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ÖRJAN STENFLO

ITERATED FUNCTION SYSTEMS CONTROLLED BY A SEMI-MARKOV CHAIN

In this paper we consider a finite set of m maps $\{w_1, w_2, \dots, w_m\}$, $w_i: X \to X$, where X is a complete metric space, together with a sequence $\{I_n\}$ of random variables taking values in the finite set $\{1, 2, \dots, m\}$. This sequence controls the dynamic system $Z_n := w_{I_{n-1}} \circ \cdots \circ w_{I_0}$. The case where $\{I_n\}$ is a sequence of independent, identically distributed random variables (or a homogeneous Markov chain) is usually referred to as an Iterated Function System IFS (or a recurrent IFS). In the present paper, we consider the more general case when the controlling sequence is a semi-Markov chain. Under "average contractivity" conditions, we obtain some ergodic theorems. In applications, these results may broaden the class of images which can be created using iterated function systems.

1. Introduction

Iterated function systems is a fast developing topic which has been studied extensively during the last decade. One major reason for this recent activity was the introduction of the fractal concept in the seventies, which led to questions on how to create new fractals. An important method is to use iterated function systems (IFS) (see Barnsley and Demko (1985)). Problems concerning more effective image building naturally led to the introduction of iterated function systems with probabilities, i.e. iterated function systems controlled by a sequence of independent, identically distributed random variables. Barnsley, Elton and Hardin (1989) generalized this model. They considered IFS controlled by a Markov chain and called them recurrent IFS.

In the present paper, we will study a more general model; iterated function systems controlled by a semi-Markov chain. Informally, we have the following dynamic structure:

Consider a set of m functions $w_j: X \to X$ where X is some measurable space and the index $j = 1, 2, \dots, m$. Choose a starting point $x_0 \in X$ and a starting index y_0. Choose at random an integer N_{y_0}. The function w_{y_0} is then iterated N_{y_0} succesive times, starting at x_0, to create the suborbit $x_0, x_1, \dots, x_{N_{y_0}}$. Now choose a new index y_1 at random, with probabilities depending on y_0. Then choose at random an integer N_{y_1} and iterate the w_{y_1} function N_{y_1} times starting at $x_{N_{y_0}}$. Continue in this fashion to create the full orbit $\{x_n\}$.

In the special case when all the N_i's ($i = 1, \dots, m$) follow a geometric distribution, i.e. N_i is the number of "cointosses until head", we have an IFS controlled by a Markov chain, which has been investigated by Barnsley et al. (1989). This model in turn generalizes iterated function systems with probabilities introduced by Barnsley and Demko (1985).

Our way to attack the problem concerning a controlling semi-Markov chain, motivates us to extend some ergodic theorems by Barnsley et al. (1989) concerning a controlling finite state space Markov chain to the countable state space case. We shall do this in

Section 1 (Theorem 1), and also improve one of their results (see Theorem 1(ii) below), in that we allow arbitrary initial distributions for the controlling Markov chain.

Results concerning a controlling semi-Markov chain are presented in Section 2 (Theorem 2).

We shall now give a more precise introduction to the models we consider: Let X be a locally compact Polish space with metric d, and let $w_j: X \to X$ be Lipschitz maps, $j \in S$, where S is a discrete (finite or countable) space. We define for such a map w, the norm

$$\|w\| = \sup_{x \neq y} \frac{d(w(x), w(y))}{d(x,y)}.$$

Let $\{I_n\}_{n=0}^{\infty}$ be a stochastic sequence with phase space S. Specify a starting point $x_0 \in X$. The stochastic sequence $\{I_n\}$ then controls the generalized dynamical system $\{Z_n^{x_0}\}_{n=0}^{\infty}$, where

$$Z_n^{x_0} := w_{I_{n-1}} \circ w_{I_{n-2}} \circ \cdots \circ w_{I_0}(x_0), \qquad n \geq 1, \quad Z_0^{x_0} = x_0.$$

What we want is to determine the behavior of the distribution of $Z_n^{x_0}$ as n tends to infinity. In doing this, one problem is that $Z_n^{x_0}$ does not in general converge pointwise, not even under strong contraction conditions. Therefore, we instead study iteration in reversed order

$$\widetilde{Z}_n^{x_0} := w_{I_0} \circ w_{I_1} \circ \cdots \circ w_{I_{n-1}}(x_0), \qquad n \geq 1, \quad \widetilde{Z}_0^{x_0} = x_0,$$

which under similar contraction conditions does converge pointwise with probability one (almost surely, a.s.).

