $(\mathcal{D}, M)$ is a model of Cart$\varpi\mathbb{T}$ and so there is a unique homomorphism $(\mathcal{C}_\mathbb{T}, M_\mathbb{T}) \longrightarrow (\mathcal{D}, M)$. In other words, there is a unique strict cartesian functor

$$F_M : \mathcal{C}_\mathbb{T} \longrightarrow \mathcal{D}$$

that maps $M_\mathbb{T}$ to M.

On the other hand, any strict cartesian functor $F : \mathcal{C}_\mathbb{T} \longrightarrow \mathcal{D}$ maps $M_\mathbb{T}$ to a model M in $\mathcal{D}$ and is the unique such F. Hence, there is a bijection between models M and strict cartesian functors F.
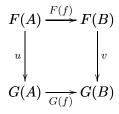
We must still prove functoriality. By Proposition 8.14, a homomorphism α between models in $\mathcal{C}$ is the same as a model in $[\vec{2}, \mathcal{D}]$, and hence corresponds to a strict cartesian functor $\mathcal{C}_\mathbb{T} \longrightarrow [\vec{2}, \mathcal{D}]$. This in turn corresponds to a natural transformation between two strict cartesian functors from $\mathcal{C}_\mathbb{T}$ to $\mathcal{D}$. One can then check the details to show that identities and composition are preserved. $\square$

We shall also prove a version that relaxes the strictness and has equivalence of the categories. We first prove a lemma.

Lemma 9.5 *Let $F, G : \mathcal{C}_\mathbb{T} \longrightarrow \mathcal{D}$ be two cartesian functors. Then for each homomorphism $\alpha : F(M_\mathbb{T}) \longrightarrow G(M_\mathbb{T})$ there is a unique natural transformation $\alpha' : F \longrightarrow G$ whose restriction to $M_\mathbb{T}$ is α.*

If α is an isomorphism, then α' is a natural isomorphism.

Proof. Let $\mathcal{E}$ be a category defined as follows. Its objects are pairs (A, u) where A is an object in $\mathcal{C}_\mathbb{T}$ and $u : F(A) \longrightarrow G(A)$ in $\mathcal{D}$. A morphism from (A, u) to (B, v) is a morphism $f : A \longrightarrow B$ in $\mathcal{C}_\mathbb{T}$ such that this square commutes.

$$
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(B) \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\
G(A) & \xrightarrow[G(f)]{} & G(B)
\end{array}
$$

One can show that $\mathcal{E}$ is cartesian, with canonical limits determined by the canonical limits in $\mathcal{C}_\mathbb{T}$. (In fact, $\mathcal{D}$ doesn't need canonical limits here.) Let $P : \mathcal{E} \longrightarrow \mathcal{C}_\mathbb{T}$ be the strict cartesian functor that, on each object (A, u), forgets u.

We now find a series of bijective correspondences. Homomorphisms $\alpha : F(M_\mathbb{T}) \longrightarrow G(M_\mathbb{T})$ correspond to $\mathbb{T}$-models M in $\mathcal{E}$ such that $P(M) = M_\mathbb{T}$. Then by Theorem 9.4 these correspond to strict cartesian functors $K : \mathcal{C}_\mathbb{T} \longrightarrow \mathcal{E}$ such that $P \circ K = \mathrm{Id}$, and these correspond to natural transformations from F to G.

If we consider a further cartesian functor $H : \mathcal{C}_\mathbb{T} \longrightarrow \mathcal{D}$, then it is clear that the process of restricting natural transformations to give model homomorphisms preserves

composition and identities, and so the inverse process does the same. From this we deduce the statement about isomorphisms. □

Theorem 9.6 *For any cartesian category $\mathcal{D}$, there is an equivalence of categories between $\mathbb{T}\text{-PMod}(\mathcal{D})$ and $\text{Cart}(\mathcal{C}_{\mathbb{T}}, \mathcal{D})$.*

Proof. $\text{StrCart}(\mathcal{C}_{\mathbb{T}}, \mathcal{D})$ is a full subcategory of $\text{Cart}(\mathcal{C}_{\mathbb{T}}, \mathcal{D})$ isomorphic to $\mathbb{T}\text{-PMod}(\mathcal{D})$. Hence we already have a functor

$$\Phi : \mathbb{T}\text{-PMod}(\mathcal{D}) \longrightarrow \text{StrCart}(\mathcal{C}_{\mathbb{T}}, \mathcal{D}) \longrightarrow \text{Cart}(\mathcal{C}_{\mathbb{T}}, \mathcal{D})$$

