Notes on $\mathbb{C}$*omplex Analysis*

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Complex numbers

Consider the (very simple) map on pairs of real numbers

$$(x, y) \mapsto (-y, x).$$

Note that applying it twice gives

$$(x, y) \mapsto (-y, x) \mapsto (-x, -y).$$

So if z is the pair (x, y) then the tranformation applied twice gives $-z$. Let i denote this transformation. We have $i \circ i = -\mathrm{id}$ where id is the identity. It is easy to see that i^{-1} is given by

$$(x, y) \mapsto (y, -x).$$

Also, it is easy to see that the set

$$x \,\mathrm{id} + yi, \quad x, y \in \mathbb{R}, \quad xy \neq 0$$

forms an abelian group under composition because it is closed under it, i.e.

$$(x \,\mathrm{id} + yi) \circ (x' \,\mathrm{id} + y'i) = (xx' - yy') \,\mathrm{id} + (x'y + xy')i,$$

(and this identity is invariant if we replace x, y by x', y', respectively), it is associative, it has id as neutral element and every element has an inverse as can be checked by the unique solution x', y' of

$$\begin{aligned} xx' - yy' &= 1 \\ x'y + xy' &= 0 \end{aligned}$$

in terms of x, y, as long as $xy \neq 0$. It is even easier to see that $x \,\mathrm{id} + yi, \quad x, y \in \mathbb{R}$ is an abelian group under addition. Also, composition distributes over addition because we are talking about linear transformations. Hence this set is a field which includes the real numbers if we think of an $x \in \mathbb{R}$ as $x \,\mathrm{id} + 0i$. We call the elements of of this field complex numbers, we denote the field by $\mathbb{C}$, and we change notation and denote id by 1, so that $x \,\mathrm{id} + yi = x1 + yi$ or, simply $x + yi$ or $x + iy$.

Since $\mathbb{C}$ is a 2-parameter field, we can think of it as $\mathbb{R}^2$, i.e. as a vector space over $\mathbb{R}$, in which case it has dimension 2. But if we think of it as a vector space over itself then it has dimension 1.

The algebra on $\mathbb{C}$ is most easily aided by the introduction of conjugation: if $z = x + iy$ then its conjugate $\overline{z}$ is given by

$$\overline{z} = x - iy.$$

We also let
$$|z| = \sqrt{x^2 + y^2}.$$
We can then express z^{-1} as
$$z^{-1} = z\overline{z}/|z|^2.$$

We define z^n, inductively, by $z^n = z{\circ}z^{n-1}$, $n \geq 1$, where $z^0 := 1$, and $z^{-1} := 1/z$. Since $i{\circ}i = -1$, we define
$$\sqrt{-1} = i.$$
We can now solve quadratic equations in $\mathbb{C}$. For example, to solve $z^2 = w$, where $w = u+iv$, we write $z = x + iy$ and observe that
$$z^2 = (x^2 - y^2) + i2xy$$
so we must solve
$$\begin{aligned} x^2 - y^2 &= u \\ 2xy &= v. \end{aligned}$$
These are easily solved and yield two solutions:
$$\begin{aligned} x &= \pm\sqrt{\frac{u^2 + \sqrt{u^2+v^2}}{2}} \\ y &= \pm \operatorname{sgn} u \sqrt{\frac{u^2 + \sqrt{u^2+v^2}}{2}} \end{aligned}$$
where $\operatorname{sgn} u = u/|u|$ if $u \neq 0$ and $\operatorname{sgn} 0 = 0$. We let $z = \sqrt{w}$ denote one of the solutions, e.g. the one with the $+$ sign.

Next we can solve more general quadratic equations For example, $az^2 + bz + c = 0$, with $a \neq 0$ always has a root. Its roots are $(-b \pm \sqrt{b^2 - 4ac})/2a$.

It will later be proved that any polynomial equation
$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$$
has exactly n roots in $\mathbb{C}$. Incidentally, division amongst polynomials works as in $\mathbb{R}$: If $P(z)$, $Q(z)$ are polynomials and $Q(z)$ is not trivial, then there are unique polynomials $A(z), R(z)$ such that
$$P(z) = A(z)Q(z) + R(z)$$
and the degree of $R(z)$ is less than the degree of $Q(z)$. Therefore, if z_0 is a root of $P(z) = 0$ we have $P(z) = (z - z_0)P_1(z)$ for some (necessarily unique) polynomial $P_1(z)$.

Note that a polynomial has real coefficients if and only if $\overline{P(z)} = P(\overline{z})$.

When $z = x + iy$, we refer to x as the real part $\Re z$ and to y as the imaginary part $\Im z$, observing also that $\Re z = \frac{1}{2}(z + \overline{z})$, $\Im z = \frac{1}{2i}(z - \overline{z})$.

Analysis on $\mathbb{C}$

By identifying $\mathbb{C}$ with R^2, we give $\mathbb{C}$ the usual topology of $\mathbb{R}^2$. Moreover, we give $\mathbb{C}$ the metric structure of $\mathbb{R}^2$ with the Euclidean norm: If $z = x + iy$ then

$$|z| := \sqrt{x^2 + y^2}.$$

Thus, a sequence z_n converges to z iff $|z_n - z| \to 0$ and, by completeness of $\mathbb{R}^2$, z_n convrerges to some element of $\mathbb{C}$ iff for all $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that $|z_n - z_m| < \varepsilon$ if $m, n \geq N$.

We can talk of functions $f : \mathbb{C} \to \mathbb{C}$. We can talk of the continuity of f at z_0 and by this we mean that for any $\varepsilon > 0$ there is $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ if $|z - z_0| < \delta$. We can talk by the continuity of f on an open set U by requiring that it is continuous at any point of U. Continuity of f on an arbitrary set $A \subset \mathbb{C}$ means that there is an open set $U \supset A$ such that f is continuous on U. We lose nothing, when speaking about continuity (except, perhaps, ink), if we think of any $f : \mathbb{C} \to \mathbb{C}$ as $f : \mathbb{R}^2 \to \mathbb{R}^2$.

So far, Analysis on $\mathbb{C}$ is identical to that of $\mathbb{R}^2$ and even Geometry on $\mathbb{C}$ is identical to that of $\mathbb{R}^2$ considered as Euclidean space.

The one-point compactification $\overline{\mathbb{C}}$ is important and will be constructed later explicitly by means of a stereographic projection. The geometry of $\mathbb{C}$ then changes.

Euclidean Geometry

The elements of $\mathbb{C}$ are called points. A straight line is a set of the form $\{(x, y) : \alpha x + \beta y = \gamma\}$ for some $\alpha, \beta, \gamma \in \mathbb{R}$ where $\alpha\beta \neq 0$. We can express the line $\alpha x + \beta y = \gamma$ as

$$\overline{a}z + a\overline{z} = \gamma$$

where $a = \frac{1}{2}(\alpha + i\beta)$. So the set of lines is the set $\{\overline{a}z + a\overline{z} = \gamma,\ |a| \neq 0, \gamma \in \mathbb{R}\}$.

The line $\overline{a}z + a\overline{z} = \gamma$ contains the point $\gamma/2\overline{a}$ and is parallel to the line $\overline{a}z + a\overline{z} = 0$, the latter being a line passing through 0 and containing any point of the form tia, $t \in \mathbb{R}$. We say that the line $\overline{a}z + a\overline{z} = \gamma$ passes through the point $\gamma/2\overline{a}$ and is parallel to the vector ia. Alternatively, the line $\overline{a}z + a\overline{z} = \gamma$ is the set

$$\left\{ \frac{\gamma}{2\overline{a}} + tia :\ t \in \mathbb{R} \right\}.$$

Two distinct points define a unique line: there is only one line passing through z_1, z_2 when $z_1 \neq z_2$ and this is the line $\overline{a}z + a\overline{z} = 1$ with $a = \pm(z_1 - z_2)/(z_1\overline{z}_2 - \overline{z}_1 z_2)$. Two distinct lines either have an empty intersection (and are called parallel) or intersect at a singleton (a one-point set).

A circle with center $a \in \mathbb{C}$ and radius $r > 0$ is the set

$$|z - a| = r$$

The circle $|z| = 1$ is the unit circle S^1. We can identify S^1 with $\mathbb{R}/2\pi\mathbb{Z}$ (where π is, for the time being, an arbitrary positive real number–its value will be fixed later) by the map $z \mapsto \theta(z)$ where $\theta(z)$ is any real number satisfying $\cos\theta(z) = (z + \overline{z})/2|z|$. Necessarily, any two such θ differ by an integer multiple of 2π. We let $\arg z$ denote the set of all these θ. If $z \neq 0$ then $\arg(z) = \arg(z/|z|)$. Hence $\arg(tz) = \arg z$ if $t > 0$. Also, $\arg(-z) = \arg(z) + \pi$. We can check that $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. The arg of 0 is undefined. All equalities are understood to be $\mod 2\pi$. The unique element of $\arg z$ lying in $[0, 2\pi)$ is the principal argument and, occasionally, we denote it by $\operatorname{Arg} z$. Thus, $\operatorname{Arg} z$ is a real number defined by

$$\{\operatorname{Arg} z\} = \arg z \cap [0, 2\pi).$$

The unique element of $\arg z$ lying in $(-\pi, \pi]$ is the typical element of $\arg z$ and we denote it by $\angle(z)$. The angle between z_1 and z_2 is defined by $\angle(z_1, z_2) = \angle(z_2/z_1)$. Thus, $\angle(z_1, z_2)$ is a real number defined by

$$\{\angle(z_1, z_2)\} = \arg(z_2/z_1) \cap (-\pi, \pi].$$

The tangent space $\mathbb{C}_w$ of $\mathbb{C}$ at w is the set $\mathbb{C}_w = \{(w, z) : \ z \in \mathbb{C}\}$ inheriting the structure of $\mathbb{C}$. The angle between $(z, z_1), (z, z_2)$ in $\mathbb{C}_z$ is the angle between z_1 and z_2.

Note that the equations of a circle and a line are invariant under conjugation. The lines $\overline{a}z + a\overline{z} = \gamma$ and $\overline{b}z + b\overline{z} = \delta$ are orthogonal if $\operatorname{Arg}(a/b) = \pi/2$, i.e. if a/b is purely imaginer. They are parallel if $\operatorname{Arg}(a/b) = 0$, i.e. is a/b is real.

The equation of the line passing through a point z_0 and being parallel to $u \neq 0$ is

$$z = z_0 + tu, \quad t \in \mathbb{R}$$

or, equivalently, the set of points z for which

$$\Im[(z - z_0)/u] = 0$$

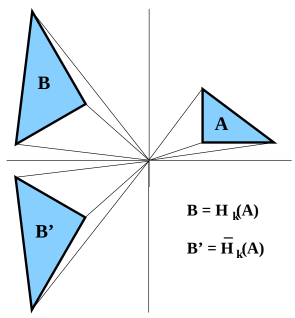
which gives

$$\overline{u}z - u\overline{z} = \overline{u}z_0 - u\overline{z}_0.$$

Notice that, for any nonzero $k \in \mathbb{C}$ the map

$$H_k : z \mapsto kz$$

preserves angles (both in magnitude and in sign) because $\angle(kz_1, kz_2) = \angle(z_1, z_2)$. It also scales distances by the same factor $|k|$ because $|k(z_1 - z_2)| = |k||z_1 - z_2|$. Such a mapping is called *homothetic transformation*. Two sets A, B are called homothetic if $B = H_k(A)$ for some k. In such a case, $A = H_{1/k}(B)$.



The map

$$\overline{H}_k : z \mapsto k\overline{z}$$

also scales distances but preserves angles only in magnitude. A similarity transformation is a map of the form H_k or $\overline{H}_k$.

Projective Geometry

The complex projective line $\mathbb{C}\mathrm{P}$ is the set $\mathbb{C}^2$ with two points (z_1, z_2), (z_1', z_2') identified if there is $z \in \mathbb{C}$, $z \neq 0$, such that $z_2 = zz_1$, $z_2' = zz_1'$. We use the notation (homogeneous coordinates) $(z_1 : z_2)$ to denote any of the points (zz_1, zz_2) when $z \in \mathbb{C}$. A linear transformation in homogeneous coordinates is given by

$$(z_1 : z_2) \mapsto (w_1 : w_2),$$

$$\begin{aligned} w_1 &= az_1 + bz_2 \\ w_2 &= cz_1 + dz_2, \end{aligned}$$

the well-posedness of which is easily verified. Since, when $z_2 \neq 0$, $(z_1 : z_2) = (z_1/z_2 : 1)$, we have that $(z_1 : z_2) \mapsto (w_1/w_2 : 1)$, where

$$\frac{w_1}{w_2} = \frac{a(z_1/z_2) + b}{c(z_2/z_2) + d},$$

in other words, the mapping $(z : 1) \mapsto (w : 1)$ is given by

$$w = S(z) = \frac{az + b}{cz + d}.$$

If this is viewed as a mapping from $\mathbb{C}$ to $\mathbb{C}$ then it is called *fractional linear transformation* or, simply, *linear transformation*, or *Möbius transformation*. Conversely, any *Möbius* transformation on $\mathbb{C}$ defines a linear transformation on $\mathbb{C}\mathrm{P}$. We will thus immediately identify $S(z)$ with the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We thus have that $S_1 \circ S_2$ of two such maps is also of the same form and the coefficients can be computed by matrix multiplication. We are interested in invertible such maps, which amounts to the determinant condition

$$ad - bc \neq 0.$$

We let $SL(2, \mathbb{C})$ be the set of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant

$$ad - bc = 1.$$

As such, it is a group under multiplication. We let $P(1, \mathbb{C})$ be the set of Möbius transformations with $ad - bc = 1$. As such it is a group under composition. Clearly, the two groups are isomorphic. We extend the action of each $S \in P(1, \mathbb{C})$ by considering the point at ∞ and by letting $S(\infty) = a/c$, $S(-d/c) = \infty$.

Three special group elements are worth identifying, both as elements of $P(1, \mathbb{C})$ and as elements $SL(2, \mathbb{C})$:

Translation: $T_b(z) = z + b, \quad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad b \in \mathbb{C},$

Homothecy: $H_a(z) = az, \quad \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad a \in \mathbb{C} - \{0\},$

Inversion: $J(z) = 1/z, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$

It can be checked that an arbitrary $S(z)$ is the composition of at most four of these special elements. A homothety H_a with $|a| = 1$ is called rotation. Note that $T_b^{-1} = T_{-b}$, $H_a^{-1} = H_{1/a}$, $J^{-1} = J$. So $(T_b, b \in \mathbb{C})$ is isomorphic to the additive group $(C, +)$, $(H_a, a \neq 0)$ is isomorphic to multiplicative group $(C - \{0\}, 0)$, whereas the subgroup $(H_a, |a| = 1)$ is isomorphic to the unit circle with the standard group structure.

Note that conjugation $z \mapsto \overline{z}$ on $\mathbb{C}$ cannot be lifted to a Möbius transformation on $\mathbb{C}$P and is thus not considered. Another 'bad' thing is, as will be seen later, that $z \mapsto \overline{z}$ is continuous but not differentiable.

Cross ratios

If we fix 3 distinct points z_2, z_3, z_4 in $\overline{\mathbb{C}}$ we can find a unique Möbius transformation S carrying them to $1, 0, \infty$, respectively. Assuming that none of the points is ∞, this transformation is given by

$$Sz = \frac{z - z_3}{z - z_4} : \frac{z_2 - z_3}{z_2 - z_4}.$$

If z_2, z_3 or $z_4 = \infty$ it is

$$Sz = \frac{z - z_3}{z - z_4}, \quad \frac{z_2 - z_4}{z - z_4}, \quad \frac{z - z_3}{z_2 - z_3},$$

respectively.

Mnemonics: It is easy to remember how to write this transformation in all cases. First consider sending $a, b \in \mathbb{C}$ to $0, \infty$ respectively by some ransformation $z \mapsto w$. It is clear that this must be of the form

$$w = k\frac{z - a}{z - b},$$

for some $k \in \mathbb{C} - \{0\}$. We find the value of k by requiring that we send a third point $c \in \mathbb{C}$ to 1, i.e.

$$1 = k\frac{c - a}{c - b},$$

and thus

$$w = \frac{z - a}{z - b} : \frac{c - a}{c - b}$$

as required. When one of the points a, b, c is ∞ we cancel the terms that contain this point. So, if $c = \infty$, we have $w = \frac{z-a}{z-b}$, as justified, say, by taking a limit as $c \to \infty$. When $a = \infty$, we have $w = \frac{1}{z-b} : \frac{1}{c-b} = \frac{c-b}{z-b}$ as justified, again, by taking a limit as $a \to \infty$.

The cross ratio (z_1, z_2, z_3, z_4) of the 4 points is defined to be equal to Sz_1, where S is the unique Möbius transformation such that $z_2 \mapsto 1$, $z_3 \mapsto 0$, $z_4 \mapsto \infty$.

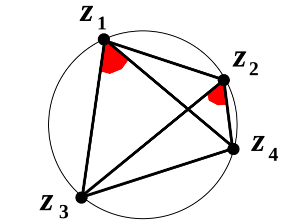
Theorem: If T is a Möbius transformation then T preserves the cross ratio:

$$(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)$$

Proof. Let $Sz := (z, z_2, z_3, z_4)$ be the unique Möbius transformation such that $(z_2, z_3, z_4) \mapsto (1, 0, \infty)$. But $T^{-1}S : (Tz_2, Tz_3, Tz_4) \mapsto (1, 0, \infty)$. So, using our notation, $ST^{-1}z = (z, Tz_2, Tz_3, Tz_4)$. In particular, $ST^{-1}(Tz_1) = (Tz_1, Tz_2, Tz_3, Tz_4)$. But $ST^{-1}(Tz_1) = Sz_1$. $\square$

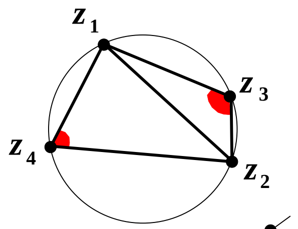
Application: There is a unique Möbius transformation carrying a triple of distinct points (a, b, c) into another triple (a', b', c') and this is given by $z \mapsto w$ where

$$(w, a', b', c') = (z, a, b, c).$$

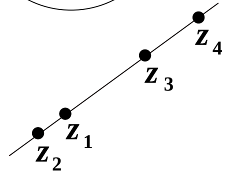


For example the only Möbius transformation that carries (a, b, c) into $(0, 1, i)$ is

$$w = \frac{i(b-c)(z-a)}{(a-c)(z-b) - i(a-b)(z-c)}.$$



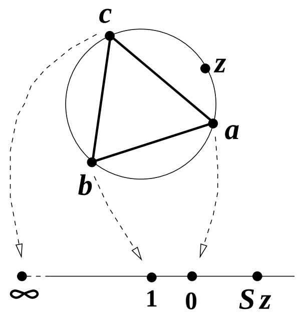
Theorem: The cross ratio (z_1, z_2, z_3, z_4) is real if and only if the 4 points are collinear or cocentral.



Proof. The cross ratio is linear if and only if its argument is 0 or $\pm\pi$. But

$$\arg(z_1, z_2, z_3, z_4) = \arg\frac{z_1 - z_3}{z_1 - z_4} - \arg\frac{z_2 - z_3}{z_2 - z_4}$$

Elementary geometry then shows the truth of the claim.



Actually if we accept the argument of elementary geometry as valid and if we agree to concider a straight line as a circle then we have shown that circles remain circles under Möbius transformations. First of all, there is only once circle C passing through a triple of distinct points a, b, c (the circle is a straight line if the points are collinear). Let $Sz = (z, a, b, c)$ be, as above, the unique Möbius transformation carrying a, b, c into $0, 1, \infty$, respectively. If z lies on the circle defined by a, b, c then Sz is a real number, i.e. Sz lies on the line L defined by $1, 0, \infty$. So the image of the circle C defined by a, b, c is the straight line L passing through $1, 0, \infty$. Because we can map any other three distinct points a', b', c' into $1, 0, \infty$ by the Möbius transformation $Tz = (z, a', b', c')$, it follows we can map the circle C' defined by $a', b'c'$ onto the straight line L. So the transformation $T^{-1}S$ maps C onto C'. Since $T^{-1}S$ is as arbitrary as any Möbius transformation it follows that any Möbius transformation maps circles into circles.

We can thus map a circle (or line) C to another circle C' by a Möbius transformation. The Möbius transformation is by no means unique. It becomes unique if we specify 3 distinct points on C and their 3 images on C'.

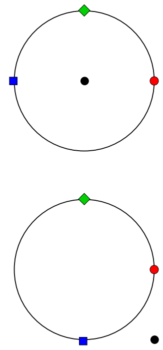
It can be seen that only Möbius transformation which maps $\mathbb{R}$ into itself must have real coefficients.

Proof. Let H be such a Möbius transformation mapping the real line into itself. Say $H(z) = \frac{az+b}{cz+d}$. Then $H(x)$ is real if x is real, i.e. $H(x) = \overline{H(x)}$, which gives

$$\frac{ax+b}{cx+d} = \frac{\overline{a}x + \overline{b}}{\overline{c}x + \overline{d}}.$$

Then
$$(a\overline{c}-\overline{a}c)x^2+(a\overline{d}+b\overline{c}-\overline{a}d-\overline{b}c)x+b\overline{d}-\overline{b}d=0.$$
This gives that $a\overline{c}$ is real, i.e., assuming $a \neq 0$, $c=\alpha/\overline{a}$ for $\alpha \in \mathbb{R}$. Also, $b\overline{d}$ is real, i.e., assuming $b \neq 0$, $d=\beta/\overline{b}$ for $\beta \in \mathbb{R}$. And $a\overline{d}+b\overline{c}$ is also real. But $a\overline{d}+b\overline{c}=\beta(a/b)+\alpha(b/a)$. So a/b must be a real multiple of b/a. This means that a/b is real or imaginary. If it is real, then $a=\lambda b$ for real λ. Combining with the above we get that $c=\alpha/\overline{a}=\alpha/\lambda\overline{b}=(\alpha/\lambda\beta)d$, i.e. $c=\mu d$ for real μ. Then $H(x)=\frac{b}{d}\frac{\lambda x+1}{\mu x+1}$ and b/d must be real, so $H(z)$ has real coefficients; etc.

