OUTLINE OF LECTURES ON THE PROOF OF THE LÉVY–KHINCHINE FORMULA & THE LÉVY–ITÔ DECOMPOSITION

1) An infinitely divisible (ID) probability measure μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ satisfies, by definition:

$$\forall n \in \mathbb{N} \quad \exists \text{ pr. measure } \mu_n \qquad \mu = \mu_n^{*n}$$

Equivalently, for each $n \in \mathbb{N}$, there exists pr. m. ν_n and a sequence $r_n \in \mathbb{N}$, $r_n \to \infty$, such that

$$\nu_n^{*r_n} \Rightarrow \mu \qquad \text{(weak convergence)}$$

2) A process $(X_t, t \geq 0)$ in $\mathbb{R}^d$ has independent increments (II) if $\forall\ 0 \leq t_1 < t_2 < \cdots < t_k$,

$$X_{t_1} - X_{t_0}, \ldots, X_{t_k} - X_{t_{k-1}} \quad \text{are independent random variables.}$$

We *always* assume that such a process is also continuous in prob:

$$\forall t \quad \forall \varepsilon > 0 \quad \lim_{s \to t} P(|X_t - X_s| > \varepsilon) = 0$$

If, in addition, the increments are stationary, viz.,

$$P(X_t - X_s \in \cdot) = P(X_{t+r} - X_{s+r} \in \cdot)$$

then the process is called Lévy. Assume, throughout, $X_0 = 0$.

3) Any II process admits a càdlàg version with no fixed jumps

The reason is that the martingale

$$\left(\frac{e^{i\theta X_t}}{E e^{i\theta X_t}}, \ t \geq 0 \right)$$

converges. The reason that the martingale is well-defined is that the characteristic function

$$\varphi_{s,t}(\theta) = E\, e^{i\theta(X_t - X_s)}$$

is nonzero for all $\theta \in \mathbb{R}$.

No fixed jumps means $\forall t \quad P(\Delta X_t \neq 0) = 1$

This is a consequence of the continuity in probability

NOTE: θX means $\langle \theta, X \rangle = \sum_{i=1}^{d} \theta^i X^i$

4) Squeezing out a Poisson random measure

Let $\Delta X_t = X_{t+} - X_{t-} = X_t - X_{t-}$

A time t for which $\Delta X_t \neq 0$ is a jump epoch.
ΔX_t is the (size of the) jump.

Let $\delta_{(t,x)}$ denote the Dirac measure

$$\delta_{(t,x)}(A) = \mathbb{1}((t,x) \in A) \qquad A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$$

that puts mass 1 at the point $(t,x) \in \mathbb{R} \times \mathbb{R}^d$

By càdlàg, there are at most countably many jumps, a.s.
Hence, $\forall \omega \in \Omega$,

$$\eta = \sum_{t : \Delta X_t \neq 0} \delta_{(t, \Delta X_t)}$$

is a measure on $\mathbb{R}_+ \times \mathbb{R}^d$

It is convenient to exclude the origin from $\mathbb{R}^d$ and consider η as a measure on the punctured space

$$\mathbb{R}_+ \times \mathbb{R}^d_0 := \mathbb{R}_+ \times (\mathbb{R}^d \setminus \{0\})$$

Note that

$$\eta(\{t\} \times \mathbb{R}^d_0) = 0 \quad \text{a.s.}, \quad \text{for all } t \in \mathbb{R}_+ .$$

(because: no fixed jumps)
and that

$\forall\, 0 \leq t_1 < t_2 < \cdots < t_k$

$$\eta((0,t_1] \times \mathbb{R}^d_0),\ \eta((t_1,t_2] \times \mathbb{R}^d_0), \ldots, \eta((t_{k-1},t_k] \times \mathbb{R}^d_0)$$

are independent random variables

By the Erlang–Lévy theorem, we have that

- η is Poisson on $\mathbb{R}_+ \times \mathbb{R}^d_0$
- $E\eta(A) < \infty$ if A is bounded Borel $\subseteq \mathbb{R}_+ \times \mathbb{R}^d_0$

4a) First property of η : $E\left[\eta\ [0,t] \times \{x \in \mathbb{R}^d : |x| \geq 1\}\right] < \infty$

Indeed, the RV inside the E is Poisson and a.s. finite (there are only finitely many jumps of magnitude ≥ 1 on each $[0,t]$); the expectation of a finite Poisson RV is, of course, finite.

5) Simple II-processes

An II-process (X_t) is simple if

$$\sum_{0 \le s \le t} 1(\Delta X_s \neq 0) < \infty \quad a.s. \quad \forall t$$

$$(\text{and hence } \forall t \text{ a.s.})$$

A simple II-process with no continuous component (see later for the def. of cont. comp.) and stationary increments is called compound Poisson.