Any cartesian functor $F : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{D}$ gives a model $F(M_{\mathbb{T}})$ in $\mathcal{D}$, and this gives us a functor

$$\Psi : \text{Cart}(\mathcal{C}_{\mathbb{T}}, \mathcal{D}) \longrightarrow \mathbb{T}\text{-PMod}(\mathcal{D})$$

From our work on the strict cartesian functors, $\Psi \circ \Phi$ is the identity on $\mathbb{T}\text{-PMod}(\mathcal{D})$. It remains only to show that if $F : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{D}$ is cartesian, then it is naturally isomorphic to the strict cartesian functor corresponding to $F(M_{\mathbb{T}})$, but this follows from the lemma. □

Our classifying category plays exactly the same role as the *syntactic categories* of [Joh02b, D 1.4]. The only reason we do not call it a syntactic category is that it is not built from the logical syntax – although its construction as a term model is extremely syntactic. The phrase "classifying category" is by analogy with the classifying topos, in which the generic model generates the rest of the category by finite limits and small colimits.

Syntactic categories for quasi-equational theories can be constructed using exactly the same techniques as in [Joh02b]. However, it is then harder to get such close control over the strict cartesian functors and their universal characterization uses equivalence of categories (as in Theorem 9.6) rather than isomorphism (Theorem 9.4). Of course, the universal characterizations are sufficient to show that the syntactic category is equivalent to our classifying category.

We conjecture that our technique using the free partial model theorem can also be applied to give analogues of other syntactic categories. Note that the technique eliminates the need for induction on structure of formulae and proofs. The induction is encapsulated in the free partial model theorem.

9.2 Quasi-equational theories for cartesian categories

Let $\mathcal{C}$ be a (small) cartesian category. We define a quasi-equational theory $\text{Th}(\mathcal{C})$ for it as follows.

For each object A of $\mathcal{C}$, there is a sort $\bar{A}$.

For each partial morphism $f = (\mathrm{d}_f, \mathrm{m}_f) : A \rightharpoonup B$ in $\mathcal{C}$, there is a corresponding function symbol $\bar{f} : \bar{A} \longrightarrow \bar{B}$, with axioms as follows.

For morphisms $f : A \longrightarrow B$ (represented as a partial morphism (Id_A, f)) and $g : B \longrightarrow C$ we have totality and functoriality axioms

(t) $\top \xmapsto{x:\bar{A}} \bar{f}(x)\downarrow$

(f1) $\top \xmapsto{x:\bar{A}} x = \overline{\mathrm{Id}_A}(x)$

(f2) $\top \xmapsto{x:\bar{A}} \bar{g}(\bar{f}(x)) = \overline{g \circ f}(x)$.

Now suppose $d : A \longrightarrow B$ is a monomorphism. Let us write $\tilde{d}$ for the partial morphism $(d, \mathrm{Id}_A) : B \rightharpoonup A$, and $\check{d}$ for (Id_A, d). We introduce axioms

(m1) $\top \xmapsto{x:\bar{A}} \overline{\tilde{d}}(\overline{\check{d}}(x)) = x$

(m2) $\overline{\tilde{d}}(y)\downarrow \xmapsto{y:\bar{B}} \overline{\check{d}}(\overline{\tilde{d}}(y)) = y$.

Next, for each partial morphism $f = (\mathrm{d}_f, \mathrm{m}_f) : A \rightharpoonup B$ we introduce axioms

(pm1) $\bar{f}(x)\downarrow \xmapsto{x:\bar{A}} \overline{\tilde{\mathrm{d}}}_f(x)\downarrow$

(pm2) $\bar{f}(x) \leftteardrop \bar{\mathrm{m}}_f(\overline{\tilde{\mathrm{d}}}_f(x))$.

For each terminal object A in $\mathcal{C}$, there is a constant symbol $u_A : \bar{A}$, with axioms as in Proposition 3.6 constraining $\bar{A}$ to be a terminal object.