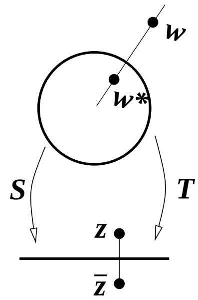
It is NOT true that a Möbius transformation which maps a circle into itself is real. To see this (and to see an example of a nontrivial transformation which is not a rotation or reflection) consider mapping the unit circle into itself by mapping $1, i, -1$ into $1, i - i$, respectively. If $z \mapsto w$ then
$$(w,1,i,-i)=(z,1,i,-1)$$
i.e.
$$\frac{w-i}{w+i}:\frac{1-i}{1+i}=\frac{z-i}{z+1}:\frac{1-i}{1+1}$$
which gives
$$w=\frac{(3i-1)z+(i+1)}{(i-1)z+(3i+1)}.$$



This map is algebraically simpler than a rotation (in that it is a ratio of linear maps) but it has screwed up the plane completely. For example, the center 0 has been moved to $1-i$. To see how much screwing is done by Möbius transformations, we have to examine families of circles called Steiner circles (see later).

Conjugation and symmetry

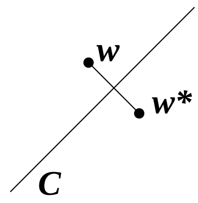
Here is another definition that extends the concept of conjugation. Recall that $\overline{z}$ and z are symmetric with respect to the real line. Let C be an arbitrary circle (or straight line). We say that
$$w \overset{C}{\sim} w^*$$
(w, w^* are symmetric with respect to C if there is a Möbius transformation $S: C \to C$ with $S(C)=\mathbb{R}$ such that $\overline{Sw}=Sw^*$. To make sure this is well-defined, we need to show that the concept does not depend on the choice of S. Let T be another Möbius transformation such that $T(C)=\mathbb{R}$. We need to show that Tw and Tw^* are conjugates of one another. But $w=S^{-1}z$ and $w^*=S^{-1}\overline{z}$. Therefore $Tw=TS^{-1}z$ and $Tw^*=TS^{-1}\overline{z}$. Now the transformarion TS^{-1} leaves the real axis invariant, i.e. $\overline{TS^{-1}z}=TS^{-1}\overline{z}$ which means that $\overline{Tw}=Tw^*$. $\square$



Hence any particular transformation which carries C onto the real line can be used to define symmetry. For example, if a, b, c are 3 points on C we can use the transformation $w \mapsto (w,a,b,c)$. Thus,
$$w \overset{C}{\sim} w^* \iff (w^*,a,b,c)=\overline{(w,a,b,c)} \quad \text{for any 3 distinct points } a,b,c \text{ on } C.$$

Case 1: $C = L$ is a straight line. Choose a, b be two points on it (they define the straight line) and $c = \infty$. Then $(w, a, b, \infty) = \frac{w-b}{a-b}$. Hence

$$\text{For } L \text{ a straight line, } w^* \overset{L}{\sim} w \iff \frac{w^*-b}{a-b} = \frac{\overline{w}-\overline{b}}{\overline{a}-\overline{b}} \quad \text{for any 2 distinct points } a, b \text{ on } L.$$



This implies that

$$|w^* - b| = |w - b|$$

for all $b \in C$ and so w, w^* are equidistant from C.

Case 2: C is a veritable circle: $|z - z_0| = R$. Let a, b, c be 3 points on C. Then for any $w \in C$, Then

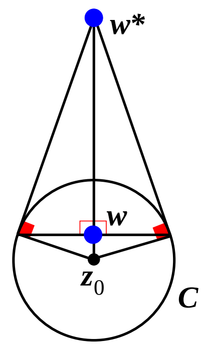
$$\begin{aligned}
\overline{(w,a,b,c)} &= \overline{(w-z_0, a-z_0, b-z_0, c-z_0)} && [z - z_0 \text{ is a Möbius transformation}] \\
&= (\overline{w-z_0}, \overline{a-z_0}, \overline{b-z_0}, \overline{c-z_0}) \\
&= \left(\overline{w-z_0}, \frac{R^2}{a-z_0}, \frac{R^2}{b-z_0}, \frac{R^2}{c-z_0}\right) && [(a-z_0)(\overline{a-z_0}) = R^2, \text{ etc.}] \\
&= \left(\frac{R^2}{\overline{w}-\overline{z}_0}, a-z_0, b-z_0, c-z_0\right) && [1/z \text{ is a Möbius transformation}] \\
&= \left(\frac{R^2}{\overline{w}-\overline{z}_0} + z_0, a, b, c\right).
\end{aligned}$$

Hence

$$w^* = \frac{R^2}{\overline{w}-\overline{z}_0} + z_0.$$

And so we have

$$\text{When } C \text{ is the circle } |z - z_0| = R, \quad w^* \overset{C}{\sim} w \iff (w^* - z_0)(\overline{w} - \overline{z}_0) = R^2.$$



And this implies, first, that

$$|w^* - z_0|\,|w - z_0| = R^2,$$

and, second, that

$$\frac{w^*-z_0}{w-z_0} = \frac{(w^*-z_0)(\overline{w}-\overline{z}_0)}{(w-z_0)(\overline{w}-\overline{z}_0)} = \frac{R^2}{|w-z_0|^2} > 0.$$

The latter means that z_0, w, w^* lie on a straight line with w, w^* on the same half line from z_0. The former means that w^* can be constructed by a simple geometrical construction.

In either case (line or circle), reflection is the result of a Möbius transformation followed by conjugation. Therefore reflection always carries (generalised) circles into (generalised) circles but is not analytic (as will be seen later, because conjugation is not analytic).

The symmetry principle: Möbius transformations preserve symmetry. So if T is a Möbius transformation,

$$w^* \overset{C}{\sim} w \iff Tw^* \overset{TC}{\sim} Tw.$$

The reason is simple. Assume $w^* \overset{C}{\sim} w$. To show that $Tw^* \overset{TC}{\sim} Tw$, pick 3 points a, b, c on TC. Since $T^{-1}a, T^{-1}b, T^{-1}c$ lie on C and since $w^* \overset{C}{\sim} w$, we have

$$(w^*, T^{-1}a, T^{-1}b, T^{-1}c) = \overline{(w, T^{-1}a, T^{-1}b, T^{-1}c)}.$$

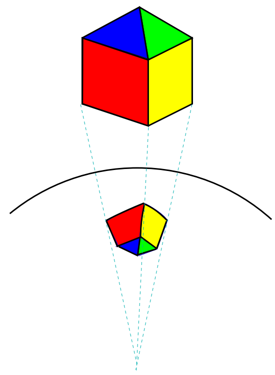
Since cross-ratios are preserved by Möbius transformations,

$$(w^*, T^{-1}a, T^{-1}b, T^{-1}c) = (Tw^*, a, b, c), \quad (w, T^{-1}a, T^{-1}b, T^{-1}c) = (Tw, a, b, c),$$

and so

$$(Tw^*, a, b, c) = \overline{(Tw, a, b, c)},$$

showing that $Tw^* \overset{TC}{\sim} Tw$.



Fact: Given 3 distinct points a, b, c there is a unique circle C passing through C such that $b = a^*$, where a^* is the symmetric of a with respect to C. To see this, notice that we can solve $(a^* - z_0)(\overline{a} - \overline{z}_0) = R^2$ uniquely for z_0 and R.

An application: To find Möbius transformations carrying a circle C onto another circle C' (such transformations are plenty) the method we described so far requires that we pick 3 distinct points a, b, c on C and 3 distinct points a', b', c' on C' and define a map $z \mapsto w$ by

$$(w, a', b', c') = (z, a, b, c).$$

But we may as well specify a point z_1 on C and two points z_2, z_3 not on C such that $z_3 = z_2^*$ with respect to C. Let w_1 be a point on C' and w_2, w_3 two points not on C' such that $w_3 = w_2^*$ with respect to C'. There is a unique Möbius transformation $T : z \mapsto w$ carrying (z_1, z_2, z_2^*) into (w_1, w_2, w_3^*):

$$(w, w_1, w_2, w_3^*) = (z, z_1, z_2, z_2^*).$$

Necessarily, this T must carry C onto C'.

Angles and orientation

An orientation of a circle C is determined by an ordered triple (z_1, z_2, z_3) of distinct points on C. For example, the triple $(1, 0, \infty)$ defines positive orientation for the 'circle' defined by the real line; notice that if the cross ratio $(z, 1, 0, \infty)$ equals z which means that $\Im(z, 1, 0, \infty) > 0$ for points on the upper half plane and < 0 for points on the lower half plane. Two triples (z_1, z_2, z_3), (z_1', z_2', z_3') determine the same orientation for C if $\Im(a, z_1, z_2, z_3)$ and $\Im(a, z_1', z_2', z_3')$ have the same sign, where a is a specific point not on C, say the centre of the circle.

A point z not on C is said to be to the right of C with respect to the triple $A = (z_1, z_2, z_3)$ if $\Im(z, z_1, z_2, z_3) > 0$ and to the left of C if $\Im(z, z_1, z_2, z_3) < 0$. Let R_A (resp. L_A) be the points to the right (resp. left) of C with respect to A. If $A' = (z_1', z_2', z_3')$ is another triple on C then either $R_A = R_{A'}$, $L_A = L_{A'}$ or $R_A = L_{A'}$, $L_A = R_{A'}$.

If the Möbius transformation S carries the oriented (through a triple $A = (z_1, z_2, z_3)$) circle C into C' then we use the triple $SA = (Sz_1, Sz_2, Sz_3)$ to orient C'. Since cross ratio is preserved by S, we have $SR_A = R_{SA}$, $SL_A = L_{SA}$.

If C is a circle, not a straight line, then we can give it the usual orientation by requiring that ∞ be to the right of C. The points that are to the right of C with respect to this orientation are the points outside of C and those that are to the left are the points inside C.

It remains to show that if C is the circle $|z-a| = R$ then the points inside C with respect to the usual orientation are the points z such that $|z-a| < R$. To show this, consider the following triple

$$(z_1, z_2, z_3) = (a+R, a+iR, a-R)$$

of points on C and notice that the cross ratio

$$(\infty, z_1, z_2, z_3) = \frac{z_1 - z_3}{z_1 - z_2} = \frac{2}{1-i} = 1+i,$$

whose imaginary part is positive and thus ∞ is to the right of C with respect to this triple. So this triple determines the usual orientation. Now let z be a point not on C such that

$$\Im(z, z_1, z_2, z_3) < 0.$$

We shall show that $|z-a| < R$. Letting $z = a + Rw$, we have

$$(z, z_1, z_2, z_3) = \frac{z - z_2}{z - z_3} : \frac{z_1 - z_2}{z_1 - z_3} = \frac{w-i}{w+1}(1+i) = \frac{(w-i)(\overline{w}+1)(1+i)}{|w+1|^2}.$$

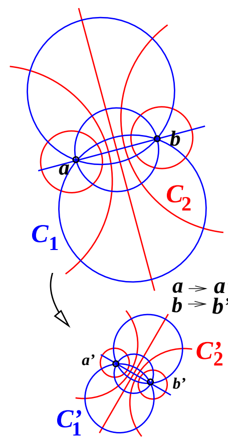
The imaginary part of (z, z_1, z_2, z_3) has thus the same sign as the imaginary part of

$$(w-i)(\overline{w}+1)(1+i) = |w|^2 + 1 + (w+\overline{w}) + i(w-\overline{w}) + i(|w|^2-1).$$

The first 3 terms on the right are real. Thus, the sign of the imaginary part is the sign of $|w|^2 - 1$ which is negative if $|w| < 1$ i.e. if $|z-a| < R$.

Steiner circles

A way to visualise a Möbius transformation is through their action on families of circles known as Steiner circles. Let $w = S(z)$ be any Möbius transformation and let a, b be two points in $\mathbb{C}$. First of all notice that if $S(a) = a'$, $S(b) = b'$ then $w = S(z)$ is of the form



$$\frac{w-a'}{w-b'} = k\frac{z-a}{z-b}.$$

Any circle C_1 passing through a, b is necessarily transformed into a circle C_1' passing through a', b'. Notice that the sets

$$\frac{|z-a|}{|z-b|} = \lambda \in \mathbb{R}$$

are also circles that do not pass through a, b. (Algebraically, it is easy to see they are circles–a geometric proof will follow.) Each such circle is called circle of Apollonius with limit points a, b. Any such circle C_2 is transformed into another Appolonius circle C_2' with limit points a', b':

$$\frac{|w-a'|}{|w-b'|} = \lambda|k|.$$

We are free to choose the points a, b (or the points a', b') in any way we like and we may even take them to be 0 or ∞.

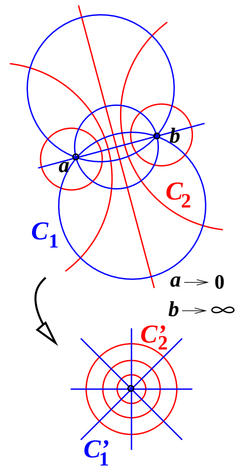
Suppose it is possible to pick a, b be such that $a' = 0, b' = \infty$, i.e. the transformation is given by

$$w = k\frac{z-a}{z-b}.$$

Then any straight line in the w-plane passing through the origin (being a circle passing through $0, \infty$) is the image of a circle passing through a, b, i.e. a circle of type C_1. On the other hand, any circle in the w-plane centred at the origin, i.e. $|w| = \rho$ for $\rho > 0$, must be the image of a circle (Möbius transformations map circles into circles) which shows that the set of points

$$\frac{|z-a|}{|z-b|} = \rho/|k|$$

is a circle (a geometric proof of the fact that any set of type C_2 ia a circle–an Appolonius circle). Since straight lines through 0 are orthogonal to any circle centred at 0, and since any Möbius transformation is conformal (i.e. it preserves angles on the whole place except, possibly, on at most two exceptional points), it follows that circles of type C_1 are orthogonal to circles of type C_2.



The collection of circles C_1 and C_2 is called the Steiner net determined by a, b. The Steiner net thus depends only on the positions of a and b. The idea is to see how it is transformed through a Möbius tranformation to getter a better picture of how the transformation screws up the plane. Some properties of the net:

To each $z_0 \in \mathbb{C}$ distinct from a, b there is exactly one C_1 and one C_2 passing through z_0. Indeed, z_0, a, b determine a unique C_1 circle. And the C_2 circle is the locus of points z such that

$$\frac{|z-a|}{|z-b|} = \frac{|z_0-a|}{|z_0-b|}.$$

Fix C_1 and C_2 and let $z \in C_2$. Then $z^* \overset{C_1}{\sim} z$ then $z \in C_2$. Furthermore, if $z^* \overset{C}{\sim} z$ for some C then C is a C_2.

Yet another choice for a, b is to take them so that they are fixed points:

$$S(a) = a, \quad S(b) = b.$$

Then $w = S(z)$ has the form

$$\frac{w-a}{w-b} = k\frac{z-a}{z-b},$$

so that the Steiner net remains invariant. However, the nature of k determines how individual circles will be mapped to one another.

If k is real the transformation is said to be *hyperbolic* and each C_1 maps into itself with the orientation preserved if $k > 0$ or reversed if $k < 0$.

If k is complex with $|k| = 1$ the transformation is said to be *elliptic* and each C_2 maps into itself.

A Möbius transformation with two distinct fixed points is the product of an elliptic and a hyperbolic transformation.

If the fixed points coincide the transformation is said to be *parabolic*.

A Möbius transformation that is neither elliptic, hyperbolic or parabolic is called *loxodromic*.

Derivative

Consider a function $f : \mathbb{C} \to \mathbb{C}$. We say that the derivative of f at z_0 exists if $(f(z) - f(z_0))/(z - z_0)$ has a limit as $z \to z_0$, when z moves on the complex plane $\mathbb{C}$, in other words if there exists a complex number A such that for all $\varepsilon > 0$ there exists $\delta > 0$ so that if $|z - z_0| < \delta$ then $|(f(z) - f(z_0))/(z - z_0) - A| < \varepsilon$. This A is denoted by $f'(z_0)$ and is the derivative of $f(z)$ at z_0:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

If f has derivative at z_0 then f is continuous at z_0 because $f(z_0 + h) - f(z_0) = h \cdot \frac{1}{h}(f(z_0 + h) - f(z_0))$ and $h \to 0$, while $\frac{1}{h}(f(z_0 + h) - f(z_0)) \to f'(z_0)$, as $|h| \to 0$, therefore $f(z_0 + h) - f(z_0) \to 0 \cdot f(z_0) = 0$.

Note that, to even define the derivative at z_0 we must have f defined on some (arbitrarily small) disk around 0. More generally, we must have f defined on some open set. Since any open set can be written as the disjoint union of its connected components, it is no loss of generality to consider open and connected sets.

A domain (or region) Ω in $\mathbb{C}$ is an open and connected set. (A simple domain is open and simply connected: more on this later.) A function f from Ω into $\mathbb{C}$ is called analytic (or holomorphic) if it has derivative on each point on which it is defined, i.e. on each $z \in \Omega$. Thus, a function which is analytic on Ω is continuous on Ω: $\mathscr{O}(\Omega) \subset \mathscr{C}(\Omega)$. In fact, we will prove the fundamental result that if f is analytic then f' is analytic and so every derivative $f^{(n)}$ is analytic. The space of analytic functions $\mathscr{O}(\Omega)$ will turn out to be a strict subset of $\mathscr{C}^\infty(\Omega)$.

The derivative $f'(z)$ at a point z of can be expressed in many ways. Let $u = \Re f$, $v = \Im f$. Moving along the real axis, we get

$$f'(z) = \frac{\partial f}{\partial x} = u_x + iv_x$$

Moving along the imaginary axis, we get

$$f'(z) = \lim_{y \to 0} \tfrac{1}{iy}(f(z + iy) - f(z)) = v_y - iu_y.$$

Consequently, the partial derivatives at $z = x + iy$ satisfy

$$u_x = v_y, \quad u_y = -v_x.$$

If u, v are $\mathscr{C}^2$ then

$$u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy},$$

because, $u_{xy} = u_{yx}$ for a $\mathscr{C}^2$ function u. Thus, the real and imaginary parts of an analytic function are harmonic functions.

Symbolics

We can formally think of analytic function $f(x, y)$ as a function of $z, \overline{z}$, through $x = \frac{1}{2}(\overline{z}+z)$ and $y = \frac{1}{2i}(\overline{z} - z)$. Since

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0,$$

it follows that f cannot depend on $\overline{z}$.

For instance, we can immediately tell that ze^z is analytic but $\overline{z}e^z$ is not.

Let $f(z)$ be analytic and consider the function $\overline{f(z)}$. This is surely not analytic (unless $f(z)$ is a constant.) Since, as a formula, $f(z)$ depends only on z, the function $\overline{f(z)}$, as a formula, depends only on $\overline{z}$. Write

$$\widetilde{f}(\overline{z}) := \overline{f(z)}.$$

But $u = \frac{1}{2}(f + \overline{f})$. Therefore

$$u(x, y) = \tfrac{1}{2}[f(x + iy) + \widetilde{f}(x - iy)].$$

This is a symbolic tautology; therefore we expect it to hold under the substitution $x = z/2$, $y = z/2i$ which cancels the argument of the second function:

$$u(z/2, z/2i) = \tfrac{1}{2}[f(z) + \widetilde{f}(0)].$$

We then have

$$f(z) = 2u(z/2, z/2i) + \text{const}.$$

The use of this is twofold: (i) it enables us to recover $f(z)$ from its real part u; (ii) it enables us to find the conjugate v of u, i.e. a function that satisfies the CR relations.

Power series

To every sequence a_n in $\mathbb{C}$ we associate the sequence of functions

$$A_k(z) := \sum_{n=0}^{k} a_n z^n$$

whose limit, as $k \to \infty$, if it exists, defines the power series

$$A(z) = \sum_{n=0}^{\infty} a_n z^n.$$

We want to study the set of z for which $A(z)$ is defined, as well as the properties of the function $A(z)$. For example, if $a_n \equiv 1$ then $\sum_{n=0}^{k} z^n = \frac{1-z^k}{1-z}$ and, if $|z| < 1$, this converges, as $k \to \infty$, to $1/(1 - z)$, i.e.

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}, \quad |z| < 1.$$

If $|z| \geq 1$, then, obviously, the series diverges.

Define $R \in [0, \infty]$ by

$$1/R := \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}.$$

If $R = 0$ then, for any $K > 0$, $|a_n| \geq K^n$, eventually, so the series $\left|\sum a_n z^n\right|$ is dominated from below by $\sum K^n |z|^n$ which diverges if $|z| \geq 1/K$. Since K is arbitrary it follows that $\left|\sum a_n z^n\right|$ diverges for all values of z other than $z = 0$.

