UPPSALA UNIVERSITET
MATEMATISKA INSTITUTIONEN
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Complex Analysis 10hp
Kand Ma2
Spring Term 2014

Home assignment 3

Note: Solutions to the problems should contain both detailed arguments and calculations (where appriopriate).

1. Assume that f is an entire function (i.e. it is analytic in the whole complex plane $\mathbb{C}$) and that it satisfies $f(z)f(-z) = f(z^2)$, $z \in \mathbb{C}$, as well as $f(1) = 1$. Show that f has no zeros in the open unit disc except, possibly, at the origin.

2. Find the Laurent expansion of $f(z) = \dfrac{1}{z^2(z+1)}$ in the region $1 < |z-1| < 2$.

3. Find all functions $f(z)$ which are analytic in the whole complex plane $\mathbb{C}$ except for a pole of order 2 with a residue -1 at $z_1 = 2$, such that $f(3) = 0$, $f(0) = \frac{3}{4}e^{4i}$, and which satisfy the condition $|f(z)| \leq 5e^{2xy}$ for $|z|$ big enough. (Note: all these conditions should be satisfied at the same time.)

4. (a) Let $\exp : \mathbb{C} \longrightarrow \mathbb{C} - \{0\}$ be the exponential mapping and let $\gamma = \gamma(t)$, $0 \leq t \leq 1$, be a closed curve in $\mathbb{C} - \{0\}$. Show that there is a (possibly non-closed) curve $\gamma' = \gamma'(t)$, $0 \leq t \leq 1$, in $\mathbb{C}$ such that $\exp(\gamma'(t)) = \gamma(t)$ for all $0 \leq t \leq 1$. Such a curve γ' is called "**a lifting** of γ to $\mathbb{C}$".

(Hint: you may have some help of the Path Covering Lemma and of the existence of a branch of $\log(z)$ in simply-connected domains not containing 0.)

(b) Given an integer $n \in \mathbb{Z}$, let Γ_n be the closed curve in $\mathbb{C} - \{0\}$ defined by $\Gamma_n(t) = e^{2\pi int}$, $0 \leq t \leq 1$. (Γ_n is the unit circle run over n turns in positive direction.) Show that **every closed curve** γ in $\mathbb{C} - \{0\}$ is, within $\mathbb{C} - \{0\}$, homotopic as a closed curve to a unique curve Γ_n.

Note: The home assignment 3 is to be left at latest on Monday, April 28th, at 5 p.m. in my mail-box.