In the simplest case when the controlling sequence $\{I_n\}$ consists of independent, identically distributed random variables, Z_n and $\widetilde{Z}_n$ (we omit the index x_0 when it is not specified) have the same distribution, so if $\widetilde{Z}_n$ converges a.s., it will follow that Z_n converges weakly.

The case when the controlling sequence is a Markov chain is more intricate since the distributions of Z_n and $\widetilde{Z}_n$ no longer coincide. However, this problem is solved by considering the reversed chain (see Section 2 or Barnsley et al. (1989)).

When the controlling sequence, $\{I_n\}$, is a semi-Markov chain (see Section 3 for a definition), there is no method of reversing the chain so the previous known methods cannot directly be used. However, we shall proceed by embedding this semi-Markov chain in a extended Markov chain ($\{I_n, \phi_n\}$ which we introduce later), use reversing results concerning Markov chains, and then "try to forget" about ϕ_n. Since the embedding chains generally will have countable state space, our method forces us to extend some results concerning the model with a controlling finite state space Markov chain, to the countable state space case.

2. IFS controlled by a Markov chain

Let $\{I_n\}$ be a homogeneous Markov chain with countable state space, and with transition matrix $\mathbf{P} = (P_{ij})$. This Markov chain controls the dynamical system $\{Z_n\}$, as well as the random sequence $\{\widetilde{Z}_n\}$ defined as above.

We will consider the case when the Markov chain $\{I_n\}$ is ergodic, i.e. irreducible and with a unique stationary probability distribution $\pi = \{\pi(i)\}_{i=1}^{\infty}$.

Define, using Kolmogorov's extension theorem, the measure P_μ on the space of trajectories of $\{I_n\}$ given on finite dimensional sets by

$$P_\mu(I_0 = i_0, I_1 = i_1, \ldots, I_n = i_n) = \mu(i_0) P_{i_0 i_1} \cdots P_{i_{n-1} i_n}.$$

If the initial distribution μ is concentrated at the point i, we use the notation P_i instead of P_μ. Furthermore, we use P_{for} instead of P_π for reasons that will be clear later.

Since all $\pi(j) > 0$, we can define the dual transition probabilities of $\mathbf{P}$,

$$Q_{ij} := \pi(j)P_{ji}/\pi(i).$$

This transition matrix corresponds to a Markov chain $\{\tilde{I}_n\}$ with the same stationary distribution, π, as $\{I_n\}$.

Define the measures $\widetilde{P}_\mu$, ($P_{bac} := \widetilde{P}_\pi$ and $\widetilde{P}_i$) in analogy with P_μ, (P_{for} and P_i), given on finite dimensional sets by

$$\widetilde{P}_\mu(\tilde{I}_0 = i_0, \tilde{I}_1 = i_1, \dots, \tilde{I}_n = i_n) := \mu(i_0)Q_{i_0 i_1} \cdots Q_{i_{n-1} i_n}.$$

It then follows that

$$P_{for}(I_0 = i_0, \dots, I_n = i_n) = P_{bac}(\tilde{I}_0 = i_n, \dots, \tilde{I}_n = i_0).$$

This explains the names of the measures P_{for} corresponding to the forward chain and P_{bac} corresponding to the backward chain. Thus, we see that for all n and all x

$$P_{for}\{\mathbf{i} = (i_0, i_1 \dots) \colon Z_n^x(\mathbf{i}) \in \cdot\} = P_{bac}\{\mathbf{i} = (i_0, i_1 \dots) \colon \widetilde{Z}_n^x(\mathbf{i}) \in \cdot)$$

where $Z_n^x(\mathbf{i}) := w_{i_{n-1}} \circ \cdots \circ w_{i_0}(x)$ and $\widetilde{Z}_n^x(\mathbf{i}) := w_{i_0} \circ \cdots \circ w_{i_{n-1}}(x)$.