Suppose $0 < R \leq \infty$. Suppose $|z| < R$. Pick ρ such that $|z| < \rho < R$. Then $\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|} < 1/\rho$ and so $|a_n| < 1/\rho^n$ eventually, i.e. $|a_n z^n| < (|z|/\rho)^n$ eventually which means that $\sum_n a_n z^n$ is absolutely dominated by $\sum_n (|z|/\rho)^n$ which converges if $|z| < \rho$. Hence $\sum_n a_n z^n$ converges if $|z| < R$. We show that the convergence is uniformin $|z| \leq \rho$ for all $\rho < R$. To this end, pick ρ' such that $\rho < \rho' < R$ and observe that $|a_n| < 1/\rho'^n$ eventually, so that if $|z| < \rho$, we have $|a_n z^n| < (\rho/\rho')^n$. But then $|A_{k+m}(z) - A_k(z)| \leq \sum_{n=k}^{k+m-1} (\rho/\rho')^n \leq (\rho/\rho')^k/(1 - (\rho'/\rho))$ which shows that $A_n(z)$ is a Cauchy sequence on $|z| \leq \rho$ and as such it converges uniformly to $A(z)$. Since each $A_n(z)$ is a continuous function we have that $A(z)$ is continuous on $|z| \leq \rho$ for each $\rho < R$. In other words, $A(z)$ is continuous on $|z| < R$. (But convergence is uniform only on compact subsets of $|z| < R$.) If $|z| > R$, then, as earlier, we can argue that $A_n(z)$ diverges.

We call R the radius of convergence of the power series.

We now look at the derived series $\sum n a_n z^n$. We have $\overline{\lim}_{n\to\infty} \sqrt[n]{n|a_n|} = 1/R$, hence the radius of convergence is the same. Let $B(z) := \sum n a_n z^n$, a function which is continuous on $|z| < R$. We will show that $A(z)$ is analytic on $|z| < R$ with $A'(z) = B(z)$. Write

$$A(z) = A_n(z) + r_n(z), \quad B(z) = B_n(z) + \tilde{r}_n(z) = A_n'(z) + \tilde{r}_n(z).$$

We then have

$$\frac{A(z+h) - A(z)}{h} - B(z) = \frac{A_n(z+h) - A_n(z)}{h} - A_n'(z) - \tilde{r}_n(z) + \frac{r_n(z+h) - r_n(z)}{h}.$$

Assume $|z|, |z+h| < \rho < R$. It is easy to see that we can make the last term $\leq \varepsilon$ if $n \geq n_0$ (regardless of z, h). There is also $n_1 = n_1(z)$ such that $|\tilde{r}_n(z)| \leq \varepsilon$ if $n \geq n_1(z)$. Fix $n \geq n_0 \vee n_1(z)$. By the definition of the derivative at the fixed point z, we can make $\left|\frac{A_n(z+h)-A_n(z)}{h} - A_n'(z)\right| \leq \varepsilon$ if $|h| < \delta = \delta(\varepsilon, n_1(z))$. This shows that $B(z) = A'(z)$. This is true for all $|z| < \rho$ for all $\rho < R$ and so $A(z)$ is analytic on $|z| < R$.

So we have shown that, if $R > 0$ then $A(z) := \sum_{n=0}^{\infty} a_n z^n$ is an analytic function on $|z| < R$ whose derivative $A'(z)$ is obtained by differentiating term by term and has the same radius of convergence. The procedure can be continued ad infinitum, showing that all derivatives $A^{(k)}(z)$ exist, they all have the same radius of convergence and they can be obtained by differentiating the terms inside the summation. By induction we can show that

$$A^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n z^n.$$

In particular, $A^{(k)}(0) = k!a_k$. In other words, $a_k = A^{(k)}(0)/k!$, so that $A(z)$ can also be written as $A(z) = \sum_{n=0}^{\infty} \frac{A^{(n)}(0)}{n!} z^n$, an expression known as Taylor-Maclaurin. We repeat that this gives a representation of a function $A(z)$ which is *a priori* defined as a power series; it is by no means a proof of the general theorem that an analytic function can be represented by power series. There is a long way to go to prove this more general theorem.

Abel's limit theorem

Suppose that $R = 1$ (no loss of generality) and that $\sum_n a_n$ converges. Then $A(z) := \sum_n a_n z^n$ converges to $A(1)$ as $z \to 1$ in such a way that $|1-z|/(1-|z|)$ remains bounded.

Proof. W.l.o.g. suppose $\sum_n a_n = 0$ (else change a_0). Let

$$A_n(z) := \sum_{k=0}^{n} a_k z^k, \quad |z| < R = 1.$$

By assumption, $A_n(1) \to 0$. Summation by parts gives

$$A_n(z) = (1-z) \sum_{k=0}^{n-1} A_k(1) z^k + A_n(1) z^n.$$

Since $|A_n(1)z^n| \leq |A_n(1)| \to 0$, we have

$$A(z) = (1-z) \sum_{n=0}^{\infty} A_n(1) z^n.$$

We will show that $A(z) \to 0$ as $z \to 1$ while $|1-z|/(1-|z|)$ remains bounded, say $|1-z| \leq K(1-|z|)$. Pick m so that $|A_m(1)| < \varepsilon$ for all $n \geq m$. Then

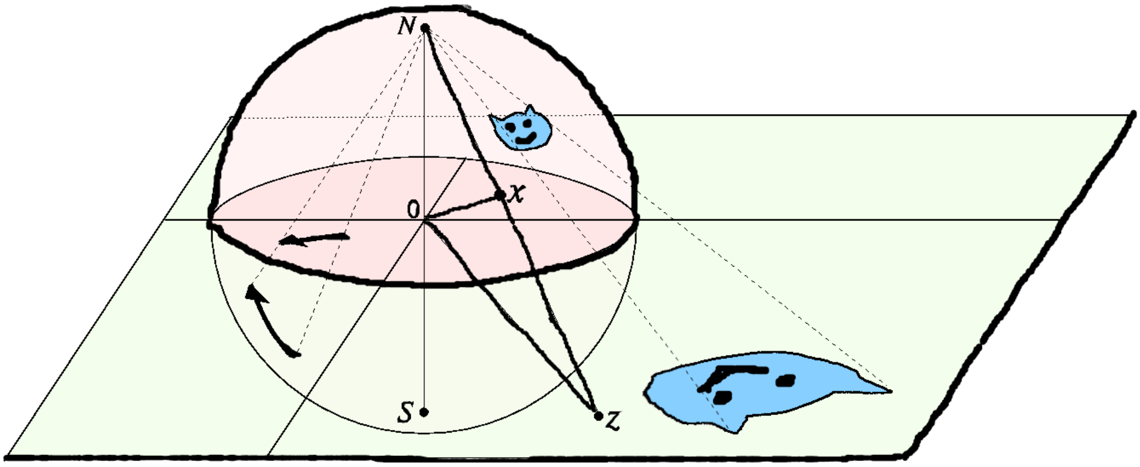
$$\begin{aligned}
|A(z)| &\leq |1-z| \left| \sum_{n=0}^{m-1} A_n(1) z^n \right| + |1-z| \sum_{n \geq m} \varepsilon |z|^n \\
&\leq |1-z| \left| \sum_{n=0}^{m-1} A_n(1) \right| + |1-z| \frac{\varepsilon}{1-|z|} \\
&\leq K_1 |1-z| + K\varepsilon.
\end{aligned}$$

As $z \to 1$, we have $\overline{\lim} |A(z)| \leq K\varepsilon$. This is true for any $\varepsilon > 0$, and so $A(z) \to 0$ as $z \to 1$, while $|1-z|/(1-|z|)$ remains bounded.

The stereographic projection

The representation of the compactification $\mathbb{C} \cup \{\infty\}$ of the complex plane as a sphere is fundamental importance.

Consider the unit sphere S^2 in 3 dimensions and let Π be a plane bisecting the sphere into two pieces. Let N, S be the points on S^2 farthest from Π (North and South pole, respectively). The stereographic projection is a map is a bijection $\varphi : S \setminus \{N\} \to \Pi$ defined as follows: If x is a point on S^2 other than N consider the straight line joining N and x and let $\varphi(x)$ be the point of its intersection with Π.



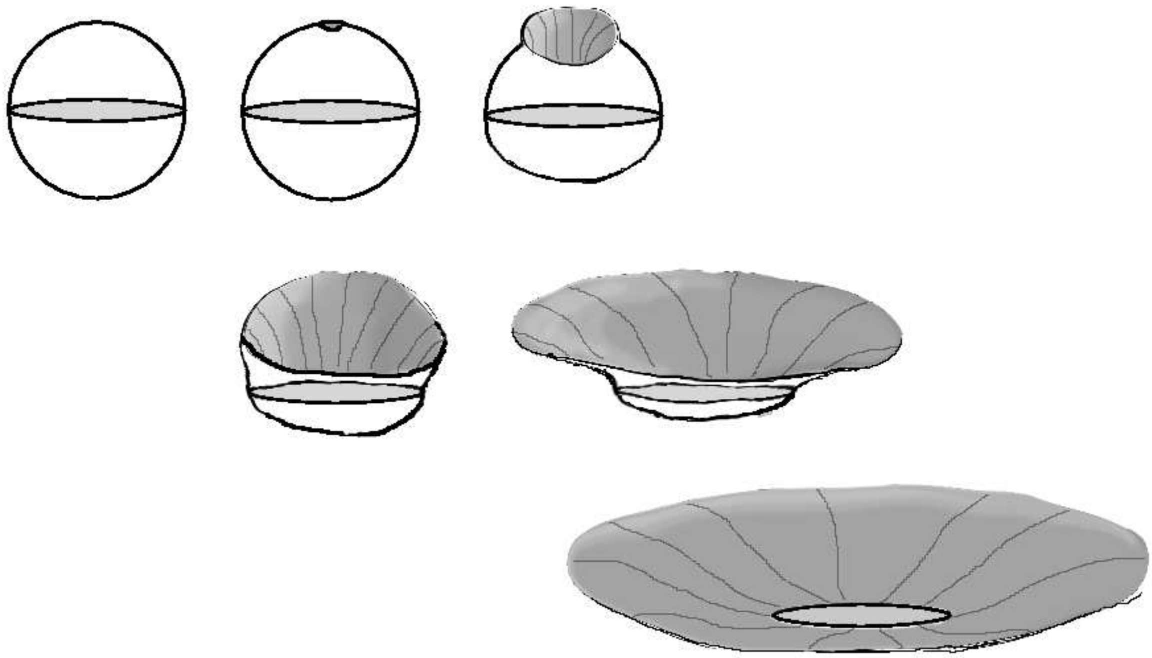
Concretely, let $S^2 \subset \mathbb{R}^3$ be the set of $x = (x_1, x_2, x_3)$ such that $x_1^2 + x_2^2 + x_3^2 = 1$, and take $N = (0, 0, 1)$. Let Π be the plane $x_3 = 0$, identified with $\mathbb{C}$. Let $z = \varphi(x)$. Then it is easy to see that

$$z = \frac{x_1 + ix_2}{1 - x_3}.$$

The inverse map is given by

$$x_1 = \frac{z + \overline{z}}{|z|^2 + 1}, \quad x_2 = \frac{z - \overline{z}}{i(|z|^2 + 1)}, \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

Note that the unit circle $|z| = 1$ remains fixed. The disk $|z| < 1$ is mapped into the southern hemisphere. The set $|z| > 1$ is mapped into the northern hemisphere. Note that orientations in the southern hemisphere are preserved. But in the northern hemisphere they are reversed.

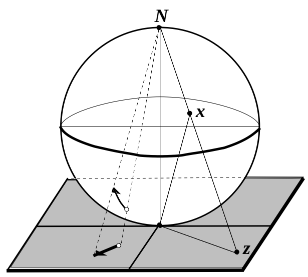


We can define the point at ∞ of $\mathbb{C}$ to be $\varphi(N)$.

In this way, we can consider $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ as a metric space with metric

$$d(z, w) = ||\varphi^{-1}(z) - \varphi^{-1}(w)||$$

where $||x|| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Note that the topology induced by d, when restricted to $\mathbb{C}$, is identical to the usual topology of $\mathbb{C}$. A sequence z_n converges to ∞ if $d(z_n, \infty) \to 0$. Note also that sets of the form $|z| \geq R$ are compact.



There are other stereographic projections. For example, if we may let Π be a tangent plane to the sphere S^2 and let N be the point on S^2 farthest from the plane. Any point $x \in S^2$ is mapped into the point z of the intersection of the straight line joining N and x with Π This projection does not reverse orientations.

The square root

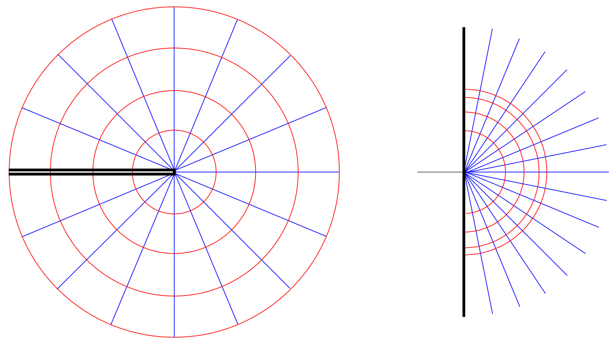
Consider the solution to $w^2 = z$. We have $w = \pm\sqrt{z}$. The function $\sqrt{z}$ is multi-valued. By letting z range over the domain

$$\Omega = \mathbb{C} - \mathbb{R}_-$$

and requiring that

$$\Re\sqrt{z} > 0$$

we obtain a branch of the square root function, namely a function from Ω into $\mathbb{H}_+$ (the open right-half plane). This function is analytic, one-to-one and onto. In other words, it is a bijection between Ω and $\mathbb{H}_+$:

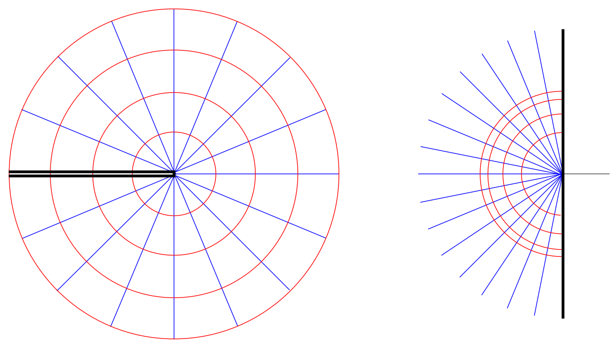


This branch of $\sqrt{z}$ is an analytic bijection $f(z)$ between $\Omega = \mathbb{C} - \mathbb{R}_-$ and the open right-half plane $\mathbb{H}_+$. It has derivative $f'(z) = 1/2\sqrt{z} \equiv 1/2f(z)$.

But we could equally well require that

$$z \in \Omega, \quad \Re\sqrt{z} < 0$$

in which case we obtain another branch:

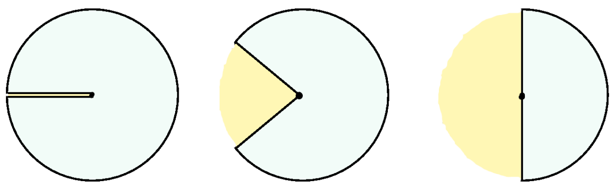


This branch of $\sqrt{z}$ is an analytic bijection between $\Omega = \mathbb{C} - \mathbb{R}_-$ and the open left-half plane $\mathbb{H}_-$. It has derivative $-1/2f(z)$.

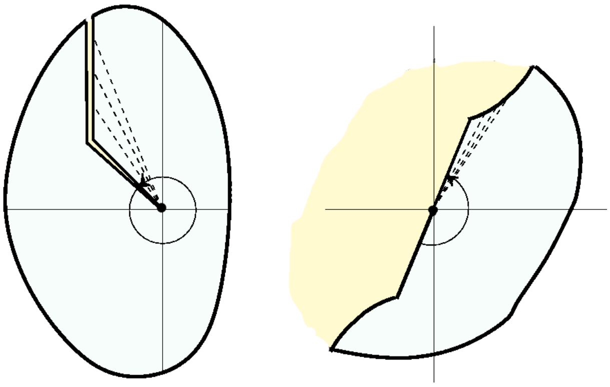
When defining a branch of a multi-valued function, we have to be careful with the restrictions on both z and its image w. So if we agree that $f(z)$ is the *principal branch* of $\sqrt{z}$, defined by the requirements $z \not\leq 0$, $\Re w > 0$, then we can be sloppy and use the notation $\sqrt{z}$ for this function. In which case, the derivative of this analytic function is $1/\sqrt{z}$.

If we choose the second branch, then $\sqrt{z}$ refers to the map $g(z) = -f(z)$ and g maps Ω onto $\mathbb{H}_-$. Its derivative is $g'(z) = -f'(z) = -1/2f(z) = 1/2g(z)$. So if we *decide* to use the sloppy notation $\sqrt{z}$ in lieu of $g(z)$ then we still get the formula $D\sqrt{z} = 1/2\sqrt{z}$.

Geometrically, it is easy to see what each of these maps does. Let us look at $f(z)$. First we cut the plane (by a pair of scissors) on its negative real axis and removed it (removing 0 as well) and then opened the cut widely. Thus the map halves the plane.



We now realise that there is nothing kosher about the two choices we considered so far. We can do many other things. For instance, let $\gamma(t)$ be a simple curve such that $\gamma(0) = 0$, $\gamma(1) = \infty$, let $[\gamma]$ be its image and let $\Omega = \mathbb{C} \setminus [\gamma]$. Let $B := f([\gamma]) \cup (-f([\gamma]))$ be the image of $[\gamma]$ under the principal branch $f(z)$ of $\sqrt{z}$ considered above together with its image under $-f(z)$. This B 'splits' the plance into two regions. By deciding to consider one of the regions we have chosen yet another branch of the function $\sqrt{z}$.



Let us call $h(z)$ the latter branch which is an analytic bijection between $U = \mathbb{C} \setminus [\gamma]$ and V one of the regions defined by $f[\gamma]$.

We maintain that the formula $h'(z) = 1/2h(z)$ still holds. Indeed, locally, $h(z)$ either equals $f(z)$ or $-f(z)$. (Check details!)

The common characteristic between the branches $f(z), g(z), h(z)$ is that they are 'maximal' in the sense that we cannot enlarge their domains without losing single-valuedness. Of course, we can restrict any of them to subdomains and, in certain subdomains they may coincide.

The analogy can be seen even in Real Analysis. Consider, for instance, the function $f(x) = \cos x$. Its inverse function is multi-valued unless we make restrictions. Let $g(y) = \arccos y$. Let us require that $-1 < y < 1$ and $0 < x < 2\pi$. Then g is a smooth bijection between $(-1, 1)$ and $(0, 2\pi)$. It is a maximal branch of g. But we can also require that

The exponential (and trigonometric) function

There is a unique function $f(z)$ that is entire and satisfies $f'(z) = f(z)$, $f(0) = 1$. To find it, assume $f(z) = \sum_{n=0}^{\infty} a_n z^n$, differentiate and equate coefficients to get, by induction, that $a_n = 1/n!$. Since $\overline{\lim}_{n\to\infty} \sqrt[n]{1/n!} = 0$, it follows that the radius of convergence is ∞. So the function defined by this power series is entire. Denote it by e^z:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

We showed that e^z is entire, $(e^z)' = e^z$, $e^0 = 1$. So we have one solution.

Notice that for any constant $c \in \mathbb{C}$, the derivative $e^z e^{c-z}$ is identically 0. Hence $e^z e^{c-z}$ is constant:

$$e^z e^{c-z} = e^c, \quad z, c \in \mathbb{C}.$$

In other words,

$$e^{a+b} = e^a e^b, \quad a, b \in \mathbb{C}.$$

In particular,

$$e^z e^{-z} = 1, \quad z \in \mathbb{C}.$$

Hence e^z is never zero. To show that there is no other entire function $f(z)$ with the properties $f'(z) = f(z)$, $f(0) = 1$, consider one such function and let $h(z) := f(z)/e^z$. But then $h'(z) = (f'(z)e^z - f(z)e^z)/e^{2z} \equiv 0$. Hence $h(z)$ is constant. But $h(1) = 1$, hence $h(z) \equiv 1$ and so $f(z) \equiv e^z$.

If $z = x \geq 0$ then e^x is precisely the real exponential function. If $z = iy$, where $y \in \mathbb{R}$ then

$$e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{n \text{ even}} \frac{(iy)^n}{n!} + \sum_{n \text{ odd}} \frac{(iy)^n}{n!}$$

Using the Taylor series expansion for the functions cos and sin from Real Analysis we obtain

$$e^{iy} = \cos y + i \sin y, \quad y \in \mathbb{R}.$$

So we have

$$e^z = e^x(\cos y + i \sin y), \quad z = x + iy \in \mathbb{C}.$$

We next define

$$\cos z := \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z := \frac{e^{iz} - e^{-iz}}{2i}, \quad z \in \mathbb{C}$$

and notice that

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots,$$

which shows that if $z = x \in \mathbb{R}$ then $\cos z = \cos x$, $\sin z = \sin x$ where $\cos x, \sin x$ are the real trigonometric functions. The functions $\cos z, \sin z$ are entire. We can easily show the algebraic identity

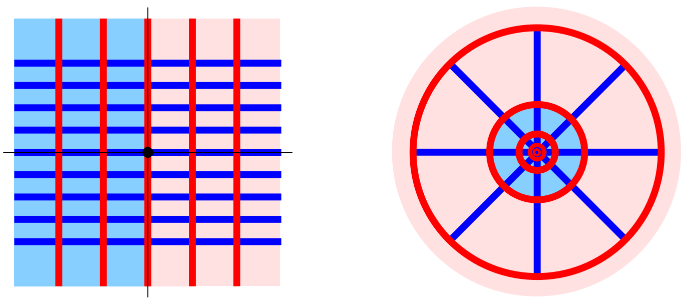
$$\cos^2 z + \sin^2 z = 1.$$

Hence $|e^{iy}| = 1$ for all $y \in \mathbb{R}$. Conversely, if $|z| = 1$ then $z = e^{iy}$ for some real y.

