TIBETAN CALENDAR MATHEMATICS

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ABSTRACT. The calculations of the Tibetan calendar are described, using modern mathematical notations instead of the traditional methods.

1. INTRODUCTION

The Tibetan calendar is derived from the Indian calendar tradition; it has the same general structure as Indian calendars, but the details differ significantly. The basis for the Tibetan calendar is the Kālacakra Tantra, which was translated from Sanskrit into Tibetan in the 11th century. (Traditional date of the translation is 1027 when the first 60 year cycle starts.) It is based on Indian astronomy, but much modified. The calendar became the standard in Tibet in the second half of the thirteenth century.

As in Indian calendars (see Dershowitz and Reingold [2]), months are lunar (from new moon to new moon) but numbered according to the corresponding solar months, i.e. the position of the sun,[1] while days are numbered by the corresponding lunar days. Since these correspondences are not perfect, there are occasionally two months with the same number, in which case the first of them is regarded as a leap month, and occasionally a skipped date or two days with the same date (then the first of them is regarded as a leap day). Unlike modern Indian calendars, there are no skipped months.

Various improvements of the calendar calculations have been suggested over the centuries, see [12], [7, Chapter VI], [38], [24], and different traditions follow different rules for the details of the calculation. There are two main versions (*Phugpa* and *Tsurphu*) of the Tibetan calendar in use today by different groups inside and outside Tibet; moreover, Bhutan and Mongolia also use versions of the Tibetan calendar. See further Appendix A. The different versions frequently differ by a day or a month, see Appendix A.13.

The description below refers to the Phugpa version, introduced 1447, which is the most common version; it is the version followed by e.g. the Dalai Lama and it can be regarded as the standard version of the Tibetan calendar. The differences in the Tsurphu version and other versions are discussed in Appendix A.

Date: 31 December, 2007 (Sunday 23, month 11, Fire–Pig year);
revised 8 January, 2014 (Wednesday 8, month 11, Water–Snake year);
typo corrected 13 February, 2022 (Sunday 12, month 12, Iron–Ox year) .

[1]Solar months are used only in the astronomical theory behind the calendar, and not in the calendar itself.

4/3 1927	28/2 1987	Fire–Rabbit	2/3 1957	27/2 2017	Fire–Bird
22/2 1928	18/2 1988	Earth–Dragon	19/2 1958	16/2 2018	Earth–Dog
10/2 1929	7/2 1989	Earth–Snake	8/2 1959	5/2 2019	Earth–Pig
1/3 1930	26/2 1990	Iron–Horse	27/2 1960	24/2 2020	Iron–Mouse
18/2 1931	15/2 1991	Iron–Sheep	16/2 1961	12/2 2021	Iron–Ox
7/2 1932	5/3 1992	Water–Monkey	5/2 1962	3/3 2022	Water–Tiger
25/2 1933	22/2 1993	Water–Bird	24/2 1963	21/2 2023	Water–Rabbit
14/2 1934	11/2 1994	Wood–Dog	14/2 1964	10/2 2024	Wood–Dragon
4/2 1935	2/3 1995	Wood–Pig	4/3 1965	28/2 2025	Wood–Snake
23/2 1936	19/2 1996	Fire–Mouse	21/2 1966	18/2 2026	Fire–Horse
12/2 1937	8/2 1997	Fire–Ox	10/2 1967	7/2 2027	Fire–Sheep
3/3 1938	27/2 1998	Earth–Tiger	29/2 1968	26/2 2028	Earth–Monkey
20/2 1939	17/2 1999	Earth–Rabbit	17/2 1969	14/2 2029	Earth–Bird
9/2 1940	6/2 2000	Iron–Dragon	7/2 1970	5/3 2030	Iron–Dog
26/2 1941	24/2 2001	Iron–Snake	26/2 1971	22/2 2031	Iron–Pig
16/2 1942	13/2 2002	Water–Horse	15/2 1972	12/2 2032	Water–Mouse
5/2 1943	3/3 2003	Water–Sheep	5/3 1973	2/3 2033	Water–Ox
24/2 1944	21/2 2004	Wood–Monkey	22/2 1974	19/2 2034	Wood–Tiger
13/2 1945	9/2 2005	Wood–Bird	11/2 1975	9/2 2035	Wood–Rabbit
4/3 1946	28/2 2006	Fire–Dog	1/3 1976	27/2 2036	Fire–Dragon
21/2 1947	18/2 2007	Fire–Pig	19/2 1977	15/2 2037	Fire–Snake
10/2 1948	7/2 2008	Earth–Mouse	8/2 1978	6/3 2038	Earth–Horse
28/2 1949	25/2 2009	Earth–Ox	27/2 1979	23/2 2039	Earth–Sheep
17/2 1950	14/2 2010	Iron–Tiger	17/2 1980	13/2 2040	Iron–Monkey
7/2 1951	5/3 2011	Iron–Rabbit	5/2 1981	3/3 2041	Iron–Bird
26/2 1952	22/2 2012	Water–Dragon	24/2 1982	21/2 2042	Water–Dog
14/2 1953	11/2 2013	Water–Snake	13/2 1983	10/2 2043	Water–Pig
4/2 1954	2/3 2014	Wood–Horse	3/3 1984	29/2 2044	Wood–Mouse
23/2 1955	19/2 2015	Wood–Sheep	20/2 1985	17/2 2045	Wood–Ox
12/2 1956	9/2 2016	Fire–Monkey	9/2 1986	6/2 2046	Fire–Tiger

Table 1. Tibetan New Year, *Losar*, (Phugpa version) and year names for the last and current 60 year cycles.

Table 1 gives the Gregorian dates of the Tibetan New Year (*Losar*) for the years in the last and current 60 year cycles (see Section 4). See also Table 9 in Appendix A. The table shows that New Year at present occurs in February or the first week of March; the extreme dates during the 20th century are 4 February (e.g. 1954) and 5 March (e.g. 1992), and during the 21st century 5 February (2019) and 7 March (2095). (The dates get slowly later in the Gregorian calendar, see Section 12.)[2]

[2]The Tibetan New Year thus frequently coincides with the Chinese New Year, which also is at new moon and always in the range 21 January – 21 February [2, Section 17.6], although often the New Years differ by a day because of the different calculation methods;

The purpose of this paper is to describe the mathematics of the Tibetan calendar, using modern mathematical notations instead of the traditional methods. After some preliminaries (Sections 2–3) and a description of the naming of years (Section 4 and Appendix B), the calculations of the Tibetan calendar are presented in two main parts. In the first part (Section 5 and Appendix C), the months are regarded as units and I discuss how they are numbered, which implies the partitioning of them into years and also shows which months that are leap months. In the second part (Sections 6–9), I discuss the coupling between months and days, including finding the actual days when a month begins and ends and the numbering of the days. Finally, some further calculations are described (Sections 10–11) and some mathematical consequences are given (Sections 12–13). Calculations for the planets and some other astrological calculations are described in Appendices D–E.

The description is based mainly on the books by Schuh [12] and Henning [7], but the analysis and mathematical formulations are often my own. (Unfortunately I do not read Tibetan, so I have to use secondary sources instead of Tibetan texts. This is of course is a serious drawback, although I have been able to check the calculations against published almanacs such as [14].) For further study I recommend in particular the detailed recent book by Henning [7], which contains much more material than this paper; see also his web site [24] with further information. I (mostly) describe only the contemporary versions and ignore the historical development; for the history of the calendar, see Schuh [12] and [38]. See also Dershowitz and Reingold [2] for a related description and computer implementation of the Tibetan calendar. Open source computer programs can be obtained from Henning [24]. Computer generated tables and calendars covering almost 1000 years are given in Schuh [12] and by Henning [24, Traditional Tibetan calendar archive]. Some further related references are Ginzel [5, pp. 403–409] (a classic, but dated and incomplete), Petri [9] (on Tibetan astronomy), Tseng [17, 18] (on astrology[3]), and further articles in the collection [1]. See also the review by Schuh [13] of the first (2007) version of the present paper (together with the books [2] and [7]), and the reply by Henning [25].[4]

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however, in about each second year (at present), the Tibetan New Year is a month after the Chinese.

[3]Tseng [17, 18] and Schuh [13] make a distinction between astrology (using planets etc.) and divination (*nag-rtsis* or *'byung-rtsis*, using elements etc.). I use "astrology" in a wider sense, encompassing both.

[4]The review by Schuh [13] is quite critical of many details. I am grateful for some of his comments, and I have tried to make some improvements in the present version. (In other cases I agree that improvements of formulations might be desirable, but I find the present version acceptable and leave it for various reasons. I also agree that several of the internet references are unreliable and not scholarly; however, I have nevertheless included them, either to illustrate actual usage among Buddhist groups or to show interesting claims that I have been unable to verify or disprove; the reader is advised to regard these with caution.) I disagree with some other comments, of which I will mention a few here. First,

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2. Notation

Mixed radix notation. Traditional Tibetan calculations are made expressing the various quantities as sequences (written as columns) of integers, which should be interpreted as rational numbers in a positional system with mixed radices;[5] the radices are fixed but different sequences of radices are used for different quantities. I usually use standard notation for rational numbers, but when quoting the traditional expressions, I use notations of the type (of varying length)[6]

$$a_0; a_1, a_2, a_3\ \ (b_1, b_2, b_3) \qquad \text{meaning} \qquad a_0 + \frac{a_1 + (a_2 + (a_3/b_3))/b_2}{b_1}.$$

Formally, we have the inductive definition

$$a_0; a_1, a_2, \dots, a_n\ \ (b_1, b_2, \dots, b_n) = a_0 + \frac{a_1; a_2, \dots, a_n\ \ (b_2, \dots, b_n)}{b_1}$$

for $n \geq 2$, and $a_0; a_1\ \ (b_1) = a_0 + a_1/b_1$. We will usually omit a leading 0 (and the semicolon) and write just e.g. $a_1, a_2, a_3\ \ (b_1, b_2, b_3)$ for numbers between 0 and 1.

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the present paper is primarily written for people who, like me, do not know Tibetan, and I therefore find it useful to use more or less standard Anglicized forms of names rather than the scientific transliterations (Wylie) used by Tibetologists (who hopefully will understand what I mean in any case), although I sometimes give the latter as well (in italics). (For example, I write "Tsurphu", used e.g. by the Tsurphu monastery itself on its English web pages and on the almanacs from Rumtek shown in [24, Open source Tsurphu calendar software].) Similarly, I have chosen to follow [7] and call the main version of the Tibetan calendar the "Phugpa version", for convenience and without going into the detailed historical background (see [12]). Furthermore, I usually use the English terminology of [7] for Tibetan terms. Schuh does not like these translations; I cannot argue with his expert linguistic remarks, but I am not convinced that they are relevant for a description of the mathematical content rather than a linguistic and cultural study of the calendar tradition (see also [25]); moreover, it seems better to use an existing English terminology [7] rather than making my own second translation from a German translation [12] of the terms.

[5]In the same way as we denote time in days, hours, minutes and seconds, with radices 24, 60, 60.

[6]This notation is taken from Henning [7], although he usually omits all or most of the radices since they are given by the context. Schuh [12], [38] uses the similar notation $[a_1, a_2, \dots, a_n]/(b_1, b_2, \dots, b_n)$ meaning either $0; a_1, a_2, \dots, a_n\ \ (b_1, b_2, \dots, b_n)$ or $a_1; a_2, \dots, a_n\ \ (b_2, \dots, b_n)$ depending on the context.

For explanations and examples of the way the traditional hand calculations are performed[7], see Henning [7], Schuh [12] and [38, Kalenderrechnung, Sandabakus].

Angular units. It will be convenient, although somewhat unconventional, to express longitudes and other angular measurements in units of full circles in our formulas. To obtain values in degrees, the numbers should thus be multiplied by 360. (A Tibetan would probably prefer multiplying by 27 to obtain lunar mansions (= lunar station = *naksatra*) and fractions thereof; this is the unit usually used for longitudes. A Western astrologer might prefer multiplying by 12 to obtain values in signs. A mathematician might prefer multiplying by 2π (radians).)

For angular measurements, full circles are often to be ignored (but see Appendix C); this means with our convention that the numbers are considered modulo 1, i.e., that only the fractional part matters.

Boolean variables. For a Boolean variable ℓ, i.e. a variable taking one of the values *true* and *false*, we use $\ell = \{\mathcal{P}\}$ to denote that $\ell = \mathit{true}$ if and only if the condition $\mathcal{P}$ holds; we further let $[\ell]$ be the number defined by $[\ell] = 1$ when $\ell = \mathit{true}$ and $[\ell] = 0$ when $\ell = \mathit{false}$.

Julian day number. The *Julian day number* (which we abbreviate by JD) for a given day is the number of days that have elapsed since the epoch 1 January 4713 BC (Julian); for days before the epoch (which hardly concern the Tibetan calendar), negative numbers are used. The Julian day numbers thus form a continuous numbering of all days by $\dots, -1, 0, 1, 2, \dots$. Such a numbering is very convenient for many purposes, including conversions between calendars. The choice of epoch for the day numbers is arbitrary and for most purposes unimportant. The conventional date 1 January 4713 BC (-4712 with astronomical numbering of years) was originally chosen by Scalinger in 1583 as the origin of the Julian period, a (cyclic) numbering of years; this was developed by 19th-century astronomers into a numbering of days. See further [3, Section 12.7], [4, Section 15.1.10].[8]

A closely related version of this idea is the *Julian date*, which is a real number that defines the time of a particular instant, meaured (in days and fractions of days) from the same epoch. The fractional part of the Julian date shows the fraction of a day that has elapsed since noon GMT (UT); thus, if n is an integer, then the Julian date is $n.0$ (i.e., exactly n) at noon GMT on the day with Julian day number n.

It is important to distinguish between the Julian day number and the Julian date, even if they are closely related. Both are extremely useful, but for different purposes, and much unnecessary confusion has been caused by

[7] Traditionally in sand rather than on paper.

[8] Dershowitz and Reingold [2] use another day number, denoted by RD, with another epoch: RD 1 is 1 January 1 (Gregorian) which is JD 1721426. Consequently, the two day numbers are related by JD = RD + 1721425.

confusing and mixing them. (We follow [3] in using different names for them, but that is not always done by other authors.[9]) For this study, and most other work on calendars, the Julian day number is the important concept. (The Julian date is essential for exact astronomical calculations, but no such calculations are used in the traditional Tibetan calendar.) Note that the Julian day number is an integer, while the Julian date is a real number. (A computer scientist would say that they have different types.) Moreover, the Julian day number numbers the days regardless of when they begin and end, while the Julian date depends on the time of day, at Greenwich. Hence, to convert a Julian date to a Julian day number, we need in practice to know both the local time the day begins and the time zone, while these are irrelevant for calculations with the Julian day number. For example, 1 January 2007 has JD 2454102, everywhere. Thus the Julian date 2454102.0 is 1 January 2007, noon GMT (UT), and the new year began at Julian date 2454101.5 in Britain, but at other Julian dates in other time zones. The Tibetan day begins at dawn, about 5 am local mean solar time (see Remark 6 below), but we do not have to find the Julian date of that instant.

Other notations. We let $\bmod$ denote the binary operation defined by $m \bmod n = x$ if $x \equiv m \pmod n$ and $0 \le x < n$ (we only consider $n > 0$, but m may be of any sign; care has to be taken with this in computer implementations).

Similarly (following [2]), we let amod denote the binary operation defined by $m \operatorname{amod} n = x$ if $x \equiv m \pmod n$ and $0 < x \le n$. This means that $m \operatorname{amod} n = m \bmod n$ except when m is a multiple of n; then $m \bmod n = 0$ and $m \operatorname{amod} n = n$. For integers m and n (the usual case, and the only one used in this paper), $m \operatorname{amod} n = 1 + (m-1) \bmod n$.

We use the standard notations $\lfloor x \rfloor$ and $\lceil x \rceil$ for the integers satisfying $x - 1 < \lfloor x \rfloor \le x$ and $x \le \lceil x \rceil < x + 1$, i.e. x rounded down and up to an integer, and $\operatorname{frac}(x) = x - \lfloor x \rfloor = x \bmod 1$ for the fractional part. (Again, care has to be taken for $x < 0$ in computer implementations.)

3. Some concepts

The calendar is based on considering several different types of days, months and years, and we give a list of the most important ones here. The calendar is in principle astronomical and based on the positions of the moon and sun in the sky, more precisely their *longitudes*[10] and in particular the *elongation* of the moon, i.e., the difference between the longitudes of the

[9]Moreover, astronomers simply define Julian day number as the integer part of the Julian date [4, Section 3.7], which is *not* the version described here [3, Section 12.7], [4, Section 15.1.10], used in historical chronology.

[10]The longitude of a celestial object is measured eastward along the ecliptic, with 0 at the first point of Aries, both in Tibetan and Western astronomy. (This is the vernal equinox, where the sun crosses the celestial equator northwards every year; however, in Tibetan astronomy, the vernal equinox is erroneously believed to be much earlier, see Footnote 71 and [7, p. 328].) The sun is always on the ecliptic (with small perturbations);

moon and the sun. The Tibetan calendar uses two different formulas to calculate each of these: a simple (linear) formula giving the *mean longitude* (assuming uniform motions of the sun and moon) and a more complicated formula for the *true longitude*; thus the "lunar" and "solar" concepts defined below have two versions, one "mean" and one "true". Note that the calendar always uses these theoretical values and not the actual astronomical positions; do not confuse the "true" longitude with the exact astronomical one. (Nowadays, the "true longitudes" actually have large errors, see Section 12.)

calendar day (*gza'* or *nyin-zhag*): Solar day or natural day; in Tibet the calendar day is from dawn to dawn. The length is thus constant, 24 hours. The calendar days are numbered by the number of the corresponding lunar day, but since the correspondence is not perfect, sometimes a number is skipped or repeated, see Section 6. Each calendar day has also a day of week, in the same way as in Western calendars, see Section 9.

lunar day (*tshes-zhag,* Sanskrit *tithi*): 1/30 of a lunar month; more precisely the time during which the elongation of the moon increases by 1/30 (= 12°). (Since the moon does not travel at uniform speed; the lunar days have different lengths, varying between about 21.5 and 25.7 hours, see Remark 15.) The lunar days are numbered by 1–30 in each lunar month, with day 1 beginning at new moon. Thus lunar day i is when the elongation is between $(i-1)/30$ and $i/30$.

calendar month (*zla-ba*): A period of 29 or 30 calendar days, approximating a lunar month. (The calendar month gets the same number as the lunar month, and they are often regarded as the same, but strictly speaking the calendar month begins at the beginning of a calendar day, at dawn, while the lunar month begins at the instant of new moon.)

lunar month (*tshes-zla*): The period from the instant of a new moon to the next new moon. (Synodic month in astronomic terminology.) The lunar months are numbered 1–12, but sometimes there is a leap month and a number is repeated, see Section 5.

solar month (*khyim-zla*): The period during which the sun travels through one sign. (Each sign is 1/12 of the ecliptic, i.e., 30°.) This is thus 1/12 of a solar year, although the lengths of the true solar months vary somewhat.

calendar year: 12 or 13 calendar months, according to the rules for inserting leap months. The length of the calendar year is 354, 355, 383, 384 or 385 days, see Remark 15. The average length is close to the length of the solar year, see Section 12; in principle, the calendar year is on the average fixed with respect to the seasons, although this is in reality not exact.

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the moon is up to 5.15° away from the ecliptic due to the inclination of its orbit, but only its longitude is relevant for the calendar. [3, §1.3, Table 15.4, p. 731]

solar year: The period during which the sun travels a full revolution around the ecliptic. (Tropical year in astronomical terminology.)

4. Numbering and naming of years

Several ways of numbering or naming Tibetan years are used, see e.g. Schuh [12, pp. 142–145], [38, Kalender]. One common method, especially among Westeners, is to simply number a Tibetan year by the Gregorian (or Julian) year in which it starts (and where most of it falls). For convenience, we will use this numbering below.

Another method is to number the years from an epoch 127 BC (the traditional ascent of the first Tibetan king); the Tibetan year starting in Gregorian year Y will then be numbered $Y + 127$.[11] Both methods are used by Tibetans; for example, the Tibetan calendar [15] has titles in both Tibetan and English; with 2003 in the English title and 2130 in the Tibetan.

Moreover, and more importantly, each year is named according to a 60 year cycle. Actually, there are two different 60 year cycles of names, one Indian and one Chinese. Of course, since the cycles have the same length, there is a 1–1 correspondence between the names. When naming years, the Chinese cycle is almost always used, and sometimes the Indian and Chinese cycle names are used together. (For example, the Chinese cycle names are used in the titles of the calendars [15] and [16].)

The Indian cycle is a list of 60 different names, in Sanskrit or Tibetan, see Appendix B. The cycle is named after its first year as Prabhava (*rab byung*). The cycles are numbered, with the first cycle beginning in AD 1027, which means that each year can be unambiguously identified by its name in the cycle and the number of the cycle; this method of naming years has sometimes been used [38, Sechzigjahreszyklen]. (See Footnote 49 and [39] for modern Mongolian examples.)

Year n in cycle m (with $1 \le n \le 60$ and, presumably, $m \ge 1$) thus corresponds to Gregorian or Julian year Y given by

$$Y = 1027 + (m - 1)60 + (n - 1) = 60m + n + 966. \qquad (4.1)$$

Conversely,

$$n = (Y - 1026) \ \operatorname{amod}\ 60 = (Y - 6) \ \operatorname{amod}\ 60, \qquad (4.2)$$

$$m = \left\lceil \frac{Y - 1026}{60} \right\rceil. \qquad (4.3)$$

For example, AD 2007 is the 21st year in the 17th Prabhava cycle, which began in 1987.

The Chinese cycle is identical to the one used in the Chinese calendar [2]. The cycles start 3 years before the Indian ones, so the first year (Prabhava, *rab byung*) in the Indian cycle is the fourth year (Fire–female–Rabbit) in

[11] This method has been used from the second half of the 20th century. Epochs 255 and 1027 have also been used. See [38, Kalender].

the Chinese cycle. The full correspondence is given in Appendix B. The last cycle thus started 1984. Hence, or by (4.2), year Y is number

$$(Y - 3) \ \text{amod}\ 60 \tag{4.4}$$

in the Chinese cycle. The Chinese cycles are not numbered in the Tibetan calendar.

The Chinese 60 year cycle is a combination of two cycles, of 10 and 12 years respectively. (Note that the least common multiple of 10 and 12 is 60; since 10 and 12 have greatest common divisor 2, only half of the 10×12 combinations are possible.)[12]

The 10 year cycle consists in China of 10 different names (proper names with no other English translation) called celestial stems. Each celestial stem is associated with an element (wood, fire, earth, iron, water) and a gender (female or male), see Table 2. (In Chinese, the two genders are the well-known *yin* and *yang*.) Note that the 2 genders are alternating and thus are given by the year mod 2, while the 5 elements are repeated 2 years each in the 10 year cycle. As a consequence, each celestial stem corresponds to a unique (element, gender) pair, and in the Tibetan calendar, the element and gender are used to name the year; the Chinese celestial stems are usually not used [7, p. 145].

It follows from (4.4) that, using the numberings in Tables 2 and 14, year Y is $z = (Y - 3)\ \text{amod}\ 10$ in the Chinese 10 year cycle, and has element $\lceil z/2 \rceil$.

The 12 year cycle is the well-known cycle of animals found in the Chinese and many other Asian calendars, see Table 3. (The English translations are taken from [7]; several easily recognized variants exist. The Tibetan names are from [2].)

It follows from (4.4) that year Y is $z = (Y - 3)\ \text{amod}\ 12$ in the 12 year animal cycle.

The Tibetan name for a year according to the Chinese cycle is thus given as Element–gender–Animal. Note that the gender, being the year mod 2, also is determined by the animal (since 2 divides 12), as shown in Table 3. Indeed, the gender is often omitted and only Element–Animal is used as the name of the year. For example, AD 2007 is the 24th year in the Chinese cycle (21st in the Indian) and is thus Fire–female–Pig or simply Fire–Pig.

The civil year starts, unsurprisingly, with month 1; note that in case there is a leap month 1, the year begins with the leap month, which precedes the regular month 1. (This happened in 2000. See also Remark 17.) The first day of the year is also celebrated as a major holiday (*Losar*; the Tibetan New Year).

Remark 1. Traditional Tibetan almanacs (such as [14]) start, however, with month 3; this month is identified with the Kālacakra month *nag pa* (Tibetan)

[12] These cycles have been used in the Chinese calendar since at least 1400 BC, first for naming days and from the Zhou dynasty (c. 1000 BC) for naming years. [10, p. 163]

year		element	gender	Tibetan	celestial stem (Chinese)
1	8	wood	male	*shing-pho*	jiǎ
2	9	wood	female	*shing-mo*	yǐ
3	10	fire	male	*me-pho*	bǐng
4	1	fire	female	*me-mo*	dīng
5	2	earth	male	*sa-pho*	wù
6	3	earth	female	*sa-mo*	jǐ
7	4	iron	male	*lcags-pho*	gēng
8	5	iron	female	*lcags-mo*	xīn
9	6	water	male	*chu-pho*	rén
10	7	water	female	*chu-mo*	guǐ

TABLE 2. The 10 year cycle. The first number on each line shows the year mod 10 counted from the start of a Chinese cycle; the second shows the year mod 10 counted from the start of a Prabhava cycle.

year		animal	gender	Tibetan
1	10	Mouse	male	*byi ba*
2	11	Ox	female	*glang*
3	12	Tiger	male	*stag*
4	1	Rabbit	female	*yos*
5	2	Dragon	male	*'brug*
6	3	Snake	female	*sbrul*
7	4	Horse	male	*rta*
8	5	Sheep	female	*lug*
9	6	Monkey	male	*spre'u*
10	7	Bird	female	*bya*
11	8	Dog	male	*khyi*
12	9	Pig	female	*phag*

TABLE 3. The 12 year cycle. The first number on each line shows the year mod 12 counted from the start of a Chinese cycle; the second shows the year mod 12 counted from the start of a Prabhava cycle.

or *Caitra* (Sanskrit), which was considered the beginning of the Kālacakra year [7, p. 194] and is the starting point in Tibetan astronomy. (*Caitra* is the first month of the year in most Indian calendars [2].) The standard numbering system is known as *Mongolian month* (*hor-zla*) (or, in recent almanacs, *Tibetan month* (*bod-zla*)), and was introduced in the 13th century when the Tibetan New Year was moved to be the same as the Mongolian[13] [7, p. 145], [38, Kalendar, Zeitmaße].