Theorem 1. *Suppose that*

A: $\{I_n\}$ *is an ergodic, aperiodic, homogeneous Markov chain.*
B: $R(x) := \sup_{i \in S} d(w_i(x), x) < \infty$ *for some* $x \in X$.
C: $\mathsf{E}_{P_{for}} \log \|w_{I_{n_0}} \circ \cdots \circ w_{I_0}\| < 0$ *for some* n_0.

Then for all $x \in X$:

(i) *An inversed ergodic theorem: For* $[P_{bac}]$ *a.a.* $\mathbf{i}$, $w_{i_0} \circ \cdots \circ w_{i_n}(x) \to \widetilde{Z}(\mathbf{i})$, *and the limit is independent of* $x \in X$.
(ii) *An ergodic theorem in average:* $P_\mu\{\mathbf{i} \colon Z_n^x(\mathbf{i}) \in \cdot\} \xrightarrow{w} \lambda(\cdot)$, *for all initial distributions* μ, *where* $\lambda(\cdot) = P_{bac}\{\mathbf{i}; \widetilde{Z}(\mathbf{i}) \in \cdot\}$.
(iii) *An individual ergodic theorem: For all* $f \in C(X)$, *the bounded continuous functions on* X, *and all initial distributions* μ,

$$\frac{1}{n+1} \sum_{k=0}^{n} f(Z_k^x(\mathbf{i})) \to \int f \, d\lambda \quad P_\mu\text{-a.s.}$$

Remark 1. Condition C can be expressed by,

$$\sum_{r=0}^{n} \sum_{i_r=1}^{\infty} \pi(i_0)P_{i_0 i_1} \cdots P_{i_{n-1} i_n} \log \|w_{i_n} \circ \cdots \circ w_{i_0}\| < 0.$$

Remark 2. (i) and (ii) hold if X is a complete metric space.

Remark 3. (iii) states informally that starting at any x, and with any initial function, the empirical distribution of a trajectory converges with probability one to λ.

The proof follows the scheme in the proof by Barnsley et al. (1989) concerning a controlling Markov chain with finite state space. The new part is to show that the limit in (ii) does not depend on μ. (Barnsley et al. considered the special case when the initial distribution of the Markov chain, μ, is the corresponding stationary measure.)

Proof. (i) Since for all n, $P_{for}\{\mathbf{i}: (i_0 i_1 \dots i_n) \in \cdot\} = P_{bac}\{\mathbf{i}: (i_n i_{n-1} \dots i_0) \in \cdot\}$, we also have

$$E_{P_{bac}} \log \|w_{I_0} \circ \cdots \circ w_{I_{n_0}}\| < 0, \quad \text{for some } n_0.$$

Using a variant of Kingman's subadditive ergodic theorem in finite measure spaces, we can show that this implies (see Stenflo (1995) for details),

$$\limsup_{n\to\infty} \frac{1}{n} \log \|w_{I_n} \circ \cdots \circ w_{I_0}\| \le -\alpha, \quad P_{for} \text{ a.s.} \tag{1}$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log \|w_{I_0} \circ \cdots \circ w_{I_n}\| \le -\alpha, \quad P_{bac} \text{ a.s.} \tag{2}$$

where α is a constant in $(0, \infty]$.

Let x be a fixed point satisfying condition B. From the definition of the norm, for P_{bac} almost all (a.a.) $\mathbf{i}$, we have

$$d(w_{i_0} \circ \cdots \circ w_{i_n}(x), w_{i_0} \circ \cdots \circ w_{i_{n+1}}(x)) \le \|w_{i_0} \circ \cdots \circ w_{i_n}\| R(x).$$

For P_{bac} a.a. $\mathbf{i}$, we may choose m_0 (depending on $\mathbf{i}$) so that $n \ge m_0 \Rightarrow \|w_{i_0} \circ \cdots \circ w_{i_n}\| < e^{-n\alpha/2}$ and thus $\sum_{n=0}^{\infty} d(w_{i_0} \circ \cdots \circ w_{i_n}(x), w_{i_0} \circ \cdots \circ w_{i_{n+1}}(x)) < \infty$.