For each pullback square P

$$
\begin{array}{ccc}
D & \xrightarrow{\ p\ } & B \\
\scriptstyle q \big\downarrow & & \big\downarrow \scriptstyle f \\
C & \xrightarrow[\ g\ ]{} & A
\end{array}
$$

in $\mathcal{C}$ there is a function symbol $r_P : \bar{B}, \bar{C} \longrightarrow \bar{D}$ with axioms

(pb1) $r_P(x, y)\downarrow \xmapsto{x:\bar{B}, y:\bar{C}} \bar{f}(x) = \bar{g}(y)$,

(pb2) $\bar{p}(r_P(x, y)) \leftteardrop x$,

(pb3) $\bar{q}(r_P(x, y)) \leftteardrop y$,

(pb4) $z \leftteardrop r_P(\bar{p}(z), \bar{q}(z))$.

There is an obvious canonical interpretation of the signature of $\mathrm{Th}(\mathcal{C})$ in $\mathcal{C}$, interpreting $\bar{A}$ and $\bar{f}$ as A and f.

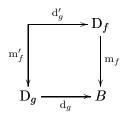**Lemma 9.7** *This canonical interpretation is a model M_0 of $\mathrm{Th}(\mathcal{C})$ in $\mathcal{C}$.*

Proof. It is routine to check the categorical meanings of the axioms. □

In [Joh02b, D 1.4.8] there is an analogous construction of a cartesian theory corresponding to the cartesian category $\mathcal{C}$. That in effect introduces function symbols $\bar{f}$ for *total* f (i.e. morphisms rather than partial morphisms), and in fact a similar construction would serve our immediate purpose (Proposition 9.9). We use a more elaborate theory, with function symbols corresponding to all partial morphisms, in order that models in $\mathcal{C}$ can correspond to theory morphisms to $\mathrm{Th}(\mathcal{C})$. This will be used (Theorem 9.11) to prove a generalization of the intial model theorem.

Axioms (t), (f1) and (f2) serve to enforce functoriality on the morphisms and then the terminal object axioms and axioms (pb1)-(pb4) enforce cartesianness. (However, we do not know how to enforce strict cartesianness in a similar way.) The remaining axioms serve to define the interpretation of symbols $\bar{f}$ for partial morphisms f in terms of the interpretation for morphisms. Axioms (m1) and (m2) deal with the case where $f = (d, \mathrm{Id})$ for a monomorphism d, and axioms (pm1) and (pm2) then deal with the general case.

Lemma 9.8 *Let $\mathcal{D}$ be a cartesian category, and let $f : A \rightarrow B$ and $g : B \rightarrow A$ be two partial morphisms satisfying conditions corresponding to axioms (m1) and (m2), with $\bar{d}$ and $\bar{\bar{d}}$ interpreted as f and g respectively. Then there is a unique monic $d : A \longrightarrow B$ such that $f = (\mathrm{Id}_A, d)$ and $g = (d, \mathrm{Id}_A)$.*

Proof. Suppose $f = (\mathrm{d}_f, \mathrm{m}_f)$ and $g = (\mathrm{d}_g, \mathrm{m}_g)$. Let there be a pullback square

$$
\begin{array}{ccc}
& \xrightarrow{\;\mathrm{d}'_g\;} & \mathbf{D}_f \\
{\scriptstyle \mathrm{m}'_f}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{m}_f} \\
\mathbf{D}_g & \xrightarrow[\;\mathrm{d}_g\;]{} & B
\end{array}
$$

The composite $g \odot f$ is then $(\mathrm{d}_f \circ \mathrm{d}'_g, \mathrm{m}_g \circ \mathrm{m}'_f)$. Axiom (m1) tells us that this is equal to $(\mathrm{Id}, \mathrm{Id})$, hence d_f and d'_g are both isomorphisms. Hence without loss of generality we can assume $\mathrm{d}_f = \mathrm{d}'_g = \mathrm{Id}_A$, so $\mathrm{m}_f = \mathrm{d}_g \circ \mathrm{m}'_f$ and $\mathrm{m}_g \circ \mathrm{m}'_f = \mathrm{Id}_A$.