The year is nevertheless considered to consist of months 1–12 in the usual way, as said above, so a traditional almanac contains the 10 last months of the year and 2 months of the next (perhaps plus a leap month).

5. Numbering of months and leap months

The Tibetan calendar months are, as said in Section 3, lunar months, that begin and end at new moon. There are usually 12 months in a year, but there is also sometimes an extra *leap* (or *intercalary*) month, so that the year then has 13 months (a *leap year*), in order to keep the calendar year roughly aligned with the (tropical) solar year. Non-leap months are called *regular*. There are thus always 12 regular months in a year.

The regular months in each year are numbered 1–12. A leap month takes the same number (and belongs to the same year) as the following month (as in Indian calendars [2]).[14] In a leap year there are thus two months having the same number and, by the rule just given, the first of these is the leap month.

5.1. **Month names.** Usually, the Tibetan months are just numbered, but month names exist too; they are printed in almanacs and are sometimes used, in particular in literature about the calendar. Several different naming systems exist as follows, see Table 4. (In all systems, a leap month gets the same name as the corresponding regular month.) See further Schuh [12, p. 145], [13] and [38, Zeitmaße], and Henning [7, pp. 147–149 and 194–196] and [24, Early epochs].

(i) The Indian system of naming months by twelve of the names of the lunar mansions, chosen to be approximatively where the full moon occurs that month (and thus opposite to the sign that the sun visits). (Cf. [2, Section 9.3]. See also Henning [7, pp. 358–359] and Petri [9, p. 92].) See Table 4, where both the Tibetan names and the Sanskrit names are given. This system was introduced with the Kālacakra calendar in the 11th century.

(ii) Animal names, by the same 12 animal cycle as for years, see Table 3. This can be extended to Element + Animal, or Element + Gender + Animal, as for years, see Appendix E.2 for details. (This is an old Chinese system, used in Tibet from the 12th century.) Note that Phugpa almanacs set month 11 = Tiger (shown in Table 4), while Tsurphu

[13]By the Tibetan spiritual and political leader Chogyel Phagspa Lodro Gyeltsen (*chos-rgyal 'Phags-pa Blo-gros rgyal-mtshan*) (1235–1280), who was advisor of Kublai Khan and Preceptor of Tibet, then part of the Mongol Empire. [40, Pakpa Lodro Gyeltsen]

[14]The original system in the Kālacakra Tantra seems to have been the opposite (although this is not stated explicitly), with a leap month taking the number of the preceding month (as in the Chinese calendar [2]), see [24, Published calendar explanation]. That system is also used in the Bhutanese version of the calendar, see Section A.4.

almanacs (Appendix A.2) set month 1 = Tiger (as in the Chinese calendar), see Table 24. This method is still important for astrological purposes.

(iii) Seasonal names, naming the 12 months as beginning, middle and end of each of the four seasons spring, summer, autumn, winter. This is the oldest system, and was used at least from the 7th century. [38, Zeitmaße]

Unfortunately, the seasonal months have been identified with the Kālacakra months in several different ways, so there are several different versions. As far as I know, none of them is really used today, but they are mentioned e.g. in almanacs, which often give two or several different versions of them, see e.g. [7, pp. 195–196]. At least the following versions exist today. (Including versions used by the Tsurphu tradition, see Appendix A.2.)

(a) Month 1 = *mchu* = late-winter, etc.; thus Month 3 = *nag pa* = mid-spring. This is the original Kālacakra identification, now used by the Tsurphu tradition.

(b) Month 1 = *mchu* = early-spring, etc.; thus Month 3 = *nag pa* = late-spring. See Table 4. This is the Phugpa version of the Kālacakra system. (Introduced c. 1200 by Drakpa Gyaltsen, see [24, Early epochs].)

(c) Identifying the animal names in (ii) by Tiger = early-spring, etc. (as in the Chinese calendar). Note that this gives different seasonal names for the Phugpa and Tsurphu versions. The Phugpa version is given in Table 4.

(d) A different system with 6 seasons with two months each. (This is an Indian system, see e.g. [5, §75 and pp. 345–346] and [10, p. 195].) Month 1 = *mchu* = early-winter; Month 3 = *nag pa* = early-spring; see [7, pp. 358–359] for the complete list.

The present numbering of months was, as said above, introduced in the 13th century.[15]

5.2. **Leap months.** The Tibetan calendar is based on the relation

$$67 \text{ lunar months} = 65 \text{ solar months}, \tag{5.1}$$

which is regarded as exact. (See Section 12.2 for the astronomical reality.) This is a fundamental relation in the Tibetan calendar, which distinguishes it from other calendars such as the Indian ones.[16] The leap months are regularly spaced in accordance with this relation, i.e., with 2 leap months for each 65 solar months. In other words, there are 2 leap months for

[15] In the Kālacakra Tantra, *nag pa* (*Caitra*) is evidently the first month, as in Indian calendars, although the months are not explicitly numbered.

[16] This relation derives, as all basic features of the calendar, from the Kālacakra Tantra, but there it is actually given as an approximation and not as an exact relation, see Appendix A.5.

	Tibetan	Sanskrit	seasonal I	animal	seasonal II
1	mchu	Māgha	early-spring	dragon	late-spring
2	dbo	Phālguna	mid-spring	snake	early-summer
3	nag pa	Caitra	late-spring	horse	mid-summer
4	sa ga	Vaiśākha	early-summer	sheep	late-summer
5	snron	Jyeṣṭha	mid-summer	monkey	early-autumn
6	chu stod	Āṣāḍha	late-summer	bird	mid-autumn
7	gro bzhin	Śrāvaṇa	early-autumn	dog	late-autumn
8	khrums	Bhādrapada	mid-autumn	pig	early-winter
9	tha skar	Āśvina	late-autumn	mouse	mid-winter
10	smin drug	Kārtikka	early-winter	ox	late-winter
11	mgo	Mārgaśīrṣa	mid-winter	tiger	early-spring
12	rgyal	Pauṣa	late-winter	rabbit	mid-spring

TABLE 4. Month names (Phugpa system), see Section 5.1: number; Tibetan and Sanskrit lunar mansion names, (i); seasonal names, (iii)b; animal names, (ii); seasonal names, (iii)c.

65 regular months, and these are regularly spaced with alternating 32 and 33 regular months between the leap months. (Thus the distance between successive leap months alternates between 33 and 34 months.)[17]

Remark 2. It follows that the leap months repeat in a cycle of 65 years. In 65 years, there are $65 \cdot 12 = 780$ regular months and therefore $2 \cdot 12 = 24$ leap months, for a total of 804 months. Since 12 and 65 are relatively prime, the leap month can occur at any place in the year; in 65 years, each leap month 1–12 occurs exactly twice.

We describe here the basic algorithmic calculations to determine leap months and thus the numbering of all months. The astronomical theory behind the rules is explained in Appendix C.

The months can be described by year, number (from 1 to 12) and, possibly, the label "leap". We can thus formally think of each month as labelled by a triple (Y, M, ℓ), where $Y \in \mathbb{Z}$, $m \in \{1, \dots, 12\}$ and $\ell \in \{\mathit{true}, \mathit{false}\}$ is a Boolean variable. (As said in Section 4, we number the Tibetan years by the Gregorian (or Julian) year in which they start.)

The Tibetan calendar calculations also use a consecutive numbering of all months (regular or leap) starting with 0 at some epoch. (This is thus a linear numbering, ignoring the division into years, unlike the standard

[17] The average length between intercalations is thus $32\frac{1}{2}$ regular months, or $33\frac{1}{2}$ months including the leap month. Tibetan authors have often interpreted this as meaning that each second intercalation really should come in the middle of a month, although it is moved to the beginning of the month, see [19], [24, On intercalary months]. As far as I know, this view has only been used in theoretical discussions and no Tibetan calendar has ever been produced with the leap month in the middle of a month. (However, leap months are inserted in this way in some Indian calendars [2, p. 277].)

cyclic numbering that starts again with each new year.) The epoch could in principle be the beginning of an arbitrarily chosen month; we assume in our formulas that the epoch is the beginning of month M_0 year Y_0.

Any epoch will give the same calendar (provided the correct initial data are used), and in calculations only a single epoch is chosen. Nevertheless, to illustrate the calendar mathematics, we give (and compare) data for three different epochs. The three epochs we use are month 3 year 806 (the traditional epoch from Kālacakra Tantra although it is several centuries before the calendar came to Tibet, used e.g. by Schuh [12]), month 3 year 1927 (used e.g. by Henning [7]) and month 3 year 1987 (used e.g. by Lai and Dolma [26] and the almanac [14]). We denote these epochs by E806, E1927, E1987.[18]

See further Remark 5 below.

Remark 3. By tradition, the epoch is always at the beginning of month 3 (*nag pa*, cf. Remark 1), so $M_0 = 3$. (But see Remark 5 for the correct interpretation.) The year Y_0 is by tradition usually (but not always) chosen to be the first year (*rab byung*) of a Prabhava cycle; it is common to use the first year of the present cycle (so for example in the almanac [14] where the epoch is 1987). (This is convenient for hand calculations because it gives smaller numbers than older epochs.)

The number of a month in the linear numbering from a given epoch is called the *true month* (*zla-dag*), and is calculated as follows for month M year Y:

First, solar months are counted starting after the epoch (month M_0, year Y_0); for month M year Y, the "number of solar months" is

$$M^* = 12(Y - Y_0) + M - M_0. \tag{5.2}$$

Next, one calculates a preliminary version of the true month as the rational number

$$\frac{67}{65}M^* + \frac{\beta^*}{65} = \frac{67\big(12(Y - Y_0) + (M - M_0)\big) + \beta^*}{65}, \tag{5.3}$$

where β^* is a constant depending on the epoch. For our three example epochs we have

$$\beta^* = 61 \qquad \text{(E806)}; \tag{5.4}$$

$$\beta^* = 55 \qquad \text{(E1927)}; \tag{5.5}$$

$$\beta^* = 0 \qquad \text{(E1987)}. \tag{5.6}$$

We write the fractional part of the true month (5.3) as $ix/65$, where the integer ix is called the *intercalation index*.[19] Thus, the (preliminary version

[18] I have chosen these three epochs just as examples. Another, historically important, traditional epoch is 1687 [7, p. 331], [24, Epoch data].

[19] We use this name from [7]; the Tibetan name *zla-bshol rtsis-'phro* simply means "remainder for the calculation of leap-month" [13], [38, Kalenderrechnung].

of the) intercalation index is

$$ix = (67M^* + \beta^*) \bmod 65 = (2M^* + \beta^*) \bmod 65, \tag{5.7}$$

with M^* given by (5.2). Note that β^* is the initial value of ix at the epoch.

The traditional Phugpa leap month rule is:

A leap month is inserted when the intercalation index $ix = 48$ *or* 49. (5.8)

For the rest of the calendar calculations, the true month is rounded to an integer by the following rule. (We follow [7] and use the same name "true month" for both the rational version (5.3) and the rounded integer version, but in order to avoid confusion, we will often call the latter "true month count".)

The true month (5.3) *is rounded down to the nearest integer if* $ix < 48$ *and rounded up if* $ix \geq 48$*, except that for a leap month (when* $ix = 48$ *or* 49*), always round down.* (5.9)

When there is a leap month, there are two months with the same number M the same year. The rule just given means that the true month is rounded down for the first of these (the leap month) and rounded up for the second (the regular month); thus the true month count will increase by 1 for each new month also immediately before and after the leap month. Similarly, it is easily checked that the true month count increases by 1 also when the intercalation index (5.7) passes 65 and drops to 0 or 1 again. Hence the true month count calculated by (5.9) is really a continuous count of months.

Remark 4. The formulation in (5.9) differs somewhat from the traditional formulation, described e.g. by Henning [7]. Traditionally, the true month is calculated by (5.3) as above, but if this yields an intercalation index > 49, one notes that there has been an earlier leap month and therefore the numbering of months has to be corrected, so the true month just calculated for month M really applies to month $M - 1$; hence the true month for such a month M (and for a regular month immediately following a leap month) is obtained by doing the calculation for $M + 1$ (which means adding $1;2\ (1,65) = 1\frac{2}{65}$ to the true month calculated for M).[20] The true month count then is always the integer part of the true month calculated in this way; i.e., the true month is always rounded down. This evidently yields the same true month count as (5.9). However, the fractional part will differ by $\frac{2}{65}$ for these months, i.e. the intercalation index will differ by 2. (The intercalation index is not used further for the main calendar calculations,

[20]This leaves a gap when the intercalation index reaches 0 or 1: If month M has true month $n;0$ or $n;1\ (1,65)$, then the calculation for $M-1$ yields true month $(n-2);63$ or $(n-2);64$, which thus applies to $M-2$. Hence no starting month yields a calculation that applies to $M-1$; the missing month $M-1$ is usually given true month $(n-1);0$ or $(n-1);1$, respectively. Note that the integer part is correct, and that the intercalation index 0 or 1 is repeated. (Sometimes 65 and 66 are used instead [Henning, personal communication].) These complications do not appear in my version (5.9).

but almanacs publish for each month the true month with its fractional part, i.e., the intercalation index, see Section 10, and it has at least one use, see Section 10.1.)

Hence, in order to get the correct intercalation index, (5.9) should be extended by the rule that when the true month is rounded up, the intercalation index is increased by 2. (When the intercalation index from (5.7) is 63 or 64, the new value can be given as either 65 or 66, or as 0 or 1, see Footnote 20.)

Since $48 = 65 - 17$, (5.9) can be written, using (5.3) and (5.2), and denoting the true month count by n,

$$n = \left\lfloor \frac{67M^* + \beta^* + 17}{65} \right\rfloor - [\ell]. \tag{5.10}$$

Remark 5. As said in Remark 3 above, the epoch is by tradition always at the beginning of month 3 (*nag pa*). However, because of the leap year rules, this has to be interpreted as follows:

The calculation of the true month uses the number of solar months after the epoch, calculated by (5.2). For month 3 at the epoch, this is 0, so the true month is by (5.3) $\beta^*/65$; thus the intercalation index is β^*. According to the rounding rule (5.9), this month thus has true month count 0 if $\beta^* < 48$ but true month count 1 if $\beta^* \geq 48$. (If $\beta^* = 48$ or 49, there is a leap month 3 and the two months 3 have true month counts 0 and 1.) Hence, it is only when $\beta^* \leq 49$ that the epoch really is at the beginning of calendar month 3; otherwise it is at the beginning of the preceding month (calendar month 2), and the nominal epoch month 3 really has true month count 1. (This means that for an epoch with $\beta^* < 48$, the true month count is the number of elapsed (lunar) months *after* month 3 in the epoch year, while if $\beta^* \geq 48$, the true month count is the number of elapsed months *starting with* month 3.)

For our three example epochs, the values of β^* in (5.4)–(5.6) show that for E806 and E1927, the epoch month with true month count 0 is month 2, while for E1987, it is month 3. (See also the discussion in [13] and [25] on the epoch of E1927.)

Note further that the epoch is not only a year and month; it is a specific day, see Remark 16 in Section 7.

5.3. **The inverse problem.** Although not needed for the traditional construction of a yearly calendar, let us also consider the inverse problem: to find the number M and the year Y of the month with true month count n, together with the indicator ℓ telling whether the month is a leap month. We begin by writing (5.10) as

$$n = \frac{67M^* + \beta^* + 17 - r}{65}, \tag{5.11}$$

where r is chosen such that the result is an integer; for a regular month, $0 \leq r \leq 64$ (r is the remainder $(67M^* + \beta^* + 17) \bmod 65$), and for a leap month,

$r = 65$ or 66 (the remainder is 0 or 1, but $[\ell] = 1$ in (5.10)). Rearranging, we get

$$M^* = \frac{65n - \beta^* - 17 + r}{67} \tag{5.12}$$

with $0 \le r \le 66$, and thus

$$M^* = \left\lceil \frac{65n - \beta^* - 17}{67} \right\rceil. \tag{5.13}$$

Recalling (5.2), and assuming $M_0 = 3$, this can be written

$$12(Y - Y_0) + M = M^* + 3 = \left\lceil \frac{65n + 184 - \beta^*}{67} \right\rceil = \left\lceil \frac{65n + \beta}{67} \right\rceil, \tag{5.14}$$

where we define

$$\beta = 184 - \beta^*. \tag{5.15}$$

For our three example epochs, this yields, by (5.4)–(5.6),

$$\beta = 123 \qquad \text{(E806)}, \tag{5.16}$$

$$\beta = 129 \qquad \text{(E1927)}, \tag{5.17}$$

$$\beta = 184 \qquad \text{(E1987)}. \tag{5.18}$$

Consequently, we can calculate (Y, M) from n by

$$x = \left\lceil \frac{65n + \beta}{67} \right\rceil, \tag{5.19}$$

$$M = x \ \text{ amod } \ 12, \tag{5.20}$$

$$Y = \frac{x - M}{12} + Y_0 = \left\lceil \frac{x}{12} \right\rceil - 1 + Y_0. \tag{5.21}$$

To complete the calculations of (Y, M, ℓ) from n, we note that a month is leap if and only if it gets the same number as the following one. Thus,

$$\ell = \left\{ \left\lceil \frac{65n + \beta}{67} \right\rceil = \left\lceil \frac{65(n+1) + \beta}{67} \right\rceil \right\} = \big\{ (65n + \beta) \bmod 67 = 1 \text{ or } 2 \big\}. \tag{5.22}$$

Note that $n = 0$ in (5.19)–(5.20) yields $M = \lceil \beta/67 \rceil$, so $\lceil \beta/67 \rceil$ is the number of the epoch month. (This is 2 or 3, see Remark 5 above.)

5.4. **A general rule.** By (5.7), (5.2), (5.8) and $M_0 = 3$, there is a leap month M year Y if and only if

$$24(Y - Y_0) + 2M - 6 + \beta^* \equiv 48 \text{ or } 49 \pmod{65}. \tag{5.23}$$

Using (5.15), $48 + 6 - \beta^* = \beta - 130 \equiv \beta \pmod{65}$ and thus (5.23) can be written

There is a leap month M in year Y if and only if

$$24(Y - Y_0) + 2M \equiv \beta \textit{ or } \beta + 1 \pmod{65}. \tag{5.24}$$

We will see later, in Appendices A and C, that the rule (5.24) holds also for other versions of the Tibetan calendar, with appropriate β.

By combining (5.7), (5.2) and $M_0 = 3$,

$$ix = \big(24(Y - Y_0) + 2M + \beta^* - 6\big) \mod 65 \tag{5.25}$$

(if necessary modified as in Remark 4) and hence the rule (5.24) says that a leap month has intercalation index

$$(\beta + \beta^* - 6) \text{ mod } 65 \qquad \text{or} \qquad (\beta + \beta^* - 5) \text{ mod } 65, \tag{5.26}$$

which is the general form of the relation between the Phugpa formulas (5.15) and (5.8).

5.5. **Leap years.** We give some further formulas for leap years and leap months, inspired by [37], that follow from the formulas above. (These formulas are not traditional, and are not needed to construct the calendar.) We begin by stating them in a general form, valid also for other versions of the calendar.

Since M may be any number $1, \dots, 12$, it follows from (5.24) that Y is a leap year if and only if

$$24(Y - Y_0) - \beta \equiv 41,\ 42, \dots,\ \text{ or } 64 \pmod{65}, \tag{5.27}$$

which also can be written as

$$\big(24(Y - Y_0) - \beta\big) \mod 65 \geq 41. \tag{5.28}$$

Since $24 \cdot 19 = 456 \equiv 1 \pmod{65}$, this can be further rewritten as

$$24(Y - Y_0 - 19\beta) \mod 65 \geq 41. \tag{5.29}$$

Hence, if we define

$$\gamma = (-Y_0 - 19\beta) \mod 65, \tag{5.30}$$

we can state the rule as:

Y is a leap year if and only if

$$24(Y + \gamma) \mod 65 \geq 41. \tag{5.31}$$

Equivalently, if we define

$$\gamma^* = 24\gamma \mod 65 = (-24Y_0 - \beta) \mod 65, \tag{5.32}$$

then:

Y is a leap year if and only if

$$(24Y + \gamma^*) \mod 65 \geq 41. \tag{5.33}$$

Furthermore, it easily seen from (5.24), cf. (5.27), that if Y is a leap year, then the leap month has number

$$M = 1 + \left\lfloor \frac{64 - \big(24(Y - Y_0) - \beta\big) \mod 65}{2} \right\rfloor \tag{5.34}$$

which, using (5.32), easily is transformed to

$$M = \left\lfloor 33 - \frac{(24Y + \gamma^*) \mod 65}{2} \right\rfloor = \left\lfloor 33 - \frac{24(Y + \gamma) \mod 65}{2} \right\rfloor; \quad (5.35)$$

moreover, if Y is not a leap year, then (5.34)–(5.35) yield an impossible value $M \geq 13$.

Finally, if $Y_1 \leq Y_2$, the number of leap years (and thus the number of leap months) in the period from Y_1 to Y_2 (inclusive) is

$$\left\lfloor \frac{24(Y_2 + 1) + \gamma^*}{65} \right\rfloor - \left\lfloor \frac{24Y_1 + \gamma^*}{65} \right\rfloor. \quad (5.36)$$

To see this it suffices to consider the case $Y_2 = Y_1$, which easily follows from (5.33).

For the standard Phugpa version, (5.30) can be written, using (5.15),

$$\gamma = (-Y_0 - 19\beta) \mod 65 = (-Y_0 + 19\beta^* + 14) \mod 65, \quad (5.37)$$

and the values of β^* in (5.4)–(5.6) yield

$$\gamma = 42 \quad (5.38)$$

and thus

$$\gamma^* = 24 \cdot 42 \mod 65 = 33. \quad (5.39)$$

(The values of γ and γ^* are the same for any epoch, since the leap years are the same. However, for other versions of the calendar, the formulas hold with other values of γ and γ^*, see Appendix A.) Thus the rule is [37]:

Y is a leap year if and only if

$$24(Y + 42) \mod 65 \geq 41, \quad (5.40)$$

or, equivalently,

Y is a leap year if and only if

$$(24Y + 33) \mod 65 \geq 41. \quad (5.41)$$

For the Phugpa version, (5.35) is

$$M = \left\lfloor 33 - \frac{(24Y + 33) \mod 65}{2} \right\rfloor. \quad (5.42)$$

6. Days

As said in Section 3, each lunar month (from the instant of new moon to the next new moon) is (as in the Indian calendars [2]) divided into 30 lunar days; these have varying length of between 21.5 and 25.7 hours, and do not correspond exactly to the calendar days of 24 hours each. During each of these lunar days, the elongation of the moon (i.e., the difference between lunar and solar longitude) increases by 1/30 (= 12°).

The calendar computations, unlike the Indian ones, do not include a function calculating the elongation at a given time; instead the computations use the inverse function, giving directly the time when the elongation has a given value. We denote this function, described in detail in Section 7, by $true\_date(d, n)$, giving the date at the end of the lunar day d in true month count n. The value of this function is a real (rational) number; traditionally it is counted modulo 7, and the integer part yields the day of week, but we will treat it as a real number so that the integer part directly gives the Julian day number JD. (A different constant m_0 below will give RD [2] instead.) The fractional part shows the time the lunar day ends; it is used to calculate some further astronomical (and astrological) information, see Section 10, but can be ignored for the present purpose.

The basic rule is:

> *A calendar day is labelled by the lunar day that is current at the beginning of the calendar day.*

In other words, a lunar day gives its name (number and month) to the calendar day where the lunar day ends. (Thus the JD of the calendar day is the integer part of *true_date* at the end of the corresponding lunar day.) There are two special cases covered by the rule above: if two lunar days end the same calendar day, then the calendar day gets the name of the first of them; if no lunar day ends a given calendar day, then that day gets the same name as the following day. The first special case occurs when a lunar day is completely contained in one calendar day; in that case no calendar day gets the number of this lunar day, so this date is skipped. (In the sense that the number is skipped in the numbering of days in the calendar. The lunar day itself exists and can be used for astrological purposes. The calendar days exist in real life and, of course, as such cannot be skipped or repeated.) The second case occurs when a lunar day completely contains a calendar day; in that case this calendar day gets the same number of the next day, so the date is repeated. (In the sense that the number is repeated.)

When a date is repeated, the first of the two days with the same number is regarded as a leap day, and denoted “Extra” in the almanacs. [13] (But see Section 11.)