Therefore, $\widetilde{Z}_n^x(\mathrm{i}) := w_{i_0} \circ \cdots \circ w_{i_{n-1}}(x)$ is a Cauchy sequence and converges to, say $\widetilde{Z}(\mathrm{i})$, for P_{bac} a.a. $\mathbf{i}$. Furthermore for P_{bac} a.a. $\mathbf{i}$,

$$d(w_{i_0} \circ \cdots \circ w_{i_n}(x), w_{i_0} \circ \cdots \circ w_{i_n}(y)) \le \|w_{i_0} \circ \cdots \circ w_{i_n}\| d(x,y) \to 0.$$

Accordingly, $\widetilde{Z}(\mathbf{i})$ does not depend on x. This proves (i) in Theorem 1.

(ii) We first note that for all n, and all $f \in C(X)$,

$$\int f(\widetilde{Z}_n^x(\mathrm{i}))\, dP_{bac}(\mathrm{i}) = \int f(Z_n^x(\mathrm{i}))\, dP_{for}(\mathrm{i}).$$

Let us choose an arbitrary point $x_0 \in X$ and define the measure,

$$\lambda_n^{x_0}(\cdot) := P_{for}\{\mathrm{i}; Z_n^{x_0}(\mathrm{i}) \in \cdot\} = P_{bac}\{\mathrm{i}: \widetilde{Z}_n^{x_0}(\mathrm{i}) \in \cdot\}.$$

For every function $f \in C(X)$, the dominated convergence theorem and statement (i) yield

$$\lim_{n\to\infty} \int f(x)\, d\lambda_n^{x_0}(x) = \lim_{n\to\infty} \int f(\widehat{Z}_n^{x_0}(\mathrm{i}))\, dP_{bac}(\mathrm{i}) = \int \lim_{n\to\infty} f(\widehat{Z}_n^{x_0}(\mathrm{i}))\, dP_{bac}(\mathrm{i})$$
$$= \int f(\widehat{Z}(\mathrm{i}))\, dP_{bac}(\mathrm{i}) = \int f(x)\, d\lambda(x).$$

That is, we have proved (ii) in the special case when the initial function index is chosen according to the stationary distribution of the Markov chain $\{I_n\}$.

If $P_{for}(\cdot) = 1$, then $P_i(\cdot) = 1$, since by definition $P_{for}(\cdot) = \sum \pi(i) P_i(\cdot)$. Thus from equation (1) in the proof of (i) we also have that

$$\limsup_{n\to\infty} \frac{1}{n} \log \|w_{i_n} \circ \cdots \circ w_{i_0}\| \le -\alpha, \quad \text{for } P_i \text{ a.a. } \mathbf{i}.$$

Using this limit result, and the definition of the norm, for any two points $x, y \in X$ we have, $d(Z_n^x(\mathrm{i}), Z_n^y(\mathrm{i})) \le \|w_{i_{n-1}} \circ \cdots \circ w_{i_0}\| d(x,y) \longrightarrow 0$, for P_i a.a. $\mathbf{i}$, and thus for a function $f \in C(X)$:

$$f(Z_n^x(\mathrm{i})) - f(Z_n^y(\mathrm{i})) \longrightarrow 0, \quad \text{for } P_i \text{ a.a. } \mathbf{i}$$

Lebesgue's dominated convergence theorem now implies,

$$\text{(3)} \qquad \lim_{n\to\infty} \int (f(Z_n^x(\mathrm{i})) - f(Z_n^y(\mathrm{i})))\,dP_i(\mathrm{i}) = \int \lim_{n\to\infty} (f(Z_n^x(\mathrm{i})) - f(Z_n^y(\mathrm{i})))\,dP_i(\mathrm{i}) = 0.$$

Let $P^{(n)}(i,\cdot)$ denote $P(I_n \in \cdot \mid I_0 = i)$ and define for a Borel set, in the space of infinite sequences, the probability measure

$$P_{I_n|i}(\cdot) := \sum_{j=1}^{\infty} P^{(n)}(i,j)P_j(\cdot).$$

Since we have

$$\int f(Z_{n+k}^{x_0}(\mathrm{i}))\,dP_i(\mathrm{i}) = \sum_{j=1}^{\infty} P^{(n)}(i,j) \int_X \int f(Z_k^x(\mathrm{i}))\,dP_j(\mathrm{i})P(Z_n^{x_0}(\mathrm{i}) \in dx \mid I_n = j, I_0 = i),$$

it follows from equation (3) and the dominated convergence theorem that for any fixed n