We can also show that there are complex numbers c such that $e^{z+c} = e^z$ for all $z \in \mathbb{C}$ and that each such c is purely imaginary. In fact, there is a smallest positive number ω_0 such that each $c = ni\omega_0$, $n \in \mathbb{Z}$. The number ω_0 is also denoted by 2π and this defines π. Hence the function e^{iz} is periodic with smallest positive period equal to 2π. So, the analytic definition of π is:

$$\pi := \min\{\omega > 0 : \ e^{i2\omega} = 1\}.$$

Using the Taylor expansion of e^z we can approximately solve the equation $e^{i2\pi} = 1$ to find $\pi = 3.14159\ldots$.



Algebraically, the function $h(t) := e^{i2\pi t}$, $h : \mathbb{Z} \to S^1$ is a homomorphism between the additive group $(\mathbb{Z}, +)$ and the multiplicative group $(S^1, \cdot)$ where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

We also define the hyperbolic cosine and sine by

$$\cosh z := \frac{e^z + e^{-z}}{2}, \quad \sinh z := \frac{e^z - e^{-z}}{2}, \quad z \in \mathbb{C}$$

These are entire but not one-to-one functions and we can study them through

$$\cosh z = \cos(iz), \quad \sinh z = -i \sin(iz).$$

From this we see that $\cosh z$ is obtained by rotating z by $\pi/2$ and then taking the cosine; and $\sinh z$ by rotating z by $\pi/2$, taking the sine followed by the inverse rotation. So, whereas cos and sin are periodic with basic period 2π, their hyperbolic counterparts cosh and sinh are periodic with basic period $i2\pi$.

The logarithm

The function e^z is not one-to-one so its inverse is not a function. However, we define a multi-valued function $w = \log z$ to be any solution of the equation $e^w = z$. We notice that, for fixed $w \neq 0$, the possible solutions of $e^w = z$ are of the form $\log|z| + i\varphi$, where $\varphi = \varphi_0 + 2\pi n$, with n ranging over the integers. and where $\log|z|$ is the real logarithm. So:

$$\log z = \{\log|z| + i(\varphi_0 + 2\pi n), \quad n \in \mathbb{Z}\}.$$

So it is only the imaginary part of $\log z$ which is multi-valued. This imaginary part is called argument of z:

$$\arg z := \Im \log z,$$

and is understood to be a set. Since the difference of any two elements of $\arg z$ is an integer multiple of 2π it follows that there is only one element of $\arg z$ on every semiopen interval of length 2π. Choosing such an interval fixes a specific element of $\arg z$ and so $\arg z$ (and thus $\log z$) becomes a single-valued function. We refer to such a function as a branch of $\arg z$ (or of $\log z$). Thus, the *principal branch* of $\arg z$ is specified by $-\pi < \arg z \leq \pi$.

Correspondingly, the *principal branch* of $w = \log z$ is specified by

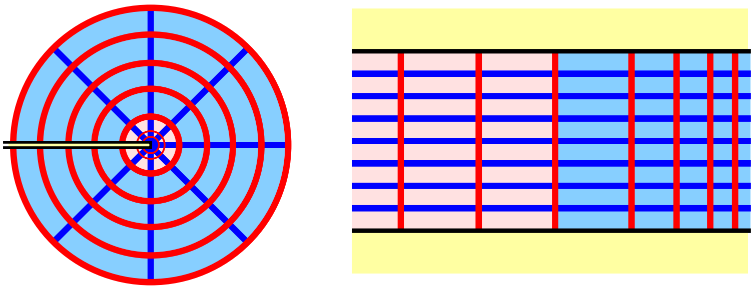
$$-\pi < \Im w < \pi.$$

Since the only numbers $z \in \mathbb{C}$ with $w = \log z$ having imaginary part $\pm\pi$ are real and negative, it follows that the above restriction immediately places the restriction $z \neq< 0$. Also, we cannot have $z = 0$. So we add (with a bit of redundancy)

$$z \not\leq 0$$

to specify the principal branch of $w = \log z$.

The principal branch of the logarithm is an analytic bijection $\ell(z)$ between $\Omega := \mathbb{C} \setminus \{z \leq 0\}$ and $S := \{-\pi < \Im w < \pi\}$. Its derivative is $\ell'(z) = 1/z$.



The algebraic property of $e^{a+b} = e^a e^b$ translates to

$$\arg(w_1 w_2) = \arg w_1 + \arg w_2,$$

and this should be interpreted as equality between sets. Alternatively, if we use a specific branch of $\arg w$, then the equality should be interpreted as

$$\arg(w_1 w_2) = \arg w_1 + \arg w_2 \mod 2\pi.$$

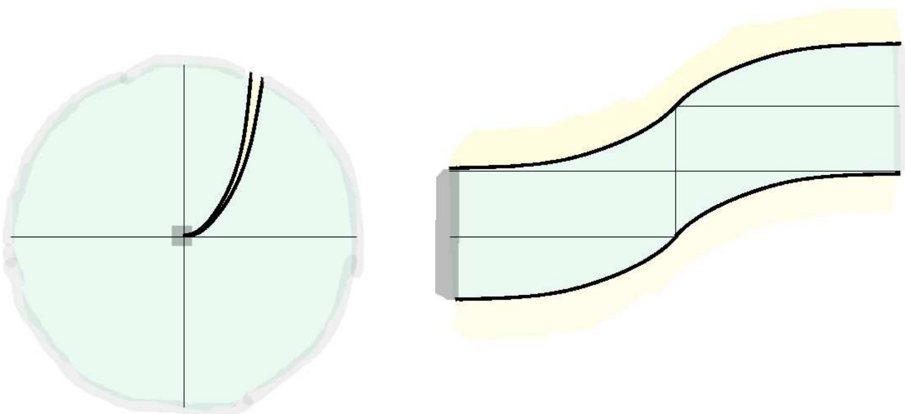
Similarly,

$$\log(w_1 w_2) = \log w_1 \log w_2$$

is equality between sets.

Later it will be shown that it is impossible to define a single-valued analytic branch of $\log w$ on certain domains. Certainly, such a domain must not contain 0. But still, this is not sufficient. For example, the annulus $1 < |z| < 2$ does not contain 0 but it is impossible to define a single-valued analytic branch of $\log z$ on this annulus.

It can be seen that we can define other branches of the logarithm as long as we exclude from its domain a simple curve that joins 0 and ∞.



Inverse cosine

Since $\cos w = \frac{1}{2}(e^{iw} + e^{-iw})$, we can define $w = \arccos z$ by solving $\cos w = z$, which is a quadratic in e^{iw}:

$$(e^{iw})^2 - 2we^{iw} + 1 = 0.$$

We obtain
$$e^{iw} = z \pm \sqrt{z^2 - 1}.$$
Hence
$$w = \arccos z = -i \log(z \pm \sqrt{z^2 - 1}).$$
Again, $\arccos w$ defines a set of numbers. If $e^{iw} = z + \sqrt{z^2 - 1}$ then, as obtained by the quadratic equation, $e^{-iw} = z - \sqrt{z^2 - 1}$. So the numbers $z + \sqrt{z^2 - 1}$, $z - \sqrt{z^2 - 1}$ are inverses of one another and so we can write the above set as

$$\arccos z = \pm i \log(z + \sqrt{z^2 - 1}).$$

We further notice that we can drop the minus sign and lose no value from the set. The reason is that if we replace z by $-z$ in $z + \sqrt{z^2 - 1}$ then the minus sign pops out. Hence,

$$\arccos z = i \log(z + \sqrt{z^2 - 1})$$

is the same set of numbers.

How can we obtain a (reasonable and maximal) analytic branch of $\arccos z$?

First, we want $\arccos z$ to reduce to the usual $\arccos x$ when $z = x$ is real, $-1 < x < 1$. We have
$$i \log(x + \sqrt{x^2 - 1}) = i \log(x + i\sqrt{1 - x^2}).$$
The complex number $x + i\sqrt{1 - x^2} \equiv \rho(x)e^{i\theta(x)}$ has modulus $\rho(x) = 1$ and argument $\theta(x)$ such that $\cos\theta(x) = x$, $\sin\theta(x) = \sqrt{1 - x^2}$ and so $\theta(x) = \arccos x$, restricted so that $0 < \theta(x) < \pi$.

For complex z, we have to first define a branch of $\sqrt{z^2 - 1}$. Let $u = z^2 - 1$. A branch of $\sqrt{u}$ is obtained by the requirements $u \in \mathbb{C} - \{u \le 0\}$, $\Re\sqrt{u} > 0$. But $u = x^2 - y^2 - 1 + i2xy$. So if we want to exclude the real and negative values of u we must exclude the numbers z for which $x^2 - y^2 - 1 \le 0$ whenever $y = 0$. So we must exclude all z for which $x^2 \le 1$. But this would not be compatible with the restriction of $\arccos z$ on $z \in \mathbb{R}$. So we take another branch obtained by the requirement $u \in \mathbb{C} - \{u \ge 0\}$. This is most easily done as follows. Let $f(u) = \sqrt{u}$ be the usual principal branch of the square root. Define the branch $f_1(u) := if(-u)$. Clearly, $f_1(u)^2 = i^2 f(-u)^2 = (-1)(-u) = u$ and if the restriction on the argument of f is that it is positive if its is real, the restriction on the argument of f_1 is that it is negative when it is real. This translates into the restriction that $|z| < 1$ when $z = x \in \mathbb{R}$, as required.

In addition to that, we have to make sure that the appropriate restrictions on the logarithm are taken care of. These may impose additional requirements. But on the domain

$$\Omega := \mathbb{C} - \{z \in \mathbb{R} : |z| \ge 1\}$$

the numbers $z^2 + \sqrt{z^2 - 1}$ are never real, so there is no additional requirement if let log be the principal branch of the logarithm.

We have thus defined a single-valued analytic branch of $\arccos z$ on the domain Ω. We can check that $\arccos(\Omega) = \{w \in \mathbb{C} :\ 0 < \Im w < \pi\}$. It is better to write

$$\arccos z = i \log(z + i\sqrt{1 - z^2})$$

where the notation used is that $\sqrt{\cdot}$ is the principal branch of the square root and $\log(\cdot)$ is the principal branch of the logarithm, So $\arccos(\cdot)$ is obtained as the composition of analytic functions. We can then comfortably differentiate it to obtain

$$D\arccos z = i\frac{1}{z+i\sqrt{1-z^2}}\left(1+i\frac{1}{2\sqrt{1-z^2}}(-2z)\right) = \frac{1}{\sqrt{1-z^2}}.$$

Conformality

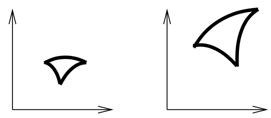
A transformation (between Euclidean spaces or Riemannian manifolds) which preserves angles locally is called *conformal* We are interested in conformal maps on $\mathbb{C}$. For example, every homothety $z \mapsto az$ with $a \neq 0$ is conformal. More generaly, every analytic function is locally conformal at any point z_0 at which $f'(z_0) \neq 0$. Indeed, let $\gamma(t)$ be a curve passing through z_0 at $t = t_0$ and let $\gamma'(t_0)$ be nonzero. Then

$$(f\circ\gamma)'(t_0) = f'(z_0)\gamma'(t_0)$$

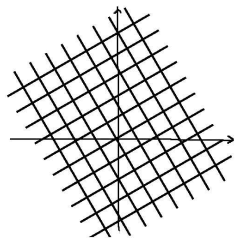
and so

$$\arg(f\circ\gamma)'(t_0) = \arg\gamma'(t_0) + \arg f'(z_0).$$

This implies that if u, v are vectors tangent to two curves passing through the point z_0 and having angle θ between them then the images of these curves have tangent vectors at $f(z_0)$ with angle again θ.

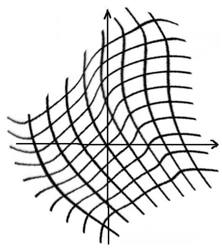


Another kind of conformality is scale-conformality in that, locally at z_0, the function scales by $|f'(z_0)|$. In other words, whenever $f'(z) \neq 0$, the analytic function f behaves like a homothety.



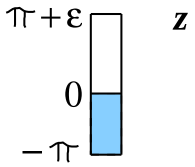
By, say, the inverse function theorem, conformality at a point z_0 of a function $f(z)$ means that f is a local diffeomorphism between a neighbourhood U of z_0 and a neighbourhood V of $f(z_0)$.

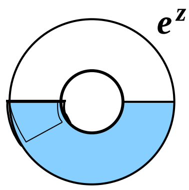
A problem of importance is to find a way to map a domain Ω_1 conformally onto another domain Ω_2.



We can aid our visualisation of a conformal map $w = f(z)$ by seeing how a certain system of coordinates in the z-plane transforms in the w-plane. If we have a system (X, Y) of orthogonal coordinates on the z-plane then, by conformality, they transform into a system (U, V) of orthogonal coordinates on the w-plane.

Standard conformal mappings

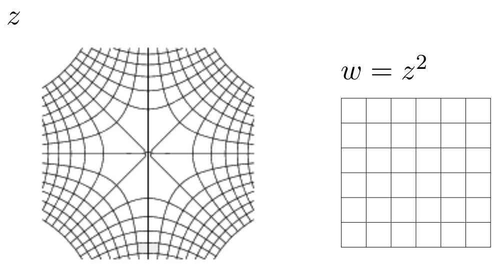




It appears that when we say we want to map a domain Ω_1 conformally into a domain Ω_2 we really mean conformally and isomorphically (in a topological sense). Indeed, a mapping can be conformal at all points of Ω_1 but may not be one-to-one (schlicht, univalent). For instance, the map $f(z) = e^z$ is entire with non-vanishing derivative, i.e. conformal everywhere, but it is not one-to-one. In other words, we are looking for global diffeomorphisms between Ω_1 and Ω_2 which, moreover, are complex analytic maps.

Some examples:

$\boxed{w = z^2}$ wraps the plane around twice because $\arg w = 2 \arg z$. So the upper half $x > 0$ plane maps onto $\mathbb{C}$ except the positive real axis. More generally, a sector $S(\varphi_1, \varphi_2)$ containing all z with $\varphi_1 < \arg z < \varphi_2$, maps into the sector $S(2\varphi_1, 2\varphi_2)$, which could be a half plane or could be covering the whole plane more than once. The Cartesian net on the w-plane is transformed into a system of two families of mutually orthogonal hyperbolas.



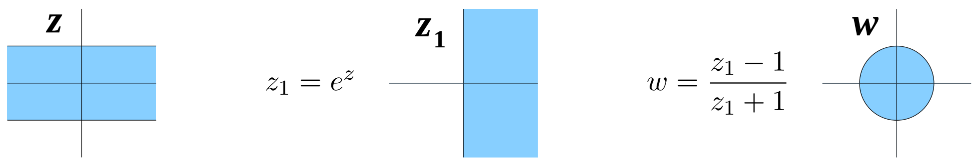
The reason is that $z^2 = (x^2 - y^2) + i2xy$ so the families of hyperbolas are

$$x^2 - y^2 = u_0, \quad 2xy = v_0.$$

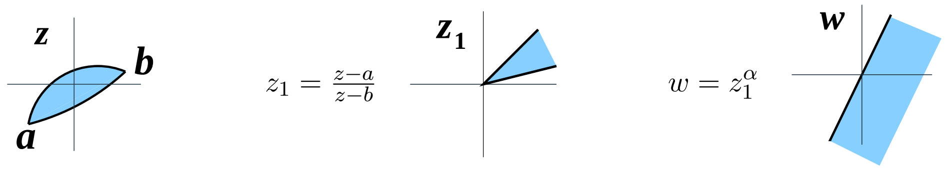
$\boxed{w = e^z}$ maps the Cartesian net on the z-plane onto polar net on the w-plane. A line $x = x_0$ is mapped into a circle $|w| = e_0^x$. A line $y = y_0$ is mapped into a half line $w = e^x e^{iy_0}$, or $w = te^{iy_0}$, $t > 0$. A horizontal strip $y_1 < y < y_2$ is mapped onto an angular sector which does not cover the plane if $y_2 - y_1 \leq 2\pi$. Otherwise, it wraps around and covers the plane. A vertical strip $x_1 < x < x_2$ maps onto an annulus $e^{x_1} < |w| < e^{x_2}$. A parallelogram $x_1 < x < x_2, y_1 < y < y_2$ is thus mapped onto a part of an annulus cut by an angular sector (which could be the whole annulus wrapped around more than once.

$\boxed{w = \frac{z-1}{z+1}}$ maps, of course, (generalised) circles into (generalised) circles. In particular, it maps the right half-plane $x > 0$ onto the disk $|w| < 1$. To see this, observe that $x = 0$ is mapped into $w = \frac{iy-1}{iy+1}$ which has modulus one, i.e. it is the unit circle $|w| = 1$. The point $z = 1$ is mapped into $w = 0$ and so $x > 0$ maps inside the circle. The upper half-plane $y > 0$ maps into itself. Indeed, $y = 0$ maps into $w = \frac{x-1}{x+1}$ which is real, i.e. the real axis. Also, $z = i$ maps into a point with positive imaginary part. We thus find that the psotive orthant $xy > 0$ maps onto the upper half-disk $|w| < 1, \Im w > 0$. The point 1 is mapped to 0 and -1 to ∞, so the transformation shrinks the right half-plane into the unit disk.

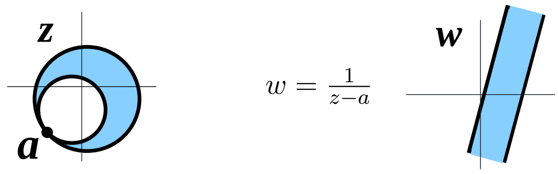
$\boxed{w = \frac{e^z - 1}{e^z + 1}}$ is the result of $z_1 = e^z$, followed by $w = \frac{z_1 - 1}{z_1 + 1}$. Consider the strip $|y| < \pi/2$. This is mapped to $\Re z_1 > 0$ and this to $|w| < 1$.



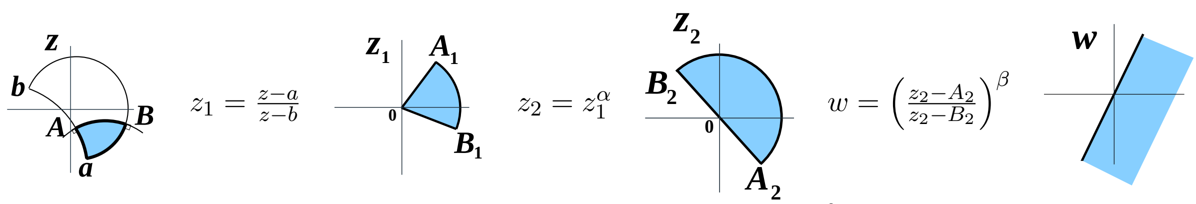
Circular wedge with endpoints a, b is contained within two circles passing through a, b. The Möbius transformation $z_1 = \frac{z-a}{z-b}$ sends a, b to $0, \infty$, and so the circular wedge is mapped into an angular sector. The subsequent map $w = z_1^\alpha$, for appropriate α maps the angular sector onto a half plane.



Tangent circles Suppose that a circle C_1 is tangent to C_2 at the point a. Then we can map the region between them onto a parallel strip by the tranformation $z_1 = 1/(z-a)$, because this is a Möbius transformation that sends a to ∞.

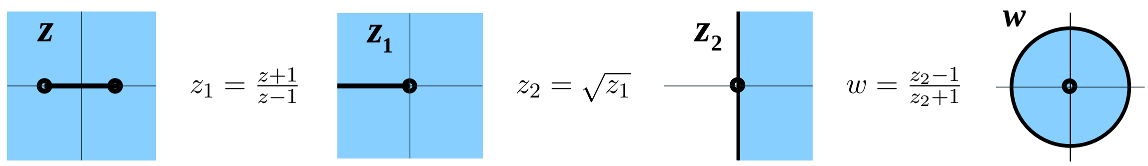


Circular triangle with two right angles Suppose that cirles C_1, C_2 meet C at right angles at the points A, B and let a, b be the common points of C_1, C_2. We can map the circular triangle aAB via $z_1 = \frac{z-a}{z-b}$ onto a circular sector $0A_1B_1$. Using $z_2 = z_1^\alpha$ we can map the circular sector onto a half circle. The half circle is a special case of a circular wedge with endpoints A_2, B_2 which can be mapped onto a half plane via $w = \left(\frac{z_2-A_2}{z_2-B_2}\right)^\beta$.



The full transformation here is: $w = \left(\frac{(A-b)^\alpha(z-a)^\alpha-(A-a)^\alpha(z-b)^\alpha}{(B-b)^\alpha(z-a)^\alpha-(B-a)^\alpha(z-b)^\alpha}\right)^\beta$.

Segment of a straight line between two point, say, $-1, +1$. We want to map this to a circle. If we first use $z_1 = \frac{z+1}{z-1}$, we send $+1, -1$ to $\infty, 0$, respectively and the segment to the negative real axis. If we then use $z_2 = \sqrt{z_1}$ we halve the plane and map it to the right half-plane. Finally, $w = \frac{z_2-1}{z_2+1}$ maps the right half-plane onto $|w| < 1$.