Recall also that each calendar day has a day of week, in the same way as in Western calendars, see Section 9. In particular, when a day is repeated, the two days with the same number are distinguished by different days of week.

Remark 6. We do not have to worry about when the day starts; the calendar day is from dawn to dawn, but the formulas take this into account (at least theoretically), and no further modification is done. In particular, no calculation of sunrise is required (as it is in Indian calendar calculations [2]). Thus, the *true_date* should be regarded as a kind of local Julian date that is offset from the standard astronomical one which assumes integer values at noon UT (= GMT), so that it instead assume integer values at local (mean) dawn. (Henning [7, pp. 10–11] and [25] specifies the start of the day as mean daybreak = 5 am local mean solar time. Since the time difference is about

6 hours (Lhasa has longitude 91° which corresponds to 6^h4^m), this is about −1 UT, i.e. 11 p.m. UT the preceding day.)

Remark 7. By definition, new moon is (exactly) at the end of lunar day 30 and full moon is at the end of lunar day 15; the rules above imply that unless the day is skipped, (true) new moon falls in calendar day 30 and (true) full moon in calendar day 15 in every month. The true elongation differs from the correct astronomical value by only about 2°, corresponding to 4 hours, see Section 12.1, so usually the same holds for the astronomcal new moon and full moon as well.

7. Astronomical functions

The *true_date* is calculated by first calculating a simpler *mean_date*, corresponding to the linear mean motion of the moon, and then adjusting it by the equations of the moon and sun, which are determined by the anomalies of the moon and sun together with tables. (The tables are really approximation to sine, suitably scaled. A similar table is used for each planet; no general sine table is used, another difference from Indian calendars [2].) The anomalies, in turn, are also calculated by linear functions.

Traditional hand calculations calculate the mean quantities first for the beginning of the month, i.e. the end of the preceding month. (This corresponds to taking day $d = 0$ below. Note that this usually gives a time during the last calendar day of the preceding calendar month.) Then the quantities are adjusted to give the values for a given day. We will combine the two steps into one, giving the values for true month n and day d directly.[21]

The Phugpa tradition uses the following functions. As said above, these give the values at the end of lunar day d in true month count n. I use below the epoch E806, but give also the corresponding constants for E1927 and E1987. (Recall that the different epochs yield the same calendar.)

The mean date (*gza' bar pa*) is (for E806)

$$mean\_date(d, n) = n \cdot m_1 + d \cdot m_2 + m_0, \tag{7.1}$$

where

$$m_1 = 29; 31, 50, 0, 480 \ \ (60, 60, 6, 707) = \frac{167025}{5656} \quad (\approx 29.530587), \tag{7.2}$$

$$m_2 = 0; 59, 3, 4, 16 \ \ (60, 60, 6, 707) = \frac{11135}{11312} \quad \Big(= \frac{m_1}{30}\Big), \tag{7.3}$$

$$m_0 = 0; 50, 44, 2, 38 \ \ (60, 60, 6, 707) + 2015501 = 2015501 + \frac{4783}{5656}. \tag{7.4}$$

Remark 8. The traditional reckoning counts days modulo 7 only, i.e. day of week (see Section 9); this is of course enough to construct a calendar month by month. My version gives the JD directly. To be precise, the traditional

[21] Lai and Dolma [26] refer to tables rather than doing multiplications to obtain the adjustments for d for *mean_date* and (without explanation) for *mean_sun*, but the results are the same.

result is obtained by adding 2 to the value in (7.1) before taking the remainder modulo 7, cf. (9.1). (Hence, the integer added to the traditional value of m_0 has to be congruent to -2 modulo 7. Indeed, $2015501 \equiv -2 \pmod 7$, and similarly for the other epochs below and in Appendix A.)

The traditional value is therefore $m_1 = 1;31,50,0,480\ (60,60,6,707)$, subtracting 28 from the value used in this paper. (It can also, and perhaps better, be regarded as $1,31,50,0,480\ (7,60,60,6,707)$, with the denominator 7 meaning that we regard the numbers as fractions of weeks, thus ignoring integer parts). Similarly, the constant 2015501 in m_0 (to get the result in JD) is my addition and not traditional. (To get RD, use 294076 instead.)

Remark 9. For E1927 [7], a simple calculation using (5.3) shows that the true month count n differs by 13866 from the value for E806; thus m_0 is instead

$$\begin{aligned} m_0 + 13866 \cdot m_1 &= 2424972 + \frac{5457}{5656} \\ &= 2015501 + 409471 + \frac{5457}{5656} \\ &= 2015501 + 409465 + 6;57,53,2,20\ \ (60,60,6,707), \end{aligned}$$

traditionally written as $6;57,53,2,20\ \ (60,60,6,707)$. (Note that $409465 = 7 \cdot 58495$ a multiple of 7, i.e. a whole number of weeks; hence this gives the correct day of week.)

Similarly, for E1987 [26], with the value of n differing by 14609, m_1 and m_2 are the same but m_0 is instead

$$\begin{aligned} m_0 + 14609 \cdot m_1 &= 2446914 + \frac{135}{707} \\ &= 2015501 + 431413 + \frac{135}{707} \\ &= 2015501 + 431410 + 3;11,27,2,332\ \ (60,60,6,707), \end{aligned}$$

traditionally written as $3;11,27,2,332\ \ (60,60,6,707)$.

Similarly, the mean longitude of the sun (*nyi ma bar pa*) is

$$mean\_sun(d,n) = n \cdot s_1 + d \cdot s_2 + s_0, \tag{7.5}$$

where,[22]

$$s_1 = 2, 10, 58, 1, 17 \ \ (27, 60, 60, 6, 67) = \frac{65}{804} \quad \Big(= \frac{65}{12 \cdot 67}\Big), \tag{7.6}$$

$$s_2 = 0, 4, 21, 5, 43 \ \ (27, 60, 60, 6, 67) = \frac{13}{4824} \quad \Big(= \frac{s_1}{30}\Big), \tag{7.7}$$

$$s_0 = 24, 57, 5, 2, 16 \ \ (27, 60, 60, 6, 67) = \frac{743}{804}. \tag{7.8}$$

Remark 10. For E1927 [7], with the base for n differing by 13866, s_1 and s_2 are the same but s_0 is instead (modulo 1, since only the fractional part matters)

$$s_0 + 13866 \cdot s_1 \equiv 25, 9, 10, 4, 32 \ \ (27, 60, 60, 6, 67) = \frac{749}{804}. \tag{7.9}$$

For E1987 [26], with the base for n differing by 14609, s_0 is instead[23]

$$s_0 + 14609 \cdot s_1 \equiv 0. \tag{7.10}$$

Thirdly, the anomaly of the moon (*ril-po dang cha-shas*) is

$$\mathit{anomaly\_moon}(d, n) = n \cdot a_1 + d \cdot a_2 + a_0, \tag{7.11}$$

where (but see Remark 14 below for an alternative for a_2)

$$a_1 = 2, 1 \ \ (28, 126) = \frac{253}{3528}, \tag{7.12}$$

$$a_2 = 1, 0 \ \ (28, 126) = \frac{1}{28}, \tag{7.13}$$

$$a_0 = 3, 97 \ \ (28, 126) = \frac{475}{3528}. \tag{7.14}$$

Remark 11. For E1927 [7], with the base for n differing by 13866, a_0 is instead (modulo 1, since only the fractional part matters)

$$a_0 + 13866 \cdot a_1 \equiv 13, 103 \ \ (28, 126) = \frac{1741}{3528}. \tag{7.15}$$

For E1987 [26], with the base for n differing by 14609, a_0 is instead

$$a_0 + 14609 \cdot a_1 \equiv 21, 90 \ \ (28, 126) = \frac{38}{49}. \tag{7.16}$$

Remark 12. For comparisons with modern astronomical calculations, note that the anomaly is measured from the Moon's apogee, while Western astronomy measures it from the perigee, which makes the values differ by a half-circle.

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[22] Note that the radices used for longitude are not the same as the radices used for time, see e.g. (7.2) and (7.6); the last radix is 67 instead of 707. This ought to cause some problems when converting between times and longitudes, but the difference is only in the last term and is usually ignored.

[23] This vanishing of the initial value, which recurs every 65th year (804th month), is called *nyi ma stong bzhugs* "sun empty enter" [26].

The equation of the moon (*zla rkang*) is calculated by

$$moon\_equ = moon\_tab(28 \cdot anomaly\_moon) \tag{7.17}$$

where $moon\_tab(i)$ is listed in the following table for $i = 0, \dots, 7$, which extends by the symmetry rules $moon\_tab(14-i) = moon\_tab(i)$, $moon\_tab(14+i) = -moon\_tab(i)$, and thus $moon\_tab(28+i) = moon\_tab(i)$; linear interpolation is used beween integer arguments.

i	0	1	2	3	4	5	6	7
$moon\_tab(i)$	0	5	10	15	19	22	24	25

(7.18)

To find the equation of the sun (*nyi rkang*), first calculate the anomaly by

$$anomaly\_sun = mean\_sun - 1/4 \tag{7.19}$$

and then take

$$sun\_equ = sun\_tab(12 \cdot anomaly\_sun) \tag{7.20}$$

where $sun\_tab(i)$ is listed in the following table for $i = 0, \dots, 3$, which extends by the symmetry rules $sun\_tab(6-i) = sun\_tab(i)$, $sun\_tab(6+i) = -sun\_tab(i)$, and thus $sun\_tab(12+i) = sun\_tab(i)$; linear interpolation is used beween integer arguments.

i	0	1	2	3
$sun\_tab(i)$	0	6	10	11

(7.21)

The date at the end of the lunar day (*gza' dag*) is finally calculated as

$$true\_date = mean\_date + moon\_equ/60 - sun\_equ/60. \tag{7.22}$$

(The half-corrected $mean\_date + moon\_equ/60$ is called semi-true date (*gza' phyed dag pa*).)

Similarly, although not needed to calculate the calendar date, the true solar longitude (*nyi dag*) is

$$true\_sun = mean\_sun - sun\_equ/(27 \cdot 60). \tag{7.23}$$

Remark 13. You will not find the factors $1/60$ and $1/(27 \cdot 60)$ explicit in the references; they are consequences of the positional system with mixed radices. Furthermore, $1/4$ in (7.19) is traditionally expressed as $6, 45\ (27, 60)$.

Remark 14. We have $m_2 = m_1/30$ and $s_2 = s_1/30$, which is very natural, since it means that the functions $mean\_date$ and $mean\_sun$ are linear functions of the lunar day count $d + 30 \cdot n$, and thus increase by the same amount every day without any jumps at the beginning of a new month.[24]

[24]Schuh [12], [38, Kalenderrechnung] gives several examples ($m = 3, 4, 5, 6, 7$ in his notation) of earlier versions of the calendar (with m_1 and s_1 slightly different from the values above), where simpler, rounded, values of m_2 and s_2 were used. Moreover, most of these versions used m_1 and s_1 only for calculating for the first month each year, and simplified value m_1' and s_1' for the increments for the successive months within the year. Some of them ($m = 3, 6, 7$) also used the simplified $a_1' = 2, 0\ (28, 126) \equiv 30a_2$ for the monthly increments of the anomaly. See Appendix A.5 for an example. I do not know any currently used versions of the calendar that use such simplifications.

For the anomaly, however, the standard value $a_2 = 1/28$ does not conform to this. Note that a_1 could be replaced by $1 + a_1 = 30, 1\ \ (28, 126)$ since we count modulo 1 here; in fact, this is the "real" value, since the astronomical anomaly increases by 1 full circle in a little less that one month, see also (12.4). Moreover, $a_2 = 1, 0\ \ (28, 126) = 1/28$ is a close approximation to $(1 + a_1)/30$; the conclusion is that one usually for convenience uses the rounded value $a_2 = 1/28$. Henning [7], however, uses instead the exact value

$$a_2 = \frac{1 + a_1}{30} = \frac{3781}{105840} = \frac{1}{28} + \frac{1}{105840} = 1, 0, 1\ \ (28, 126, 30); \qquad (7.24)$$

this is also used in his computed calendars [24, Traditional Tibetan calendar archive].[25] The value (7.24) is mathematically more natural than (7.13), since the latter value yields a (small) jump in the anomaly at the end of each month while (7.24) yields same increase every day. Nevertheless, the simpler (7.13) is usually used when calculating calendars, for example in the almanac [14].[26]

The difference 1/105840 between (7.13) and (7.24) is small, and the resulting difference in the anomaly is at most $30/105840 = 1/3528$; the difference in the argument to *moon_tab* is thus at most $28/3528 = 1/126$; since the increments in *moon_tab* are at most 5, the difference in *moon_equ* is at most 5/126; finally, by (7.22), the difference in the true date is at most $(5/126)/60 = 1/1512$ (see also Footnote 26), so when rounding to an integer (see (8.1) below), we would expect to obtain the same result except, on the average, at most once in 1512 days. We would thus expect that the two different values of a_2 would give calendars that differ for at most one day in 1512 on the average. Moreover, since this was a maximum value, and the average ought to be less by a factor of about 1/2, or more precisely $15.5/30 = 31/60$, since the difference is proportional to the day d which on the average is 15.5, and by another factor of 5/7 since the average derivative of *moon_tab* is 25/7 while we just used the maximum derivative 5. (For a sine function, the average absolute value of the derivative is $2/\pi$ times the maximum value, but the ratio is closer to one for the approximation in *moon_tab*.) Hence we would expect the average (absolute) difference in *true_date* to be about

$$\frac{31}{60} \cdot \frac{5}{7} \cdot \frac{1}{1512} \approx \frac{1}{4100}.$$

[25] The value (7.24) was proposed by Minling Lochen Dharmashri (1654–1717). [Henning, personal communication].

[26] This is witnessed by, for example, the values given for the true day of week (including fractional part), calculated by (7.22). This is given in the almanac with 3 terms (1,60,60), and the two calculations often differ (more often towards the end of the month), although typically only by 1 unit in the third term. A computer calculation for the 360 lunar days in 2013 yields 134 days with no difference in the true date (truncated to three terms), 149 days with a difference ±1 in the third term, 75 days with ±2 and 2 days with ±3.

Consequently, we expect that the two versions of a_2 will lead to different Tibetan dates for, on the average, one day in 4100, or a little less than one day in 10 years. This is confirmed by a computer search finding 9 examples in 1900–1999 and 8 examples in 2000–2099.[27] Some recent examples are 10 February 2001 (JD 2451951) and 10 May 2006 (JD 2453866); the next example is 19 November 2025 (JD 2460999). (It would be interesting to check these dates in published calendars.)

Remark 15. From one day to the next, the anomaly of the moon increases by 1/28 (or slightly more if (7.24) is used, and by the slightly larger $1,1\ (28,126) = 127/3528$ at each new month if (7.24) is not used; we can ignore these differences); this means that the arguments used in (7.17) differ by 1, and thus the resulting values of *moon_equ* differ by at most 5 (see (7.18)). Similarly, the anomaly of the sun differs by slightly less than 1/360, so by (7.20) and (7.21), the values of *sun_equ* differ by at most about $1/5 = 0.2$. The mean date increases by $m_2 = 0.98435$ each lunar day; this is thus the length of the mean lunar day, measured in calendar days, which equals $24m_2 = 23.6245$ hours. By (7.22) and the calculations just made we see that the increase of the true date from one lunar day to the next, i.e., the length of the true lunar day, differs from this by at most $\pm(5 + 0.2)/60 = 0.087$ days, or 2.1 hours. Consequently, the length of the (true) lunar day varies between, roughly, 21.5 and 25.7 hours.

A similar calculation shows that the length of a (true) lunar month varies between 29.263 and 29.798 days, or about 29^d6^h and 29^d19^h. Similarly, a lunar year of 12 lunar months has length between 354.00 and 354.74 days, and a lunar year of 13 lunar months has length between 383.67 and 384.13 days.

The length of the calendar year is one of these lengths for a lunar year rounded up or down to one of the nearest integers; the possibilities are thus 354, 355, 383, 384, 385. A computer calculation (for 10000 years) gave the frequencies:

354	355	383	384	385
42%	21%	3%	33%	1%

Remark 16. The epoch is a specific day (or instant), viz. the mean new moon at the beginning of the month with true month count 0 (see Remark 5 for this month). By (7.1), the mean date of the epoch is m_0 (since $n = d = 0$); hence the JD of the epoch is $\lfloor m_0 \rfloor$.

Our version thus encodes the epoch in m_0, and the epoch is not needed explicitly. Traditionally, with m_0 given modulo 7, see Remark 8, m_0 gives only the day of week of the epoch (with $\lfloor m_0 \rfloor$ giving the number according to Table 5, see (9.1)), and the epoch date is needed to construct the calendar.

The epoch is always close to the beginning of month 2 or 3, depending on β^* (the initial value of the intercalation index), see Remark 5. Since the

[27] On the other hand, there are 16 examples in the first 65 year cycle 1027–1091, so the occurences are irregular.

calculation is for the beginning of lunar day 1 (which equals the end of the last lunar day of the preceding lunar month), the epoch is usually day 30 of calendar month 1 or 2, respectively, cf. the rule for labelling calendar days in Section 6; however, since the true date differs from the mean date, see (7.22), it may be one of the adjacent calendar days (29/1 or 1/2, or 29/2 or 1/3). For our three example epochs, the Tibetan dates are 29/1 806, 29/1 1927 and 1/3 1987.

8. Calendrical functions

As explained in Section 6, the Julian day number JD of a given Tibetan date can be calculated as the JD of the calendar day containing the end of the corresponding lunar day, i.e.

$$\mathrm{JD} = \lfloor \mathit{true\_date} \rfloor, \tag{8.1}$$

with *true_date* given by (7.22), except that if the Tibetan date is repeated, this gives the JD of the second day; for the first we thus have to subtract 1. If we do the calculations for a day that is skipped, formula (8.1) will still give the JD of the calendar day that lunar day ends, which by the rule in Section 6 is the day with name of the preceding lunar day (since that lunar day ends the same calendar day).

The calendar is really given by the inverse of this mapping; a day is given the number of the corresponding lunar day (i.e., the lunar day ending during the calendar day). To find the Tibetan date for a given JD, we thus compute approximate true month count and day (using the mean motion in (7.1)) and then search the neighbouring lunar days for an exact match, if any, taking care of the special cases when there are 0 or 2 such lunar days. For a detailed implementation, see Dershowitz and Reingold [2].

Beginning and end of months. To find the last day of a month, we can just compute the JD for lunar day 30 of the month; this gives the correct result also when day 30 is repeated or skipped. To find the first day, however, requires a little care since lunar day 1 may end during the second day of the month (when day 1 is repeated) or during the last day of the preceding month (when day 1 is skipped); the simplest way to find the JD of first day of a month is to add 1 to the JD of the last day of the preceding month. By the comment on skipped days above, this can be computed as 1 + the JD of day 30 the preceding month, regardless of whether day 30 is skipped or not. (There seems to be errors in the tables in [12] due to this.)

Tibetan New Year. The Tibetan New Year (*Losar*) is celebrated starting the first day of the year. Since the first month may be a leap month 1 (which happens twice every 65 year cycle, the last time in 2000) and the first day may be day 2 (when day 1 is skipped), some care has to be taken to calculate the date. The simplest is to add 1 to the JD of the last day the preceding year, which thus is 1 + JD of the last day of (regular) month 12 the preceding year (and can be calculated as 1 + JD of day 30 of (regular) month 12 the preceding year).

Remark 17. Holidays are otherwise usually not celebrated in leap months. However, the Tibetan New Year is really the first day of the year even in a year that begins with a leap month 1.[28]

9. Day of week

Each calendar day is, as explained in Sections 6 and 8, given a number in the range 1,. . . ,30. It also has a day of week, which as in the Gregorian and many other calendars simply repeats with a period of 7. The day of week thus corresponds uniquely to the Western day of week.

The days of week are numbered 0,. . . ,6, and they also have names. Each day of week corresponds to one of the seven "planets" (including sun and moon), and the name of the day is more or less identical to the name of the planet.[29] The correspondence with the English names is given in Table 5. (For the last column, see Appendix E.)

	English	Tibetan	planet	element
0	Saturday	spen ma	Saturn	earth
1	Sunday	nyi ma	Sun	fire
2	Monday	zla ba	Moon	water
3	Tuesday	mig dmar	Mars	fire
4	Wednesday	lhag pa	Mercury	water
5	Thursday	phur bu	Jupiter	wind
6	Friday	pa sangs	Venus	earth

Table 5. Days of week.

The day of week is simply calculated from the Julian day number found in (8.1) by

$$day\_of\_week = (\mathrm{JD} + 2) \mod 7. \tag{9.1}$$

The date (i.e., the number of the day) and the day of week are often given together. This resolves the ambiguity when days are repeated. (It also helps to resolve most ambiguities when different rules of calculation may have been used.)

[28]This is verified by [36] (written by a representative of the personal monastery of the Dalai Lama), according to which *Losar* was celebrated on Sunday, February 6th (2000), which was the first day of leap month 1.

[29]The correspondence between days of week and planets in Tibetan is the same as in Latin and (incompletely) in many modern European languages (e.g., Sunday, Monday and Saturday in English). This correspondence goes back to Roman astrology, see [10, pp. 268–273 and 391–397] and [3, §12.12], and came to Tibet through India.

10. Further calculations

Tibetan almanacs traditionally also contain further information, mainly for astrological purposes, see Henning [7, Chapter IV]. (See also his extensive computer calculated examples [24, Traditional Tibetan calendar archive] and programs with explanations [24, Open source Tibetan calendar software], [24, Open source Tsurphu calendar software].) In fact, the calendar is known as "the five components" (*lnga-bsdus*),[30] where the five components are: the day of week, the lunar day, the lunar mansion, the yoga and the karaṇa (for these, see below).[31] Nevertheless, a complete Tibetan almanac typically contains not only these five but also some further data. (The following description is based on [7] and the almanac [14], see also the examples in [7, pp. 202–203], [13] and [38, Kalenderrechnung]. There are certainly minor variations between different almanacs.)

The daily data in an almanac include (typically) the following. See further Appendix E. The numbers are usually truncated to 3 significant terms, so not all radices given below are used.

(i) The true day of week (*gza' dag*), i.e. the day of week and fractional part of the day when the lunar day ends. This is $(true\_date + 2) \mod 7$, calculated by (7.22) and written with the radices (60,60,6,707). The integer part (i.e., the first digit) of the true day of week is thus the day of week; its name (given by Table 5) is also given in letters, together with the (lunar) date.

(ii) The (true) longitude of the moon at the end of the lunar day (*tshes 'khyud zla skar*), calculated for lunar day d, true month n by

$$moon\_lunar\_day(d, n) = true\_sun(d, n) + d/30 \qquad (10.1)$$

and written in lunar mansions with the radices (27,60,60,6,67). (The rationale for (10.1) is that the moon's elongation, i.e. the difference between lunar and solar longitude, by definition is $d/30$ at the end of lunar day d, see Section 3.)

(iii) The (true) longitude of the moon at the beginning of the calendar day (*res 'grogs zla skar*) calculated from the values in (10.1) and (7.22) by (recalling that $\mathrm{frac}(x)$ denotes the fractional part of x)

$$moon\_calendar\_day = moon\_lunar\_day - \mathrm{frac}(true\_date)/27, \qquad (10.2)$$

and written in lunar mansions with the radices (27,60,60,6,67). (The idea here is that $\mathrm{frac}(true\_date)$ is the time from the beginning of the calendar day to the end of the lunar day, so this formula is really an approximation assuming that the moon moves with constant speed

[30] Just as in Indian calendars (*panchang*), see [2, Chapter 18, p. 312].

[31] [38, Kalenderrechnung] has a slightly different interpretation, including the longitude of the sun as one of the five and omitting the day of week (which is seen as given).

and making a full circle in 27 days.[32] The division by 27 is convenient since it is simply a shift of the terms in the mixed radix notation.)

(iv) The name of the lunar mansion (Sanskrit *naksatra*), which is determined by the Moon's longitude (10.2) with fractions of mansions ignored. More precisely, numbering the lunar mansions from 0 to 26, the number of the lunar mansion is

$$\lfloor 27 moon\_calendar\_day \rfloor. \tag{10.3}$$

(In the traditional notation, this is the first term of the lunar longitude (10.2).) The names of the mansions (in Tibetan and Sanskrit) are listed in [7, Appendix I].

(v) The (true) longitude of the sun (*nyi dag*), given by *true_sun* and written in mansions with the radices (27,60,60,6,67). (Note that *true_sun* really is computed for the end of the lunar day, but is regarded as valid also for the calendar day; the motion of the sun during the day is thus ignored and no correction as in (10.2) is made.[33])

(vi) The *yoga* "longitude" (*sbyor ba*) is the sum of the longitudes of the sun and the moon, calculated by[34]

$$yoga\_longitude = moon\_calendar\_day + true\_sun \pmod 1, \tag{10.4}$$

and written in mansions with the radices (27,60,60,6,67).

(vii) The name of the yoga, which is determined by the yoga longitude with fractions of mansions ignored. More precisely, numbering the yogas from 0 to 26, the number of the yoga is

$$\lfloor 27 yoga\_longitude \rfloor. \tag{10.5}$$

The names of the yogas (in Tibetan and Sanskrit) are listed in [7, Appendix I]. (The names differ from the names of the lunar mansions.)