The composite $f \odot g$ is $(\mathrm{d}_g, \mathrm{m}_f \circ \mathrm{m}_g)$, and the second axiom tells us that $\mathrm{m}_f \circ \mathrm{m}_g = \mathrm{d}_g$. Hence $\mathrm{d}_g = \mathrm{d}_g \circ \mathrm{m}'_f \circ dm_g$, so d_g monic implies $\mathrm{m}'_f \circ dm_g = \mathrm{Id}$, so m'_f is an isomorphism. Hence

$$ g = (\mathrm{d}_g \circ \mathrm{m}'_f, \mathrm{m}_g \circ \mathrm{m}'_f) = (\mathrm{m}_f, \mathrm{Id}_A). $$

□

Proposition 9.9 *Let $\mathcal{D}$ be a cartesian category. Then $\mathrm{Cart}(\mathcal{C}, \mathcal{D}) \cong \mathrm{Th}(\mathcal{C})\text{-PMod}(\mathcal{D})$.*

Proof. If $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a cartesian functor, then for any quasi-equational theory F transforms partial models in $\mathcal{C}$ to partial models in $\mathcal{D}$. Hence $F(M_0)$ is a partial model of $\mathrm{Th}(\mathcal{C})$ in $\mathcal{D}$. Moreover, this is functorial — any natural transformation between two cartesian functors gives a homomorphism between the corresponding partial models.

Now suppose M is a partial model of $\mathrm{Th}(\mathcal{C})$ in $\mathcal{D}$. We define a cartesian functor $F_M : \mathcal{C} \longrightarrow \mathcal{D}$ as follows. For each object A of $\mathcal{C}$, $F_M(A) = M \bar{A}$. Now suppose $f : A \longrightarrow B$ is a morphism in $\mathcal{C}$. In $\mathcal{D}$ we have a partial morphism $M \bar{f} : M \bar{A} \longrightarrow M \bar{B}$, but by axiom (t) this is total and can be expressed in the form $(\mathrm{Id}_{M \bar{A}}, F_M(f))$ for a unique morphism $F_M(f) : M \bar{A} \longrightarrow M \bar{B}$. Axioms (f1) and (f2) tell us that F_M is a functor, and the axioms for terminal objects, and (pb1) and (pb2), tell us that it is cartesian (though not necessarily strict). The transformation from M to F_M is functorial.

We show that $M = F_M(M_0)$.

$$F_M(M_0) \bar{A} = F_M(M_0 \bar{A}) = F_M(A) = M \bar{A}$$

Now suppose $f = (d, m) : A \rightharpoonup B$ in $\mathcal{C}$. By (pm1) and (pm2), and using Lemma 9.8,

$$
\begin{aligned}
M \bar{f} &= M \bar{m} \circ M \overline{\bar{d}} \\
&= (\mathrm{Id}_D, F_M(m)) \odot (F_M(d), \mathrm{Id}_A) \\
&= (F_M(d), F_M(m)) \\
&= F_M(M_0 \bar{f}) = F_M(M_0) \bar{f}.
\end{aligned}
$$

Finally, if $F : \mathcal{C} \longrightarrow \mathcal{D}$ is cartesian, we must show that $F_{F(M_0)} = F$. They clearly agree on objects, and if $f : A \longrightarrow B$ in $\mathcal{C}$ then $F(M_0) \bar{f} = (\mathrm{Id}_{F(A)}, F(f))$ and so $F_{F(M_0)}(f) = F(f)$. $\square$

9.3 Generalized theory morphisms

Our definitions (Section 5) of signature morphism and theory morphism were completely bound to the syntax: they transformed sorts to sort, function symbols to function symbols and so on. One important role of the classifying category is to provide a presentation-independent form of the theory, in which the objects and morphisms are understood as *derived* sorts and functions.

Suppose $\mathbb{T}$ and $\mathbb{T}'$ are two quasi-equational theories, and suppose we have a uniform method, applicable in any cartesian category, that transforms partial models of $\mathbb{T}'$ into partial models of $\mathbb{T}$. In particular, we can apply the method to the generic model of $\mathbb{T}'$ to obtain a partial model of $\mathbb{T}$ in $\mathcal{C}'_{\mathbb{T}}$.

On the other hand, given such a model, we can exploit the equivalence between partial models and strict cartesian functors out of the classifying category. Then (see below) composition of the functors provides a uniform method of transforming partial models of $\mathbb{T}'$ into partial models of $\mathcal{C}_{\mathbb{T}}$.

Definition 9.10 *Let $\mathbb{T}$ and $\mathbb{T}'$ be two quasi-equational theories. A* generalized theory morphism *from $\mathbb{T}$ to $\mathbb{T}'$ is a model of $\mathbb{T}$ in $\mathcal{C}_{\mathbb{T}'}$.*

If M is a generalized theory morphism from $\mathbb{T}$ to $\mathbb{T}'$, then for any cartesian category $\mathcal{D}$ we get a forgetful functor

$$U_M : \mathbb{T}'\text{-PMod}(\mathcal{D}) \longrightarrow \mathbb{T}\text{-PMod}(\mathcal{D})$$

It is such that if M corresponds to $F : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}_{\mathbb{T}'}$ and a partial model N of $\mathbb{T}'$ in $\mathcal{D}$ corrsponds to $G : \mathcal{C}_{\mathbb{T}'} \longrightarrow \mathcal{D}$, then $U_M(N)$ corresponds to $G \circ F$.