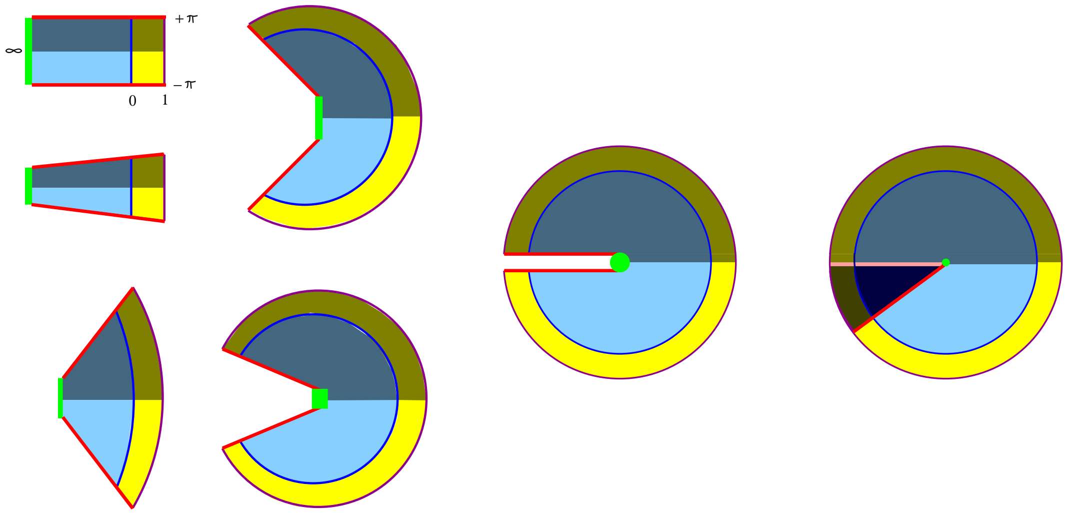


Riemann surfaces

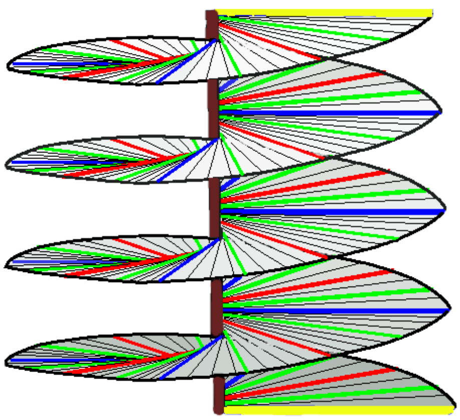
A Riemann surface is a 2-dimensional real analytic manifold that is used, roughly speaking, to transform a multi-valued function into a single-valued one.

It is easier to give examples that involve single-valued but not schlicht (univalent, one-to-one) finctions $w = f(z)$. For such functions, we extend their range from $\mathbb{C}$ to some appropriate Riemann surface so that the extended function becomes schlicht and hence invertible.

A picture is worth 1000 words, so here is how the Riemann surface for the range of the non-schlicht function $w = \exp z$ is obtained.



The picture should be read column-wise, like a motion picture. The first image is the strip $-\pi < \Im Z < \pi$, $-\infty < \Re z < 1$ in the z-plane. This strip is transformed into the region $|w| < 1$ but whith the points $\{w : \Im w = 0,\ \Re w < 0\}$ excluded; see the pre-last image. To "obtain" this, we can envision a transformation of the strip that opens up while bringing the point $-\infty$ (green area) into a finite point and, finally, mapping it onto the origin. The red lines represent the upper and lower boundaries of the strip; they rotate arounf until rhey finally coincide as the negative real axis becomes the so-called branch-cut for the Riemann surface. If we extend the strip above its upper level then the final image keeps turning and starts overlapping. Instead of overlapping on the same plane we move it on a higher plane and start creating a helicoidal surface. This is the Riemann surface that extends the range of w. If we let z range over the strip extended up and down indefinitely (i.e. $-\infty < \Im z < + < \infty$, $-\pi < \Im z < + < \pi$) we obtain a helix with infinitely many spirals. If finally we let $\Re z$ extend to the right beyond the point $+1$, we obtain a helix with unbounded spirals.



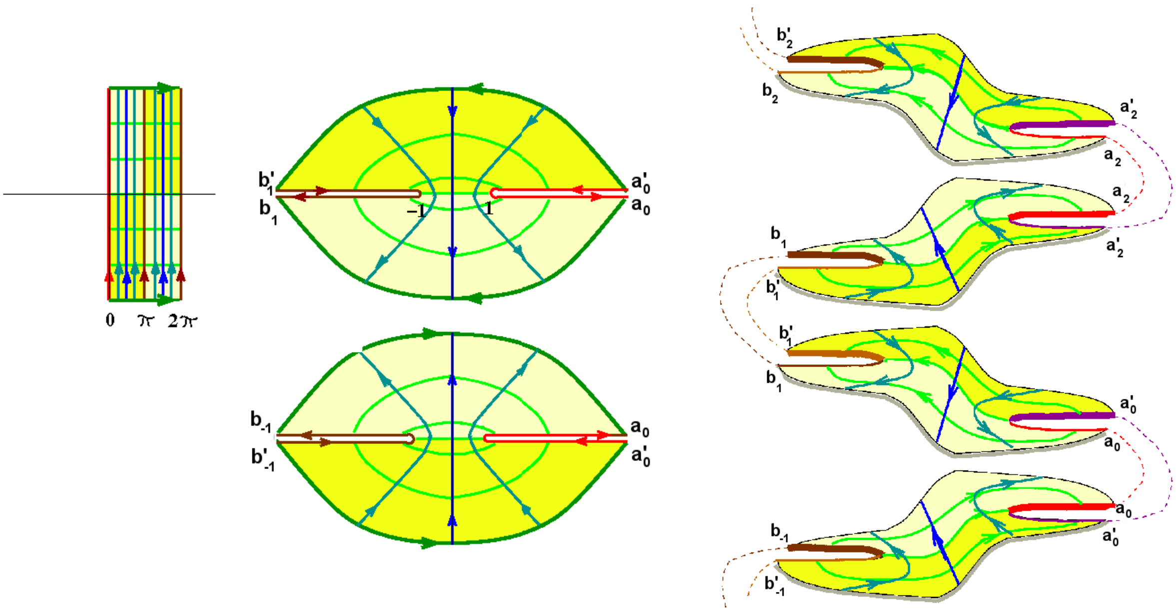
We now describe the Rieman surface for $w = \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$. To find out what is going on, we cover the z-plane with two families of curves and see how they are transformed in the w-plane.

Notice that $\cos z$ is periodic with fundamental period 2π and that $\cos(z + \pi) = -\cos z$. So it suffices to see what is going on in the strips $0 \le x \le \pi$ and $\pi \le x \le 2\pi$.

Consider the curve $x = 0$. It is transformed to $w = \frac{1}{2}(e^{-y} + e^{y}) = \cosh y$, $y \in \mathbb{R}$. Similarly, $x = \pi$ is transformed to $w = -\cosh y$, $y \in \mathbb{R}$.

Consider now the curves $x = x_0$ for $0 < x_0 < \pi$. They transformed into $w = \frac{1}{2}(e^{ix_0}e^{-y} + e^{-ix_0}e^{y})$, $y \in \mathbb{R}$. In other words, $u = \cos x_0 \cosh y$, $v = -\sin x_0 \sinh y$. Since $0 < x_0 < \pi$, we have $\cos x_0 > 0$, $\sin x_0 > 0$ and so $dv/du \approx 1$ as $y \to -\infty$, while $dv/du \approx -1$ as $y \to +\infty$. For $\pi < x_0 < 2\pi$, the curve $x = x_0$ is transformed into the mirror image of the above with respect to the imaginary axis.

Next consider $y = y_0$. This is transformed into $w = \frac{1}{2}(e^{ix}e^{-y_0} + e^{-ix}e^{y_0})$, $x \in \mathbb{R}$. If $y_0 = 0$ we obtain $w = \cos x$, $x \in \mathbb{R}$, whose image is the segment $-1 \le u \le 1, v = 0$. When $y = y_0 > 0$ we obtain a curve in the upper half-plane that starts from the right and moves to the left. When $y = y_0 < 0$ we obtain a curve in the upper half-plane that starts from the left and moves to the right.



It is clear that the region $0 < x < \pi, y < 0$ is transformed into the half-plane, while $0 < x < \pi, y > 0$ into the lower half-plane. The opposite happens with the regions $\pi < x < 2\pi, y < 0$ and $\pi < x < 2\pi, y > 0$.

So the strip $0 < x < \pi$ is transformed into the whole plane. Continuing with the strip $\pi < x < 2\pi$ we see that it is transformed into the whole plane again, but "in reverse orientation".

Each strip $S_k = \{z : (k-1)\pi < x < k\pi\}$ is a *fundamental region* for $\cos z$ in that it maps onto the whole plane with the exception of a cuts. By sewing the regions $\cos(S_k)$, $k \in \mathbb{Z}$, appropriately together, we obtain the Riemann surface for $\cos z$. It is impossible to do the sewing in $\mathbb{R}^3$ in an honest way. We need an extra dimension for this. In other words, the Riemann surface for $\cos z$ is embaddable only in 4 dimensions and up.

Integration

A curve γ is a function from some interval $[a, b] \subset \mathbb{R}$ into $\mathbb{C}$. The point $\gamma(a)$ is the initial point of the curve, while $\gamma(b)$ is the final point. There is a natural ordering of the elements of $\gamma[a, b]$: we say that $\gamma(s)$ precedes $\gamma(t)$ if $s \le t$. Let π be a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b]$. For each such π define

$$L(\pi) = \sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})|$$

The length of the curve is defined as $L = \sup_\pi L(\pi)$. We say that γ is rectifiable if $L < \infty$. Equivalently, γ is rectifiable iff $\Re\gamma$ and $\Im\gamma$ are functions of bounded variation. A curve is differentiable of its real and imaginary parts are differentiable. We let $\gamma'(t)$ be the derivative of a differentiable curve at the point t. Then its length is given by $L = \int_a^b |\gamma'(t)|dt$. A curve is piecewise differentiable if it is differentiable everywhere except a finite number of points; its length is given by the same expression. For a continuous function $f : \Omega \to \mathbb{R}$, a curve $\gamma : [a, b] \to \Omega$, and a partition π of $[a, b]$, define

$$I(f, \pi) := \sum_{k=1}^{n} f(t_k)[\gamma(t_k) - \gamma(t_{k-1})].$$

Let also $||\pi||$ be the largest $t_k - t_{k-1}$, for $k = 0, \ldots, n$. Then there is a number $A \in \mathbb{C}$ such that for all $\varepsilon > 0$ there is $\delta > 0$ such that if $||\pi|| \le \delta$ then $|I(f, \pi) - A| \le \varepsilon$. The number A is denoted by $\int_\gamma f(z)dz$ and is called the complex integral of f along γ. We can prove that if we defined $I(f, \pi) = \sum_{k=1}^{n} f(t_{k-1})[\gamma(t_k) - \gamma(t_{k-1})]$ the limit is the same. If the curve is piecewise differentiable, we have

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Many results regarding curves do not depend explicitly on the parametrisation of the curve. The integral is one of them. A parametrisation of the curve is an order preserving piecewise differentiable map τ from some $[\alpha, \beta]$ into $[a, b]$. Then

$$\int_a^b f(\gamma(t))\gamma'(t)dt = \int_\alpha^\beta f(\gamma(\tau(s)))\gamma'(\tau(s))\tau'(s)ds = \int_\alpha^\beta f((\gamma\circ\tau)(s))(\gamma\circ\tau)'(s))ds.$$

We let $-\gamma$ be the curve obtained by an order-reversing parametrisation, e.g. by $\tau(t) = -t$. Then

$$\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz.$$

When we have a curve γ and pick points $\gamma(t_0), \ldots, \gamma(t_n)$, where $a = t_0 < \cdots < t_n = b$, we can split γ into the subcurves γ_k with endpoints $\gamma(t_{k-1})$, $\gamma(t_k)$, $k = 1, \ldots, n$. We write, symbolically,

$$\gamma = \sum_{k=1}^{n} \gamma_k$$

and then we have

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)dz.$$

When p, q are real-valued functions on $\mathbb{C}$, we can define of the integral

$$\int_\gamma pdx + qdy$$

as the integral of the 1-form $\omega = pdx + qdy$, where x, y are cartesian coordinates of $\mathbb{R}^2$, over the curve γ through the pullback operation:

$$\int_\gamma \omega = \int_{(a,b)} \gamma^* \omega,$$

i.e. by referring to the natural parameter space (=time). Associated to a continuous function $f = u + iv$ we have two 1-forms:

$$\omega := udx - vdy, \quad \eta := udy + vdx.$$

We can then easily show that

$$\int_\gamma f(z)dz = \int_\gamma \omega + i \int_\gamma \eta.$$

Formally, this is due to

$$fdz = (u + iv)(dx + idy) = (udx - vdy) + i(vdx + udy).$$

We can also define integral with respect to the measure induced by the length. In case of a piecewise differentiable curve, we have:

$$d|\gamma| := |\gamma'(t)| \ dt.$$

It is customary to write

$$\int_\gamma f \ |dz| := \int_\gamma f \ d|\gamma|.$$

We can then easily prove the inequality

$$\left| \int_\gamma fdz \right| \leq \int_\gamma |f| \ |dz|.$$

Finally, we define

$$\int_\gamma f \ \overline{dz} := \overline{\int_\gamma \overline{f} dz}$$

and

$$\int_\gamma fdx = \frac{1}{2} \left(\int_\gamma fdz + \int_\gamma f \ \overline{dz} \right)$$
$$\int_\gamma fdy = \frac{1}{2i} \left(\int_\gamma fdz - \int_\gamma f \ \overline{dz} \right)$$

Dependence on endpoints

Suppose $f(z)$ is continuous on Ω. When does $\int_\gamma f dz$ depend only on then endpoints of the curve γ for every γ in Ω? The answer is that both forms $udx - vdy$ and $vdx + udy$ must be closed, i.e. there must exist functions A, B such that $udx - vdy = dA$, $vdx + udy = dB$, or $u = A_x, -v = A_y$, $v = B_x, u = B_y$, which necessitates that $A_x = B_y$, $-A_y = B_x$. So A, B form a conjugate pair. Equivalently then, there must be an analytic function $F = A + iB$ on Ω, such that $u + iv = A_x + B_x = B_y - iA_y$, which further means that $f(z) = F'(z)$. We have thus proved:

> *$\int_\gamma f dz$ depends only on then endpoints of the curve γ for every γ in Ω if and only if there exists $F(z)$, analytic on Ω, such that $f(z) = F'(z)$.*

This statement is further equivalent to:

> *For every closed curve γ in Ω, $\int_\gamma f dz = 0$.*

Notes: We must stress that this result does not require that Ω be simply connected. Whether an anlytic function $f(z)$ has a primitive (i.e. is the derivative of an analytic function) is of importance and depends both on $f(z)$ and on Ω. For instance, the function $f(z) = 1/z$ has no primitive on $\{|z| < 1\}$ because $\int_\gamma (1/z)dz = 2\pi i$ (see below) for any circle γ enclosing the origin. However, if we take $\Omega = \{|z-2| < 1\}$, say, then $1/z$ does have a primitive on Ω. On the other hand, any other integer power of z, i.e. any function of the form $f(z) = z^p$ with $p \in \mathbb{Z} - \{-1\}$ has an antiderivative on a punctured neighbourhood $\{0 < |z| < 1\}$ of zero. If, moreover, $p \in \mathbb{Z}_+$ then z^p has antiderivative everywhere.

Some simple integrals

Since $(z-a)^n$, for $n \in \mathbb{Z}_+$, is the derivative of the analytic function $(n+1)^{-1}(z-a)^{n+1}$, we have, for every closed curve γ,

$$\int_\gamma (z-a)^n dz = 0.$$

For $n = -1$, the function $(z-a)^{-1}$ is not the derivative of an analytic function (it is the derivative of $\log(z-a)$ which is neither analytic–it has a singularity at a–not a function–it is multivalued). So the integral can only be decided, for the time being, by computation. For instance, if $\gamma = C(a, r)$, a circle with centre a and radius r, we have

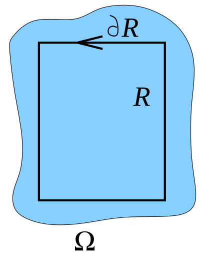
$$\int_{C(a,r)} \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{it}}{re^{it}} dt = 2\pi i.$$

If $n = -m \leq -2$ then $(z-a)^{-m}$ is the derivative of $-(m-1)^{-1}(z-a)^{-(m-1)}$. This functions is analytic everywhere except a. Therefore it is analytic on the domain $\Omega - \{a\}$. So if γ is a curve in Ω, not passing through a, then, for any integer $m \geq 2$,

$$\int_\gamma \frac{dz}{(z-a)^m} = 0.$$

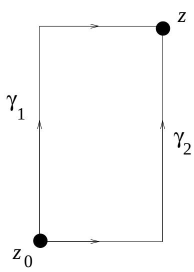
The difference with the $m = -1$ case is that the $\log(z-a)$ cannot be made analytic by the removal of a, simply because it will not be a function.

Simple form of Cauchy's theorem



We first prove that
A. If $f(z)$ is analytic on a closed rectangle R (meaning that it is analytic on some domain containing R) then $\int_{\partial R} f(z)dz = 0$.
We shall use this to prove that
B. If $f(z)$ is analytic on an open disk Δ then $\int_\gamma f(z)dz = 0$ for any closed curve γ in Δ.

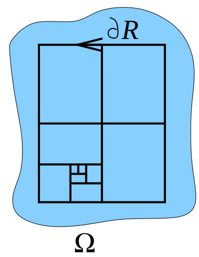
We first indicate the passage from A to B. Suppose A has been proved and that $f(z)$ is analytic on Δ. We prove that $f(z) = F'(z)$ for some analytic function $F(z)$ on Ω. This is precisely the theorem that says that if a form is closed then it is exact. So we may view the fact that $\int_{\partial R} f(z)dz = 0$ over each rectangle R as that both forms $udx - vdy$, $vdx + udy$ are closed and therefore exact, which means that $f(z)$ is the derivative of some analytic $F(z)$, as argued above, which further means that $\int_\gamma f(z)dz = 0$.



Alternatively, we can ignore the forms point of view and do a bare-hands proof. Fix $z_0 = x_0 + iy_0, z = x + iy \in \Delta$ and consider the rectangle R with vertices z_0, z and sides parallel to the axes. Let γ_1 be the curve consisting of the segment joining z_0 to (x_0, y), followed by the segment joining (x_0, y) to z. Let $F(x, y) = \int_{\gamma_1} f(\zeta)d\zeta$. Let γ_2 be the curve consisting of the segment joining z_0 to (x, y_0), followed by the segment joining (x, y_0) to z. The fact that $\int_{\partial R} f(\zeta)d\zeta = 0$ translates into $F(x, y) = \int_{\gamma_1} f(\zeta)d\zeta = \int_{\gamma_2} f(\zeta)d\zeta$. Taking derivative with respect to x in $F(x, y) = \int_{\gamma_1} f(\zeta)d\zeta$ we find that $f(z) = \frac{\partial F}{\partial x}(x, y)$ and taking derivative with respect to y in $F(x, y) = \int_{\gamma_2} f(\zeta)d\zeta$ we find that $f(z) = -i\frac{\partial F}{\partial y}(x, y)$. Hence $\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$, which means that $\Re F$, $\Im F$ satisfy the CR relations and so $F(z)$ is analytic; furthermore, $f(z) = F'(z)$.

To prove A we argue as follows: First, let

$$\eta(S) = \int_{\partial S} f dz$$



for any rectangle S. Split R into 4 equal subrectangles S_1, S_2, S_3, S_4 and write

$$\eta(R) = \sum_{j=1}^{4} \eta(S_j),$$

whence $|\eta(S_j)| \geq \frac{1}{4}|\eta(R)|$, for some $j = 1, 2, 3, 4$. Split S_j into 4 further subrectangles S_{jk}, $k = 1, 2, 3, 4$ and conclude that $|\eta(S_{jk})| \geq \frac{1}{4}|\eta(S_j)| \geq \frac{1}{4^2}|\eta(R)|$ for some $k = 1, 2, 3, 4$. Continuing this way, we have: There is a sequence of nested rectangles $R \supset R_1 \supset R_2 \supset \cdots$ such that

$$|\eta(R_n)| \geq \frac{1}{4^n}|\eta(R)|, \quad n \in \mathbb{N}.$$

Furthermore, $\cap_n R_n$ is a singleton $\{a\}$. Since $f(z)$ is analytic at a, if we fix $\varepsilon > 0$, we can find $\delta(\varepsilon)$ such that

$$|f(z) - f(a) - (z - a)f'(a)| \leq \varepsilon|z - a|$$

provided that $z \in B(a, \delta(\varepsilon))$. In fact, choose ε so small so that $B(a, \delta(\varepsilon)) \subset \Omega$, and choose an n so large so that $R_n \subset B(a, \delta(\varepsilon))$. Then write

$$\eta(R_n) = \int_{\partial R_n} f(z)dz = \int_{\partial R_n} [f(z) - f(a) - (z - a)f'(a)]dz,$$

simply because $\int_{\partial R_n} z^k dz = 0$, for $k = 0, 1$. We then have

$$|\eta(R_n)| \le \varepsilon \int_{\partial R_n} |z-a|\ |dz| \le \varepsilon\ \operatorname{diam}(R_n)\ \operatorname{length}(\partial R_n).$$

But $\operatorname{diam}(R_n) = 2^{-n}\operatorname{diam}(R)$, $\operatorname{length}(\partial R_n) = 2^{-n}\operatorname{length}(\partial R)$. Hence

$$|\eta(R_n)| \le \varepsilon \int_{\partial R_n} |z-a|\ |dz| \le \varepsilon 4^{-n}\ \operatorname{diam}(R)\ \operatorname{length}(\partial R).$$

Combining with the first inequality, we have

$$|\eta(R)| \le \varepsilon\ \operatorname{diam}(R)\ \operatorname{length}(\partial R),$$

and since ε is arbitrarily small, we have $\eta(R) = 0$.