(viii) The *karaṇa* (*byed pa*) in effect at the start of the calendar day. Each lunar day is divided into two halves, and each half-day is assigned one of 11 different karaṇas. There are 4 "fixed" karaṇas that occur once each every month: the first half of the first lunar day, the second half of the 29th lunar day, and the two halves of the 30th day; the remaining 7 karaṇas, called "changing", repeat cyclically for the remaining 56 half-days. In other words, lunar day D consists of half-days $2D-1$ and $2D$, and half-day H has one of the fixed karaṇas if $H = 1, 58, 59, 60$, and otherwise it has the changing karaṇa number $(H-1)$ amod 7.

[32]This is of course not exact, but it is a rather good approximation since the tropical (or sidereal; the difference is negligible) month is 27.322 days [3, Table 15.3]. In the calculation, one may also ignore that in the traditional notations, the last radix differs betwen the true date and the longitudes, cf. Footnote 22.

[33]The motion of the sun is much slower than the motion of the moon; it is about $1/365 \approx 1^\circ$ each day.

[34]As noted by [7], this is inconsistent, since we add one longitude at the beginning of the calendar day and one longitude at the end of the lunar day. On the other hand, this addition has in any case no physical or astronomical meaning.

The names of the karaṇas (in Tibetan and Sanskrit) are listed in [7, Appendix I]. (The exact rule for determining the time in the middle of the lunar day that divides it into two halves is not completely clear to me. Henning [7] divides each lunar day into two halves of equal lengths, but there might be other versions.)

(ix) The mean longitude of the sun (*nyi bar*) in signs and degrees (and minutes), i.e. *mean_sun* (7.5) written with the radices (12,30,60).

(x) The (true) longitude of the moon at the beginning of the calendar day (*zla skar*) as in (iii), but calculated according to the karaṇa[35] calculations, see Appendix A.5.[36] This is, usually at least, calculated for the correct calendar day, regardless of whether its date is the same in the Phugpa and karaṇa versions or not; so for example in the almanac [14]. (There are exceptions, possibly mistakes, for example in the page from a 2003 calendar shown in [7, p. 202].)

(xi) The Gregorian date. In modern almanacs written in Western (European) numerals.

As an example, the almanac [14] lists for each day, in addition to the Tibetan (lunar) date and the Gregorian date (xi), the six numbers (i), (iii), (v), (vi), (x), (ix) above[37]; there is further (astrological) information as text.[38]

Note that (i), (ii), (v), (ix), refer to (the end of) the lunar day while the others refer to (the beginning of) the calendar day. When a day is skipped (i.e., there is a lunar day without corresponding calendar day), the almanac usually still gives (i) and (v) for the skipped lunar day; when a day is repeated, so there are two calendar days corresponding to the same lunar day, the data are given for both days, with the data referring to the lunar day thus repeated, but with modifications for the first day: In [14], (i) (otherwise the end of the lunar day) for the first day is given as the end of the calendar day, written as $x; 60, 0$ where x is the day of week; (iii) is obtained as $moon\_lunar\_day - 1/27$,[39] cf. (10.2); (vi) is calculated separately for both days; (ix) is given only for the second day. Thus only (v) is identical for the two days.

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[35] The (Sanskrit) word karaṇa has a different meaning here than in (viii). In contrast, the standard calculations described above are called siddhānta. See further Appendix A.5.

[36] This means that all calendar calculations have to be done also for the karaṇa version, in order to find this value.

[37] These numbers are truncated (not rounded) to 3 terms, with radices $(1, 60, 60)$ or (27,60,60).

[38] The same daily values are given in the 2003 almanac shown in [7, p. 202], see further the discussion there.

[39] This is a reasonable approximation, since the end of the lunar day is just a little more than one calendar day later than the beginning of the first calendar day, cf. Footnote 32. However, the resulting longitude is often *smaller* than the longitude at the end of the preceding lunar day, a short time earlier. (The latter longitude is not printed in the almanac, so the contradiction is not visible without further calculations.)

There is also further information at the beginning of each month, including the mean date (*gza' bar*) (7.1) and the mean solar longitude (*nyi bar*) (7.5) (both given with all 5 terms), and also the lunar anomaly (*ril cha*) (7.11), all calculated for the beginning of the lunar month (day $d = 0$ in the formulas above); furthermore, the true month (*zla dag*) is given, with its fractional part (the intercalation index), see Remark 4. These monthly values (mean date, mean solar longitude, lunar anomaly and true month) are also given for the karaṇa calculation (see Appendix A.5). There is also data on the position of the planets (see Appendix D).

10.1. **Some special days.** Furthermore, the almanac gives extra information on some special days, for example the days when the mean solar longitude passes certain values, including the three series (with 30° intervals)

- $0°, 30°, 60° \dots$ (when the mean sun enters a new sign);
- $8°, 38°, 68° \dots$ (the definition points (*sgang*), see Appendix C.2);
- $23°, 53°, 83° \dots$ (the midpoints between the definition points (*dbugs*)).

(There are also 4 further such days, with longitudes 66°, 132°, 147°, 235° in the 2013 almanac [14].) In all cases, the almanac gives the longitude and the mean date for the instant *mean_sun* reaches this value. This is easily found from (7.5) and (7.1), now letting d be an arbitrary rational number denoting the lunar day including a fractional part to show the exact instant.[40] If we assume that $m_2 = m_1/30$ and $s_2 = s_1/30$, as is the case in the Phugpa version and in all other modern versions that I know of, see Remark 14, then (7.5) shows that the mean solar longitude equals a given value λ in the year Y at lunar date

$$d = \frac{\lambda + Y - Y_0 - s_0}{s_2} = 30\frac{\lambda + Y - Y_0 - s_0}{s_1} \tag{10.6}$$

after the epoch, where Y_0 is the epoch year (so $Y - Y_0 + \lambda$ can be regarded as the desired longitude of the sun measured on a linear scale from the epoch); however, in order for this to give the correct year, s_0 has to be chosen such that s_0 is close to 0; for E1987 we thus take $s_0 = 0$ by (7.10), but the values (7.8) and (7.9) for E806 and E1927 have to be decreased by 1 (recall that the integer part of s_0 was irrelevant earlier); moreover, the integer part of λ should similarly be adjusted so that $0 \le \lambda < 1$, except at the beginning of the year (before longitude 0°) when one should subtract 1 so that $-1 < \lambda < 0$. (Cf. Appendix C for similar considerations regarding the mean solar longitude as a real number on a linear scale.) The mean date is then obtained from (7.1) as

$$d \cdot m_2 + m_0 = \frac{m_2}{s_2}(\lambda + Y - Y_0 - s_0) + m_0 = \frac{m_1}{s_1}(\lambda + Y - Y_0 - s_0) + m_0. \tag{10.7}$$

The traditional calculation is somewhat different [Henning, personal communication]. First the month is determined (or guessed, for possible later

[40] This is the only case that I know when the calculations involve fractions of a lunar day.

correction) – this is easy since the mean sun always passes the definition point $((M-3)\cdot 30+8)^\circ$ during month M, see Appendix C. The lunar date of a new sign ($k\cdot 30^\circ$, for integer k) is given by $6ix/13$, where ix is the intercalation index of the month, see (5.7) and Remark 4.[41] The definition point 8° later comes $8;3,1\ \ (13,5) = 8\frac{16}{65}$ lunar days later and the midpoint 7° earlier comes $7;2,4\ \ (13,5) = 7\frac{14}{65}$ lunar days earlier. Finally, the mean dates for these lunar dates are calculated by a multiplication equivalent to (7.1); tables exist to assist with the multiplication by fractional parts of a lunar day.

To verify that this method works, we first note that $8\frac{16}{65} = 8\cdot 1\frac{2}{65}$ and $7\frac{14}{65} = 7\cdot 1\frac{2}{65}$, and that during $1\frac{2}{65}$ lunar day, the mean sun moves, see (7.6)–(7.7),

$$1\frac{2}{65}\cdot s_2 = \frac{67}{65}\cdot s_2 = \frac{67}{65}\cdot\frac{s_1}{30} = \frac{67}{65}\cdot\frac{65}{67\cdot 12}\cdot\frac{1}{30} = \frac{1}{360} = 1^\circ; \tag{10.8}$$

hence the mean sun indeed moves 8° and 7° during these periods.

To verify the entry into signs, it is convenient to use the epoch E1987, since then both $\beta^* = 0$ (5.6) and $s_0 = 0$ (7.10). Consider month M, year Y, and let as in (5.2) $M^* = 12(Y-Y_0)+(M-M_0)$ be the number of solar months since the epoch (year $Y_0 = 1987$, month $M_0 = 3$). If this month has true month count n and intercalation index ix, then the rule above yields a lunar date (recalling that each month has 30 lunar days)

$$x = 30n + \frac{6ix}{13} = 30\Big(n+\frac{ix}{65}\Big) \tag{10.9}$$

lunar days after the epoch. Note that $n + ix/65$ is the true month. If the intercalation index $ix < 48$, then $n + ix/65$ is thus given by (5.3), and is thus $\frac{65}{67}M^*$ (since $\beta^* = 0$), so the lunar date (10.9) is

$$x = 30\cdot\frac{65}{67}M^* \tag{10.10}$$

and consequently the mean solar longitude then is, by (7.5) (with $s_0 = 0$) and (7.6)–(7.7),

$$x\cdot s_2 + s_0 = 30\cdot\frac{65}{67}\cdot M^*\cdot\frac{s_1}{30} + 0 = s_1\cdot\frac{65}{67}\cdot M^* = \frac{1}{12}\cdot M^* = M^*\cdot 30^\circ, \tag{10.11}$$

showing that the mean sun enters the sign with longitude $M^*\cdot 30^\circ \equiv (M-3)\cdot 30^\circ$ (since longitudes are measured modulo 360°). If $ix \geq 48$, then the true month is increased by $1\frac{2}{65}$, see Remark 4, which corresponds to an increase in the mean solar longitude of $\frac{65}{67}s_1 = \frac{1}{12} = 30^\circ$, so the mean sun instead enters the next sign, with longitude $(M-2)\cdot 30^\circ$ and the entry into the sought sign is found (by the same rule) in the preceding month.[42] (If $ix = 48$ or 49, this means leap month M.)

---

[41] Note that $0 \leq 6ix/13 \leq 6\cdot 64/13 = 384/13 < 30$.

[42] It is easily verified that $\frac{6\cdot 47}{13} + 8\frac{16}{65} < 30 < \frac{6\cdot 48}{13} + 8\frac{16}{65}$, which implies that in any case, the definition point $((M-3)\cdot 30+8)^\circ$ is reached during month M as said above, see further Appendix C.

Remark 18. A complication for the almanac maker is that the lunar day found in this way sometimes corresponds to a calendar day that is one day earlier or later than the calendar day given by (the integer part of the) mean date found above, since this correspondence uses the true date and not the mean date. In this case, the data are entered in the almanac straddling both days, see [13] for details and examples.

The days discussed here have been defined by the mean solar longitude, including day when it is 0, i.e., the mean sun passes the first point of Aries. The day when the true solar longitude is 0 is also marked in the almanac; in this case the true date is given.[43] I do not know exactly how this is calculated, but I assume that it is by some more complicated version of the rule above, including adjustments for the equations of sun and moon.

11. Holidays

A list of holidays, each occuring on a fixed Tibetan date every year, is given in [7, Appendix II]. If a holiday is fixed to a given date, and that date is skipped, the holiday is on the preceding day. If the date appears twice, the holiday is on the first of these (i.e., on the leap day, see Section 6). (These rules are given by [21], but I have not checked them against published calendars.)

Holidays are usually not celebrated in leap months, but see Remark 17.

12. Mean lengths and astronomical accuracy

The mean length of the month is

$$m_1 = 29;31,50,0,480\ \ (60,60,6,707) = \frac{167025}{5656} \approx 29.530587 \text{ days}, \tag{12.1}$$

which is essentially identical to the modern astronomical value of the synodic month 29.5305889 (increasing by about $2 \cdot 10^{-7}$ each century) [3, (12.11-2)].

Consequently, the mean length of the lunar day is

$$\frac{m_1}{30} = \frac{167025}{30 \cdot 5656} = \frac{11135}{11312} \approx 0.98435 \text{ days}. \tag{12.2}$$

The mean length of the year is, cf. (5.1),

$$\frac{1}{s_1} \text{ months} = \frac{m_1}{s_1} = \frac{804}{65} m_1 = \frac{6714405}{18382} \approx 365.270645 \text{ days}, \tag{12.3}$$

which is 0.02846 days more than the modern astronomical value of the tropical year 365.24219 [3, (12.11-1) and Table 15.3], and 0.02815 days longer than the mean Gregorian year 365.2425 days. (It is also longer than the sidereal year 365.25636 days [3, Table 15.3].) Hence the Tibetan year lags

[43] The latter ("true") day is about 2 days before the first ("mean"), because of the equation of the sun, see (7.23); both are over a month later then the astronomical vernal equinox when the real sun has longitude 0, see Section 12.1.

behind and starts on the average later, compared to the seasons or the Gregorian year, by almost 3 days per century or almost a month (more precisely, 28 days) per millennium.

The year has thus drifted considerably since it was introduced, and even more since the Kālacakra Tantra was written; the drift during the 1200 years since the epoch 806 is 34 days, see further Section 12.1.

The mean length of the anomalistic month is

$$\frac{1}{1+a_1} \text{ months} = \frac{3528}{3781} \cdot \frac{167025}{5656} = \frac{10522575}{381881} \approx 27.55459 \text{ days.} \qquad (12.4)$$

This agrees well with the modern astronomical value 27.55455 days [3, Table 15.3]; the drift is about 1 day in 2000 years.

See also Petri [9].

12.1. **Accuracy of longitudes.** As noted above, the mean length of the month is essentially equal to the exact astronomical value. Indeed, the times (true date) for new moons as calculated by the Tibetan calendar are close to the exact astronomical times. (Recall that the new moons signify the end of lunar day 30 in each month and the beginning of a new month; in the almanac the time is thus given as the true date of day 30 in the month.) If we (following [7]) regard the Tibetan day as starting at mean daybreak, ca. 5 a.m. local mean solar time, and set this equal to −1 UT (GMT), see Remark 6, then the true dates given by Tibetan calculations are about 4 hours earlier than the exact astronomical values.[44] Since the elongation increases by about 12° each day, see Section 6, this corresponds to an error in the elongation of about 2° too large.

As also noted above, the mean length of the year is less accurate, and the year has drifted 34 days since the epoch 806. Indeed, the solar longitude as computed by *true_sun* above passes 0° during the 16th day in the third Tibetan month 2013, which equals April 26, 37 days after the astronomical vernal equinox on March 20, which agrees well with this drift. Another way to see this is to note that the true sun calculated at the (astronomical) vernal equinox is only about 324°, which means that the solar longitude is about 36° too small.[45] Since the elongation was seen above to be about 2° too large, the Tibetan longitude of the moon is about 34° (= 2.5 mansion) too small.

12.2. **The ratio between solar and lunar months.** The ratio of the exact astronomical lengths of solar and lunar months is (with values correctly

[44] A calculation for the 12 new moons in the Gregorian year 2013 and comparison with a Swedish almanac gave differences between 3^h16^m and 4^h21^m.

[45] This varies slightly over the year, since the equation of the sun given by (7.20) is not astronomically exact, both because it is intrinsically an approximation only, and because the anomaly is by (7.19) also about 36° wrong. However, the error does not vary by more than 1–2° over the year and is always about 36–37°.

rounded for both today and 1000 years ago)

$$\frac{365.2422}{12 \cdot 29.53059} \approx 1.030689. \tag{12.5}$$

The standard way to find good rational approximations of a real number is to expand it as a continued fraction, see e.g. [6]:

$$1.030689 = 1 + \cfrac{1}{32 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\dots}}}}} \tag{12.6}$$

and then calculate the partial quotients obtained by truncating the continued fraction. In this case the first partial quotients are (after 1)

$$1 + \frac{1}{32} = \frac{33}{32} = 1.03125 \tag{12.7}$$

$$1 + \frac{1}{32 + \frac{1}{1}} = \frac{34}{33} \approx 1.030303 \tag{12.8}$$

$$1 + \frac{1}{32 + \frac{1}{1+\frac{1}{1}}} = \frac{67}{65} \approx 1.030769 \tag{12.9}$$

$$1 + \frac{1}{32 + \frac{1}{1+\frac{1}{1+\frac{1}{2}}}} = \frac{168}{163} \approx 1.030675 \tag{12.10}$$

$$1 + \frac{1}{32 + \frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{2}}}}} = \frac{403}{391} \approx 1.030690. \tag{12.11}$$

The first two approximations 33/32 and 34/33 have errors of about $5 \cdot 10^{-4}$ and $4 \cdot 10^{-4}$, or about 1 month in 200 years. The next approximation is 67/65, used in the Tibetan calendar, which has an error of about $8 \cdot 10^{-5}$, or as said above about 1 month in 1000 years. The next approximation is 168/163, with an error of only $1 \cdot 10^{-5}$, about 1 month in 6 000 years, and 403/391 has an error $2 \cdot 10^{-6}$, about 1 month in 50 000 years[46].

The Tibetan approximation 67/65, see (5.1), is thus a good choice from a mathematical point of view, if we want a good approximation with small numbers. (168/163 or 403/391 would yield more accurate but more complicated calendars. As far as I know, they are not used in any calendar.)

Remark 19. An approximation used in several other calendars (e.g. the Jewish, and for calculation of the Christian Easter), but not in the Tibetan, is the relation 235 lunar months ≈ 228 solar months (19 solar years), known as Meton's cycle [2]. We have 235/228 = 1.030702, with an error of $1 \cdot 10^{-5}$, about 1 month in 6000 years. This does not appear in the list of partial quotients above, and indeed, 168/163 with smaller numbers is about as accurate. However, the Metonic cycle has the advantage that it comprises a

[46]The latter accuracy is only fictional; the error 50 000 years from now is much larger, 10^{-4}, because of changes in the lengths of the month and the year.

whole number of years. (It appears as one of the partial quotients for the continued fraction expansion of the number of lunar months in a solar year.)

13. Period

The *mean_date* given by (7.1) repeats after 5656 months, but *true_date* depends also on *anomaly_moon* and *true_sun*, which repeat after 3528 and 804 months, respectively. The least common multiple of 5656, 3528 and 804 is $p = 23873976$; consequently, all astronomical functions above repeat after p months, which equals $m_1 p = 705012525$ days or $s_1 p = 1930110$ years (these are necessarily integers). Note further that this period contains an integral number of leap year cycles (804 months = 65 years), see Section 5, so the numbering of the months repeats too. In other words:

The calendar (days and months) repeats after

$$705\,012\,525 \ \textit{days} = 23\,873\,976 \ \textit{months} = 1\,930\,110 \ \textit{years}.$$

Moreover, the number of days is divisible by 7, so also the day of week repeats after this period.

However, $1930110 \equiv 6 \pmod{12}$, so to repeat the animal names of the years in the 12-year cycle, or the names in the 60-year cycle, we need two of these periods, i.e. 3 860 220 years.

Of course, the period is so long that the calendar will be completely out of phase with the tropical year and thus the seasons long before, and it will move through the seasons hundreds of times during one period.

If we also consider the planets, see Appendix D, the period becomes a whooping 2 796 235 115 048 502 090 600 years, see Henning [7, pp. 332–333] and [38, Weltzeitalter].

Appendix A. Different versions

Different traditions follow different rules for the details of the calculation of the Tibetan calendar, and as said in Section 1, there are two main versions in use today. (Several other versions survive according to [7, p. 9], but no details are given.) Schuh [12] gives historical information, including many versions of the constants in Section 7 above used or proposed during the centuries, but says nothing about the different versions today. See also [38, Kalenderrechnung]. Henning [7] discusses the Phugpa and Tsurphu versions in detail, and also one recent attempt at reform. He gives epoch data for several versions in [24, Epoch data]. (Another source on different versions today is Berzin [21]; he gives the impression that several versions are in use. However, since [21] discusses the calendar and astrology together, it is possible that some of these versions actually use the same calendar but differ in other, astrological, calculations or interpretations.) See also the web pages of Nitartha [32].

A.1. **Phugpa.** (Phukluk, *phug-lugs*.) This is the most widespread version, and is regarded as the official Tibetan calendar. (Also by at least some followers of the rival Tsurphu tradition [32].) It was started in 1447 (the first year, *rab byung*, of the 8th Prabhava cycle) by Phugpa Lhundrub Gyatso (*phug-pa lhun-grub rgya-mtsho*), and was used by the Tibetan government from at least 1696 to 1959 [38, Phugpa-Schule], [7, pp. 8 and 321–337], [12, p. 139], [13]. The Phugpa version is used by the Gelug, Sakya, Nyingma and Shangpa Kagyu traditions of Tibetan buddhism, including the Dalai Lama, and is used e.g. in the calendars published in Dharamsala in India, where the Tibetan exile government resides, see [14; 22]. The Bon calendar is the same as the Phugpa [21]. The Phugpa calendar is described in detail in the main body of the present paper.

A.2. **Tsurphu.** (Tsurluk.) This version was also introduced in 1447, by Jamyang Dondrub Wozer (*mtshur-phu 'jam-dbyangs chen-po don-grub 'od-zer*) and derives from 14th century commentaries to Kālacakra Tantra by the 3rd Karmapa Rangjung Dorje of Tsurphu monastery (the main seat of the Karma Kagyu tradition of Tibetan buddhism) [21], [7, pp. 9 and 337–342], [38, Tshurphu-Schule]. This version is used by the Karma Kagyu tradition, and it is used e.g. in calendars published by the Rumtek monastery in India, the main exile seat of the Karmapa (the head of Karma Kaygyu) [24, Open source Tsurphu calendar software]. Also the calendar published by Nitartha in USA [16; 32] gives the Tsurphu version (from 2004 the Phugpa version too is given).

The Tsurphu tradition uses the astronomical functions in Section 7 with the same values as given there for the Phugpa version of the constants m_1, s_1 and a_1 (and m_2, s_2 and a_2, see Remark 14) for mean motions, while the epoch values m_0, s_0 and a_0 are different. See further [7, pp. 337–342].

Remark 20. Traditional Tsurphu calculations use, however, somewhat different sets of radices that the Phugpa versions, with one more term (and thus potentially higher numerical accuracy in the calculations). The same constants are thus written differently:

$$m_1 = 29;31,50,0,8,584 \ \ (60,60,6,13,707) = \frac{167025}{5656}, \tag{A.1}$$

$$s_1 = 2,10,58,1,3,20 \ \ (27,60,60,6,13,67) = \frac{65}{804}, \tag{A.2}$$

although m_1 is traditionally given as $1;31,50,0,8,584 \ \ (60,60,6,13,707)$, as always calculating modulo 7, and I have added 28, see Remark 8.

Two epochs given in classical text [7, p. 340] are, with m_0 modified to yield JD, see Remark 8,

$$\mathrm{JD} = 2353745 \quad \text{(Wednesday, 26 March 1732 (Greg.))}$$

$$m_0 = 4;14,6,2,2,666 \ \ (60,60,6,13,707) + 2353741 = 2353745 + \frac{1795153}{7635600}, \tag{A.3}$$

$$s_0 = -\big(1,29,17,5,6,1 \ \ (27,60,60,6,13,67)\big) = -\frac{5983}{108540}, \tag{A.4}$$

$$a_0 = 14,99 \ \ (28,126) = \frac{207}{392}. \tag{A.5}$$

and 1485 months later (equivalent and giving the same calendar)

$$\mathrm{JD} = 2397598 \quad \text{(Monday, 19 April 1852)}$$

$$m_0 = 2;9,24,2,5,417 \ \ (60,60,6,13,707) + 2397596 = 2397598 + \frac{1197103}{7635600}, \tag{A.6}$$

$$s_0 = 0,1,22,2,4,18 \ \ (27,60,60,6,13,67) = \frac{23}{27135}, \tag{A.7}$$

$$a_0 = 0,72 \ \ (28,126) = \frac{1}{49}. \tag{A.8}$$

We denote these by E1732 and E1852. (See also [24, Epoch data], where also a third epoch 1824 is given.)

To find the leap months, the true month is calculated by (5.3) with a constant $\beta^* = 59$ (E1732) or 14 (E1852) (the epoch value of the intercalation index); the reason being that counting backwards to the Kālacakra epoch *nag pa* (*Caitra*) 806 yields the intercalation index 0, as in the Kālacakra Tantra. (But unlike the Phugpa version, see (5.4).) Moreover, instead of the Phugpa rule (5.8), the Tsurphu version uses the simpler rule:

A leap month is inserted when the intercalation index $ix = 0$ *or* 1. (A.9)

This implies that for a regular month, the true month count n is obtained from the true month (5.3) by rounding down to the nearest integer; for a leap month we further subtract 1. (The Tsurphu rules are thus simpler and more natural than the Phugpa rules (5.8) and (5.9) in Section 5. They also follow the original Kālacakra Tantra, cf. Appendix A.5. In particular, if we extend the calendar backwards, there was a leap month at the Kālacakra epoch *nag pa* (*Caitra*) 806, in agreement with the Kālacakra Tantra.)

Remark 21. This means that for the Tsurphu version, the intercalation index calculated in this paper always agree with the traditional definition without further correction; cf. Remark 4 for the Phugpa version.

By (5.26), (A.9) agrees with the general rule (5.24) if β is defined such that

$$\beta + \beta^* \equiv 6 \bmod 65. \tag{A.10}$$

As we will see in Appendix C.3, the values are $\beta = 142$ (E1732) and 187 (E1852), in accordance with (A.10) and the values for β^* given above.