In terms of non-strict cartesian functors, suppose $G : \mathcal{C}_{\mathbb{T}'} \longrightarrow \mathcal{D}$ is cartesian. Its corresponding partial model of $\mathbb{T}'$ in $\mathcal{D}$ is $G(M_{\mathbb{T}'})$. Applying U_M to this gives $(G \circ F)(M_{\mathbb{T}}) = G(M)$.

We can now generalize the Free Partial Model Theorem (5.4). As before, we prove it only for models in **Set**. It is not true for models in a cartesian category in general, since the cartesian structure is not enough to construct the free models.

Theorem 9.11 *Let $\mathbb{T}$ and $\mathbb{T}'$ be quasi-equational theories and let $M : \mathbb{T} \longrightarrow \mathbb{T}'$ be a generalized theory morphism. Then the forgetful functor $U_M : \mathbb{T}'\text{-PMod} \longrightarrow \mathbb{T}\text{-PMod}$ has a left adjoint.*

Proof. First, note that the original Free Partial Model Theorem easily allows for a mild generalization of theory morphism. Let Σ and Σ' be the signatures of $\mathbb{T}$ and $\mathbb{T}'$, and let $\rho : \Sigma \longrightarrow \Sigma'$ be a signature morphism. Suppose that for each axiom $\varphi \vdash^{\vec{x}:\vec{A}} \psi$ of $\mathbb{T}$, the ρ-translation $\varphi^\rho \vdash^{\vec{x}:\vec{A}^\rho} \psi^\rho$ is a consequence of the axioms of $\mathbb{T}'$. (Previously it had to be one of the axioms.) If $\mathbb{T}''$ is defined to be $\mathbb{T}'$ with the addition of all these ρ-translations as axioms, then clearly $\mathbb{T}''\text{-PMod}$ is equal to $\mathbb{T}'\text{-PMod}$, and ρ is a theory morphism from $\mathbb{T}$ to $\mathbb{T}''$.

Now consider a generalized theory morphism M. This gives a signature morphism ρ from Σ to the signature of $\mathrm{Th}(\mathcal{C}_{\mathbb{T}'})$. By completeness, if a sequent is valid in the classifying category then it is derivable, so ρ is a theory morphism in the mildly generalized sense described above. It follows that

$$U_\rho : \mathbb{T}'\text{-PMod} \longrightarrow \mathbb{T}\text{-PMod}$$

has a left adjoint. We have an isomorphism

$$\mathrm{Cart}(\mathcal{C}_{\mathbb{T}'}, \mathbf{Set}) \cong \mathrm{Th}(\mathcal{C}_{\mathbb{T}'})\text{-PMod}(\mathbf{Set})$$

and an equivalence

$$\mathrm{Cart}(\mathcal{C}_{\mathbb{T}'}, \mathbf{Set}) \simeq \mathbb{T}'\text{-PMod}(\mathbf{Set})$$

it suffices to show that U_M and U_ρ act in the same way modulo the equivalence. Let $G : \mathcal{C}_{\mathbb{T}'} \longrightarrow \mathbf{Set}$ be cartesian. An easy calculation shows that its images in $\mathbb{T}\text{-PMod}(\mathbf{Set})$ via U_M and via U_ρ are both $G(M)$.

We have glossed over the fact that **Set** is not a small category, but it can always be replaced by a small subcategory in these calculations. $\square$

9.4   Practical conversions

The proof of the equivalence between quasi-equational theories and cartesian theories, with its explicit use of a classifying category, gives some insight into the theory but is not very useful in practice when one wants to convert from one form to another. We now give an alternative proof that shows an explicit conversion.

Theorem 9.12 *Let $\mathbb{T}$ be a cartesian theory. Then there is an equivalent quasi-equational theory.*

Proof. We show how to construct a quasi-equational theory $\mathbb{T}'$ and axioms $\mathbb{U}$ over the union of the signatures for $\mathbb{T}$ and $\mathbb{T}'$ so that $\mathbb{T} \cup \mathbb{T}' \cup \mathbb{U}$ is equivalent to both $\mathbb{T}$ and to $\mathbb{T}'$. The basic idea is similar to that of Proposition 3.6, but some extra work is needed for the unique existential quantification.