(4)

$$\begin{aligned}
&\limsup_{k\to\infty} \left| \int f(Z_{n+k}^{x_0}(\mathrm{i}))\,dP_i(\mathrm{i}) - \int f(Z_k^{x_0}(\mathrm{i}))\,dP_{I_n|i}(\mathrm{i}) \right| \\
&\le \sum_{j=1}^{\infty} P^{(n)}(i,j) \int_X \int \lim_{k\to\infty} |f(Z_k^x(\mathrm{i})) - f(Z_k^{x_0}(\mathrm{i}))|\,dP_j(\mathrm{i})P(Z_n^{x_0} \in dx \mid I_n = j, I_0 = i) \\
&= 0.
\end{aligned}$$

Then, using A we obtain, the following inequalities (uniformly for all k),

$$\text{(5)} \qquad \begin{aligned}
&\left| \int f(Z_k^{x_0}(\mathrm{i}))\,dP_{I_n|i}(\mathrm{i}) - \int f(Z_k^{x_0}(\mathrm{i}))\,dP_{for}(\mathrm{i}) \right| \\
&\le \left| \sum_{j=1}^{\infty} (P^{(n)}(i,j) - \pi(j)) \int f(Z_k^{x_0}(\mathrm{i}))\,dP_j(\mathrm{i}) \right| \\
&\le \left| \sup_{x\in X} f(x) \right| \sum_{j=1}^{\infty} |P^{(n)}(i,j) - \pi(j)| \longrightarrow 0 \quad \text{as } n \mapsto \infty.
\end{aligned}$$

Given $\epsilon > 0$, we may choose fixed positive integers N and $K(N)$ sufficiently large so that for all $k > K(N)$,

$$\begin{aligned}
&\left| \int f(Z_{N+k}^{x_0}(\mathrm{i}))\,dP_i(\mathrm{i}) - \int f(Z_k^{x_0}(\mathrm{i}))\,dP_{for}(\mathrm{i}) \right| \\
&\le \underbrace{\left| \int f(Z_{N+k}^{x_0}(\mathrm{i}))\,dP_i(\mathrm{i}) - \int f(Z_k^{x_0}(\mathrm{i}))\,dP_{I_N|i}(\mathrm{i}) \right|}_{<\varepsilon/2 \text{ due to (4)}} \\
&+ \underbrace{\left| \int f(Z_k^{x_0}(\mathrm{i}))\,dP_{I_N|i}(\mathrm{i}) - \int f(Z_k^{x_0}(\mathrm{i}))\,dP_{for}(\mathrm{i}) \right|}_{<\varepsilon/2 \text{ due to (5)}} < \varepsilon.
\end{aligned}$$

However, we have proved that

$$\int f(Z_k^{x_0}(\mathrm{i}))\,dP_{for}(\mathrm{i}) \longrightarrow \int f(x)d\lambda(x).$$

Thus

$$\begin{aligned}\limsup_{k\to\infty}\int f(Z_k^{x_0}(\mathrm{i}))\,dP_i(\mathrm{i}) &= \limsup_{k\to\infty}\int f(Z_{N+k}^{x_0}(\mathrm{i}))\,dP_i(\mathrm{i})\\ &\le \limsup_{k\to\infty}\int f(Z_k^{x_0}(\mathrm{i}))\,dP_{for}(\mathrm{i})\\ &\quad+\limsup_{k\to\infty}\left|\int f(Z_{N+k}^{x_0}(\mathrm{i}))\,dP_i(\mathrm{i})-\int f(Z_k^{x_0}(\mathrm{i}))\,dP_{for}(\mathrm{i})\right|\\ &\le \int f d\lambda+\varepsilon,\end{aligned}$$

and

$$\begin{aligned}\liminf_{k\to\infty}\int f(Z_k^{x_0}(\mathrm{i}))\,dP_i(\mathrm{i}) &= \liminf_{k\to\infty}\int f(Z_{N+k}^{x_0}(\mathrm{i}))\,dP_i(\mathrm{i})\\ &\ge \liminf_{k\to\infty}\int f(Z_k^{x_0}(\mathrm{i}))\,dP_{for}(\mathrm{i})\\ &\quad-\limsup_{k\to\infty}\left|\int f(Z_{N+k}^{x_0}(\mathrm{i}))\,dP_i(\mathrm{i})-\int f(Z_k^{x_0}(\mathrm{i}))\,dP_{for}(\mathrm{i})\right|\\ &\ge \int f d\lambda-\varepsilon.\end{aligned}$$