By (5.30) and (5.32), the rules (5.31), (5.33) and the formula (5.36) hold with $\gamma = 55$ and $\gamma^* = 20$.

Remark 22. Tsurphu almanacs give essentially the same information as almanacs for the Phugpa version, see Section 10, but one difference is that they by tradition give the true solar longitude for each day calculated by the karaṇa calculation, see Appendix A.5, instead of the calculation above. (Cf. (x) in Section 10.)

In some Tsurphu almanacs, the solar equation from the karaṇa solar longitude calculation has also been used to calculate the (siddhānta) *true_date* in (7.22).[47] In other words, the values (A.26) and (A.30) are used in the calculations instead of the values for s_1 and s_0 given above. See [24, Open source Tsurphu calendar software]. This will lead to a slightly different *true_date*, and occasionally a different repeated or skipped day.[48] However, the traditional version seems to be to use the values above for the calculation of the true date, but to also calculate separately the karaṇa version of *true_sun* and publish it [Henning, personal communication].

A.3. **Mongolia.** The Mongolian calendar is a version of the Tibetan. It became the official calendar in Mongolia in 1911 when Mongolia declared independence from China after the Chinese revolution that overthrew the last Chinese emperor; however, it was replaced in the 1920s (under Communist rule) by the Gregorian calendar (officially in 1948) and the authorities even tried to abolish the traditional celebrations of the Mongolian New Year. The calendar has had a revival together with other traditions after the end of Communist rule in the 1990s, although the Gregorian calendar remains

[47]This seems only to have been a time saving device, as usually the siddhānta and karaṇa calculations are kept seperate.

[48]Calculations similar to the ones in Remark 14 suggest that this will happen about 5 times per year, and that the New Year will differ by a day about 2 times per century. One example is day 13, month 6, 2013, for which the two versions yield 20 and 21 July.

the official calendar and is used for everyday civil use;[49] in particular, the Mongolian New Year (*Tsagaan Sar*) is again celebrated as a major national holiday. See [11], [27], [35, 4.1.3]. (I guess that otherwise the Mongolian calendar is mainly used for religious purposes and astrology.) In 2012, a second public holiday was declared that is calculated by the Mongolian calendar (the other holidays have fixed dates in the Gregorian calendar [35]), viz. Genghis Khan's birthday, the first day in the first winter month (month 10)[50] [34], [23], [35, 4.1.8].

According to [21], the Buryats and Tuvinians of Siberia (in Russia) follow the New Genden version (i.e., the version described here), while the Kalmyk Mongols in Russia follow the Phugpa version. (Inner Mongolia, in China, uses instead the Chinese "yellow system", see Section A.11.)

The information below is mainly based on Berzin [21], Henning [24, Epoch data] and Salmi [37], but I have no reliable Mongolian sources, and not even a printed calendar as an example. An example of its use is in the daily horoscope at [31], but as noted by [37], this contains some obvious errors and is thus not reliable.[51] A list of Gregorian dates of the Mongolian New Years 1896–2008 is given in [33], and the same list extended to 2013 in [28].[52] [53] See [37] for many further references.

The Mongolian calendar follows, see [21], [37], [27] and [28], a version of the Tibetan calendar known as New Genden (Mongolian *Tögs buyant*,

[49]The first democratic constitution came into force "from the horse hour of the auspicious yellow horse day of the black tiger first spring month of the water monkey year of the seventeenth 60-year cycle". [noon 9/1 = 12 February 1992] [11, p. 240]

[50]His actual date of birth is not known, and there are different dates in different sources; for example Terbish [39] believes in day 16 in month 4, 1162. (However, this was 500 years before the present vesion of the calendar was introduced, so the calculation of the corresponding Julian date (1 May) in [39] seems uncertain.)

[51]In, for example, May, June and September 2012, some Mongolian dates are out of order: others are missing. Nevertheless, most of the skipped or repeated dates in [31] for March 2011 – August 2013 agree with the rules below, as does the the leap month 6 in 2011. (Note that the skipped and repeated days are sensitive to also small changes in the constants. For example, they usually differ between the New Genden version and the Phugpa version, see the examples in Table 8.) The discrepancies that exist may be due to further errors in their dates. It is also possible, although less likely, that different versions of the Mongolian calendar are used.

[52]These dates seem to be calculated by L. Terbish, who seems to be the present expert on the calendar in Mongolia (see e.g. [28]). The dates all agree with the rules below. Note also that the list of New Years immediately yields the leap years. Since the leap months are regularly spaced, see Section 5, the sequence of leap years for 65 consecutive years uniquely determines the leap months for all years, and they agree with the leap year rule below.

[53]However, there are also other, contradictory lists, for example in [11] and on various unreliable web sites. When this is written, Mongolian Wikipedia [43, Tsagaan Sar] gives two lists, one 1989–2013 (partial) and one 2000–2099; of the 13 common years, only 4 agree. (The first list is attributed to Terbish, and agrees with the formulas here, except for a possible typo. The second list agrees with neither the New Genden version, the Phugpa version, nor the Chinese calendar.)

"Very virtuous" [37]) which was created by Sumpa Khenpo Yeshe Paljor[54] (*sum-pa mkhan-po ye-shes dpal-'byor*; Mongolian Sümbe Khamba Ishbaljir) in 1786[55].

The constants m_1, s_1 and a_1 for mean motions are the same in the New Genden version as in the Phugpa version, see Section 7, and thus so are m_2, s_2 and a_2, see Remark 14, while the epoch values m_0, s_0 and a_0 are different and given by, see [24, Epoch data]:

$$\begin{aligned} \mathrm{JD} &= 2359237 \quad \text{(Sunday, 9 April 1747 (Greg.))} \\ m_0 &= 1;55,13,3,31,394 \ \ (60,60,6,67,707) + 2359236 \\ &= 2359237 + \frac{2603}{2828}, \qquad \text{(A.11)} \\ s_0 &= 26,39,51,0,18 \ \ (27,60,60,6,67) = \frac{397}{402}, \qquad \text{(A.12)} \\ a_0 &= 24,22 \ \ (28,126) = \frac{1523}{1764}. \qquad \text{(A.13)} \end{aligned}$$

I have here, as in (7.4), modified m_0 to yield the JD, see Remark 8.

Remark 23. In traditional calculations, one uses a different sequence of radices than the standard one, but the mean values can be expressed exactly also with the standard Phugpa radices; the constant m_0 in (A.11) above equals $1;55,13,3,333 \ (60,60,6,707)$. Similarly, m_1 is given by Yeshe Paljor as $1;31,50,0,45,345 \ (60,60,6,67,707)$, which equals the standard value $1;31,50,0,480 \ (60,60,6,707)$; as always these traditional values are interpreted modulo 7, see Remark 8, and we add 28 and use the value in (7.2).

Remark 24. For comparison, the constants for the Phugpa version for this epoch, which thus give the standard Phugpa calendar, can be calculated to be

$$\begin{aligned} m_0 &= 1;52,41,2,524 \ \ (60,60,6,707) + 2359236 = 2359237 + \frac{4967}{5656}, \qquad \text{(A.14)} \\ s_0 &= 26,9,37,3,45 \ \ (27,60,60,6,67) = \frac{779}{804}, \qquad \text{(A.15)} \\ a_0 &= 24,19 \ \ (28,126) = \frac{3043}{3528}. \qquad \text{(A.16)} \end{aligned}$$

To find the leap months, the true month is calculated by (5.3) with a constant $\beta^* = 10$ (the epoch value of the intercalation index). However, the

[54] An prominent 18th century Tibetan monk of Mongolian origin. [40]

[55] According to [21], who comments that the starting point is the 40th year of the 60 year cycle and claims that "Because of this difference, the Mongolian calendar works out to be unique." However, as said in Remark 5, the choice of epoch in itself has no importance. Furthermore, Yeshe Paljor used the epoch 1747 (the beginning of the corresponding 60 year cycle) [24, Epoch data].

Phugpa rule (5.8) is modified to[56]

A leap month is inserted when the intercalation index $ix = 46$ *or* 47. (A.17)

(There is also a corresponding change of (5.9), with 48 replaced by 46. Similarly, Remark 4 applies again, with 49 replaced by 47.)

By (5.26), (A.17) agrees with the general rule (5.24) if β is defined such that

$$\beta + \beta^* \equiv 52 \bmod 65. \tag{A.18}$$

For the epoch above, with $\beta^* = 10$, we take $\beta = 172 \equiv 42 \pmod{65}$, see Appendix C.4.

By (5.30) and (5.32), the rules (5.31), (5.33) and the formula (5.36) hold with $\gamma = 55$ and $\gamma^* = 20$, cf. [37]. (The same constants as for the Tsurphu version in Appendix A.2, since as said above, these two versions have the same leap months.)

The years in the Mongolian calendar are named by Element + Animal as described in Section 4, but often the element is replaced by the corresponding colour according to the correspondence in Table 14 [11].[57][58] The Tibetan 60-year *rab byung* cycle, see Section 4, is recognized in Mongolia too (called *jaran* in Mongolian), and the cycles are numbered as in Tibet (with the first starting in 1027), see [27], [33] and [28] (both with formulas similar to (4.1)–(4.3)), and the example in Footnote 49.

The months are not numbered, but are named.[59] Two different systems are used. (Both have been used earlier in Tibet, see Section 5.1.) One method is by Animal, or Colour (or perhaps Element?) + Animal, by the same rules as described for the Tsurphu version in Appendix E.2 below (which are the rules of the Chinese calendar); thus month 1 is Tiger, etc. The months are also named as beginning, middle and end of each of the four seasons spring, summer, autumn, winter, with month 1 beginning of spring (as in the Chinese calendar, and in Section 5.1(iii)c) [11, pp. 155, 241–242]. See Footnote 49 for an example where both methods are used together.

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[56]The reason seems to be that counting backwards to the Kālacakra epoch *nag pa* (*Caitra*) 806 yields the intercalation index 46 and thus a leap month, agreeing with the Kālacakra Tantra (as in the Tsurphu version but not in Phugpa). As a consequence, the Mongolian version has the same leap months as the Tsurphu version (although one may start a day before the other), see Table 7.

[57]The Mongolians use the colours in parenthesis in Table 14 [11, p. 155]. (I suspect that this is a difference in translation of the colours more than an actual difference in colour.

[58]According to [11, p. 155], it is common to use the element in male years and the colour in female years, which would give the cycle Wood, Blue, Fire, Red, Earth, Yellow, Iron, White, Water, Black, see Table 14. However, I have not seen this confirmed in other sources. For example, the lists (in Mongolian) of New Years in [33] and [28] use Element + Animal for all years.

[59]Numbers are used for Gregorian months. See [11, p. 106] and, for examples, [35].

The calendar days can also be named by Colour (or Element?) + Animal, in a simple 60-day cycle, see Appendix E.4 and the example in Footnote 49 [11, p. 242].

A.4. **Bhutan.** Bhutan uses a version of the Tibetan calendar as an official calendar; see e.g. the government's web page [30] for an example, where a calendar with both Bhutanese and Gregorian dates is shown. In official documents such as acts, both the Bhutanese and Gregorian dates are given (in both the Dzongkha and English versions), see many examples at [29, Acts]. Of the public holidays, some have fixed dates in the Bhutanese calendar (New Year and some dates connected to Buddha and Buddhism); some have fixed dates in the Gregorian calendar (e.g. the National Day and the King's Birthday); finally, the Winter Solstice is a holiday, but it is calculated according to the Bhutanese calendar (see below); it usually occurs at the Gregorian date 2 January. (See [24, Open source Bhutanese calendar software] and [30, Public holidays] for a complete list.)

The version was described by Lhawang Lodro (*lha-dbang blo-gros*) in the 18th century [24, Open source Bhutanese calendar software], but is said to be older.

An important difference from the other main versions of the Tibetan calendar is that a leap month is given the number of the *preceding* month insted of the next month, see [24, Open source Bhutanese calendar software]. (This is the system in the Chinese calendar, but not in Indian calendars [2], see Remark 27; it seems to have been the original Kālacakra Tantra system, see Appendix A.5.)

A unique feature of the Bhutanese calendar is that its day of week differs by one day from Tibet (and the rest of the world, see Footnote 29). The names of the days of week are the same as in Tibetan[60], but day 0 (Saturday) is *spen pa* in Tibet but *nyi ma* (really meaning sun) in Bhutan, day 1 (Sunday) is *nyi ma* in Tibet but *zla ba* (really meaning moon) in Bhutan, etc., see Table 5. (This difference is lost when writing dates in English, since of course the correct English day of week is chosen in both cases, for example in the calendar at [30].) See further [24, Bhutan calendar problem].

The constants m_1, s_1 and a_1 for mean motions are the same in the Bhutanese version as in the Phugpa version (and in the Tsurphu and Mongolian versions), see Section 7, and thus so are m_2, s_2 and a_2, see Remark 14, while the epoch values are different and given by, see Henning [24, Epoch

[60]Dzongkha (the national language of Bhutan) is, depending on one's view, a dialect of Tibetan or a language closely related to Tibetan; it is written with Tibetan script.

data]:

$$\text{JD} = 2361807 \quad (\text{Monday, 22 April 1754 (Greg.)})$$

$$m_0 = 2;4,24,552 \ (60,60,707) + 2361805 = 2361807 + \frac{52}{707}, \tag{A.19}$$

$$s_0 = 0,24,10,50 \ (27,60,60,67) = \frac{1}{67}, \tag{A.20}$$

$$a_0 = 3,30 \ (28,126) = \frac{17}{147}. \tag{A.21}$$

Again, m_0 is here modified to yield the JD, see Remark 8.

Remark 25. The traditional calculations use different sequences of radices than the standard ones (with fewer radices), see (A.19)–(A.20), but the standard values can be expressed exactly also with these radices. The constant m_1 is in this tradition given by $1;31,50,80$ $(60,60,707)$, which equals the standard value (7.2) after our usual addition of 28; similarly, s_1 is given by $2,10,58,14$ $(27,60,60,67) = 65/804$ which equals (7.6). [24, Epoch data].

To find the leap months, the true month is calculated by (5.3) with a constant $\beta^* = 2$ (the epoch value of the intercalation index) [24, Epoch data]. Furthermore, recalling that in Bhutan a leap month gets the number of the preceding month, the Phugpa rule (5.8) is modified to

A month is a leap month if and only if its intercalation index $ix = 59$ *or* 60. (A.22)

Here have to use the correct (traditional) intercalation index, see Remark 4, which is (5.7) increased by 2 for the leap month (and all later months until ix passes 65). A computationally simpler, but more clumsy, formulation where our simplified definition (5.7) can be used directly is:

Regular month M in year Y is followed by a leap month if and only if its intercalation index $ix = 57$ *or* 58. (A.23)

There is a corresponding change of (5.9), with 48 replaced by 59, and the true month rounded up for a leap month.

Comparing (A.23) with (5.26) (and taking into account that now a leap month M comes after the regular month M), or better by (A.22) and (C.62) below, (A.22) and (A.23) agree with the general rule (5.24) if β is defined such that

$$\beta + \beta^* \equiv 63 \bmod 65. \tag{A.24}$$

For the epoch above, with $\beta^* = 2$, we take $\beta = 191 \equiv 61 \pmod{65}$, see Appendix C.6.

By (5.30) and (5.32), (5.31) and (5.33) hold with $\gamma = 12$ and $\gamma^* = 28$.

The winter solstice is, as said above, a holiday in Bhutan. It is defined as the day the mean solar longitude (7.5) reaches 250°, which at present occurs on 2 January, see [24, Open source Bhutanese calendar software].[61] (The date drifts slowly, by almost 3 days per century, see Section 12. It will be 3 January for the first time in 2020. It coincided with the astronomical winter solstice about 400 years ago, in the early 17th century, which was before the present version of the Bhutanese calendar was described.)

The months in Bhutan are numbered, as in Tibet. (Names exist as in Tibet, see Section 5.1, but are usually not used.) The years are named by Element + Gender + Animal, see Section 4. (See the examples in [29, Acts], where the Bhutanese dates are given in this way, while the Gregorian dates are given with the number of the year.)

A.5. **Kālacakra Tantra (Karaṇa).** The original Kālacakra Tantra calculations are explained by Schuh [12, Chapter 5] and Henning [7, Chapter V], [8]; they are (as far as I know) not used to produce calendars today; however, complete Tibetan almanacs usually give (by tradition, and for no other obvious reason) some values calculated by this version in addition to values calculated by one of the versions above, see Section 10. The Kālacakra Tantra version is called *karaṇa* (*byed rtsis*), while the Phugpa and other similar versions are called *siddhānta* (*grub rtsis*).

The Kālacakra Tantra (karaṇa) version uses different values of the mean motions m_1 and s_1 than the versions described above but the same a_1 (see also [24, Epoch data]):

$$m_1 = 29;31,50 \ \ (60,60) = \frac{10631}{360} = 29 + \frac{191}{360}, \tag{A.25}$$

$$s_1 = 2,10,58,2,10 \ \ (27,60,60,6,13) = \frac{1277}{15795}, \tag{A.26}$$

$$a_1 = 2,1 \ \ (28,126) = \frac{253}{3528}; \tag{A.27}$$

the epoch values are

$$\text{JD} = 2015531 \quad \text{(Monday, 23 March 806 (Julian))} \tag{A.28}$$

$$m_0 = 2;30,0 \ \ (60,60) + 2015529 = 2015531 + \frac{1}{2}, \tag{A.29}$$

$$s_0 = 26,58 \ \ (27,60) = \frac{809}{810}, \tag{A.30}$$

$$a_0 = 5,112 \ \ (28,126) = \frac{53}{252}. \tag{A.31}$$

[61] The correct astronomical definition is 270°, for the (astronomical) true longitude; the correct Gregorian date is (at present) 21 or 22 December.

Modern karaṇa calculations (e.g. in the almanac [14]) use (at least in the cases I know) the natural values $m_1/30$ and $s_1/30$ for the daily increments m_2 and s_2, while $a_2 = 1/28$ is given by (7.13) as for the versions above. However, the Kālacakra Tantra gives instead the following simplified (rounded) values for m_2 and s_2, cf. Remark 14, which thus not are used today:

$$m_2 = 0;59 \ \ (60) = \frac{59}{60}, \tag{A.32}$$

$$s_2 = 0,4,20 \ \ (27,60,60) = \frac{13}{4860}, \tag{A.33}$$

Furthermore, the Kālacakra Tantra gives also another computational simplification (perhaps suggested as an alternative?), also not used today (as far as I know, and certainly not in the almanac [14]), see [7, p. 232–233], [8, Chapter 2, v. 30], where the values m_1, s_1, a_1 above are used only for the first month of a calendar calculation (i.e, month 3, *nag pa*), and for each following month one adds instead the simpler[62]

$$m_1' = 29;32,0 \ \ (60,60) = \frac{443}{15} = 29 + \frac{8}{15}, \tag{A.34}$$

$$s_1' = 2,11 \ \ (27,60) = \frac{131}{1620}, \tag{A.35}$$

$$a_1' = 2,0 \ \ (28,126) = \frac{1}{14}. \tag{A.36}$$

The Kālacakra Tantra calculates the true month by (5.3) with an epoch value $\beta^* = 0$. The true month is simply rounded down, as in the later Tsurphu version, see Appendix A.2. This would correspond to the intercalation rule (A.9) if a leap month is given the same number as the following month, as in Indian and later Tibetan calendars. However, it seems that the Kālacakra system instead gave a leap month the same name as the month preceding it, as in the Bhutanese calendar in Section A.4 (and the Chinese calendar [2]), see [24, Published calendar explanation]. In this case, the leap month is inserted before the month with intercalation index 0 or 1; in the traditional formulation, these indices are repeated (see Remark 4) and thus the leap month gets the same intercalation index, so the rule (A.9) nevertheless holds. Cf. the rules (A.22) and (A.23) for the Bhutanese version.[63]

[62] It seems surprising that anyone should bother about these minor simplifications, once each for each month, which are very minor compared to the mass of calculations for every day.

[63] The Karaṇa calculations in modern almanacs do not give any names or numbers, or years, for the months, as far as I know, so the question of leap months is irrelevant to them. However, in e.g. [14], the true month count and intercalation index are given for each month, for both Phugpa and karaṇa calculations. This almanac includes a month with karaṇa intercalation index given as 65, in accordance with the rules presented here; this month would thus be a leap month (a second *Māgha* = *mChu*, or month 1, 2014 if we anachronistically use the same numbering as for the other versions, although this was introduced long after the Kālacakra Tantra, see Section 5.1).

More precisely, this means that there is a leap month (Y, M) when the regular month (Y, M) has intercalation index (5.7) 63 or 64, i.e., its true month is $n+63/65$ or $n+64/65$ where n is the true month count. The month $(Y, M+1)$ then has a true month that is $67/65 = 1 + 2/65$ larger, i.e. $n+2$ or $n+2+1/65$; hence it has true month count that is $n+2$ and intercalation index 0 or 1. There is thus a gap, which is filled by a leap month which is given the missing true month count $n+1$, and (conventionally) intercalation index 65 or 66; this month is thus leap month (Y, M). We thus have the rule, cf. (A.23),

Regular month M in year Y is followed by a leap month if and only if its intercalation index $ix = 63$ or 64. (A.37)

Furthermore, we can say, cf. (A.23),

A month is a leap month if and only if its intercalation index $ix = 65$ or 66. (A.38)

This, however, is a tautology since we use the exceptional values 65 and 66 only for leap months. We cannot replace them by 0 and 1 in (A.38), since the following (regular) month has intercalation index 0 or 1.

Summarizing, in all cases, the true month count is given by (5.3) rounded down, and increased by 1 in the case of a leap month; the intercalation index is given by (5.7), increased by 2 (to 65 or 66) for a leap month. (Cf. Remark 4.)

For the epoch 806 above, with $\beta^* = 0$, it follows from (5.7) and the discussion above that (A.37) is equivalent to (5.24) with $\beta \equiv 4 \pmod{65}$, see also (C.62) below.

By (5.30) and (5.32), (5.31) and (5.33) hold with $\gamma = 28$ and $\gamma^* = 22$.

The value (A.25) of m_1 (the mean length of the month) is ≈ 29.530556 days, about 2.7 seconds shorter than the mean length (12.1) for the standard versions. The Kālacakra value differs from the modern astronomical value by about 2.9 seconds, and is thus less exact than the value (12.1), but the difference amounts to less than 1 hour every century.

The mean length of the year is, cf. (12.3),

$$\frac{m_1}{s_1} = \frac{3731481}{10216} \approx 365.258516 \text{ days}, \tag{A.39}$$

which is better than the value in (12.3) for the later versions; it is 0.01633 days longer than the astronomical value of the tropical year (365.24219 days) and only 0.002156 days longer than the sidereal year (365.25636 days), see Section 12 and [3, Table 15.3]. (Nevertheless, the year is intended to be a tropical year [7, p. 298].)

A.6. **Further historical versions.** Several further historical versions of the Tibetan calendar are described by Schuh [12]. (I do not know to which extent these were actually used.)

A proposed (but never adopted) version by Zhonnu Pal (*gzhun-nu dpal*) from 1443 is described by Henning [7, pp. 307–321] and [24, Error correction system].

A.7. **Sherab Ling.** A modern attempt at a reformed Tibetan calendar developed by Kojo Tsewang Mangyal (Tsenam) at the Sherab Ling monastery in Bir, India, is described in [7, pp. 342–345], see also [24, A reformed Tibetan calendar] and [24, Epoch data]. It uses the value of m_1 in Section 7, but uses

$$s_1 = 2, 10, 58, 2, 564, 5546 \ (27, 60, 60, 6, 707, 6811) = \frac{3114525}{38523016}. \qquad \text{(A.40)}$$

This deviation from the traditional value, and from the equivalent fundamental relation (5.1), means that leap months no longer will be regularly spaced in the traditional pattern with 2 leap months every 65 regular months. (However, the difference is small; the average number of leap months for 65 regular months is 1.9978, about 0.1% less than in the standard calendars, which means that on the average a leap month is delayed a month after about 75 years.)

A.8. **Sarnath.** A version of the Sherab Ling calendar (see A.7) is published by the Jyotish Department of the Central University of Tibetan Studies at Sarnath, India, see [7, p. 346], [24, A reformed Tibetan calendar]. This version differs from all other Tibetan calendars that I have heard of in that the months begin at full moon (as in Indian calendars in northern India [2]).

A.9. **Sherpa.** The Sherpas (a minority in Nepal) have a Tibetan calendar called the Khunu almanac. (The official Nepalese calendar is of the Indian type.) The Sherpa calendar for 2002 I found on the web [41] is identical to the Phugpa version, and this seems to be the general rule.

A.10. **Drikung Kagyu.** According to [21], the Drikung Kagyu tradition follows a system that combines the Tsurphu and Phugpa traditions.

A.11. **Yellow calculations.** (Chinese-style) According to [21], the yellow system is like the Chinese in that it has no repeated or skipped days. Months have 29 or 30 days, numbered consecutively "and determined according to several traditions of calculation". The way it adds leap months is similar to, but not equivalent to, the Chinese. Unlike the Chinese calendar, it uses the basic calculations from Kālacakra Tantra. This seems to be a version of the Chinese calendar (or astrology) rather than the Tibetan; it might also be the Chinese calendar (without modification), but then some of the statements just made are incorrect. See also [27] and [42]. Inner Mongolia (in China) follows the yellow system [21].

A.12. **An astronomical version.** Henning [24] has constructed a reformed Tibetan calendar combining the principles of the Kālacakra Tantra with modern astronomical calculations.