$\mathbb{T}'$ is constructed as follows.

First, $\mathbb{T}'$ has the sorts and function symbols of $\mathbb{T}$.

Second, give $\mathbb{T}'$ a new sort U with constant u and axioms as in the proof of Proposition 3.6.

Third, for each predicate symbol $R \rightarrowtail \vec{S}$ in $\mathbb{T}$, $\mathbb{T}'$ has a (partial) function symbol $f_R : \vec{S} \longrightarrow U$. The only axioms of $\mathbb{U}$ are corresponding axioms

$$f_R(\vec{x}) \downarrow \;\vdash^{\vec{x}}\; R(\vec{x}), \tag{30}$$

which define each of R and f_R in terms of the other.

Now for each cartesian formula or subformula $(\vec{x}.\varphi)$ in the axioms of $\mathbb{T}$, we define a formula φ° in $\mathbb{T}'$ in the same context as follows.

- $(s = t)^\circ \equiv (s = t)$

- $R(\vec{t})^\circ \equiv f_R(\vec{t}) \downarrow$

- $\top^\circ \equiv \top$

- $(\varphi \wedge \psi)^\circ \equiv (\varphi^\circ \wedge \psi^\circ)$

Now suppose φ is a unique existential quantification $(\exists! y)\psi$ where ψ is in context $\vec{x}, y$. Then we give $\mathbb{T}'$ a new function symbol $W_\psi : \sigma(\vec{x}) \longrightarrow \sigma(y)$ (the witness function) and an axiom

$$W_\psi(\vec{x}) = y \;\vdash^{\vec{x}, y}\; \psi^\circ, \tag{31}$$

which defines W_ψ in terms of previous ingredients of $\mathbb{T}'$. Note that it is the cartesianness, the uniqueness of the existential quantification, that guarantees that in any model of $\mathbb{T}$ there is such a partial function W_ψ. This is essential in showing that $\mathbb{T}$ is equivalent to $\mathbb{T} \cup \mathbb{T}' \cup \mathbb{U}$. We define

$$((\exists! y)\psi)^\circ \equiv W_\psi(\vec{x}) \downarrow.$$

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Now for each axiom $\varphi \stackrel{\vec{x}}{\longmapsto} \psi$ in $\mathbb{T}$, we give $\mathbb{T}'$ an axiom

$$\varphi^\circ \stackrel{\vec{x}}{\longmapsto} \psi^\circ. \tag{32}$$

The main property needed is that

$$\varphi^\circ \stackrel{\vec{x}}{\longmapsto\!\!\!\dashv} \varphi \tag{33}$$

should be PHL provable from $\mathbb{T} \cup \mathbb{T}' \cup \mathbb{U}$. This is used in two directions. First, suppose we are given a model of $\mathbb{T}$. Then one shows by induction on the axioms and their subformulae that the symbols of $\mathbb{T}'$ can be interpreted in a unique way that validates the axioms of $\mathbb{T}'$ (except for (32)) and $\mathbb{U}$. The induction also proves that property (33) holds, and this validates axiom (32), completing the proof that we have a model of $\mathbb{T} \cup \mathbb{T}' \cup \mathbb{U}$. Next suppose we are given a model of $\mathbb{T}'$. The predicates of $\mathbb{T}$ can be interpreted uniquely in accordance with Axiom (30) of $\mathbb{U}$, and then by induction one can show the other axioms of $\mathbb{U}$, property (33), and the axioms of $\mathbb{T}$. $\square$

9.5 Essentially algebraic theories

Essentially algebraic theories were introduced in [Fre72]. From our point of view, they are best defined as follows.

Definition 9.13 A quasi-equational theory is *essentially algebraic* if

1. the set of function symbols has a well-founded partial order;

2. for each function symbol $f : \vec{A} \longrightarrow B$, there is an axiom

$$f(\vec{x})\!\downarrow \stackrel{\vec{x}}{\longmapsto\!\!\!\dashv} \vec{s} = \vec{t},$$

   where $\vec{s}$ and $\vec{t}$, term sequences in context $\vec{x}$, are constructed from function symbols preceding f in the ordering; and

3. all other axioms are of the form $s \leftrightharpoons t$.