Thus, since ϵ is arbitrary we have

$$\lim_{k\to\infty}\int f(Z_k^{x_0}(\mathrm{i}))\,dP_i(\mathrm{i})=\int f(x)d\lambda(x)$$

for all i. This completes the proof of (ii).

(iii) The proof by Barnsley et al. (1989) concerning a controlling finite state space Markov chain based on the result by Elton (1987), may be generalized whithout change to this countable state space case. □

3. IFS controlled by a semi-Markov chain

In this section, we shall formalize the ideas that were informally presented in the introduction. That is, we shall define a semi-Markov chain and show how we can embed this chain in a Markov chain. Finally, we arrive at the main theorem of this paper through this embedding idea. (For references concerning semi-Markov chains, see for instance Çinlar (1975) or Medhi (1982).)

Consider a (time-homogeneous) Markov renewal process $\{Y_n, L_n\}_{n=0}^{\infty}$ with state space $\{1,2,\ldots,m\}\times \mathbf{Z}^+$, i.e. a homogeneous two-component Markov chain with transition probabilities $P\{Y_{n+1}=j, L_{n+1}=k|Y_n=i, L_n=l\}=q_{ij}(k)$. As is known, the first component of the Markov renewal process $\{Y_n\}$ is also a Markov chain with transition probabilities $p_{ij}=\sum_k q_{ij}(k)$. We shall study the case when Y_{n+1} and L_{n+1} are conditionally independent given Y_n. That is, $q_{ij}(k)=p_{ij}f^{(i)}(k)$ where $f^{(i)}(k)$ is a shorter notation for $P\{L_{n+1}=k|Y_n=i\}$. Define the time for the nth renewal $T_n:=\sum_{i=1}^n L_i$, $n\ge 1$, $(T_0:=0)$. We can now define a semi-Markov chain as

$$I_n := Y_l \quad \text{for } T_l\le n<T_{l+1},\ l=0,1,2,\ldots.$$

This sequence controls the dynamical system $\{Z_n=w_{I_{n-1}}\circ\cdots\circ w_{I_0}\}$ as well as the random sequence $\{\tilde{Z}_n=w_{I_0}\circ\cdots\circ w_{I_{n-1}}\}$.

Let $N_n:=\max\{k: T_k\le n\}$ denote the number of renewals before time n, and define the time since the last renewal, $\phi_n:=n-T_{N_n}$.

Then $\{I_n, \phi_n\}_{n=0}^{\infty}$ will form a Markov chain on $\{1,\ldots,m\} \times \mathbf{N}$ with transition probabilities given by

$$P_{(i,k),(j,l)} = \begin{cases} \frac{f^{(i)}(k)}{\sum_{n=k}^{\infty} f^{(i)}(n)} p_{ij}, & \text{if } l = 0, \\ \frac{\sum_{n=k+1}^{\infty} f^{(i)}(n)}{\sum_{n=k}^{\infty} f^{(i)}(n)}, & \text{if } i = j \text{ and } l = k+1 \\ 0, & \text{otherwise.} \end{cases}$$

Let the Markov chain $\{Y_n\}$ be irreducible. Then there exists a stationary probability distribution $\nu = \{\nu_i\}_{i=1}^{m}$.