A.13. **A comparison.** We give some examples of the (small) differences between the four different versions of the Tibetan calendar in Appendices A.1–A.4.

The epoch values given above for the different versions use different years. To enable an easy comparison, Table 6 gives the epoch values for the four versions calculated for the same epoch, here chosen as the Kālacakra Tantra epoch at the beginning of *nag pa* (*Caitra*) 806. (In all four versions, the epoch, i.e., the mean new moon, is at JD 2015531, Monday 23 March 806 (Julian), cf. Remark 16.)[64] Since all four versions have the same mean motions m_1, s_1, a_1, the differences between them for mean dates, solar longitudes and lunar anomalies will be constant. For example, the Tsurphu mean solar longitude is always $0.018261 - 0.004975 = 0.013286 = 4.78°$ larger than the Phugpa value, and the mean dates differ by 0.046, about 1/20. The differences in true date and true solar longitude will be varying because of the corrections in (7.22) and (7.23), but the average difference is the same as for the mean values. Thus the Phugpa and Tsurphu dates will differ for, on the average, about one day in 20, i.e., typically one or two days in a month. (See Table 8 for some examples, with 0–4 days differing each month.)

	Phugpa	Tsurphu	Mongolia	Bhutan
m_0	$2; 22, 34, 2, 518$	$2; 25, 20, 2, 352$	$2; 25, 6, 3, 327$	$2; 24, 37, 5, 431$
	2.376238	2.422338	2.418494	2.410537
s_0	$0, 8, 3, 3, 33$	$0, 29, 34, 5, 37$	$0, 38, 17, 0, 6$	$0, 28, 12, 3, 15$
	0.004975	0.018261	0.023632	0.017413
a_0	$5, 98$	$5, 112$	$5, 101$	$6, 22$
	0.206349	0.210317	0.207200	0.220522

Table 6. Epoch data for four versions of the Tibetan calendar for JD 2015531 (23 March 806), given both in Tibetan form with radices $(60, 60, 6, 707)$, $(27, 60, 60, 6, 67)$, $(28, 126)$ and with 6 decimals. We have here given m_0 in the traditional form modulo 7; to get the result in JD as in this paper, add 2015529.

Remark 26. The differences in the correction terms can be estimated as follows: Consider two versions whose epoch values m_0, s_0, a_0 differ by Δm_0, Δs_0, Δa_0. By (7.17), the arguments used in the table *moon_tab* differs by $28\Delta a_0$; since the maximum derivative (slope) in the table (7.18) is 5, the difference in *moon_equ* satisfies $|\Delta moon\_equ| \le 140|\Delta a_0|$; this correction is divided by 60 in (7.22), so the corrections to the true date differ by at most $\frac{7}{3}|\Delta a_0|$. Similarly, by (7.19), (7.20) and (7.21), the difference in *sun_equ* satisfies $|\Delta sun\_equ| \le 6 \cdot 12|\Delta s_0|$; this correction is divided by 60

[64]This happens to be the last day of the preceding month in all four versions; this is day 30 in month 2 for the Phugpa and Bhutanese versions, and day 30 in leap month 3 in the Tsurphu and Mongolian versions.

in (7.22) and by $27 \cdot 60$ in (7.23), so the corrections to the true date and true solar longitude differ by at most $1.2|\Delta s_0|$ and $0.045|\Delta s_0|$. In particular, the difference between the true solar longitudes differ always by between 0.95 and 1.05 times the mean difference Δs_0; for Phugpa and Tsurphu the difference is thus between 4.57° and 5.00°.

For the date, the mean difference Δm_0 between Phugpa and Tsurphu is as said above 0.046, and the maximum differences in the correction terms $\frac{1}{60}\Delta moon\_equ$ and $\frac{1}{60}\Delta sun\_equ$ are $\frac{7}{3}\Delta a_0 = 0.009$ and $1.2\Delta s_0 = 0.016$, respectively; hence the difference of the true dates is $0.046 \pm 0.009 \pm 0.016$, i.e., between 0.021 and 0.071, with Tsurphu always larger.[65] As a consequence, the Phugpa version is slightly behind the Tsurphu, so when the Phugpa and Tsurphu dates of a calendar day differ, the Phugpa date is always larger (by 1). (See the example of differences in Table 8.)

Similarly, the difference in mean date between the Mongolian and Tsurphu versions is only $\Delta m_0 = 0.0038$; the differences in solar longitude and lunar anomaly are $\Delta s_0 = -0.0054$ and $\Delta a_0 = 0.0031$; hence the true date differs by $0.0038 \pm 0.0073 \pm 0.0064$, i.e., between -0.010 and 0.018. (So we expect a difference of the calendar date only a few days each year, cf. Table 8.)

Table 7 shows leap months for a range of (Gregorian) years. Since all four versions have a simple periodic pattern with alternating 32 or 33 regular months between the leap months, the same pattern repeats for ever. Note that the Tsurphu and Mongolian versions have the same leap months, as said before. We see also that leap months in Bhutan come 2 or 3 months after the Phugpa leap months (3 or 4 months later if we take into account the different numbering of Bhutanese leap months), and the Tsurphu and Mongolian leap months come an additional 4 (really 3) months later.

Table 8 shows all repeated and skipped days during 2012 for the four versions. (This year is chosen since none of the versions has a leap month. The versions also all start this year on the same day.) It is seen that four versions are very similar; often the same day is repeated or skipped, but it also frequently happens that different versions differ by a day (meaning that some days get different dates). We can see that this year each month has the same length in all four versions.

Table 9 shows the Gregorian dates of New Year for a range of (Gregorian) years. We see that most years, all four calendars coincide. However, some times the Phugpa and Bhutanese versions, or just the Phugpa version, is a month later (due to the difference in leap months); sometimes there is also a difference of a day between two versions.[66]

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[65]The two correction terms are periodic, with periods the anomalistic month (12.4) and the year (12.3); the maximum values will thus coincide several times each year.

[66]Bhutan actually had New Year 3/3 2003, with day 1 of month 1 repeated 3/3 and 4/3, according to the government web site calendar [30] [Henning, personal communication]; the calculations described here yield (as do the ones by Henning [24, Open source Bhutanese

In particular, we do not see any differences between the Tsurphu and Mongolian New Year in Table 9; a computer search reveals that the last time these versions had different New Year was 1900 (31/1 vs 1/2) and the next will be 2161 (26/2 vs 25/2), and only four more differences were found for the present millenium.

	Phugpa	Tsurphu	Mongolia	Bhutan
2000	1	8	8	4
2001				
2002	10			12
2003		4	4	
2004				
2005	6			9
2006		1	1	
2007				
2008	3	9	9	5
2009				
2010	11			
2011		6	6	2
2012				
2013	8			10
2014		2	2	
2015				
2016	4	11	11	7
2017				
2018				
2019	1	7	7	3
2020				

TABLE 7. Leap months for four versions of the Tibetan calendar.

calendar software]) 3/3 instead as a repeated day 30 in the last (leap) month of 2002. I have no explanation for this discrepancy.

month	Phugpa	Tsurphu	Mongolia	Bhutan
1	5, -19	4, -20	4, -20	4, -19
2	9, -12, -25, 27	8, -13	8, -13	8, -13
3	-17	-17	-17	-17
4	3, -10	2, -11	2, -11	2, -10
5	-13, 29	-14, 28	-14, 28	-13, 28
6	-6	-6	-6	-6
7	-9, 25	-9, 25	-9, 25	-9, 24
8	-1	-2	-2	-1
9	-5, 20, -29	-6, 19, -29	-6, 20, -29	-5, 19, -29
10				
11	-3, 13, -27	-3, 12, -28	-4, 12, -28	-3, 12, -27
12	17, -21	15, -22	15, -22	15, -21

TABLE 8. Repeated and skipped (marked with -) days in each month 2012 for four versions of the Tibetan calendar.

	Phugpa	Tsurphu	Mongolia	Bhutan
2000	6/2	6/2	6/2	6/2
2001	24/2	24/2	24/2	24/2
2002	13/2	13/2	13/2	13/2
2003	**3/3**	2/2	2/2	**4/3**
2004	21/2	21/2	21/2	21/2
2005	9/2	9/2	9/2	9/2
2006	**28/2**	**30/1**	**30/1**	**28/2**
2007	18/2	18/2	18/2	18/2
2008	**7/2**	8/2	8/2	8/2
2009	25/2	25/2	25/2	25/2
2010	14/2	14/2	14/2	14/2
2011	**5/3**	3/2	3/2	3/2
2012	22/2	22/2	22/2	22/2
2013	11/2	11/2	11/2	11/2
2014	**2/3**	**31/1**	**31/1**	**2/3**
2015	19/2	19/2	19/2	19/2
2016	9/2	9/2	9/2	9/2
2017	27/2	27/2	27/2	27/2
2018	16/2	16/2	16/2	16/2
2019	5/2	5/2	5/2	5/2
2020	24/2	24/2	24/2	24/2
2021	12/2	12/2	12/2	12/2
2022	**3/3**	**2/2**	**2/2**	**3/3**
2023	21/2	21/2	21/2	21/2
2024	10/2	10/2	10/2	10/2
2025	**28/2**	**1/3**	**1/3**	**28/2**
2026	18/2	18/2	18/2	18/2
2027	7/2	7/2	7/2	7/2
2028	26/2	26/2	26/2	26/2
2029	14/2	14/2	14/2	14/2
2030	**5/3**	3/2	3/2	3/2

Table 9. Gregorian dates for New Year (*Losar*) for four versions of the Tibetan calendar. Dates differing from the majority in boldface.

Appendix B. The 60 year cycle

The Indian 60 year cycle is derived from an Indian cycle of 60 names for the "Jovian years".[67] Table 10 (taken from [7]) gives the full list of names, in Tibetan and Sanskrit, together with the corresponding year in the Chinese 60 year cycle and the Gregorian years in the last and current cycle. (Somewhat different transliterations of the Sanskrit names are given in [2].)

See also Table 1, which gives the Gregorian dates of New Year for the years in the last and current cycles.

year		element–animal	Tibetan	Sanskrit		
1	4	Fire–Rabbit	rab byung	prabhava	1927	1987
2	5	Earth–Dragon	rnam byung	vibhava	1928	1988
3	6	Earth–Snake	dkar po	suklata	1929	1989
4	7	Iron–Horse	rab myos	pramadi	1930	1990
5	8	Iron–Sheep	skyes bdag	prajapati	1931	1991
6	9	Water–Monkey	anggi ra	ankira	1932	1992
7	10	Water–Bird	dpal gdong	srimukha	1933	1993
8	11	Wood–Dog	dngos po	bhava	1934	1994
9	12	Wood–Pig	na tshod ldan	yuvika	1935	1995
10	13	Fire–Mouse	'dzin byed	dhritu	1936	1996
11	14	Fire–Ox	dbang phyug	isvara	1937	1997
12	15	Earth–Tiger	'bru mang po	vahudhvanya	1938	1998
13	16	Earth–Rabbit	myos ldan	pramadi	1939	1999
14	17	Iron–Dragon	rnam gnon	vikrama	1940	2000
15	18	Iron–Snake	khyu mchog	brisabha	1941	2001
16	19	Water–Horse	sna tshogs	citra	1942	2002
17	20	Water–Sheep	nyi ma	bhanu	1943	2003
18	21	Wood–Monkey	nyi sgrol byed	bhanutara	1944	2004
19	22	Wood–Bird	sa skyong	virthapa	1945	2005
20	23	Fire–Dog	mi zad	aksaya	1946	2006
21	24	Fire–Pig	thams cad 'dul	sarvajit	1947	2007
22	25	Earth–Mouse	kun 'dzin	sarvadhari	1948	2008
23	26	Earth–Ox	'gal ba	virodhi	1949	2009
24	27	Iron–Tiger	rnam 'gyur	vikrita	1950	2010
25	28	Iron–Rabbit	bong bu	khara	1951	2011
26	29	Water–Dragon	dga' ba	nanda	1952	2012
27	30	Water–Snake	rnam rgyal	vijaya	1953	2013

[67] It takes Jupiter almost 12 years to orbit the sun. A 1/12 of the orbital period can be called a Jovian year, and the traditional Indian Jovian cycle gives each Jovian year a name (*samvatsara*) from a list of 60 names, so the names repeat with a cycle of 5 revolutions. The solar years are given the same names, based on the calculated position of Jupiter at the beginning of the year; hence sometimes (every 85 or 86 years) a name is skipped (expunged) from the list. In southern India this has from the 9th century been simplified to a simple 60 year cycle of names, and the same is done in Tibet. [2], [7, p. 143f].

28	31	Wood–Horse	rgyal ba	jaya	1954	2014
29	32	Wood–Sheep	myos byed	mada	1955	2015
30	33	Fire–Monkey	gdong ngan	durmukha	1956	2016
31	34	Fire–Bird	gser 'phyang	hemalambha	1957	2017
32	35	Earth–Dog	rnam 'phyang	vilambhi	1958	2018
33	36	Earth–Pig	sgyur byed	vikari	1959	2019
34	37	Iron–Mouse	kun ldan	sarvavati	1960	2020
35	38	Iron–Ox	'phar ba	slava	1961	2021
36	39	Water–Tiger	dge byed	subhakrita	1962	2022
37	40	Water–Rabbit	mdzes byed	sobhana	1963	2023
38	41	Wood–Dragon	khro mo	krodhi	1964	2024
39	42	Wood–Snake	sna tshogs dbyig	visvabandhu	1965	2025
40	43	Fire–Horse	zil gnon	parabhava	1966	2026
41	44	Fire–Sheep	spre'u	pravamga	1967	2027
42	45	Earth–Monkey	phur bu	kilaka	1968	2028
43	46	Earth–Bird	zhi ba	saumya	1969	2029
44	47	Iron–Dog	thun mong	sadharana	1970	2030
45	48	Iron–Pig	'gal byed	virobhakrita	1971	2031
46	49	Water–Mouse	yongs 'dzin	paradhari	1972	2032
47	50	Water–Ox	bag med	pramadi	1973	2033
48	51	Wood–Tiger	kun dga'	ananda	1974	2034
49	52	Wood–Rabbit	srin bu	raksasa	1975	2035
50	53	Fire–Dragon	me	anala	1976	2036
51	54	Fire–Snake	dmar ser can	vingala	1977	2037
52	55	Earth–Horse	dus kyi pho nya	kaladuti	1978	2038
53	56	Earth–Sheep	don grub	siddhartha	1979	2039
54	57	Iron–Monkey	drag po	rudra	1980	2040
55	58	Iron–Bird	blo ngan	durmati	1981	2041
56	59	Water–Dog	rnga chen	dundubhi	1982	2042
57	60	Water–Pig	khrag skyug	rudhirura	1983	2043
58	1	Wood–Mouse	mig dmar	raktaksi	1984	2044
59	2	Wood–Ox	khro bo	krodhana	1985	2045
60	3	Fire–Tiger	zad pa	ksayaka	1986	2046

Table 10: The Chinese and Indian 60 year cycles of names, with the names from the Indian cycle both in Tibetan and in Sanskrit. The first number on each line shows the number in the Prabhava cycle; the second shows the number in the Chinese cycle. The last two numbers show the Gregorian years in the last and current cycles.

Appendix C. Leap months and the mean sun

We give here an explanation of the leap month rules in Section 5, based on Tibetan astronomy. (This appendix is based on the description in Henning [7]; see also [24, On intercalary months]. For a detailed historical discussion of different leap month rules, see Schuh [12, pp. 107–117]; see further Yamaguchi [19] (which also contains tables with actually observed leap months from historical data) and Henning [24, Early epochs].[68])

C.1. **General theory.** The key is the position of the mean sun, which is a fictitious version of the sun that travels along the ecliptic with uniform speed. Its longitude (the mean solar longitude) at the beginning of true month n was denoted by $mean\_sun(0,n)$ in Section 7; we use here the simplified notation $mean\_sun(n)$. It is by (7.5) given by the linear formula

$$mean\_sun(n) = s_1 n + s_0, \tag{C.1}$$

where $s_0 = mean\_sun(0)$ is the mean solar longitude at the epoch. The Tibetan constant s_1, the mean motion of the sun per (lunar) month, is given by

$$s_1 = \frac{65}{804} = \frac{65}{67 \cdot 12}, \tag{C.2}$$

see (7.6). (This can also be expressed as 65/67 signs, or $29\frac{7}{67}^\circ$.)

Note that (C.2) says that the mean sun goes around the ecliptic exactly 65 times in 804 lunar months, i.e.,

$$804 \text{ lunar months } = 65 \text{ years.} \tag{C.3}$$

Since $804 = 67 \cdot 12$, this is equivalent to the relation (5.1), and it explains the leap year cycle of 65 years, see Remark 2.

Moreover, the zodiac contains 12 evenly spaced *definition points*[69] (*sgang*), one in each sign. Let us denote these (and their longitudes) by $p_1, \dots, p_{12}$. The rule for naming months is, see [7]:

> *A month where the mean sun passes a definition point p_M is given number M.* (C.4)

Since $s_1 < \frac{1}{12}$, the spacing of the definition points, the mean sun can never pass two definition points in one month, but sometimes it does not pass any of them; in that case the month is designated as a leap month, and is given the number of the next month. In both cases, the number of the month is thus given by the first definition point p_M that comes after $mean\_sun(n)$.

[68] In particular, according to Schuh [38, Kalenderrechnung], see also [12], the definition points described here were introduced (for the Phugpa version) in 1696. However, some similar method to align the year seems to have been used earlier; the principle to use the position of the mean sun goes back to early Indian calendars and the calculations described in Section 5 are only minor modifications of the ones in the Kālacakra Tantra which thus seem to be based on considerations related to the ones given here, although possibly in different formulations. See also the discussion in Section C.7.

[69] We use this term from Henning [7].

(We do not have to worry about the exact definition in the ambiguous case when $mean\_sun(n)$ exactly equals some p_M; the constants in the Phugpa system are such that this never will happen, see Remark 30 below. I have therefore just chosen one version in the formulas below.) The leap month rule is thus:

A month where the mean sun does not pass any definition point is a leap month. (C.5)

Remark 27. The rule above for numbering the months is the same as in many Indian calendars [2, Chapters 9 and 17], and almost the same as the rule in the Chinese calendar [2, Chapters 16]; however, in these calendars, the definition points are beginnings of the zodiacal signs, i.e., multiples of $30^\circ = 1/12$ while in the Tibetan calendar, the definition points are shifted, see Section C.2. (In the Chinese calendar, the definition points are called *(major) solar terms.* In most (but not all) Indian calendars, month 1 (New Year) is defined by the vernal equinox, i.e., $p_1 = 0$ in our notation; the exact rule in the Chinese calendar is that the winter solstice ($270^\circ = 3/4$) occurs in month 11, which corresponds to $p_{11} = 3/4$, but there are some complications for other months.) Note however, that the (present) Chinese and Indian calendars use the true motion of sun and moon, while the Tibetan uses the mean motion, leading to regularly spaced leap months in the Tibetan calendar, but not in the Chinese and Indian ones. (It also leads to skipped months sometimes in the Indian calendars.) Note further that the numbering of leap months differs in the Chinese calendar, where a leap month is given the number of the preceding month.

The longitude in (C.1) is naturally taken modulo 1, i.e., considering only the fractional part. But a moment's consideration shows that the integer part shows the number of elapsed full circles of the sun, i.e., the number of years; in this appendix we thus use $mean\_sun(n)$ for the real number defined by (C.1). Let, as in Section 5, Y and M be the year and the number of the month with true month count n, and let the Boolean variable ℓ indicate whether the month is a leap month. The rule (C.4) then yields the following relations determining (Y, M, ℓ), with $p_0 = p_{12} - 1$,

$$Y - Y_0 + p_{M-1} < mean\_sun(n) \le Y - Y_0 + p_M, \tag{C.6}$$

$$\ell = [mean\_sun(n+1) \le Y - Y_0 + p_M]. \tag{C.7}$$

Furthermore, the points p_M are evenly spaced, so $p_M = p_0 + M/12$.

Remark 28. The initial longitude s_0 is, as any longitude, really defined modulo 1, i.e., only the fractional part matters. However, when we regard $mean\_sun(n)$ as a real number, we have to make the right choice of integer part of s_0. Since the epoch is assumed to be in year Y_0, with a month M satisfying $1 \le M \le 12$, taking $n = 0$ in (C.6) shows that we must have

$$p_0 < s_0 \le p_{12} = p_0 + 1. \tag{C.8}$$

Equivalently, α defined in (C.12) below must satisfy

$$0 < \alpha \le 12. \tag{C.9}$$

(This follows also by taking $n = 0$ and $Y = Y_0$ in (C.14)–(C.16) below, see Remark 29.) Consequently, for the formulas for month numbers and leap months below, we have to assume that the integer part of s_0 is chosen such that (C.8)–(C.9) hold; this sometimes means adding 1 to the traditional value (which does not affect the solar longitude seen as an angle, i.e. modulo 1).[70]

Let us for simplicity write $Y' = Y - Y_0$. Then (C.6) can be rewritten

$$Y' + \frac{M-1}{12} + p_0 < \mathit{mean\_sun}(n) \le Y' + \frac{M}{12} + p_0 \tag{C.10}$$

or

$$12Y' + M - 1 < 12(\mathit{mean\_sun}(n) - p_0) \le 12Y' + M. \tag{C.11}$$

Hence, if we use (C.1) and further define

$$\alpha = 12(s_0 - p_0), \tag{C.12}$$

we have

$$12Y' + M = \lceil 12(\mathit{mean\_sun}(n) - p_0) \rceil = \lceil 12s_1 n + \alpha \rceil. \tag{C.13}$$

Consequently, we can calculate (Y, M) from n by

$$x = \lceil 12s_1 n + \alpha \rceil, \tag{C.14}$$

$$M = x \ \text{ amod } \ 12, \tag{C.15}$$

$$Y = \frac{x - M}{12} + Y_0. \tag{C.16}$$

To complete the calculations of (Y, M, ℓ) from n, we find similarly from (C.7), or simpler by (C.13) because a month is leap if and only if it gets the same number as the following one,

$$\ell = \big\{ \lceil 12s_1(n+1) + \alpha \rceil = \lceil 12s_1 n + \alpha \rceil \big\}. \tag{C.17}$$

For the Tibetan value of s_1 in (C.2), we have $12s_1 = \frac{65}{67}$, and thus (C.14) can be written

$$x = \left\lceil \frac{65}{67} n + \alpha \right\rceil = \left\lceil \frac{65n + 67\alpha}{67} \right\rceil = \left\lceil \frac{65n + \beta}{67} \right\rceil, \tag{C.18}$$

where we define the integer β to be 67α rounded up, i.e.,

$$\beta = \lceil 67\alpha \rceil. \tag{C.19}$$

Note that (C.18) and (C.15)–(C.16) are the same as (5.19)–(5.21), which shows that the algorithmic calculations in Section 5 yield the same correspondence between (Y, M) and true month count n as the rule (C.4) used in this appendix, provided the value of β is the same; in particular, the two methods yield the same leap months.

[70]For calculations one can use any s_0 and instead normalize α in (C.12) modulo 12 so that (C.9) holds.

Remark 29. By (C.14)–(C.16), the epoch month with true month count $n = 0$ is month $M = \lceil \alpha \rceil$ in the epoch year. (Recall that we have $0 < \alpha \leq 12$ by (C.9), so this yields a value $1 \leq M \leq 12$. Conversely, we see again that (C.9) is necessary for Y_0 to be the epoch year.) By Remark 5, the traditional choices of epoch always yield $M = 2$ or 3 for $n = 0$. There are thus only two possibilities: either $1 < \alpha \leq 2$ and the epoch month with true month count 0 is month 2 (so the nominal epoch month 3 has true month count 1), or $2 < \alpha \leq 3$ and the true month count is 0 for month 3.

Remark 30. To verify the assertion above that *mean_sun*(n) never equals some definition point, note that the calculations above show that this would happen if and only if $12s_1n + \alpha$ would be an integer. Since $12s_1 = \frac{65}{67}$, this can happen only if $67\alpha = 804(s_0 - p_0)$ is an integer (and in that case it would happen for some n); we will see in (C.30) below that for the Phugpa version, this is not the case ($804\,s_0$ is an integer by (7.8) but $804\,p_0$ is not). Similarly, it will never happen for the other versions of the Tibetan calendar with the definition points defined for them below.

Conversely, given (Y, M, ℓ), we can find the true month count n by the theory in this appendix from (C.10), noting that if there are two possible values of n, then we should choose the smaller one if $\ell = \mathit{true}$ (a leap month) and the larger one if $\ell = \mathit{false}$ (a regular month). If $\ell = \mathit{false}$, then (C.10) and (C.1) thus show that n is the largest integer such that

$$s_1 n + s_0 - p_0 \leq Y' + \frac{M}{12} \tag{C.20}$$

or, recalling (C.12),

$$s_1 n \leq Y' + \frac{M - \alpha}{12} = \frac{12(Y - Y_0) + M - \alpha}{12}; \tag{C.21}$$

if $\ell = \mathit{true}$, this value of n should be decreased by 1. Hence, in all cases, n can be computed from (Y, M, ℓ) by

$$n = \left\lfloor \frac{12(Y - Y_0) + M - \alpha}{12 s_1} \right\rfloor - [\ell]. \tag{C.22}$$

(For the Phugpa version, it is easily verified directly that this is equivalent to (5.10), using (5.2) and (C.36) below.)