Removable singularities

Suppose $f(z)$ is analytic on $\Omega - \{z_0\}$. We say that $f(z)$ has a removable singularity at z_0 if

$$\lim_{z\to z_0} (z - z_0) f(z) = 0.$$

The point is that $f(z)$ may fail to be analytic on z_0 but the lack of analyticity is of a very benign nature. For instance, $f(z) = sin(z)/z$ is not defined at 0 but has a removable singularity at 0. We will later see that a function with removable singularity at a point z_0 is the restriction of a unique analytic function on Ω. But, given that we don't know this, we need to prove that functions with removable singularities still satisfy Cauchy's theorem.

First, let R be a rectangle and $f(z)$ a function which is analytic on R with the exception of removable singularities $\zeta_1, \ldots, \zeta_k$ in the interior of R. We show that

$$\int_{\partial R} f dz = 0.$$

Suffice to prove this if we assume that there is only one removable singularity ζ. Let $R_0 \subset R$ be a rectangle centred at ζ, so small that

$$|f(z)| \le \frac{\varepsilon}{|z-\zeta|},$$

for all $z \in R_0 - \{\zeta\}$. But

$$\int_{\partial R} f dz = \int_{\partial R_0} f dz,$$

and so

$$\left|\int_{\partial R} f dz\right| \le \varepsilon \int_{\partial R_0} \frac{|dz|}{|z-\zeta|}$$

Let α be the length of the side of R_0. Then $|z - \zeta| \ge \alpha/2$, for all $z \in \partial R_0$, while the length of ∂R_0 is 4α. Thus

$$\int_{\partial R_0} \frac{|dz|}{|z-\zeta|} \le \frac{4\alpha}{\alpha/2} = 8.$$

We thus have that $\left|\int_{\partial R} f dz\right| \le 8\varepsilon$ for all $\varepsilon > 0$ and we are done.

The theorem can be adopted to the case of a general closed curve in a disk:

If $f(z)$ is analytic on an open disk Δ with the exception of finitely many removable singularities, then for any closed curve γ in Δ, we have $\int_\gamma f dz = 0$.

Curve windings

Suppose that γ is a (piecewise differentiable) closed curve. Then for any a not lying on the curve,

$$\int_\gamma \frac{dz}{z-a} = 2\pi i n,$$

for some $n \in \mathbb{Z}$.

Proof. Suppose $\gamma : [\alpha, \beta] \to \mathbb{C}$. Then

$$\int_\gamma \frac{dz}{z-a} = \int_\alpha^\beta \frac{\gamma'(s)ds}{\gamma(s)-a}.$$

Let

$$h(t) := \int_\alpha^t \frac{\gamma'(s)ds}{\gamma(s)-a}, \quad \alpha \le t \le \beta.$$

We have

$$h'(t) = \frac{\gamma'(t)}{\gamma(t)-a},$$

and so

$$\frac{d}{dt} e^{-h(t)} \, (\gamma(t)-a) = 0.$$

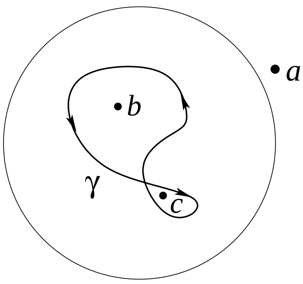
Thus, $e^{-h(t)} \, (\gamma(t)-a)$ is a constant. So $e^{-h(\beta)} \, (\gamma(\beta)-a) = e^{-h(\alpha)} \, (\gamma(\alpha)-a)$. But $\gamma(\alpha) = \gamma(\beta)$ and $h(\alpha) = 0$, so

$$e^{h(\beta)} = 1.$$

This means that $h(\beta)$ is $2\pi i$ times an integer. □

This motivates the definition of the index (or winding number) of a closed curve γ with respect to a point a:

$$\mathfrak{n}(\gamma, a) := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}.$$



$\mathfrak{n}(\gamma, a) = 0$
$\mathfrak{n}(\gamma, b) = 1$
$\mathfrak{n}(\gamma, c) = -1$

Clearly, $\mathfrak{n}(-\gamma, a) = -\mathfrak{n}(\gamma, a)$. If γ lies in a disk Δ and a lies in the interior of the complement of Δ then $\mathfrak{n}(\gamma, a) = 0$ because $1/(z-a)$ is then analytic on Δ. The connected components of the interiors of the sets $\{a \in \mathbb{C} : \mathfrak{n}(\gamma, a) = n\}$ for various values of $n \in \mathbb{Z}$ are regions determined by γ. We prove that $\mathfrak{n}(\gamma, a_1) = \mathfrak{n}(\gamma, a_2)$ if the curve does not meet the segment with endpoints a_1, a_2. Indeed, the function $\log[(z-a_1)/(z-a_2)]$ is single-valued and analytic off this segment. Hence its derivative must integrate to zero over γ:

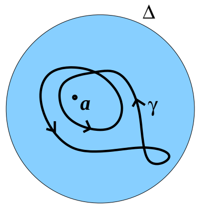
$$\int_\gamma \left(\frac{1}{z-a_1} - \frac{1}{z-a_2} \right) dz = 0,$$

which gives what we want. One of the regions determined by γ is unbounded. So if a is in this region then γ is contained in an open disk Δ and a is contained in the complement of the closure of Δ. And so $\mathfrak{n}(\gamma, a) = 0$ for all a in the unbounded region.

Integral formula

If $f(z)$ is analytic on an open disk Δ and γ is a closed curve in Δ then, for any a not lying on γ,

$$\int_\gamma \frac{f(z)}{z-a}dz = 2\pi i\, \mathfrak{n}(\gamma, a) f(a).$$



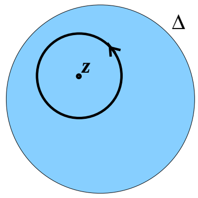
Proof. Consider the function $F(z) = \frac{f(z)-f(a)}{z-a}$. Since $(z-a)F(z) = f(z) - f(a) \to 0$ as $z \to a$, the point a is a removable singularity for $F(z)$, therefore $\int_\gamma F(z)dz = 0$. $\square$

N.B. The integral formula holds even in presence of removable singularities as long as they do not belong to the curve γ nor the point a is one of them.

Representation formula

For any z in the domain of analyticity of $f(z)$ and any closed curve γ for which $\mathfrak{n}(\gamma, z) = 1$, we have

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z}.$$



In particular, if $\gamma = C(z, r)$, we have

$$f(z) = \frac{1}{2\pi i}\int_{C(z,r)} \frac{f(\zeta)d\zeta}{\zeta - z} = \frac{1}{2\pi}\int_0^{2\pi} f(z + re^{it})dt.$$

Derivatives

If $f(z)$ is analytic then the representation formula holds. We can differentiate the representation formula with respect to z and pass the derivative inside the integral sign, by DCT. This gives a representation formula for the derivative:

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2}.$$

This immediately shows that the function $f'(z)$ is also analytic and its derivative satisfies

$$f''(z) = \frac{1}{\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^3}.$$

Continuing in the same manner we have that an analytic function possesses derivatives of all orders, which are all analytic functions and

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^{n+1}}.$$

Cauchy's estimate:

$$|f^{(n)}(z)| \le \frac{n!}{2\pi}\int_C |f(\zeta)| \frac{|d\zeta|}{|\zeta - z|^{n+1}} \le \frac{n!}{2\pi}\max\{|f(\zeta)| : \zeta \in C\}\frac{\text{length}(C)}{r^{n+1}} = \frac{n!}{r^n}\max_{|\zeta - z| = r\}}|f(\zeta)|.$$

Morera's theorem

If $f(z)$ is a continuous function on some domain Ω such that $\int_\gamma f dz = 0$ for all closed curves γ in Ω then $f(z)$ is analytic.

Proof. If $\int_\gamma f dz = 0$ for all closed curves γ in Ω then $f(z)$ is the derivative of an analytic function and hence it is itself analytic. Conversely, if $f(z)$ is analytic then $\int_\gamma f dz = 0$ for all closed curves γ in Ω.

Liouville's theorem

If $f(z)$ is entire (analytic on the whole $\mathbb{C}$) and bounded then it is a constant.

Proof. Let M be a global bound for $f(z)$. Then, by Cauchy's estimate,

$$|f'(z)| \leq \frac{1}{r}M,$$

for all $r > 0$. Hence $f'(z) = 0$, i.e. $f(z)$ is a constant.

The fundamental theorem of Algebra

If $P(z)$ is a nonconstant polynomial then it has a zero.

Proof. If not, then $1/P(z)$ is an entire function (analytic on $\mathbb{C}$). If m is the degree of $P(z)$ then, for $|z| > 1$ we have $|P(z)| \geq c|z|$ for some $c \in \mathbb{C}$. Therefore $P(z) \to \infty$ as $z \to \infty$ and so $1/P(z) \to 0$. The function $1/P(z)$ is thus bounded on $\mathbb{C}$ and so it must–by Liouville's theorem–be constant, in violation to the assumption. $\square$

Of course, we can continue and show that every polynomial of degree m has exactly m zeros (not necessarily distinct) $a_1, \ldots, a_m$, and that $P(z) = b(z - a_1) \cdots (z - a_m)$. Indeed, by the fundamental theorem of Algebra, $P(z)$ has a zero a and so $P(z) = (z - a)Q(z)$ where $Q(z)$ is a polynomial of degree $m - 1$. The result follows by induction.

The removable singularities can be removed

Suppose $f(z)$ is analytic on $\Omega - \{a\}$ where a is a removable singularity. Then there exists a unique analytic function on Ω which coincides with $f(z)$ on $\Omega - \{a\}$.

Proof. Let r be such that $B = \{z : |z - a| < r\} \subseteq \Omega$. Define

$$g(z) := \frac{1}{2\pi i} \int_{\partial B} \frac{f(\zeta) d\zeta}{\zeta - z}, \quad z \in B,$$

a function which is analytic on B, and

$$F(z) := \begin{cases} g(z), & z \in B \\ f(z), & z \in \Omega \setminus B \end{cases},$$

a function which is analytic on Ω. Since the integral formula is valid in presence of removable singularities, we have

$$f(z) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(\zeta) d\zeta}{\zeta - z} = g(z), \quad z \in B - \{a\}.$$

Hence $f(z) = F(z)$ for all $z \in \Omega - \{a\}$. $\square$

Taylor's formula

If $f(z)$ is analytic in a neighbourhood Ω of a point a then, for all $n \in \mathbb{N}$, there is an analytic function $f_n(z)$ such that

$$f(z) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (z - a)^k + f_n(z)(z - a)^n.$$

Proof. Let

$$F(z) = (f(z) - f(a))/(z - a), \quad z \in \Omega - a.$$

Then a is a removable singularity for $F(z)$. Since $F(z) \to f'(a)$ as $z \to a$, the function

$$f_1(z) := \begin{cases} \frac{f(z) - f(a)}{z - a}, & z \in \Omega - a \\ f'(a), & z = a \end{cases}$$

is an analytic extension of $F(z)$. By induction, define, for $n = 2, 3, \ldots$,

$$f_n(z) := \begin{cases} \frac{f_{n-1}(z) - f_{n-1}(a)}{z - a}, & z \in \Omega - a \\ f'_{n-1}(a), & z = a \end{cases},$$

functions which are all analytic on Ω. Unraveling this, we have

$$\begin{aligned} f(z) &= f(a) + (z - a) f_1(z) = f(a) + (z - a)[f_1(a) + (z - a) f_2(z)] \\ &= f(a) + (z - a) f_1(a) + (z - a)^2 f_2(z) = f(a) + (z - a) f_1(a) + (z - a)^2 [f_2(a) + (z - a) f_3(z)] \\ &= f(a) + (z - a) f_1(a) + (z - a)^2 f_2(z) + (z - a)^3 f_3(z) \\ &\cdots \\ &= f(a) + (z - a) f_1(a) + (z - a)^2 f_2(z) + \cdots + (z - a)^n f_n(z). \end{aligned}$$

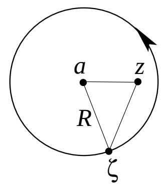
Differentiate this expression n times and set $z = a$ to get

$$f^{(n)}(a) = n! f_n(a),$$

whence the result. $\square$

Note that, since $f_n(z)$ is continuous at a, the last term (=remainder) can be written as $O(|z-a|^n)$ or as $o(|z-a|^{n-1})$, as $z \to a$.

There is a usefule formula for the remainder, obtained by applying the representation formula for $f_n(z)$. Let $B_R := \{\zeta : |\zeta - a| < R\} \subseteq \Omega$. Then



$$f_n(z) = \frac{1}{2\pi i} \int_{\partial B_R} \frac{f_n(\zeta) d\zeta}{\zeta - z}, \quad z \in B.$$

But

$$f_n(\zeta) = \frac{f(\zeta)}{(\zeta-a)^n} - \frac{f(a)}{(\zeta-a)^n} - \frac{f_1(a)}{(\zeta-a)^{n-1}} - \cdots - -\frac{f_{n-1}(a)}{\zeta - a}.$$

We have

$$\int_{\partial B_R} \frac{d\zeta}{(\zeta-a)(\zeta-z)} = \frac{1}{z-a} \int_{\partial B_R} \left(\frac{d\zeta}{\zeta - z} - \frac{d\zeta}{\zeta - a} \right) = 0$$

The LHS is an analytic function of a and so if we differentiate k times, we still get 0, showing that

$$\int_{\partial B_R} \frac{d\zeta}{(\zeta-a)^k(\zeta-z)} = 0.$$

Therefore,

$$\boxed{f_n(z) = \frac{1}{2\pi i} \int_{\partial B_R} \frac{f(\zeta) d\zeta}{(\zeta-a)^n(\zeta-z)}.}$$

The structure of zeros and poles

Suppose $f(z)$ is analytic on the domain Ω. We say that $a \in \Omega$ is a zero of $f(z)$ if $f(a) = 0$.

Suppose $f(z)$ is analytic on $\Omega - a$. The point a is a POLE if $f(z) \to \infty$ as $z \to a$.

By definition, a pole is an isolated singularity (i.e., by definition, there are no other singularities in a neighbourhood of a pole). A pole is a singularity, but it is not a bad singularity because the assertion says that the limit exists (it is infinitity but, at least, it exists). Since the function $f(z)$ is continuous and tends to ∞ as $z \to a$, we have that $f(z) \neq 0$ in a neighbourghood Ω' of a. Hence the function $1/f(z)$ is analytic on $\Omega' - a$, with a zero at a. We can thus treat poles of $f(z)$ as zeros of $1/f(z)$ in a sufficiently small neighbourhood of the pole.

We show that if a is a zero of $f(z)$ and the function $f(z)$ is not trivial then only finitely many derivatives of f can vanish at a. That is,
If a is a zero then

$$h := \min\{n \in \mathbb{N} : \; f^{(n)}(a) \neq 0\}$$

is a well-defined natural number, called the order of the zero.
Correspondingly, we may talk about the order of a pole.

Proof. Suppose that a is a zero for which $f^{(n)}(a) = 0$ for all n. We will show that $f(z)$ is the constant function. By Taylor's theorem, we have

$$f(z) = (z-a)^n f_n(z), \quad \forall n \in \mathbb{N},$$

with $f_n(z)$ being all analytic on Ω. Consider the integral expression for $f_n(z)$. Let M be the maximum of $|f(\zeta)|$ on ∂B_R. Then, for $z \in B_R$, $\zeta \in \partial B_R$, we have $|\zeta - a| = R$, and $|\zeta - z| \geq |\zeta - a| - |z-a| = R - |z-a|$, so

$$|f_n(z)| \leq \frac{M}{R^{n-1}(R - |z-a|)}, \quad z \in B_R.$$

Hence

$$|f(z)| = |f_n(z)(z-a)^n| \leq \left(\frac{|z-a|}{R}\right)^n \frac{MR}{R - |z-a|}, \quad z \in B_R, \quad n \in N.$$

Letting $n \to \infty$, we get $f(z) = 0$, for all $z \in B_R$. □

So if a is a zero of $f(z)$ of order h then $f(z) = (z-a)^h f_h(z)$, where $f_h(a) \neq 0$, and since f_h is continuous, it is nonzero in a neighbourhood of a, meaning that $f(z)$ is nonzero in a neighborhood of a, meaning that there can be no other zero in a sufficiently small neighbourhood of a. In other words, the zeros of an analytic function in Ω are isolated in Ω.

Incidentally, this provides another proof of the fact that if a polynomial $P(z)$ has a zero at a then $z-a$ divides $P(z)$. Indeed, $P(z)$ being an analytic function, we have, by the above, that $P(z) = (z-a)Q(z)$ where $Q(z)$ is also anlytic. Necessarily, $Q(z)$ must be a polynomial.

Hence: If $f(z)$, $g(z)$ are analytic function such that $f(z) = g(z)$ for all $z \in A$, where A has an accumulation point in Ω, then f coincides with g. Proof: If not, then the seto of zeros of the function $f(z) - g(z)$ would have an accumulation point in Ω and this is not possible.

A function is called *meromorphic* on Ω if it is analytic except for poles (which are isolated singularities, by definition). Linear combinations of meromorphic functions are meromorphic. Products and quotients of meromorphic functions are meromorphic.

Let a be a point in Ω such that

$$\lim_{z \to a} (z-a)^p f(z) = 0$$

for some $p \in \mathbb{R}$. Then $\lim_{z \to a}(z-a)^{p+\varepsilon} f(z) = 0$ for all $\varepsilon > 0$. So for some $m \in \mathbb{Z}$, $\lim_{z \to a}(z-a)^m f(z) = 0$. Then $(z-a)^m f(z)$ has a removable singularity at a and vanishes at a. Unless $f(z)$ is trivial, $(z-a)^m f(z)$ has a zero at a of finite order k. Thus, $(z-a)^m f(z) = (z-a)^k g(z)$ where $g(z)$ is analytic, $g(a) \neq 0$. Hence $f(z) = (z-a)^{k-m} g(z)$. So $\lim_{z \to a}(z-a)^p f(z) = 0$ for all $p > m-k$, while $\lim_{z \to a}(z-a)^p f(z) = \infty$ for all $p < m-k$. The point here is that if $\lim_{z \to a}(z-a)^p f(z) = 0$ for some $p \in \mathbb{R}$ then

> (*) there exists $h \in \mathbb{Z}$ such that the function $(z-a)^h f(z)$ is analytic and does not vanish at a and so $\lim_{z \to a}(z-a)^{-h+\varepsilon} f(z) = 0$, $\lim_{z \to a}(z-a)^{-h-\varepsilon} f(z) = \infty$, for all $\varepsilon > 0$.

Let a be a point in Ω such that $f(z)$ is analytic on $\Omega - a$ and

$$\lim_{z \to a} (z-a)^p f(z) = \infty$$

for some $p \in \mathbb{R}$. The same conclusion (*) holds. The number h can be any integer. If $h = 0$ there is nothing to say. If $h < 0$ then a is a zero of order $|h|$. If $h > 0$ then a is a pole of order h and because $(z-a)^h f(z)$ is analytic, we have, by Taylor's theorem,

$$(z-a)^h f(z) = B_h + B_{h-1}(z-a) + \cdots + B_1(z-a)^{h-1} + \varphi(z)(z-a)^h,$$

where $\varphi(z)$ is analytic. Thus,

$$f(z) = B_h(z-a)^{-h} + \cdots + B_1(z-a)^{-1} + \varphi(z).$$

This is the development of $f(z)$ around a pole of order h. It is the sum of an analytic function $\varphi(z)$ and a so-called singular part (a rational function of order h).

Suppose that $f(z)$ is defined and analytic on a neighbourhood of ∞. We say that ∞ is a removable singularity (resp. zero or pole) for $f(z)$ if 0 is a removable singularity (resp. zero or pole) for $f(1/z)$.

Essential singularities

If $f(z)$ is analytic on $\Omega - a$ then a is called essential (isolated) singularity if neither $\lim_{z \to a}(z-a)^p f(z) = 0$ for some $p \in \mathbb{R}$ nor $\lim_{z \to a}(z-a)^p f(z) = \infty$ for some $p \in \mathbb{R}$. For instance, $z = 0$ is an essential singularity for $\sin(1/z)$.

The theorem of Weierstraß says that if a is an essential singularity for $f(z)$ then for any open set U containing a we have $f(U) = \mathbb{C}$.

Suppose that $f(z)$ is defined and analytic on a neighbourhood of ∞. We say that ∞ is an essential singularity for $f(z)$ if 0 is an essential singularity for $f(1/z)$. The theorem of Weierstraß applies in this case as well.

Number of zeros

Suppose $f(z)$ is a nontrivial analytic function on a disk Δ and let $z_1, z_2, \ldots$ be its zeros (each zero counted as many times as its order). Then for any closed curve γ in Δ not passing through a zero we have

$$\sum_j \mathfrak{n}(\gamma, z_j) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz,$$

and the sum has only finitely many terms.