Define $M_i := E(L_{n+1}|Y_n = i) = \sum_{k=1}^{\infty} k f^{(i)}(k)$. If in addition to the irreducibility of the Markov chain $\{Y_n\}$, all $M_i < \infty$, $i = 1,\ldots,m$, then the Markov chain $\{I_n, \phi_n\}_{n=1}^{\infty}$ will be ergodic with stationary probability distribution

$$m(i,k) = \nu_i \sum_{n=k}^{\infty} f^{(i)}(n) \left(\sum_{j=1}^{m} \nu_j M_j \right)^{-1}.$$

Using Kolmogorov's extension theorem we define the measure P_f on the space of trajectories of $\{I_n\}$ determined on finite dimensional sets by

$$\begin{aligned} &P_f(I_0 = i_0, I_1 = i_1, \ldots, I_n = i_n) \\ &= \sum_{r=0}^{n} \sum_{k_r=0}^{\infty} m(i_0,k_0) P_{(i_0,k_0),(i_1,k_1)} P_{(i_1,k_1),(i_2,k_2)} \cdots P_{(i_{n-1},k_{n-1}),(i_n,k_n)}. \end{aligned}$$

We are now going to describe how to "reverse time", following the method of the previous section. Consider the transition probability

$$Q_{(i,k),(j,l)} = \frac{m(j,l)}{m(i,k)} P_{(j,l),(i,k)},$$

which corresponds to a Markov-chain on $\{1,2,\ldots,m\} \times \mathbf{N}$ with the same stationary distribution as the Markov chain governed by P, $m(i,k)$, embedding the "backward running" chain $\{\tilde{I}_n\}$. Define P_b in analogy with the construction of P_f (but now use Q instead of P) we then get:

$$\begin{aligned} &P_f(I_0 = i_0, I_1 = i_1, \ldots, I_n = i_n) \\ &= \sum_{r=0}^{n} \sum_{k_r=0}^{\infty} m(i_0,k_0) P_{(i_0,k_0),(i_1,k_1)} P_{(i_1,k_1),(i_2,k_2)} \cdots P_{(i_{n-1},k_{n-1}),(i_n,k_n)} \\ &= \sum_{r=0}^{n} \sum_{k_r=0}^{\infty} m(i_0,k_0) \frac{m(i_1,k_1)}{m(i_0,k_0)} Q_{(i_1,k_1),(i_0,k_0)} \cdots \frac{m(i_n,k_n)}{m(i_{n-1},k_{n-1})} Q_{(i_n,k_n),(i_{n-1},k_{n-1})} \\ &= \sum_{r=0}^{n} \sum_{k_r=0}^{\infty} m(i_n,k_n) Q_{(i_n,k_n),(i_{n-1},k_{n-1})} \cdots Q_{(i_1,k_1),(i_0,k_0)} \\ &= P_b(\tilde{I}_0 = i_n, \tilde{I}_1 = i_{n-1}, \ldots, \tilde{I}_n = i_0). \end{aligned}$$

Define on the space of trajectories of $\{I_n\}$, using Kolmogorov's extension theorem, for an arbitrary probability distribution, μ, on $\{1,\ldots,m\}$ the measure $P_{(\mu)}$, determined on finite dimensional sets by

$$P_{(\mu)}(I_0 = i_0, \ldots, I_n = i_n) := \mu(i_0) \sum_{r=1}^{n} \sum_{k_r=0}^{\infty} P_{(i_0,0),(i_1,k_1)} \cdots P_{(i_{n-1},k_{n-1}),(i_n,k_n)}.$$

(This measure is the one generated through the procedure given informally in the introduction.)

We have now introduced all necessary definitions and concepts, to be able to state our theorem concerning a controlling semi-Markov chain:

Theorem 2. *Suppose that*

A: $\{Y_n\}$ *is irreducible.*

B: $M_i < \infty, i = 1, \ldots, m$.

C: *The embedding double Markov chain,* $\{I_n, \phi_n\}$*, is aperiodic.*

D: $E_{P_f} \log \|w_{I_{n_0}} \circ \cdots \circ w_{I_0}\| < 0$ *for some* n_0.

Then for all $x \in X$:

(i) *An inversed ergodic theorem: For* $[P_b]$ *a.a.* $\mathbf{i}, w_{i_0} \circ \cdots \circ w_{i_n}(x) \to \widetilde{Z}(\mathbf{i})$, *and the limit is independent of* $x \in X$.

(ii) *An ergodic theorem in average:* $P_{(\mu)}\{\mathbf{i}\colon Z_n^x(\mathbf{i}) \in \cdot\} \xrightarrow{w} \sigma(\cdot)$, *for all initial distributions* μ, *where* $\sigma(\cdot) = P_b\{\mathbf{i}\colon \widetilde{Z}(\mathbf{i}) \in \cdot\}$.