An alternative formula, which has the advantage that if there is no leap month M in year Y, then (Y, M, true) gives the same result as (Y, M, false), is given by

$$n = \left\lfloor \frac{12(Y - Y_0) + M - \alpha - (1 - 12 s_1)[\ell]}{12 s_1} \right\rfloor; \tag{C.23}$$

to see this, note that if $\ell = \mathit{false}$, then (C.23) and (C.22) give the same result, while if $\ell = \mathit{true}$, then (C.23) gives 1 more than (C.22) applied to $(Y, M-1, \mathit{false})$, which is the month preceding leap month M (also if $M = 1$, when this really is $(Y - 1, 12, \mathit{false})$).

Let us write $M' = 12(Y - Y_0) + M$; this can be interpreted as a solar month count from the beginning of year Y_0 (or we can interpret month M year Y as month M' year Y_0); note that M^* in Section 5 is given by, by (5.2),

$$M^* = M' - M_0 = M' - 3. \tag{C.24}$$

Since $12s_1 = 65/67$, we can write (C.22) as

$$n = \left\lfloor \frac{67}{65}(M' - \alpha) \right\rfloor - [\ell] = \left\lfloor \frac{67M' - 67\alpha}{65} \right\rfloor - [\ell], \tag{C.25}$$

and since $67M'$ is an integer, this value is not affected if 67α is replaced by the integer $\beta = \lceil 67\alpha \rceil$, see (C.19). We thus have

$$n = \left\lfloor \frac{67M' - \beta}{65} \right\rfloor - [\ell] = M' + \left\lfloor \frac{2M' - \beta}{65} \right\rfloor - [\ell]. \tag{C.26}$$

There is a leap month M in year Y if and only if the true month count for (Y, M, false) jumps by 2 from the preceding regular month $(Y, M-1, \mathit{false})$. By (C.26), this happens exactly when $2M' - \beta$ just has passed a multiple of 65, i.e., when $2M' - \beta \equiv 0$ or $1 \pmod{65}$. Thus, the general leap month rule (C.5) is equivalent to:

There is a leap month M in year Y if and only if

$$2M' \equiv \beta \text{ or } \beta + 1 \pmod{65}, \tag{C.27}$$

where $M' = 12(Y - Y_0) + M$.

Since this is the same as (5.24), we see again that (C.5) leads to the same results as the rules in Section 5.

C.2. **Phugpa definition points.** In the Phugpa version, the first definition point p_1 is $23;6\ (60)$ mansions, i.e.

$$p_1 = 23, 6\ (27, 60) = \frac{77}{90} \tag{C.28}$$

and thus

$$p_0 = p_1 - \frac{1}{12} = \frac{139}{180}; \tag{C.29}$$

in degrees, this is $p_0 = 278°$, $p_1 = 308°$, and so on, with intervals of 30°, i.e. 8 degrees after the beginning of each sign, see [7], [13].[71]

Recall that in the formulas above, the integer part of s_0 has to be chosen such that (C.8) holds, i.e., $139/180 < s_0 \le 319/180$. This is satisfied by the values (7.8) and (7.9) for the epochs E806 and E1927, but for E1987 the

[71]For other purposes, Phugpa astronomy regards the winter solstice to be at solar longitude $18, 31, 30\ (27, 60, 60) = 247/360 = 247°$, one degree earlier than $p_{11} = 248°$, and similarly for the summer solstice, see [7, p. 322–328], [24, On intercalary months], [12, pp. 114–115] and [13, pp. 223, 225]. I have no explanation for this difference, but it does not affect the calendar which uses the values above.

value $s_0 = 0$ in (7.10) has to be replaced by 1. (Recall that this does not matter in Section 7.) The constant α defined by (C.12) equals thus

$$\alpha = 12(s_0 - p_0) = \frac{1832}{1005} = 1 + \frac{827}{1005} \qquad \text{(E806);} \tag{C.30}$$

or

$$\alpha = 12(s_0 - p_0) = \frac{1922}{1005} = 1 + \frac{917}{1005} \qquad \text{(E1927);} \tag{C.31}$$

$$\alpha = 12(s_0 - p_0) = \frac{41}{15} = 2 + \frac{11}{15} \qquad \text{(E1987).} \tag{C.32}$$

Hence, (C.19) yields

$$\beta = 123 \qquad \text{(E806),} \tag{C.33}$$

$$\beta = 129 \qquad \text{(E1927),} \tag{C.34}$$

$$\beta = 184 \qquad \text{(E1987),} \tag{C.35}$$

in agreement with (5.16)–(5.18). Consequently, we have verified that the arithmetic calculations in Section 5 and the astronomical theory in this appendix yield the same result.

Recall that in Section 5, β was defined by (5.15), so we see, using (C.12) and (C.19), that the initial value β^* in the true month calculation (5.3) is related to the definition points by

$$\beta^* = 184 - \beta = 184 - \lceil 67\alpha \rceil = 184 + \lfloor 804(p_0 - s_0) \rfloor. \tag{C.36}$$

This yields directly

$$\beta^* = 61 \qquad \text{(E806),} \tag{C.37}$$

$$\beta^* = 55 \qquad \text{(E1927),} \tag{C.38}$$

$$\beta^* = 0 \qquad \text{(E1987),} \tag{C.39}$$

in agreement with (5.4)–(5.6).

Note also that by Remark 29, the values of α in (C.30)–(C.32) show again that for E806 and E1927, the epoch month with true month count 0 is month 2, while for E1987, it is month 3.

C.3. **Tsurphu definition points.** The values for β given in Appendix A.2 are consistent with, cf. (C.28) for Phugpa,

$$p_1 = 23, 1, 30 \ \ (27, 60, 60) = \frac{307}{360} = 307^\circ, \tag{C.40}$$

which gives

$$p_0 = \frac{277}{360} = 277^\circ. \tag{C.41}$$

To see this, note first that the values of s_0 in (A.4) and (A.7) both have to be increased by 1 in order to satisfy (C.8), see Remark 28. Thus we now take, for E1732 and E1852, respectively,

$$s_0 = 1 - \frac{5983}{108540} = \frac{102557}{108540}, \tag{C.42}$$

$$s_0 = 0, 1, 22, 2, 4, 18 \ \ (27, 60, 60, 6, 13, 67) + 1 = 1 + \frac{23}{27135}, \tag{C.43}$$

which together with (C.41) yield

$$\alpha = 12(s_0 - p_0) = 2 + \frac{1903}{18090} \qquad \text{(E1732)}, \tag{C.44}$$

$$\alpha = 12(s_0 - p_0) = 2 + \frac{14053}{18090} \qquad \text{(E1852)}. \tag{C.45}$$

By (C.19) this yields the values $\beta = 142$ (E1732) and 187 (E1852), as said in Appendix A.2.

Note that 67α is not an integer, so (as for Phugpa) *mean_sun*(n) never equals some definition point, see Remark 30.

Remark 31. Any p_1 satisfying

$$\frac{92432}{27 \cdot 4020} < p_1 < \frac{92567}{27 \cdot 4020} \tag{C.46}$$

would give the same β and thus by (C.27) the same calendar. The value (C.40) is from Henning [personal communication]; it is determined from calendars and not explicitly taken from any text.

C.4. **Mongolian (New Genden) definition points.** As far as I know, Yeshe Paljor does not discuss definition points explicitly for the New Genden version, but the value of β^* is consistent with increasing the definition point p_1 from (C.28) to

$$p_1 = 23, 9 \ \ (27, 60) = \frac{463}{540} = 308\tfrac{2}{3}^\circ, \tag{C.47}$$

and thus

$$p_0 = \frac{209}{270} = 278\tfrac{2}{3}^\circ, \tag{C.48}$$

which together with (A.12) yields, by (C.12) and (C.19),

$$\alpha = \frac{7724}{3015} = 2 + \frac{1694}{3015} \tag{C.49}$$

and $\beta = 172 \equiv 42 \pmod{65}$, as said in Appendix A.3. (Again, note that 67α is not an integer, cf. Remark 30.)

In fact, any value of p_1 with

$$\frac{689}{804} < p_1 < \frac{690}{804} \tag{C.50}$$

gives the same β, and thus the same calendar. The value (C.47) is just our choice within this range.

We do not know whether such definition points are used at all in the Mongolian calendar, or whether the rule (A.17) is used as it is without further justification.

C.5. **When the leap month is the second.** In the Bhutanese version of the calendar (Appendix A.4, see also Appendix A.5), a leap month takes the number of the *preceding* month, so the leap month is the second of the two months with the same number. For such versions, the theory above has to be modified.

The basic rules (C.4) and (C.5) remain unchanged, but they now imply that the number of a month is the number of the last definition point before the end of the month. Hence (C.6)–(C.7) are replaced by

$$Y - Y_0 + p_M < mean\_sun(n+1) \le Y - Y_0 + p_{M+1}, \tag{C.51}$$

$$\ell = [mean\_sun(n) > Y - Y_0 + p_M]. \tag{C.52}$$

Note that (C.51) is the same as (C.6) with M and n replaced by $M+1$ and $n+1$;[72] thus (C.10)–(C.13) hold with the same modification, which yields, recalling $12s_1 = \frac{65}{67}$,

$$12Y' + M = \lceil 12s_1(n+1) + \alpha \rceil - 1 = \lceil 12s_1 n + \alpha - \tfrac{2}{67} \rceil. \tag{C.55}$$

Hence we now define, in addition to (C.12),

$$\alpha' = \alpha - \tfrac{2}{67} = 12(s_0 - p_0) - \tfrac{2}{67} \tag{C.56}$$

and have, instead of (C.13),

$$12Y' + M = \lceil 12s_1 n + \alpha' \rceil. \tag{C.57}$$

Consequently, (C.14)–(C.17) hold with α replaced by α'.

We now change the definition (C.19) to

$$\beta = \lceil 67\alpha' \rceil = \lceil 67\alpha \rceil - 2; \tag{C.58}$$

then (C.57) can be written, cf. (C.18),

$$12Y' + M = \left\lceil \frac{65}{67} n + \alpha' \right\rceil = \left\lceil \frac{65n + 67\alpha'}{67} \right\rceil = \left\lceil \frac{65n + \beta}{67} \right\rceil. \tag{C.59}$$

Hence, with our new definition of β, (5.19)–(5.21) are valid and yield the same correspondence between (Y, M) and true month n as before (but we have to remember that the leap month now is the later of two months with the same number).

For the converse problem, to find the true month n given (Y, M, ℓ), note first that if the month is regular ($\ell = \mathit{false}$), then the month before has number $M - 1$ (interpreted modulo 12) and true month count $n - 1$, and thus by (C.51)

$$mean\_sun(n) \le Y - Y_0 + p_M < mean\_sun(n+1). \tag{C.60}$$

[72]This implies that (C.8) and (C.9) have to be slightly modified to

$$p_1 < s_0 + s_1 \le p_{13} = p_1 + 1 \tag{C.53}$$

and, using (C.56),

$$0 < \alpha' = \alpha - \frac{2}{67} \le 12; \tag{C.54}$$

this is no difference in practice since the epochs traditionally are chosen at the beginning of month 3 (*nag pa*), so α (or rather $\alpha - 2/67$ in this case) is between 1 and 3, see Remark 5.

Hence, n is the largest integer such that (C.20)–(C.21) hold, just as in Appendix C. For a leap month ($\ell = true$), this value of n now should be increased by 1. Hence, in all cases, (C.22) holds if $-[\ell]$ is changed to $+[\ell]$. Similarly, (C.23) is modified to

$$n = \left\lfloor \frac{12(Y - Y_0) + M - \alpha + (1 - 12s_1)[\ell]}{12s_1} \right\rfloor; \tag{C.61}$$

as before this gives the same result for $(Y, M, true)$ and $(Y, M, false)$ if there is no leap month M. (If $\ell = false$, (C.61) agrees with (C.22); if $\ell = true$ it gives 1 less than (C.22) applied to $(Y, M + 1, false)$.)

Also (C.25) holds with if $-[\ell]$ is changed to $+[\ell]$. There is now a leap month M in year Y if and only if the true month count jumps by 2 from the regular month $(Y, M, false)$ to the next $(Y, M+1, false)$. It follows from (the modified) (C.25), cf. (C.26) and the argument after it, that this happens exactly when $2(M'+1) - \lceil 67\alpha \rceil \equiv 0$ or $1 \pmod{65}$. With our new definition (C.58) of β, this means that the rules (C.27) and the equivalent (5.24) still hold.

If there is a leap month (Y, M), then (5.25) yields the intercalation index for the preceding regular month, so for the leap month we have to increase the result by 2, yielding $(2M' + \beta^* - 4) \bmod 65$. By (C.27), this shows that (5.26) is modifed when the leap month comes after the regular month with the same number; a leap month now has intercalation index

$$(\beta + \beta^* - 4) \bmod 65 \qquad \text{or} \qquad (\beta + \beta^* - 3) \bmod 65. \tag{C.62}$$

Finally, (5.27)–(5.36) were derived from (5.24), and thus hold in the present case too.

C.6. **Bhutanese definition points.** The value $\beta = 191 \equiv 61 \pmod{65}$ for the Bhutanese version of the calendar in Appendix A.4 (and the epoch used there) is consistent with the definition points defined by, for example,[73]

$$p_1 = 23, 10, 30\ \ (27, 60, 60) = \frac{103}{120} = 309° \tag{C.63}$$

and thus

$$p_0 = p_1 - \frac{1}{12} = 20, 55, 30\ \ (27, 60, 60) = \frac{31}{40} = 279°. \tag{C.64}$$

Again, in accordance with Remark 28 and (C.8), we have to add 1 to the value of s_0 given in (A.20); we thus now use

$$s_0 = 0, 24, 10, 50\ \ (27, 60, 60, 67) + 1 = 1 + \frac{1}{67}, \tag{C.65}$$

which together with (C.64) yields, by (C.12),

$$\alpha = \frac{1929}{670} = 2 + \frac{589}{670} \tag{C.66}$$

[73]This is our own choice. Any p_1 with $\frac{690}{804} < p_1 < \frac{691}{804}$ yields the same result.

and thus by (C.58) $\beta = 193 - 2 = 191$ as asserted above. By (A.24), this is consistent with the value $\beta^* = 2$ used in Bhutan (for the epoch above).

However, this is *not* consistent with the original text by Lhawang Lodro and published calendars, which give the definition points as [Henning, personal communication]

$$p_1 = 23, 15 \ (27, 60) = \frac{31}{36} = 310^\circ \tag{C.67}$$

and thus

$$p_0 = p_1 - \frac{1}{12} = 21, 0 \ (27, 60) = \frac{7}{9} = 280^\circ, \tag{C.68}$$

which together with (A.20) would yield, by (C.12) and (C.58),

$$\alpha = \frac{572}{201} = 2 + \frac{170}{201} \tag{C.69}$$

and $\beta = 189 \equiv 59 \pmod{65}$, which does not agree with (A.24), nor with actual leap months.[74] I have no explanation for this. One possibility is that at some time, either the definition points or the epoch value for true month and intercalation index was adjusted, ignoring the fact that they in theory are connected and that one cannot be changed without changing the other. (Cf. the similar problem in Footnote 71.)

Note also that, as said in Appendix A.4, the winter solstice is defined as when the mean solar longitude is 250°; this is the definition point p_{11} if we use (C.67), but not if we use (C.63).

C.7. **Karaṇa definition points.** The leap month rule (A.37) may seem to be in accordance with the definition points given by

$$p_1 = 22, 30 \ (27, 60) = \frac{5}{6} = 300^\circ. \tag{C.70}$$

Thus the definition points are at the beginning of zodiacal signs. In particular, the definition point for *nag pa* (*Caitra*) is $p_3 = 0^\circ$, the first point of Aries.[75] This would give

$$p_0 = \frac{9}{12} = \frac{3}{4} = 270^\circ \tag{C.71}$$

[74]For example, month 5 was a leap month in 2008. (This is shown by the Election act [29, National Assembly, Acts] who is enacted the "26th Day of the Second 5th Month of the Earth Male Rat Year corresponding to the 28th Day of the 7th Month of the Year 2008".) A simple calculation shows that at the beginning of the second (leap) month 5, the mean solar longitude (7.5) was $5, 14, 19, 47 \ (27, 60, 60, 67) = 13/67 = 69.851^\circ$, so if the definition point $p_5 = 5, 15 \ (27, 60) = 70^\circ$, as implied by (C.67), then the mean sun reaches p_5 just after the beginning of the month, which therefore would not be a leap month. However, with (C.63) we have $p_5 = 69^\circ$, passed at the end of the preceding month, and the mean sun does not pass any definition point during the month; hence the month is (correctly) a leap month. (The next definition point is $p_6 = 99^\circ$ and the mean sun has longitude 98.955° at the end of the month.)

[75]This agrees with Indian calendars, see Remark 27, and seems to be the most natural choice.

and (A.30), (C.12) and (C.58) would yield

$$\alpha = \frac{403}{135} = 2 + \frac{133}{135} \tag{C.72}$$

and $\beta = 199 \equiv 4 \pmod{65}$, in accordance with Appendix A.5, see also (C.62).

However, the theory of definition points above is based on the standard value (7.6) for the mean solar motion s_1, and is thus logically inconsistent with the slightly larger value (A.26) used in the Kālacakra Tantra; the mean sun will move a little faster, and if the leap year rule (A.37) is used, the mean sun will advance relative to the position indicated by the number of the month. (But the difference is small (0.003%), and amounts to about 1 sign (= 1 month) in 2600 years.)[76] Conversely, if the leap month rule (C.5) is used with any fixed definition points and s_1 is the karaṇa value (A.26), then the leap months will not follow a strict 65 year cycle. See further [24, On intercalary months]. In order to use the formulas in Section C.1, or rather their modifications described in Section C.5 with leap months numbered by the preceding month, we thus have to use the siddhānta value $s_1 = 65/804$ from (7.6) and not the karaṇa value (A.26), which seems illogical.

As far as I know, definition points are not mentioned explicitly in the Kālacakra Tantra. Nevertheless, it is obvious that the idea was to insert leap months so that the year on the average agrees with the tropical solar year, and thus the sun has more or less the same longitude for the, say, first month any year. The simple relation (5.1) was at some stage chosen as an approximation, perhaps believed to be exact, and the mean solar motion s_1 was at some stage chosen to be the siddhānta value 65/804; these choices are logically connected, but I do not know whether that was realized when these values were introduced in the calendar, and whether these choices were made together or at different times (as the karaṇa system suggests).

[76] The relation (5.1) is thus only an approximation in the original Kālacakra Tantra version, but it has been treated as exact in later versions of the calendar.

Appendix D. Planets

The positions of the planets are, for astrological purposes, calculated by the following procedure, also based on the Kālacakra Tantra. The calculations yield the longitudes at the end of the calendar (=solar) day.

In modern terms, the (geocentric) longitude of a planet is found by first calculating the heliocentric longitude and the longitude of the sun, and then combining them (using trigonometry). The Tibetan calculations do effectively this, although the theory behind (which is not explicitly mentioned) rather is an old Indian system using epicycles similar to the Ptolemaic system [7, p. 57]. As in the calculations of *true_date* in Section 7, the calculations use special tables as substitutes for trigonometrical calculations. For a more detailed description of the traditional calculation, including conversions between different radices, se Henning [7, Chapter II], which also includes discussions of the astronomical background and geometry, and of the accuracy of the formulas.

The constants given below are (as the main text of the present paper) for the Phugpa tradition; the Tsurphu and other versions use their own slightly different epoch values, see [7, p. 340] and [24, Epoch data].

D.1. **General day.** The general day (*spyi zhag*) is a count of (solar) days since the epoch. It is thus simply given by

$$general\_day = \mathrm{JD} - epoch. \tag{D.1}$$

For E1927 (as in [7]), the epoch is JD 2424972 (Friday, 1 April 1927; RD 703547) and thus

$$general\_day = \mathrm{JD} - 2424972. \tag{D.2}$$

Remark 32. The traditional method to calculate the general day is to first find the true month n as in Section 5, see (5.3) and (5.9) or (5.10), and then use it to find the number of elapsed lunar days since the epoch, ld say, by $ld = 30n + D$, where D is the date.

Next, ld is multiplied by the ratio 11135/11312 in (12.2) (the mean length of a lunar day). A small constant gd_0 is subtracted; for E1927, $gd_0 = 2,178\ (64,707) = 199/5656$. This constant is the fraction of the solar day remaining at the end of the mean lunar day at the epoch. The difference $\frac{11135}{11312} ld - gd_0$ is rounded up to the nearest integer, and one sets provisionally

$$general\_day = \left\lceil \frac{11135}{11312} ld - gd_0 \right\rceil. \tag{D.3}$$

This gives an approximation of the general day, but may be off by a day because only the mean motion of the moon is considered (this is equivalent to ignoring the corrections in (7.22)). Hence the day of week is calculated by the simple formula

$$\big(general\_day + wd_0\big) \mod 7 \tag{D.4}$$

where the constant wd_0 is the day of week at the epoch; for E1927, $wd_0 = 6$. If the value in (D.4) differs from the correct day of week, then *general_day* is adjusted by ± 1 (the error cannot be larger) so that (D.4) becomes correct. (Since this final check and correction has to be done, one could as well ignore the subtraction of gd_0 above, but it is traditionally done.)

D.2. **Mean heliocentric motion.** The mean heliocentric position of a planet is represented by an integer for each plane called *particular day* (*sgos zhag*), calculated as

$$particular\_day = \begin{cases} (100 \cdot general\_day + pd_0) \bmod R, & \text{Mercury,} \\ (10 \cdot general\_day + pd_0) \bmod R, & \text{Venus,} \\ (general\_day + pd_0) \bmod R, & \text{Mars, Jupiter, Saturn,} \end{cases} \tag{D.5}$$

where the modulus R and the epoch value pd_0 are given in Table 11. The periods of the planets are thus, exactly, 87.97, 224.7, 687, 4332 and 10766 days, respectively. (Modern astronomical values are 87.9684, 224.695, 686.93, 4330.6, and 10746.9 [3, Table 15.6].)

The particular day is 0 at the first point of Aries (i.e., when the longitude is 0), and thus the mean heliocentric longitude is

$$mean\_helio\_long = \frac{particular\_day}{R} \pmod 1. \tag{D.6}$$

This is traditionally expressed in the radices $(27, 60, 60, 6, r)$ with the final radix r depending on the planet and given in Table 11. (Note that r always is a divisor of R.)

	Mercury	Venus	Mars	Jupiter	Saturn
R	8797	2247	687	4332	10766
pd_0 (E1927)	4639	301	157	3964	6286
r	8797	749	229	361	5383
birth_sign	11/18	2/9	19/54	4/9	2/3
trad. (27,60)	16,30	6,0	9,30	12,0	18,0

TABLE 11. Constants for planets.

D.3. **Mean longitude of the sun.** The mean longitude of the sun (at the end of the calendar day) is calculated from scratch and not using the calculation of *mean_sun* (at the end of the lunar day) in Section 7, but the results are consistent [7, pp. 87–88]. (The Tsurphu tradition uses the calculation of *mean_sun*, and in one version *true_sun* [7, p. 341].) The formula used is

$$mean\_solar\_long = s_1' \cdot general\_day + s_0', \tag{D.7}$$

where

$$s_1' = 0,4,26,0,93156\ (27,60,60,6,149209) = \frac{18382}{6714405} \quad \Big(= \frac{11312}{11135} s_2\Big) \tag{D.8}$$

and the epoch value is, for E1927,

$$s_0' = 25,9,20,0,97440\ (27,60,60,6,149209) = 1 - \frac{458672}{6714405}. \tag{D.9}$$

	Mercury	Venus	Mars	Jupiter	Saturn
0	0	0	0	0	0
1	10	5	25	11	22
2	17	9	43	20	37
3	20	10	50	23	43

TABLE 12. Equation for planets.

	Mercury	Venus	Mars	Jupiter	Saturn
0	0	0	0	0	0
1	16	25	24	10	6
2	32	50	47	20	11
3	47	75	70	29	16
4	61	99	93	37	20
5	74	123	114	43	24
6	85	145	135	49	26
7	92	167	153	51	28
8	97	185	168	52	28
9	97	200	179	49	26
10	93	208	182	43	22
11	82	202	171	34	17
12	62	172	133	23	11
13	34	83	53	7	3

TABLE 13. Final correction for planets.