Proof. Suppose there are only finitely many zeros: $z_1, \ldots, z_n$. Then

$$f(z) = (z - z_1) \cdots (z - z_n) g(z),$$

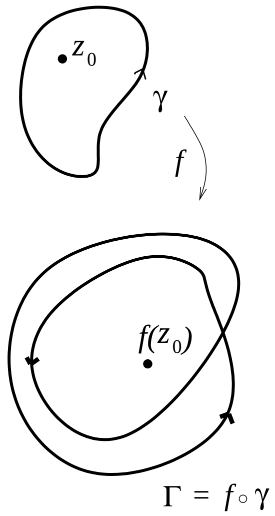
where $g(z)$ is analytic with no zeros in Δ. Then

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \cdots + \frac{1}{z - z_n} + \frac{g'(z)}{g(z)},$$

for all $z \neq z_j$. Let γ be a cclosed curve in Δ not passing through a zero. By Cauchy's theorem, $\int_\gamma g'(z)/g(z)\ dz = 0$, while $\int_\gamma dz/(z - z_j) = 2\pi i\, \mathfrak{n}(\gamma, z_j)$. If there are infinitely many zeros then since they have no accumulation points in Δ, we can find disk $\Delta' \subseteq \Delta$ containing γ such that only finitely many zeros lie in Δ' and apply the same reasoning. $\square$

Consider now the mapping $z \mapsto w = f(z)$, under which $z_j \mapsto 0$. Let $\Gamma = f \circ \gamma$. Then Γ is a closed curve and, by a change of variables,

$$\int_\gamma \frac{f'(z)}{f(z)} dz = \int_\Gamma \frac{dw}{w}.$$



If the order of z_0 as a zero of the function $f(z) - f(z_0)$ is n then any curve γ around z_0 has an image $f \circ \gamma$ which winds n times around $f(z_0)$.

But $\int_\Gamma \frac{dw}{w} = \mathfrak{n}(\Gamma, 0)$ the number of times that Γ winds around zero. Thus, the theorem above can be interpreted as:

$$\mathfrak{n}(\Gamma, 0) = \sum_j \mathfrak{n}(\gamma, z_j)$$

meaning that if γ is a closed curve then its image $\Gamma = f \circ \gamma$ winds around 0 as many times as the zeros contained in γ. In particular, if γ encloses no zeros of $f(z)$ then Γ does not encircle zero.

Another application is:

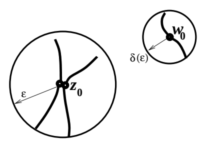
$$\#\{\text{roots of } f(z) = a \text{ enclosed by } \gamma\} = \frac{1}{2\pi i} \int_\gamma \frac{f'(z) dz}{f(z) - a}.$$

This also reads:

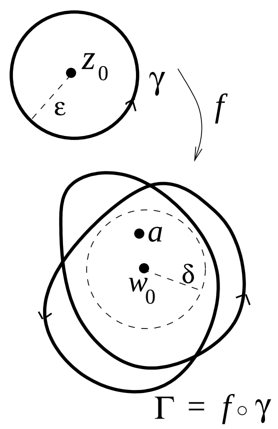
$$\#\{\text{times } \Gamma = f \circ \gamma \text{ winds around } a\} = \#\{\text{roots of } f(z) = a \text{ enclosed by } \gamma\}.$$

It is necessary to assume, in both cases, that γ does not pass through any root of the equation $f(z) = a$.

Local correspondence



If z_0 is a zero of order n of the function $f(z) - f(z_0)$ then a small perturbation does not change the number of zeros. Precisely, letting $w_0 = f(z_0)$ we have that $\forall \varepsilon > 0\ \exists \delta = \delta(\varepsilon) > 0$ such that if $|a - w_0| < \delta(\varepsilon)$ then the equation $f(z) = a$ has exactly n roots in the disk $|z - z_0| < \varepsilon$.



Proof. Suppose the order of z_0 as a zero of $f(z) - w_0$ is n. Since zeros are isolated, let ε be so small that the circle γ with radious ε and centred at z_0 contains no other zero of $f(z) - w_0$. Then $\Gamma = f \circ \gamma$ winds n times around w_0. Choose δ so that the δ-neighbourhood of w_0 is not intersected by Γ. Pick a in this neighbourhood. Then $\mathfrak{n}(\Gamma, a) = n$, also. Since this neighbourhood is contained in the image of the interior of the circle γ under f, we have that the equation $f(z) = a$ has roots inside γ. In fact, if we pick ε so small that $f'(z)$ has no zero for $0 < |z - z_0| < \varepsilon$ (possible because the zeros of $f'(z)$ are isolated) then whatever roots the equation $f(z) = a$ has they must be simple.

Further properties of analytic functions

Any analytic function is an open map, i.e. it maps open sets into open sets.

Proof. Let $w_0 = f(z_0)$. Let ε be so small we can find an open disk $\Delta(z_0, \varepsilon)$ so that z_0 is the only zero of $f(z) = w_0$ and $f'(z)$ does not vanish on this disk. Then there is a $\delta > 0$ such that for each $a \in \Delta(w_0, \delta)$ the $f(z) - a$ has exactly one zero in $\Delta(z_0, \varepsilon)$. $\square$

If $f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$ then f maps a neighbourhood of z_0 conformally and diffeomorphically onto some domain.

Proof. As argued above, the inverse function f^{-1} exists locally. It is thus analytic. And its derivative at $f(z_0)$ is also nonzero. Hence f^{-1} is also conformal. $\square$

The maximum principle

If $f(z)$ is analytic on Ω then $|f(z)|$ has no maximum on Ω.
First proof. It is due to the fact that Ω is an open map. If there is $z_0 \in \Omega$ such that $|f(z_0)| \geq |f(z)|$ for all $z \in \Omega$, then consider an open disk $\Delta(z_0, \varepsilon) \subseteq \Omega$. Its image V is an open set containing $f(z_0)$. Therefore there is a point $w \in V$ such that $|w| > |f(z_0)|$. Since $w \in V = f(\Delta(z_0, \varepsilon))$, there a $z \in \Omega$ such that $w = f(z)$. So $|f(z)| > |f(z_0)|$ and this is impossible.

More generally,

> *No open map f from $\mathbb{R}^d$ into $\mathbb{R}^d$ can be such that $|f|$ attain a maximum in the interior of an open set.*

Second proof of the maximum principle. Let $\Delta(z_0, r)$ be a disk centred at $z_0 \in \Omega$ with radius r such that $\Delta(z_0, r) \subset \Omega$. Consider the representation formula on the boundary of the disk:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta.$$

If $|f(z_0)|$ were a maximum then

$$|f(z_0 + re^{i\theta})| \leq |f(z_0)|, \quad 0 \leq \theta < 2\pi.$$

If, even for a single θ the equality were strict, then, by continuity and compactness, the inequality would be strict for all θ, and so

$$\int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta < 2\pi |f(z_0)|.$$

But

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta,$$

and so we would obtain the contradiction $|f(z_0)| < |f(z_0)|$. $\square$

Lemma of Schwarz

Suppose $f(z)$ is analytic on $\{|z| < 1\}$ and
(i) $|f(z)| \leq 1$ for all $|z| < 1$;
(ii) $f(0) = 0$;
(iii) $|f'(0)| \leq 1$.
If $|f(z)| = |z|$ for some $z \neq 0$, or if $|f'(0)| = 1$, then

$$f(z) = cz$$

for some $c \in \mathbb{C}$ with $|c| = 1$.

Proof. Let

$$f_1(z) := \begin{cases} \dfrac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}.$$

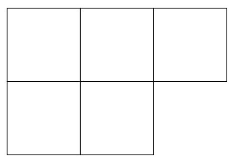
If $|z| = r < 1$ then $|f_1(z)| = |f(z)|/r \leq 1/r$. Then $|f_1(z)| \leq 1/r$ for $|z| \leq r$ as well. As $r \to 1$, we obtain $|f_1(z)| \leq 1$ for $|z| < 1$. If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, or if $|f'(0)| = 1$, then $|f_1(z_0)| = 1$ of $f_1(0)| = 1$. Thus, at some point, $|f_1(z)|$ attains its maximum value 1. By the maximum principle, $f_1(z) \equiv c$, where c is a constant. Necessarily, $|c| = 1$. $\square$

Chains and cycles

Recall that a *chain* in Ω is a formal linear combination

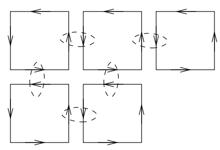
$$\gamma = \alpha_1\gamma_1 + \cdots + \alpha_n\gamma_n$$

of distinct curves γ_j in Ω, with coefficients $\alpha_j \in \mathbb{N}$. Note that $\gamma_1 - 2\gamma_2$ stands for $\gamma_1 + 2(-\gamma_2)$, so there is no restriction in requiring all α_j to be positive.



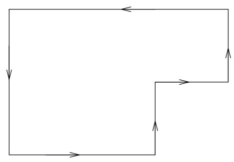
A *cycle* is a chain such that all the γ_j are closed curves. Integration over chains is defined by

$$\int_{\sum_j \alpha_j\gamma_j} f dz := \sum_j \alpha_j \int_{\gamma_j} f dz.$$



The index of a cycle γ with respect to a point $a \in \mathbb{C}$ is defined by

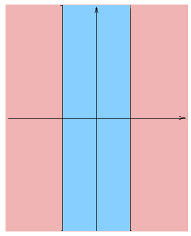
$$\mathfrak{n}(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - a}$$

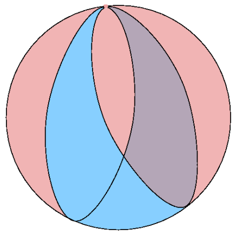


and so, by the definition of the integral over a chain and the index of a closed curve, we have

$$\mathfrak{n}(\gamma, a) = \sum_j \alpha_j\, \mathfrak{n}(\gamma_j, a), \qquad \text{if } \gamma = \sum_j \alpha_j\gamma_j.$$

Simple connectivity





A special way, valid only in 2 dimensions, to define simple connectivity of a domain Ω is to require that Ω^c be connected, i.e. that it cannot be written as the disjoint union of two nontrivial closed sets. As a matter of fact, this is a definition valid on a sphere. If $\Omega \subset \mathbb{C}$ then we have to think of $\mathbb{C}$ as the image of the sphere under the Riemann projection, *including* the point at infinity. Thus, a strip $\Omega = (a, b) \times \mathbb{R}$ is simply connected.

Theorem: A domain Ω is simple connected if and only if

$$\mathfrak{n}(\gamma, a) = 0, \qquad \text{for all cycles } \gamma \text{ in } \Omega, \text{ and all } a \in \Omega^c.$$

Proof. Suppose Ω is simply connected, i.e. Ω^c is connected. Let γ be a closed curve in Ω. Consider the regions $R_1, R_2, \ldots, R_m$ (disjoint open sets) determined by γ. All except one of them are bounded. Let R_1 be the unbounded region. Since the curve belongs to Ω we have that $\Omega^c \subset R_1 \cup R_2 \cup \cdots \cup R_m$. Since $\infty \in \Omega^c$ and $\infty \in R_1$ and since Ω^c is connected, we have $\Omega^c \subset R_1$. Therefore, if $a \in \Omega$ then $a \in R_1$. But for any a in the unbounded component of the curve, we have $\mathfrak{n}(\gamma, a) = 0$. Since the result holds for a closed curve, it also holds for a cycle contained in Ω.
Conversely, assume that $\mathfrak{n}(\gamma, a) = 0$ for all cycles in Ω, and assume that $\Omega^c = A \cup B$, where A, B are closed sets, $A \cap B = \emptyset$. We show that this is impossible. Let

$$\delta = \min\{|z - w| : z \in A, w \in B\}.$$

Consider a grid G of closed squares of side $\delta/\sqrt{2}$ extending over the whole space. Call the squares Q_j. Chose the grid so that a lies in the centre of a square, say square Q_1. Each square Q_j has a boundary denoted by ∂Q_j which can be considered as a closed curve oriented in the positive direction. Now consider the cycle

$$\gamma = \sum_{j : Q_j \cap A \neq \emptyset} \partial Q_j.$$

Hence

$$\mathfrak{n}(\gamma, a) = \sum_{j : Q_j \cap A \neq \emptyset} \mathfrak{n}(\partial Q_j, a) = \mathfrak{n}(\partial Q_1, a) + \sum_{j \neq 1 : Q_j \cap A \neq \emptyset} \mathfrak{n}(\partial Q_j, a) = 1,$$

because $\mathfrak{n}(\partial Q_1, a) = 1$ but $\mathfrak{n}(\partial Q_j, a) = 0$ for every Q_j not containing a. But now observe that the cycle γ is actually a closed curve (after we perform all cancellations). By the choice of δ this closed curve does not meet A and it does not meet B either. Hence γ lies in Ω. But this violates our hypothesis that the index of any cycle in Ω with respect to a point not in Ω is zero. $\square$

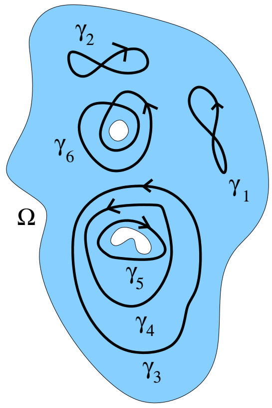
Homology

Let γ be a cycle in an open set Ω. Say that γ is homologous to 0 with respect to Ω if $\mathfrak{n}(\gamma, a) = 0$ for all $a \in \Omega^c$, and write

$$\gamma \sim 0 \mod \Omega.$$

If γ_1, γ_2 are two cycles in Ω we say that they are homologous with respect to Ω if $\gamma_1 - \gamma_2 \sim 0 \mod \Omega$ i.e. if $\mathfrak{n}(\gamma_1, a) = \mathfrak{n}(\gamma_2, a)$ for all $a \in \Omega^c$, and write this as

$$\gamma_1 \sim \gamma_2 \mod \Omega.$$



$\gamma_1 \sim \gamma_2$
$\gamma_3 \sim \gamma_4$
$\gamma_4 \not\sim \gamma_5$
$\gamma_6 \not\sim \gamma_3$

Of course, if $\gamma_1 \sim 0 \mod \Omega$ and $\gamma_2 \sim 0 \mod \Omega$ then $\gamma_1 \sim \gamma_2 \mod \Omega$ but the point is that the former is only a sufficient condition for the latter. Note that, for all integers λ, λ',

$$\gamma_1 \sim \gamma_2 \quad \& \quad \gamma_1' \sim \gamma_2' \quad \Rightarrow \quad \lambda\gamma_1 + \lambda'\gamma_1' \sim \lambda\gamma_2 + \lambda'\gamma_2'.$$

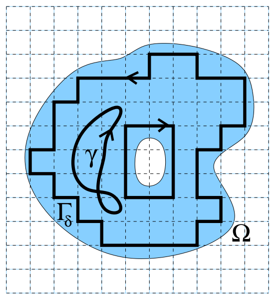
Indeed, for all $a \in \Omega^c$, $\mathfrak{n}(\lambda\gamma_1 + \lambda'\gamma_1', a) = \lambda\,\mathfrak{n}(\gamma_1, a) + \lambda'\,\mathfrak{n}(\gamma_1', a) = \lambda\,\mathfrak{n}(\gamma_2, a) + \lambda'\,\mathfrak{n}(\gamma_2', a) = \mathfrak{n}(\lambda\gamma_2 + \lambda'\gamma_2', a)$. Also,

$$\gamma \sim 0 \mod \Omega \quad \Rightarrow \quad \gamma \sim 0 \mod \Omega', \quad \text{if } \Omega' \supset \Omega.$$

The idea is that $\gamma_1 \sim \gamma_2 \mod \Omega$ if γ_1 can be continuously transformed to γ_2 within Ω.

General form of Cauchy's theorem

Let Ω be an open set and $f(z)$ analytic on Ω. If γ is a cycle in Ω such that $\gamma \sim 0 \mod \Omega$ then $\int_\gamma f(z)dz = 0$. The same remains true in presence of removable singularities.



Proof. Suppose that Ω is a bounded open set. Let $\delta > 0$ and let $\mathscr{Q}_\delta = \{Q_1, Q_2, \ldots\}$ be the collection of all closed squares of the form $[m\delta, (m+1)\delta] \times n\delta, (n+1)\delta]$, where $m, n \in \mathbb{Z}$. If δ is sufficiently small then

$$J := \{j : Q_j \in \mathscr{Q}_\delta, Q_j \subset \Omega\}.$$

Also note that, due to the boundedness of Ω, the set J is finite. Let

$$\Gamma_\delta := \sum_{j \in J} \partial Q_\delta, \quad \Omega_\delta := \left(\cup_{j \in J} Q_j\right)^o.$$

Also, let γ be a cycle in Ω such that $\gamma \sim 0 \mod \Omega$. Let δ be even smaller if necessary so that γ is a cycle in Ω_δ. Then $\gamma \sim 0 \mod \Omega_\delta$, i.e. $\mathfrak{n}(\gamma, a) = 0$ for all $a \notin \Omega_\delta$. In particular,

$$\mathfrak{n}(\gamma, \zeta) = 0, \quad \text{for all } \zeta \text{ on } \Gamma_\delta.$$

Let $f(z)$ be analytic on Ω. Then

$$\int_{\Gamma_\delta} \frac{f(\zeta)d\zeta}{\zeta - z} = \sum_{j \in J} \int_{\partial Q_j} \frac{f(\zeta)d\zeta}{\zeta - z}$$

If $z \in \cup_{j \in J} Q_j^0$ then, since the union is disjoint, there is only one $j \in J$ so that $z \in Q_j^o$ and, for this j,

$$\int_{\partial Q_j} \frac{f(\zeta)d\zeta}{\zeta - z} = 2\pi i f(z),$$

whereas $\int_{\partial Q_j} \frac{f(\zeta)d\zeta}{\zeta - z} = 0$ for all other j, from the simple form of Cauchy's theorem on a rectangle. Therefore,

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(\zeta)d\zeta}{\zeta - z}.$$

for all $z \in \cup_{j\in J} Q_j^0$. Since both sides are continuous functions of z, we have that this equality remains true for all $z \in \Omega_\delta$.
(Comment: Essentially what he have done is this: Regardless of the connectedness properties of Ω we argued that there is a (simple) closed curve (Γ_δ) for which $\frac{f(\zeta)}{\zeta - z}$ integrates to zero as ζ moves on this curve. We did that by taking a regular δ-covering of Ω and looked at the boundary of the largest region of it contained in Ω. Having found one closed curve over which the integral of this function is zero, we will pass on to arbitrary closed curved γ which are homologous to 0.)
Consider then the closed curve γ in Ω_δ which is homologous to 0. We integrate both sides of the above over γ:

$$\int_\gamma f(z)dz = \frac{1}{2\pi i}\int_\gamma dz \int_{\Gamma_\delta} \frac{f(\zeta)d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_{\Gamma_\delta} f(\zeta)d\zeta \int_\gamma \frac{dz}{\zeta - z},$$

where integrals were interchanged by virtue of the joint continuity of $f(\zeta)/(\zeta - z)$, $\zeta \in \Gamma_\delta$, $z \in \gamma$. But

$$\int_\gamma \frac{dz}{\zeta - z} = -\,\mathfrak{n}(\gamma, \zeta) = 0,$$

since $\zeta \in \Gamma_\delta$. If Ω is unbounded, then let Δ be a disk that contains γ in its interior and let $\Omega^* = \Omega \cap \Delta$. If $a \in \Omega^{*c}$ then $\mathfrak{n}(\gamma, a) = 0$, so $\gamma \sim 0 \mod \Omega^*$. Hence, by the above argument, $\int_\gamma f dz = 0$. □

Cauchy's theorem on simply connected domains

If Ω is open and simply connected then, as we proved, any cycle γ is homologous to 0 and so $\int_\gamma f dz = 0$ for any analytic function $f(z)$.

One consequence of this is that any analytic function $f(z)$ on a simple domain Ω has an analytic antiderivative, i.e. there is an analytic (single-valued!) function $F(z)$ on Ω such that $F'(z) = f(z)$ for all $z \in \Omega$.

A consequence of the consequence is that if $f(z)$ is analytic on a simple domain Ω and $f(z) \neq 0$ for all $z \in \Omega$, then we can pick a single-valued analytic, on Ω, branch of $\log f(z)$. Indeed, under our assumptions, $f'(z)/f(z)$ is analytic on Ω. Hence it has an antiderivative $F(z)$, i.e. there is an analytic, on Ω, function $F(z)$ such that $F'(z) = f'(z)/f(z)$. Define $h(z) := f(z)e^{-F(z)}$. Then $h'(z) = f'(z)e^{-F(z)} - F'(z)f(z)e^{-F(z)} = f'(z)e^{-F(z)} - f'(z)e^{-F(z)} = 0$, for all $z \in \Omega$. This means that $h(z)$ is constant. Fix $z_0 \in \Omega$ Hence $h(z) = h(z_0)$ for all $z \in \Omega$, i.e. $f(z)e^{-F(z)} \equiv f(z_0)e^{-F(z_0)}$, or $f(z) \equiv e^{F(z)-F(z_0)}f(z_0)$. Pick $\log f(z_0)$ to be one of the infinitely many possible values for this specific point z_0. We have $f(z_0) = e^{\log f(z_0)}$. So $f(z) \equiv e^{F(z)-F(z_0)+\log f(z_0)}$. Therefore, we DEFINE, $\log f(z) \equiv F(z) - F(z_0) + \log f(z_0)$. The right-hand side is well-defined (because we picked a specific value for the constant $\log f(z_0)$) and is analytic because $F(z)$ is analytic. Therefore $\log f(z)$ is well-defined for all $z \in \Omega$, single-valued, and analytic. It is a branch of the logarithm because, by the way it was defined, the exponential of the right-hand side equals $f(z)$.