In other words, each function symbol comes with an equational characterization of its domain of definition, and the other axioms can be expressed as equations under the interpretation that the equation holds if both sides are defined. For an example, see the theory of cartesian categories in Section 6.1.

By definition, every essentially algebraic theory is quasi-equational. For the converse, we work via cartesian categories $\mathcal{C}$. Recall the theory $\mathrm{Th}(\mathcal{C})$ of Subsection 9.2, and consider a subtheory $\mathrm{Th}'(\mathcal{C})$ that has operators $\bar{f}$ only for *total* morphisms f, and omits the axioms (m1), (m2), (pm1) and (pm2). $\mathrm{Th}'(\mathcal{C})$ is essentially algebraic, and Proposition 9.9 still holds with $\mathrm{Th}(\mathcal{C})$ replaced by $\mathrm{Th}'(\mathcal{C})$ – in fact the proof is somewhat simpler.

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9.6 Summary

We have now seen five different kinds of theory that can be used to describe models in cartesian categories.

- cartesian categories (so models are strict cartesian functors)

- partial Horn theories, quasi-equational theories and essentially algebraic theories, all using partial structures

- cartesian theories, using total structures

[Joh02b] shows that cartesian theories are equivalent to cartesian categories – that is to say, they are able to describe the same classes of models. Proposition 3.6 showed that partial Horn theories are equivalent to quasi-equational theories. In this Section we have shown that cartesian categories, quasi-equational theories and essentially algebraic theories are all equivalent, and we also gave a more concrete description of how to convert cartesian theories into quasi-equational theories.

Hence all five types have the same expressive power in cartesian categories.

They each have their advantages and disadvantages.

The advantages of the cartesian categories, the classifying categories, are largely theoretical. The classifying category is a direct, presentation-independent embodiment of the theory and it is often useful in abstract discussion. On the other hand it is normally infinite and to describe it in practice it is often much more convenient to present it by generators and relations. That is what the other kind of theory in effect do.

The advantage of the cartesian theories is that they work within the standard logic of total functions and terms. The disadvantage is that this then compels the use of predicates in order to describe partial functions such as pullbacks, and the use of unique existential quantifications in the theory. The resulting complications in the theory, with a logical quantifier whose use has to be justified by side-proofs, make it difficult to use cartesian theories as a theoretical device.

The three that use partial terms have the disadvantage that they have to work in a modified logic. However, they avoid all the problems of the unique existential quantification. Of the three, we have found the quasi-equational theories the most effective in both practical and theoretical use, giving an excellently short proof of the Initial Model Theorem.

We have mentioned sketches [BW84] briefly in the introduction. They provide a graphical, rather than logical, approach. In particular the finite limit sketches (or, in [BW84], the LE-sketches or left exact sketches) are equivalent to cartesian categories, and hence to the five kinds of theory we listed above. The LE-theories described in [BW84] are the analogues of our classifying categories.

10 Conclusions

The theories whose logic corresponds to the structure of cartesian categories have long been known to be important. For instance, since at least as far back as [Ken68] it

has been known that free models can be constructed for these theories. Various ways have been devised to present these theories, notably the cartesian theories using unique existential quantification, the essentially algebraic theories using equationally described domains of definition, and the finite limit sketches using graph-theoretic devices.

The partial Horn logic presented here works by a single change to ordinary logic, amending the substitution rule to allow for partial terms, with definedness represented by self-equality. Once that is done, the cartesian theories correspond to the minimal kind of theory in that logic, the quasi-equational theories. The simplicity of the quasi-equational theories makes it just about as easy to prove the free partial model theorem predicatively as it is in the better known algebraic case: form terms, and factor out a partial congruence.

Once proved, the free partial model theorem is extremely powerful since it is the basis of inductive proofs based on the structure of terms, formulae and proofs. In particular, we use it to prove the existence of classifying categories for quasi-equational theories. We conjecture that the same technique will readily extend to other logics in the following way. Suppose we are given a cartesian theory $\mathbb{C}$ whose models are categories with a certain kind of structure, and suppose $\mathbb{T}$ is a theory in a logic (in general not cartesian) that can be interpreted in categories that model $\mathbb{C}$. We conjecture that in many useful cases there is then a cartesian theory $\mathbb{C}\varpi\mathbb{T}$ whose initial model is a classifying $\mathbb{C}$-category for the theory $\mathbb{T}$.

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