(iii) *An individual ergodic theorem: For all* $f \in C(X)$, *the bounded continuous functions on* X, *and all initial distributions* μ,

$$\frac{1}{n+1}\sum_{k=0}^{n} f(Z_k^x(\mathbf{i})) \to \int f d\sigma \quad P_{(\mu)} \text{ a.s.}$$

Remark 4. Condition D can be written as

$$\sum_{r=0}^{n_0}\sum_{i_r=1}^{m}\sum_{k_r=0}^{\infty} m(i_0, k_0) P_{(i_0,k_0),(i_1,k_1)} \cdots P_{(i_{n_0-1},k_{n_0-1}),(i_{n_0},k_{n_0})} \log \|w_{i_{n_0}} \circ \cdots \circ w_{i_0}\| < 0.$$

Proof. Since the conditions in Theorem 1 are satisfied for the embedding double Markov chain, $\{I_n, \phi_n\}$, and since P_f and P_b are just the restrictions of the measures P_{for} and P_{bac}, respectively, given in Theorem 1 to the space of trajectories of the first component $\{I_n\}$, the present theorem is actually a corollary of Theorem 1. □

4. Example

Consider the IFS $X = \mathbf{R}^2$, together with the four functions:

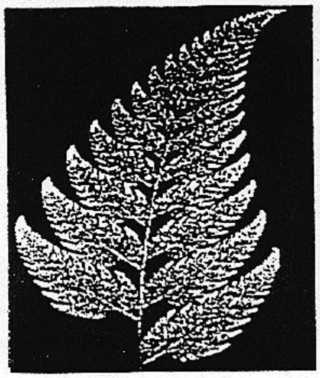
$$w_1\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0.16 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad w_2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix},$$

$$w_3\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.2 & -0.26 \\ 0.23 & 0.22 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}, \quad w_4\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0.44 \end{bmatrix}.$$

Let $x_0 = (0, 0)$ be the starting point. We can generate an orbit using these four affine maps and an independent, identically distributed controlling sequence (with $P(I_n = 1) = 0.01$, $P(I_n = 2) = 0.85$, $P(I_n = 3) = 0.07$ and $P(I_n = 4) = 0.07$ for all n; These functions and probabilities are given in Barnsley's book "Fractals everywhere" (1988). If we plot the points on the orbit (except the first, say 1000, points), we can "draw a picture" of the invariant measure using Theorem 1(iii), or Theorem 2(iii). This procedure generates the left hand picture (Barnsley's fern).

Alternatively, if we use a controlling semi-Markov chain, with "index controlling matrix"

$$\mathbf{P} = \begin{bmatrix} 0 & 0.85/0.99 & 0.07/0.99 & 0.07/0.99 \\ 0.01/0.15 & 0 & 0.07/0.15 & 0.07/0.15 \\ 0.01/0.93 & 0.85/0.93 & 0 & 0.07/0.93 \\ 0.01/0.93 & 0.85/0.93 & 0.07/0.93 & 0 \end{bmatrix},$$



and geometric "timespending distributions", $\{f^{(i)}(k)\}$, $i = 1, \ldots, 4$, with parameters 0.01, 0.85, 0.07, 0.07 respectively, we have exactly the above model in "semi-Markov setting". Thus, this more general procedure also generates the left hand picture.

If we choose "timespending distributions" other than geometrical, we can, by using the same functions, create "new" pictures. For instance we can obtain the right hand "nordic leaf" picture, with $f^{(1)}(1) = 1$, $f^{(2)}(1) = 0.1$, $f^{(2)}(2) = 0.3$, $f^{(2)}(3) = 0.3$, $f^{(2)}(4) = 0.3$, $f^{(3)}(1) = 0.3$, $f^{(3)}(2) = 0.3$, $f^{(3)}(3) = 0.4$, $f^{(4)}(1) = 0$, $f^{(4)}(2) = 0.1$, $f^{(4)}(3) = 0.1$, $f^{(4)}(4) = 0.1$, $f^{(4)}(5) = 0.7$.

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Department of Mathematics, Umeå University, S-901 87 Umeå, Sweden
E-mail: stenflo@abel.math.umu.se