Locally exact 1-forms

A 1-form $\omega = pdx + qdy$ is locally 1 exact in a domain Ω if for each open disk there are functions U, V such that $p = \frac{\partial U}{\partial x}$ $q = \frac{\partial V}{\partial y}$ on the disk. In general, p, q are allowed to be complex-valued.

It is easy to see that a 1-form is locally eaxact iff $\int_{\partial R} \omega = 0$ for all rectangles $R \subset \Omega$.

If $f(z)$ is analytic on Ω such that, for each disk, it has analytic antiderivative $F(z)$ on the disk then $\omega = f(z)dz$ is exact. This is a consequence of (the simple form of) Cauchy's theorem: $\int_{\partial R} f(z)dz = 0$ for all rectangles $R \subset \Omega$. It is useful to notice

$$f(z)dz = (u + iv)dx + (-v + iu)dy.$$

Also,

$$f(z)dz = F'(z)dz = F'(z)dx + iF'(z)dy.$$

Since $F'(z) = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial (iy)}$, we have

$$f(z)dz = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy.$$

Now, if $F = U + iV$ then $F_x = U_x + iV_x$, $F_y = U_y + iV_y$. But $U_x = V_y$, $U_y = -V_x$. Hence $F_x = U_x - iU_y$, $F_y = U_y + iU_x$. So we can also write

$$\begin{aligned} f(z)dz &= (U_x - iU_y)dx + (U_y + iU_x)dy \\ &= (U_x dx + U_y dy) + i(U_x dy - U_y dx). \end{aligned}$$

Using V instead of U we also have

$$f(z)dz = (V_y dx - V_x dy) + i(V_x dx + V_y dy).$$

So we have

$$f(z)dz = \xi + i\eta$$

where the real and imaginary parts,

$$\omega = U_x dx + U_y dy, \quad \eta = V_x dx + V_y dy,$$

are both (real) exact differentials.

Conversely, given an exact real differential $\omega = pdx + qdy = U_x dx + U_y dy$ we can find another exact real differential $\eta = V_x dx + V_y dy$ so that $\xi + i\eta = f(z)dz$, where $f(z)$ is analytic, provided that U is harmonic: $\Delta U = 0$. But this is too restrictive for an aribtrary exact differential.

Nevertheless, we can prove that if ω is any locally exact differential then for any cycle γ in Ω such that $\gamma \sim 0$,

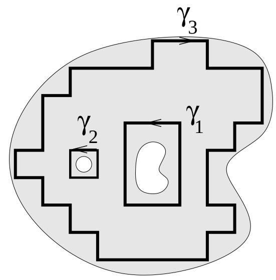
$$\int_\gamma \omega = 0.$$

Thus, we can lift the local property $\int_{\partial R} \omega = 0$ to a global one.

Multiply connected regions

We say that a domain Ω has finite connectivity n if Ω^c has n connected components. Let $A_1, \ldots, A_n$ be the components, with A_n being the unbounded one.

As above, we can find cycles $\gamma_1, \ldots, \gamma_{n-1}$ such that



$$\mathfrak{n}(\gamma_j, a) = \begin{cases} 1 & \text{if } a \in A_j \\ 0 & \text{if } a \in A_k, k \in \{1, \ldots, n\} - \{j\}. \end{cases}$$

Indeed, the cycle Γ_δ constructed earlier is written simply as

$$\Gamma_\delta = \gamma_1 + \cdots + \gamma_{n-1} + \gamma_n,$$

and the cycles we are interested in are the first $n-1$. Notice that any cycle γ in Ω satisfies

$$\gamma \sim \sum_{j=1}^{n-1} c_j \gamma_j \qquad \text{mod } \Omega$$

where

$$c_j := \mathfrak{n}(\gamma_j, a), \quad a \in A_j.$$

Therefore,

$$\int_\gamma f dz = \sum_{j=1}^{n-1} c_j \int_{\gamma_j} f dz.$$

We say that $\gamma_1, \ldots, \gamma_{n-1}$ is a homology basis for Ω. The numbers

$$P_j := \int_{\gamma_j} f dz$$

are called periods. Notice that P_j does not depend on the specific choice for γ_j, but only on the function f. So, to compute an integral over a cycle, we need (a) to find the indices of the cycle with respect to the components A_j and (b) to find the periods of f.

Next consider a curve $\gamma(z_0, z)$ with endpoints z_0, z. It is easy to see that, if $\gamma'(z_0, z)$ is another such curve, then

$$\int_{\gamma(z_0,z)} f(\zeta) d(\zeta) - \int_{\gamma'(z_0,z)} f(\zeta) d(\zeta) = \sum_{j=1}^{n-1} \ell_j P_j.$$

That is, the function $F(z) := \int_{\gamma(z_0,z)} f(\zeta) d(\zeta)$ is determined up to multiples of the periods. In other words, the dependence of $F(z) := \int_{\gamma(z_0,z)} f(\zeta) d(\zeta)$ is through the endpoints and through the windings of the curve. If the periods vanish (and this depends only on the function) then $\int_{\gamma(z_0,z)} f(\zeta) d(\zeta)$ depends only on endpoints and thus a single-valued analytic indefinite integral is determined on Ω.

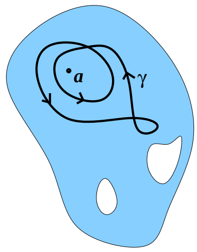
Example: an annulus

Suppose that $\Omega := \{z : r_1 < |z| < r_2\}$. Then Ω is doubly connected since $\Omega^c = A_1 + A_2$, $A_1 = \{z : |z| \le r_1\}$, $A_2 = \{z : |z| \ge r_2\}$. Any homology basis has one element, for example the circle C with center 0 and radius $r \in (r_1, r_2)$. Any cycle γ satisfies $\gamma \sim nC$, where $n \in \mathbb{Z}$. So for any analytic function $f(z)$ on Ω, we have $\int_\gamma f dz = nP$, where $P = \int_C f dz$. If $P = 0$ then an indefinite integral $F(z)$ exists on Ω. This does not mean that the function can be analytically extended on the disk $\{z : |z| < r_2\}$. Indeed, let, for example, $f(z) = 1/z^2$ on the 'annulus' $0 < |z| < \infty$. Let C be the circle around 0 with radius 1. Then $P = \int_C f(z)dz = \int_0^{2\pi} e^{-2it} i e^{it} dt = i \int_0^{2\pi} e^{-it} dt = 0$. So $f(z) = F'(z)$ for some function $F(z)$, analytic on $0 < |z| < \infty$. Of course, we know that $F(z) = -1/z$. Neither $f(z)$ nor $F(z)$ are analytic on $\mathbb{C}$.

Integral formula in a multiply connected domain

Let Ω be an n-connected domain and let γ be a cycle such that $\gamma \sim 0 \mod \Omega$. Then, for any function $f(z)$, analytic on Ω,

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)dz}{z-a} = \mathfrak{n}(\gamma, a) f(a).$$

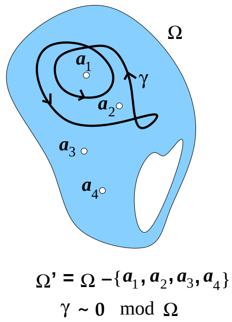


The proof of this is as in the case of a disk: Let $F(z) := \frac{f(z)-f(a)}{z-a}$. This is analytic on $\Omega - a$ with removable singularity at a. Hence Cauchy's theorem holds: $\int_\gamma F(z)dz = 0$, and this gives the integral formula.

Calculus of residues

Consider the case where $f(z)$ is analytic on

$$\Omega' = \Omega \setminus \{a_1, \ldots, a_n\},$$



where Ω is a domain. Let C_j be a circle with centre a_j and radius $\delta_j > 0$, small enough to be contained in Ω without wnclosing any other point a_k. Any cycle γ in Ω', such that $\gamma \sim 0 \mod \Omega$ must satisfy

$$\gamma \sim \sum_{j=1}^{n} \mathfrak{n}(\gamma, a_j) C_j \mod \Omega'.$$

Therefore if $f(z)$ is analytic on Ω',

$$\boxed{\int_\gamma f dz = \sum_{j=1}^{n} \mathfrak{n}(\gamma, a_j) \int_{C_j} f dz.}$$

The periods $P_j = \int_{C_j} f dz$, $j = 1, \ldots, n$ are what need to be computed.

Note that the function

$$f_j(z) := f(z) - \frac{P_j/2\pi i}{z-a}$$

satisfies

$$\int_{C_j} f_j(z)dz = 0$$

and therefore $f_j(z)$ has an indefinite intergral on the annulus $0 < |z - a_j| < \delta_j$, i.e. there is a function $F_j(z)$, analytic on $0 < |z - a_j| < \delta_j$, such that $F_j'(z) = f_j(z)$ on $0 < |z - a_j| < \delta_j$. It is customary to define *the residue of $f(z)$ at an isolated singularity a_j* by

$$\operatorname{Res}_{z=a_j} f(z) := \frac{1}{2\pi i}\int_{C_j} f dz.$$

In view of this notation, the previous display is written

$$\boxed{\frac{1}{2\pi i}\int_{\gamma} f dz = \sum_{j=1}^{n} \mathfrak{n}(\gamma, a_j)\operatorname{Res}_{z=a_j} f(z).}$$

Computing residues

Recall that if a is an isolated singularity of $f(z)$ then $R = \operatorname{Res}_{z=a} f(z)$ is such that $f(z) = \frac{R}{z-a}$ has analytic antiderivative on $0 < |z - a| < \delta$ for some $\delta > 0$.

If a is a pole of order h then $(z - a)^h f(z)$ is analytic around a and so, by Taylor-expanding $(z - a)^h f(z)$ around a we obtain

$$(z-a)^h f(z) = B_h + B_{h-1}(z-a) + \cdots + B_h(z-a)^{h-1} + (z-a)^h\varphi(z),$$

where $\varphi(z)$ is analytic around a, and so

$$f(z) = B_h(z-a)^{-h} + \cdots + B_1(z-a)^{-1} + \varphi(z),$$

Since all terms $(z - a)^{-k}$ with $k \neq 1$ have analytic antiderivative on $0 < |z - a| < \delta$, we see if we subtract $B_1(z - a)^{-1}$ from $f(z)$ we obtain a function which has analytic antiderivative on $0 < |z - a| < \delta$. Hence, here,

$$\operatorname{Res}_{z=a} f(z) = B_1.$$

Alternatively, since B_1 is the $(h - 1)$-th coefficient in the Taylor expansion of $(z - a)^h f(z)$ we have

$$\operatorname{Res}_{z=a} f(z) = B_1 = \frac{1}{(h-1)!}\,\frac{d^h}{dz^{h-1}}(z-a)^h f(z)\Big|_{z=a}.$$

In particular, for a simple pole ($h = 1$), $\operatorname{Res}_{z=a} f(z) = (z - a)f(z)\big|_{z=a}$.

Special cases of the residue theorem

1. The integral formula: If $f(z)$ is analytic on the domain Ω then, for $a \in \Omega$, $f(z)/(z - a)$ has a pole at a with residue $f(a)$. So $\frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-a} = 1$, for any cycle γ in Ω, homologous to 0, not passing through a.

2. Number of zeros of analytic function on a disk: If $f(z)$ is analytic on the disk Δ and if a_j are its zeros, then $f'(z)/f(z)$ is meromorphic on Δ with poles at a_j. The residue of $f'(z)/f(z)$ at a_j equals the order of the zero. Hence, for anyy for any cycle γ in Ω, homologous to 0, not passing through any a_j, we have that $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz$ equals the sum of the residues, i.e. the total number of zeros in Δ (counting multiplicities).

The argument principle

If $f(z)$ is meromorphic on Ω with zeros a_j and poles b_k then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_j \mathfrak{n}(\gamma, a_j) - \sum_k \mathfrak{n}(\gamma, b_k),$$

for any cycle γ in Ω not passing through any zeros or poles, and such that $\gamma \sim 0 \mod \Omega$.

Proof. If $f(z)$ is meromorphic on Ω with zeros A_j (distinct) and poles B_k (distinct) then we can write

$$f(z) = \prod_j (z - A_j)^{h_j} \prod_k (z - B_k)^{-d_k} g(z),$$

where $g(z)$ is analytic with no zeros and where h_j is the order of A_j and d_k the order of B_k. Hence

$$\frac{f'(z)}{f(z)} = \sum_j \frac{h_j}{z - A_j} - \sum_k \frac{d_k}{z - B_k} + \frac{g'(z)}{g(z)}.$$

The function $f'(z)/f(z)$ is also meromorphic on Ω with poles A_j, B_k. The residue at A_j is h_j and at B_k is $-d_k$. So, by the residue theorem, if γ is any cycle in Ω homologous to 0, not passing through any A_j or B_k, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_j \mathfrak{n}(\gamma, A_j)h_j - \sum_k \mathfrak{n}(\gamma, B_k)d_k.$$

If (a_j) is a listing of the (A_j) counting their multiplicities (similarly, (b_k) for (B_k)) then the result follows. $\square$

Rouché's theorem

Suppose $\gamma \sim 0 \mod \Omega$ such that $\mathfrak{n}(\gamma, z) \in \{0, 1\}$ for all $z \in \Omega - \gamma$. Suppose $f(z), g(z)$ are analytic on Ω such that

$$|f(z) - g(z)| < |f(z)|, \quad z \in \gamma.$$

Then $f(z)$ and $g(z)$ have the same number of zeros enclosed by γ.

Weirstraß' theorem

Suppose that $f_n(z)$ is analytic on Ω_n and there is a function $f(z)$ defined on some domain Ω such that $f_n(z) \to f(z)$ as $n \to \infty$ uniformly on compact subsets of Ω. Then $f(z)$ is analytic. Moreover, $f_n'(z)$ converges to $f'(z)$ uniformly on compact subsets of Ω.

Note: It is implicitly understood that, for each compact $K \subset \Omega$, we have $K \subset \Omega_n$ eventually.

Weirstraß' theorem for series

If $f_n(z)$ are analytic on Ω and if $\sum_{n=1}^{\infty} f_n(z) = f(z)$ uniformly on compact subsets of Ω then $f(z)$ is analytic and $f'(z) = \sum_{n=1}^{\infty} f_n'(z)$.

Hurwitz' theorem

If $f_n(z)$ are zero-free, analytic on Ω, and $f_n(z) \to f(z)$ uniformly on compact subsets of Ω then either $f(z)$ is zero-free or identically equal to zero.

Taylor series

If $f(z)$ is analytic on Ω, $z_0 \in \Omega$, then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z - z_0)^n$$

uniformly on each closed disk centred at z_0 and contained in Ω.

Proof. All we have to do is show that the remainder in Taylor's formula converges to 0 uniformly over compact circles contained in Ω and centred at a. Taylor's formula reads

$$f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(z-a)^k + f_{n+1}(z)(z-a)^{n+1},$$

where

$$f_{n+1}(z) = \frac{1}{2\pi i} \int_{\partial B_R} \frac{f(\zeta) d\zeta}{(\zeta - a)^{n+1}(\zeta - z)}.$$

where $B_R = \{|z - a| \leq R\}$. Then, for all $z \in B_R$,

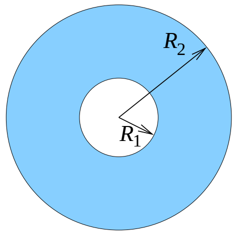
$$|f_{n+1}(z)(z-a)^{n+1}| \leq \frac{M|z-a|^{n+1}}{R^n(R - |z-a|)} = \frac{MR}{R - |z-a|}\left(\frac{|z-a|}{R}\right)^{n+1}$$

For each $r < R$, the last term tends to 0 uniformly over $|z - a| \leq r$. Hence $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z - z_0)^n$ uniformly over compact circles contained in Ω and centred at a. $\square$

Laurent series

A Laurent series is an expression of the form

$$\sum_{n\in\mathbb{Z}} a_n z^n,$$



i.e. a power series with positive and negative powers, understood to be defined for the values of z for which the series

$$\sum_{n\geq 0} a_n z^n, \quad \sum_{n<0} a_n z^n$$

converge. The first series converges u.o.c. in $|z| < R_2$, where R_2 is the radius of convergence. The second series can be considered as a power series in the variable $1/z$, and so it converges when $|1/z| < R_1$. Hence a Laurent series always converges on an *annulus*

$$R_1 < |z| < R_2$$

where $R_1, R_2 \in [0, \infty]$. If $R_1 = \infty$ or $R_2 = 0$ then the Laurent series does not converge at all. More generally, we can, of course, consider Laurent series around a point z_0, i.e.

$$\sum_{n\in\mathbb{Z}} a_n (z - z_0)^n.$$

Such a series converges on an annulus $R_1 < |z - z_0| < R_2$.

Now let $f(z)$ be an analytic function on some domain Ω and assume that

$$\{z : R_1 < |z - z_0| < R_2\} \subset \Omega.$$

We shall show that $f(z)$ can be represented as a Laurent series. Let C_r be the circle with centre z_0 and radius $r \in (R_1, R_2)$ and define

$$g(z) = \frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad |z - z_0| < r.$$

Clearly, $g(z)$ is analytic and $g(z)$ does not depend on r as long as $|z - z_0| < r < R_2$. Hence $g(z)$ is analytic on $|z - z_0| < R_2$. Define also

$$h(z) = \frac{1}{2\pi i}\int_{-C_r} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad |z - z_0| > r.$$

Clearly, $h(z)$ is analytic and $h(z)$ does not depend on r as long as $|z - z_0| > R_1$. Hence $g(z)$ is analytic on $|z - z_0| > R_1$.

Since $g(z)$ is analytic on $\Omega_g = \{|z - z_0| < R_2\}$ and $h(z)$ is analytic on $\Omega_h = \{|z - z_0| > R_1\}$ we have that $g(z) + h(z)$ is analytic on $\Omega_g \cap \Omega_h = \{z : R_1 < |z - z_0| < R_2\}$. We show that $g(z)+h(z) = f(z)$. Let z be such that $R_1 < |z-z_0| < R_2$. Pick $R_1 < r_1 < |z-z_0| < r_2 < R_2$. Since $C_{r_1} - C_{r_2} \sim 0 \mod \Omega_g \cap \Omega_h$ we have, by the general integral formula,

$$\mathfrak{n}(C_{r_1} - C_{r_2}, z) f(z) = \frac{1}{2\pi i}\int_{C_{r_1} - C_{r_2}} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

But $\mathfrak{n}(C_{r_1} - C_{r_2}, z) = 1$. On the other hand, $g(z) = \frac{1}{2\pi i}\int_{C_{r_1}} \frac{f(\zeta)}{\zeta - z} d\zeta$, $h(z) = \frac{1}{2\pi i}\int_{-C_{r_2}} \frac{f(\zeta)}{\zeta - z} d\zeta$. We have

$$g(z) = \sum_{n=0}^{\infty} A_n (z - z_0)^n, \quad A_n = \frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta.$$

Since $h(z)$ is analytic on $\Omega_h = \{|z - z_0| > R_1\}$, we change variable by

$$z = z_0 + \frac{1}{z'}.$$

Then Ω_h is mapped onto $|z'| < 1/R_1$. The function

$$h'(z') = h(z_0 + 1/z')$$

From the definition of $h(z)$, note that $h(z) \to 0$ as $z \to \infty$, so $h'(0) = 0$. By ordinary Taylor's theorem,

$$h'(z') = \sum_{n=1}^{\infty} B_n z'^n, \quad |z'| < 1/R_1, \quad B_n = \frac{1}{2\pi i}\int_{|\zeta'| = 1/r} \frac{h'(\zeta')d\zeta'}{\zeta'^{n+1}}.$$

Changing variables we have

$$h(z) = \sum_{n=1}^{\infty} B_n (z - z_0)^{-n}, \quad |z - z_0| > R_1, \quad B_n = \frac{1}{2\pi i}\int_{|\zeta - z_0| = r} f(\zeta)(\zeta - z_0)^{n-1} d\zeta.$$

Combining the above, we have (by setting $A_n := B_{-n}$ for $n < 0$):

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z - z_0)^n, \quad R_1 < |z - z_0| < R_2, \quad A_n = \frac{1}{2\pi i}\int_{C_r} f(\zeta)(\zeta - z_0)^{-n-1} d\zeta, \quad n \in \mathbb{Z}.$$

□

Partial fractions

The ratio of two polynomials (rational function) can be written as a sum of partial fractions (by applying the algorithm of division). If we specify the zeros and the poles (together with their orders) for a rational function then we have specified (up to a constant) the function. A similar result is possible for meromorphic functions.

Suppose that $f(z)$ is meromorphic on Ω. Thus it has isolated poles. Each pole b has finite order $o(b)$. By considering a Laurent development around b we have $f(z) = \sum_{n=1}^{o(b)} B_n (z - b)^{-n} + f_+(z)$, where $f_+(z)$ is analytic around b. The first part is called singular part of $f(z)$ at $z = b$. So, for each pole b_ν there is a polynomial P_ν such that $f(z) - P_\nu(1/(z - b_\nu))$ is analytic around b_ν. It is tempting to stipulated that if, from $f(z)$, we subtract the sum of all singular parts then we will obtain a function that is analytic on Ω, i.e. that $f(z) - \sum_\nu P_\nu(1/(z - b_\nu))$ is analytic on Ω. Unfortunately, this is not true in general.