D.4. **Slow longitude and step index.** The remaining calculations are based on the mean heliocentric longitude and the mean solar longitude, but these quantities are treated differently for the inner (or "peaceful") planets Mercury and Venus and for the outer (or "wrathful") planets Mars, Jupiter, Saturn. The reason is that the mean motion of an outer planet is given by the mean heliocentric longitude, while the mean motion of an inner planet is given by the mean longitude of the sun. In both cases, this main term is called the *mean slow longitude* (*dal ba*), while the other quantity is called the *step index* (*rkang 'dzin*). In other words, for the inner planets

$$\begin{aligned} mean\_slow\_long &= mean\_solar\_long, \\ step\_index &= mean\_helio\_long \end{aligned}$$

and for the outer planets

$$\begin{aligned} mean\_slow\_long &= mean\_helio\_long, \\ step\_index &= mean\_solar\_long. \end{aligned}$$

Next, cf. the calculations for the moon and sun in Section 7, the anomaly is calculated by

$$anomaly = mean\_slow\_long - birth\_sign \pmod 1, \tag{D.10}$$

where the "birth-sign" (*skyes khyim*) is given in Table 11, both as a rational number and in the traditional form in mansions (27,60). The anomaly is used to find the equation from

$$equ = planet\_equ\_tab(12 \cdot anomaly), \tag{D.11}$$

where $planet\_equ\_tab(i)$ is given in Table 12 for $i = 0, \dots, 3$, which extends by the symmetry rules $planet\_equ\_tab(6 - i) = planet\_equ\_tab(i)$ and $planet\_equ\_tab(6 + i) = -planet\_equ\_tab(i)$; linear interpolation is used beween integer arguments. Finally, the true slow longitude (*dal dag*) is given by

$$true\_slow\_long = mean\_slow\_long - equ/(27 \cdot 60). \tag{D.12}$$

D.5. **Geocentric longitude.** The final step is to combine the true slow longitude and the step index. First, the difference of these is found:

$$diff = step\_index - true\_slow\_long. \tag{D.13}$$

This is used to find a correction by another table look-up:

$$corr = planet\_corr\_tab(27 \cdot diff) \pmod 1, \tag{D.14}$$

where $planet\_corr\_tab(i)$ is given in Table 13 for $i = 0, \dots, 13$, which extends by the symmetry rules $planet\_corr\_tab(27 - i) = -planet\_corr\_tab(i)$ and $planet\_corr\_tab(27 + i) = planet\_corr\_tab(i)$; as always, linear interpolation is used beween integer arguments.

Finally the (geocentric, or *fast*) longitude (*myur ba*) is given by

$$fast\_long = true\_slow\_long + corr/(27 \cdot 60). \tag{D.15}$$

D.6. **Rahu.** *Rahu* is the name of the nodes of the lunar orbit, i.e., the intersections of the orbit and the ecliptic. More precisely, the ascending node is called the *Head of Rahu* and the descending node is called the *Tail of Rahu*. In Tibetan (as in Indian) astrology, Rahu is treated as a planet, or perhaps two planets. Rahu is further essential for prediction of eclipses [7, Chapter III].

Rahu has a slow motion that is retrograde (i.e., to the west, with decreasing longitude, unlike the real planets). In the Tibetan system, the period is exactly 230 lunar months = 6900 lunar days.

Remark 33. In calendar (solar) days, this is, cf. (12.2),

$$\frac{11135}{11312} \cdot 6900 = \frac{19207875}{2828} \approx 6792.035 \text{ days} \tag{D.16}$$

or, cf. (12.3),

$$\frac{11135}{11312} \cdot 6900 s_2 = 6900 \cdot \frac{s_1}{30} = \frac{7475}{402} \approx 18.5945 \text{ Tibetan years.} \tag{D.17}$$

The modern astronomical value is 6798 days = 18.61 Gregorian years [3, Table 15.4].

To find the position of Rahu at day D, month M, year Y, one first calculates the true month count n, for example by (5.2) and (5.10). Next, the number x of elapsed lunar days since the Head of Rahu had longitude 0 is calculated by, cf. the calculation of ld in Remark 32,

$$x = 30(n + rd_0) + D = 30n + D + 30rd_0 = ld + 30rd_0, \tag{D.18}$$

where rd_0 is an epoch value; $rd_0 = 187$ for E1927. (For E1987, $rd_0 = 10$.) Finally, the longitudes of the head and tail of Rahu are given by

$$rahu\_head\_long = -\frac{x}{6900} \pmod 1, \tag{D.19}$$

$$rahu\_tail\_long = rahu\_head\_long + \frac{1}{2} \pmod 1. \tag{D.20}$$

Remark 34. Since Rahu has a retrograde motion, these are decreasing. In traditional calculations, $-rahu\_head\_long = x/6900 \pmod 1$ is called the longitude of the *Source of Rahu.* The longitudes are traditionally expressed in $(27, 60, 60, 6, 23)$, and $1/6900$ is then written as $0, 0, 14, 0, 12$.

Appendix E. Further astrological calculations

As explained in Section 4, each year has a name consisting of an element and an animal. For astrological[77] purposes (in the Chinese or elemental astrological system), there are many further associations, assigning a year, month, lunar day or solar day one of the 12 animals in Table 3, one of the 5 elements in Table 14, one of the 8 trigrams in Table 15, or one of the 9 numbers $1, \dots, 9$ in Table 16; as shown in Tables 3, 14, 15, 16, these have further associations to, for example, numbers, colours and directions. There are also further attributes in the Indian system. This appendix is only a brief introduction, and only some of the attributes and their connections are mentioned here. See Tseng [17, 18] and Henning [7] for further details.

Remark 35. The order of the directions in Table 16 may seem jumbled, yet there is method in it. The 9 numbers are often arranged in a 3×3 square according to the directions as in Table 17 (upside down the standard Western orientation), and then the numbers form a magic square with all rows, columns and diagonals summing to 15.

In formulas below, Y is the Gregorian number of the year, see Section 4. By Section 4, year Y has in the Chinese 60 year cycle number

$$(Y-3) \ \text{amod} \ 60, \tag{E.1}$$

and hence numbers $(Y-3) \ \text{amod} \ 10$ and $(Y-3) \ \text{amod} \ 12$ in the Chinese 10 and 12 year cycles.

E.1. **Attributes for years.**

Elements. Each year is given 4 or 5 elements: the power element (*dbang thang*), life element (*srog*), body element (*lus*), fortune element (*klung rta*), and sometimes also the spirit element (*bla*).

The *power element* is the element associated to the celestial stem, see Table 2; it is thus repeated in a cycle of 10 years, with each element repeated 2 consecutive years, in the standard order wood, fire, earth, iron, water (the order in Table 14). As said above, year Y is $(Y-3) \ \text{amod} \ 10$ in the Chinese 10 year cycle, and by Tables 2 and 14, its power element has number

$$\left\lceil \frac{(Y-3) \ \text{amod} \ 10}{2} \right\rceil = \left\lceil \frac{Y-3}{2} \right\rceil \ \text{amod} \ 5. \tag{E.2}$$

The *life element* is repeated in a cycle of 12 years, and is thus determined by the animal name of the year. The list is given in Table 18. Note that each third year is earth (the years $\equiv 2 \bmod 3$); the remaining four elements come repeated 2 years each, in the same (cyclic) order wood, fire, iron, water as the power element.

The *fortune element* is repeated in a cycle of 4 years, in the order wood, water, iron, fire. (Earth is not used. Note that the order of the four used

[77]See Footnote 3.

number	element	colour
1	wood	green (blue)
2	fire	red
3	earth	yellow
4	iron	white
5	water	dark blue (black)

Table 14. The 5 elements and the associated numbers and colours.

	binary	trigram	Tibetan	Chinese	direction	element
1	5	☲	li	lí	S	fire
2	0	☷	khon	kūn	SW	earth
3	6	☱	dwa	duì	W	iron
4	7	☰	khen	qián	NW	sky
5	2	☵	kham	kǎn	N	water
6	1	☶	gin	gèn	NE	mountain
7	4	☳	zin	zhèn	E	wood
8	3	☴	zon	xùn	SE	wind

Table 15. The 8 trigrams with some attributes. The ordering is the standard (King Wen, Later Heaven) order; the numbering is perhaps not traditional, and the binary coding (reading the trigrams bottom-up) is mathematically natural but not traditionally used.

	colour	element	direction
1	white	iron	N
2	black	water	SW
3	blue	water	E
4	green	wood	SE
5	yellow	earth	Centre
6	white	iron	NW
7	red	fire	W
8	white	iron	NE
9	red	fire	S

Table 16. The 9 numbers and their attributes.

4	9	2
3	5	7
8	1	6

SE	S	SW
E	C	W
NE	N	NW

Table 17. A magic square of numbers and their directions

year		animal	life element	spirit element
1	10	Mouse	water	iron
2	11	Ox	earth	fire
3	12	Tiger	wood	water
4	1	Rabbit	wood	water
5	2	Dragon	earth	fire
6	3	Snake	fire	wood
7	4	Horse	fire	wood
8	5	Sheep	earth	fire
9	6	Monkey	iron	earth
10	7	Bird	iron	earth
11	8	Dog	earth	fire
12	9	Pig	water	iron

TABLE 18. The 12 year cycle of life elements. The first number on each line shows the year mod 12 counted from the start of a Chinese cycle; the second shows the year mod 12 counted from the start of a Prabhava cycle.

elements is different from the order used for the power and life elements.) Since 4 is a divisor of 12, the fortune element is determined by the animal name of the year. See Table 19.

year		animals	fortune element
1	2	Mouse, Dragon, Monkey	wood
2	3	Ox, Snake, Bird	water
3	4	Tiger, Horse, Dog	iron
4	1	Rabbit, Sheep, Pig	fire

TABLE 19. The 4 year cycle of fortune elements. The first number on each line shows the year mod 4 counted from the start of a Chinese cycle; the second shows the year mod 4 counted from the start of a Prabhava cycle.

The *body element* is calculated in two steps. (See Henning [7] for traditional ways of doing the calculations.) First, an element is determined by the animal name; I do not know any name for this intermediate element so let us call it x. The element x is repeated in a cycle of 6 years, with the 3 elements wood, water, iron repeated 2 years each, see Table 20. Then count the number of steps from x to the power element y, or equivalently, replacing the elements by the corresponding numbers in Table 14, calculate $y - x \bmod 5$. Finally, this difference determines the body element by Table 21. (Note that the order in this table is not the standard one.)

Since both x and the power element y are repeated 2 consecutive years each, the same is true for the body element. If we consider only the even

year		animals	element	number
1	4	Mouse, Horse	wood	1
2	5	Ox, Sheep	wood	1
3	6	Tiger, Monkey	water	5
4	1	Rabbit, Bird	water	5
5	2	Dragon, Dog	iron	4
6	3	Snake, Pig	iron	4

TABLE 20. The 6 year cycle of the element x and its number in the body element calculation. The first number on each line shows the year mod 6 counted from the start of a Chinese cycle; the second shows the year mod 6 counted from the start of a Prabhava cycle.

$y - x \mod 5$	body element
0	iron
1	water
2	fire
3	earth
4	wood

TABLE 21. The final step in the body element calculation.

years, say, then x is by Table 20 given by 1, 0, −1, 1, 0, −1, ... (mod 5), while y by Table 2 is given by 1, 2, 3, 4, 5, 1, 2, 3, ... Hence, $y - x \mod 5$ repeats in a cycle of length 15: 0, 2, 4, 3, 0, 2, 1, 3, 0, 4, 1, 3, 2, 4, 1 (with differences $+2, +2, -1, +2, +2, -1, \dots$).

Consequently, the body element repeats in a cycle of 30 years, with each element repeated for 2 consecutive years. The full cycle is given in Table 22.

The *spirit element* is the element preceding the life element in the standard order, see Table 18. Thus it too is repeated in a cycle of 12 years, and is determined by the animal name of the year. Each third year is fire (the years $\equiv 2 \bmod 3$) and the remaining four elements come repeated 2 years each, in the standard (cyclic) order.

Since all elements for the year have periods dividing 60, they all repeat in the same order in each 60 year cycle as is shown in Table 23.

Numbers. Each year is also associated with a set of three numbers, in the range $1, \dots, 9$, the *central number*, the *life number* and the *power number*. (These numbers are used for persons born that year. The central number is also called the *body number* or *birth number*.) The numbers are associated with elements and directions according to Table 16. The numbers decrease by 1 (mod 9) for each new year, and thus repeat in a cycle of 9 years; they are given simply by, for Gregorian year Y:

$$\text{central number} = (2 - Y) \ \text{amod}\ 9, \tag{E.3}$$

$$\text{life number} = (\text{central number} - 3) \ \text{amod}\ 9 = (8 - Y) \ \text{amod}\ 9, \tag{E.4}$$

$$\text{power number} = (\text{central number} + 3) \ \text{amod}\ 9 = (5 - Y) \ \text{amod}\ 9. \tag{E.5}$$

Since 9 does not divide 60, these numbers do not follow the 60 year cycle. The period for repeating all elements and numbers is 180 years, i.e., 3 cycles of 60 years.

year		name (power)	life	body	fortune	spirit
1	58	Wood–Mouse	water	iron	wood	iron
2	59	Wood–Ox	earth	iron	water	fire
3	60	Fire–Tiger	wood	fire	iron	water
4	1	Fire–Rabbit	wood	fire	fire	water
5	2	Earth–Dragon	earth	wood	wood	fire
6	3	Earth–Snake	fire	wood	water	wood
7	4	Iron–Horse	fire	earth	iron	wood
8	5	Iron–Sheep	earth	earth	fire	fire
9	6	Water–Monkey	iron	iron	wood	earth
10	7	Water–Bird	iron	iron	water	earth
11	8	Wood–Dog	earth	fire	iron	fire
12	9	Wood–Pig	water	fire	fire	iron
13	10	Fire–Mouse	water	water	wood	iron
14	11	Fire–Ox	earth	water	water	fire
15	12	Earth–Tiger	wood	earth	iron	water
16	13	Earth–Rabbit	wood	earth	fire	water
17	14	Iron–Dragon	earth	iron	wood	fire
18	15	Iron–Snake	fire	iron	water	wood
19	16	Water–Horse	fire	wood	iron	wood
20	17	Water–Sheep	earth	wood	fire	fire
21	18	Wood–Monkey	iron	water	wood	earth
22	19	Wood–Bird	iron	water	water	earth
23	20	Fire–Dog	earth	earth	iron	fire
24	21	Fire–Pig	water	earth	fire	iron
25	22	Earth–Mouse	water	fire	wood	iron
26	23	Earth–Ox	earth	fire	water	fire
27	24	Iron–Tiger	wood	wood	iron	water
28	25	Iron–Rabbit	wood	wood	fire	water
29	26	Water–Dragon	earth	water	wood	fire
30	27	Water–Snake	fire	water	water	wood
31	28	Wood–Horse	fire	iron	iron	wood
32	29	Wood–Sheep	earth	iron	fire	fire
33	30	Fire–Monkey	iron	fire	wood	earth

34	31	Fire–Bird	iron	fire	water	earth
35	32	Earth–Dog	earth	wood	iron	fire
36	33	Earth–Pig	water	wood	fire	iron
37	34	Iron–Mouse	water	earth	wood	iron
38	35	Iron–Ox	earth	earth	water	fire
39	36	Water–Tiger	wood	iron	iron	water
40	37	Water–Rabbit	wood	iron	fire	water
41	38	Wood–Dragon	earth	fire	wood	fire
42	39	Wood–Snake	fire	fire	water	wood
43	40	Fire–Horse	fire	water	iron	wood
44	41	Fire–Sheep	earth	water	fire	fire
45	42	Earth–Monkey	iron	earth	wood	earth
46	43	Earth–Bird	iron	earth	water	earth
47	44	Iron–Dog	earth	iron	iron	fire
48	45	Iron–Pig	water	iron	fire	iron
49	46	Water–Mouse	water	wood	wood	iron
50	47	Water–Ox	earth	wood	water	fire
51	48	Wood–Tiger	wood	water	iron	water
52	49	Wood–Rabbit	wood	water	fire	water
53	50	Fire–Dragon	earth	earth	wood	fire
54	51	Fire–Snake	fire	earth	water	wood
55	52	Earth–Horse	fire	fire	iron	wood
56	53	Earth–Sheep	earth	fire	fire	fire
57	54	Iron–Monkey	iron	wood	wood	earth
58	55	Iron–Bird	iron	wood	water	earth
59	56	Water–Dog	earth	water	iron	fire
60	57	Water–Pig	water	water	fire	iron

Table 23: The 60 year cycle of combinations of different elements. The first number on each line shows the year mod 60 counted from the start of a Chinese cycle; the second shows the year mod 60 counted from the start of a Prabhava cycle. The power element is the first part of the name.

E.2. **Attributes for months.** Each regular calendar month is given attributes as follows. A leap month is given the same attributes as the regular month with the same number.

Animals. The 12 months are assigned one each of the 12 animals, in standard order but with different starting points in the Phugpa and Tsurphu traditions, see Section 5(ii). The full lists are given in Table 24. (The Tsurphu system is the same as the Chinese, see e.g. [5, §126] and [20].[78])

[78]The Chinese astrological system in [20] assigns animals, gender and elements to solar months, defined by the minor solar terms, and not to the (lunar) calendar months. However, there are different astrological traditions in China.

year		power	body element
1	28	wood	iron
2	29	wood	iron
3	30	fire	fire
4	1	fire	fire
5	2	earth	wood
6	3	earth	wood
7	4	iron	earth
8	5	iron	earth
9	6	water	iron
10	7	water	iron
11	8	wood	fire
12	9	wood	fire
13	10	fire	water
14	11	fire	water
15	12	earth	earth
16	13	earth	earth
17	14	iron	iron
18	15	iron	iron
19	16	water	wood
20	17	water	wood
21	18	wood	water
22	19	wood	water
23	20	fire	earth
24	21	fire	earth
25	22	earth	fire
26	23	earth	fire
27	24	iron	wood
28	25	iron	wood
29	26	water	water
30	27	water	water

TABLE 22. The 30 year cycle of the body element. The first number on each line shows the year mod 30 counted from the start of a Chinese cycle; the second shows the year mod 30 counted from the start of a Prabhava cycle.

Gender. Each month is given the gender (male or female) associated to its animal. As Tables 3 and 24 show, this simply means (both in the Phugpa and Tsurphu traditions) that odd-numbered months are male and even-numbered female. (This is in accordance with the general Chinese principle that odd numbers are male (*yang*) and even numbers female (*yin*).)

Elements. Each month is assigned one of the 5 elements in Table 14.

In the Tsurphu version, the months cycle continuously through the cycle in Table 2, year after year. This combines with the 12 month cycle for animals to a 60 month cycle, exactly as for years, see Section 4 and Tables 1 and 10. Since each element is repeated 2 months in the 10 month cycle, the element of the first month advances one step in the list Table 14 each year. More precisely, the first month (Tiger) Gregorian year Y has month element number $(Y-2)$ amod 5. (This is exactly as in the Chinese calendar, see e.g. [5, §126] and [20].)

In the Phugpa version, the Tiger month (the first in the Chinese calendar), which is month 11 *the preceding year* (see Table 24 and Section 5(ii)), gets the element following the element of the year given in Table 2, using the standard element order in Table 14. (The element following another, x, in this cyclic order is called the *son* of x.) By (E.2), this is element $\lceil (Y-1)/2 \rceil$ amod 5. Having determined the element of the Tiger month, the elements for the 12 month period starting with it are assigned in the same pattern as for years in Table 2: each element is repeated 2 months, the first male and the second female, and then followed by the next element. (This is the same sequence as for Tsurphu, but only for 12 month periods.)

We thus obtain the following formulas for the number of the element for month M year Y.

Phugpa:

$$\begin{cases} \left(\left\lceil \frac{Y-1}{2} \right\rceil + \left\lfloor \frac{M+1}{2} \right\rfloor\right) \text{ amod } 5, & 1 \le M \le 10, \\ \left(\left\lceil \frac{Y}{2} \right\rceil + \left\lfloor \frac{M-11}{2} \right\rfloor\right) \text{ amod } 5, & 11 \le M \le 12. \end{cases} \tag{E.6}$$

Tsurphu:

$$\left(Y - 2 + \left\lfloor \frac{M-1}{2} \right\rfloor\right) \text{ amod } 5. \tag{E.7}$$

month	Phugpa	Tsurphu	gender
1	Dragon	Tiger	male
2	Snake	Rabbit	female
3	Horse	Dragon	male
4	Sheep	Snake	female
5	Monkey	Horse	male
6	Bird	Sheep	female
7	Dog	Monkey	male
8	Pig	Bird	female
9	Mouse	Dog	male
10	Ox	Pig	female
11	Tiger	Mouse	male
12	Rabbit	Ox	female

Table 24. The animals for the months.

Numbers. Only Tsurphu calendars give one of the 9 numbers to each month. The number decreases by 1 (mod 9) for each month (except leap months), and thus by $12 \equiv 3 \pmod 9$ for each year. Month M year Y has number

$$\big(3 - (12Y + M)\big) \ \operatorname{amod}\ 9. \tag{E.8}$$

The Tsurphu rules for animal, gender and element agree with the rules in the Chinese calendar, see [20]. Mongolia uses also the same rules, see Appendix A.3.

E.3. **Attributes for lunar days.**

Animal. Each lunar day has an animal. These repeat in the usual cycle of 12, see Table 3, with each odd-numbered month starting with Tiger (number 3 in Table 3) and each even-numbered month starting with Monkey (number 9 in Table 3); since there are exactly $30 \equiv 6 \pmod{12}$ lunar days in each month, the animals thus repeat in a continuous cycle broken only at leap months, where the animals are repeated in the same order in the leap month and the following regular month and there is a discontinuity between the two months. The number (in Table 3) of the animal for lunar day D in month M is thus

$$(D + 30M + 8) \ \operatorname{amod}\ 12 = (D + 6M + 8) \ \operatorname{amod}\ 12. \tag{E.9}$$

Element. Each lunar day has an element; these cycle inside each month in a cycle of 5 in the usual order given in Table 14 (without repetitions as for years and months), beginning with the element following the element of the month calculated above. If the month has element x, then lunar day D of the month thus has element $(x + D)$ amod 5. (There is thus often a jump in the sequence at the beginning of a new month.)

Trigram. The trigrams for lunar days cycle in the usual cycle of 8 in Table 15; as for the animals, this is continuous across months except for leap months. A Tiger month begins with trigram number 1, Li. Thus the trigram for lunar day D in a month with animal number A (in Table 3) has number

$$(D+30(A-3)) \ \operatorname{amod}\ 8 = (D-2A-2) \ \operatorname{amod}\ 8 = (D+6A+6) \ \operatorname{amod}\ 8. \tag{E.10}$$

Hence months Tiger, Horse, Dog begin with Li; Rabbit, Sheep, Pig begin with Zin; Mouse, Dragon, Monkey begin with Kham; Ox, Snake, Bird begin with Dwa.

Number. The nine numbers (with their colours) for lunar days cycle forward in the usual cycle of 9 in Table 16; as for the animals, this is continuous across months except for leap months. A Tiger month begins with number 1 (white). Thus the number for lunar day D in a month with animal number A (in Table 3) has number

$$(D + 30(A - 3)) \ \operatorname{amod}\ 9 = (D + 3A) \ \operatorname{amod}\ 9. \tag{E.11}$$

Hence, months Tiger, Snake, Monkey, Pig begin with 1 (white); Mouse, Rabbit, Horse, Bird begin with 4 (green); Ox, Dragon, Sheep, Dog begin with 7 (red).

E.4. **Attributes for calendar days.** As explained in Sections 6, 8 and 9, each calendar (solar) day has a number (the date) and a day of week.

Each (calendar) day is also given an element (and its colour according to Table 14, gender, animal, trigram and number from the Chinese system; these are simple cyclic with periods 10, 2, 12, 8, 9, respectively, with the elements repeated twice each as for years in Table 2. At least the element, gender and animal are the same as in the Chinese calendar for the same day, see [20].

Element. The element corresponds to number JD amod 10 in the (Chinese) cycle in Table 2. The number of the element in Table 14 is thus

$$\left\lceil \frac{\mathrm{JD} \ \ \mathrm{amod} \ \ 10}{2} \right\rceil = \left\lceil \frac{\mathrm{JD}}{2} \right\rceil \ \ \mathrm{amod} \ \ 5. \tag{E.12}$$

Gender. The gender is male when JD is odd, and female when JD is even.[79]

Animal. The animal has number $(\mathrm{JD}+2)$ amod 12 in the (Chinese) cycle in Table 3.

Trigram. The trigram has number $(\mathrm{JD}+2)$ amod 8 in Table 15.

Number. The number in Table 16 is $(-\mathrm{JD})$ amod 9.

Remark 36. Henning [7, pp. 208–209] describe these using different, and presumably traditional, numberings for the 10 day cycle of elements and the 12 day cycle of animals; his numbers (for the same element and animal as given above) are $(\mathrm{JD}-2)$ amod 10 and $(\mathrm{JD}-2)$ amod 12. He further calculates the number as $10-((\mathrm{JD}+1)$ amod $9)$.

The element calculated in (E.12) is the power element of the day. Exactly as for years, see above, further elements (life, fortune, body, spirit) can be calculated from the animal–element pair; these elements thus follow a cycle of 60 days, which is equal to the cycle in Table 23. A day has the elements given in Table 23 on line $(\mathrm{JD}-10)$ amod 60.

Remark 37. Tsurphu calendars use a different method to assign the nine numbers; the first Wood–Mouse day after the (true astronomical) winter solstice is 5 (yellow); then the numbers increase by 1 (mod 9) each day, until the first Wood–Mouse day after the (true astronomical) summer solstice, which is 4 (green), and then the numbers decrease by 1 (mod 9) each day for half a year until the first Wood–Mouse day after the next winter solstice. This requires accurate astronomical calculations of the solstices, which is

[79] This is in accordance with the Chinese rule mentioned above that odd numbers are male and even numbers female, but this is just a coincidence since Julian Day Numbers were not invented as part of Chinese astrology.

a central part of the Chinese calendar system, but foreign to the Tibetan calendar calculations.

Elemental yoga. In the Indian system, each day of week has an associated element from the set {earth, fire, water, wind}, see Table 5. Further, each lunar mansion is also associated to an element from the same set, see Henning [7, Appendix I] for a list. Each calendar day is thus given a combination of two elements, for the day of week and for the lunar mansion (calculated in (9.1) and (10.3)); the order of these two elements does not matter and the combination is regarded as an unordered pair. There are thus 10 possible different combinations (*yogas*), each having a name, see [7, p. 204].

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