6) Sums of big jumps are independent of the rest

Fix an II-process (X_t) and some $\varepsilon > 0$. Define

$$X_t^\varepsilon = \sum_{0 \le s \le t} \Delta X_s \, 1(|\Delta X_s| > \varepsilon)$$

Since there are only finitely many jumps of magnitude $> \varepsilon$ on each $[0,t]$, the sum consists of finitely many terms. So it is well-defined. Let

$$Y_t^\varepsilon = X_t - X_t^\varepsilon$$

Since $|\Delta X_t^\varepsilon| \cdot |\Delta Y_t^\varepsilon| = 0$, since $(X_t^\varepsilon, Y_t^\varepsilon)$ has independent increments (in $\mathbb{R}^d \times \mathbb{R}^d$), and since none of the two has fixed jumps, it follows that

$$X^\varepsilon, Y^\varepsilon : \text{are independent processes}$$

7) Second property of $E\eta$:

$$\int_{[0,t] \times \mathbb{R}_0^d} |x|^2 \, E\eta(ds, dx) < \infty \qquad \forall t$$

This follows from an estimate of the exponent of the characteristic function of X_t^ε when $\varepsilon \to 0$.

First note that

$$\left|E e^{i\theta X_t}\right| = \left|E e^{i\theta X_t^\varepsilon} \cdot E e^{i\theta Y_t^\varepsilon}\right| \le \left|E e^{i\theta X_t^\varepsilon}\right|$$

and recall that $E e^{i\theta X_t}$ never vanishes.

Second, write

$$X_t^\varepsilon = \sum_{0\le s\le t} \Delta X_s \, 1(|\Delta X_s| > \varepsilon) = \int_{\mathbb{R}_+\times\mathbb{R}_0^d} x \, 1(|x|>\varepsilon)\, 1(0\le s\le t)\, \eta(ds, dx)$$

$$=: \int f(s,x)\, \eta(ds, dx)$$

and recall the formula for the characteristic functional of a Poisson random measure:

$$E\, e^{i\theta \int f \, d\eta} = e^{\int (e^{i\theta f} - 1)\, E d\eta}$$

to obtain that

$$\left| E e^{i\theta X_t^\varepsilon} \right| = \left| \exp \int_{\substack{0\le s\le t \\ |x|>\varepsilon}} (e^{i\theta x} - 1) \; E\eta(ds, dx) \right|$$

$$= \exp\left\{ - \int_{\substack{0\le s\le t \\ |x|>\varepsilon}} (1 - \cos \theta x)\; E\eta(ds, dx) \right\}$$

If $\int_{0\le s\le t} |x|^2 \, E\,\eta(ds, dx) = \infty$ then, owing to $1-\cos\theta x = O(|x|^2)$, we have that $\lim_{\varepsilon\to 0} E e^{i\theta X_t^\varepsilon} = 0$: contradiction!

8) Compensated sums of small jumps

Whereas sums of big jumps, say of magnitude > 1,

$$X_t^1 = \sum_{0\le s\le t} \Delta X_s \; 1\left(|\Delta X_s| > 1\right) = \int_{\substack{0\le s\le t \\ |x|>1}} x \;\; \eta(ds, dx)$$

can be defined without problem, this is not the case with small jumps: If we formally replace $1(\cdot > 1)$ by $1(\cdot \le 1)$ we can find examples that make no sense. However,

$$Y_t^1 = \int_{\substack{0\le s\le t \\ |x|\le 1}} x \;\left[\eta(ds, dx) - E\eta(ds, dx)\right]$$

makes sense, owing to the second property of $E\eta$.

To abbreviate, write $X^1 = \int_{|x|>1} x \, d\eta$, $Y^1 = \int_{|x|\le 1} x \left[d\eta - dE\eta\right]$

Since <u>we cannot</u> write

$$X^1 + Y^1 = \int_{|x|>1} x \, d\eta + \int_{|x|\le 1} x \, d\eta - \int_{|x|\le 1} x \, dE\eta = \int_{\mathbb{R}^d_0} x \, d\eta - \int_{|x|\le 1} x \, dE\eta$$

(from which it would have been obvious that X^1+Y^1 has the same jumps as X),

we must make sure that X^1+Y^1 has the same jumps as X.

To this end, note that

$$Y^{\varepsilon,1}_t = \int_{\substack{0\le s\le t \\ \varepsilon<|x|\le 1}} x \left[\eta(ds,dx) - E\eta(ds,dx)\right]$$

is a simple II-process that converges to Y^1_t in the sense that

$$E \sup_{0\le s\le t} |Y^{\varepsilon,1}_s - Y^1_s|^2 \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

because

$$E \sup_{0\le s\le t} |Y^{\varepsilon,1}_s - Y^1_s|^2 \le 2 \, E\, |Y^{\varepsilon,1}_t - Y^1_t|^2 \qquad \text{(Doob's inequality)}$$

$$= 2 \int_{\substack{0\le s\le t \\ |x|\le\varepsilon}} |x|^2 \, E\eta(ds,dx) \qquad \text{(variance formula for Poisson random measures)}$$

$$\longrightarrow 0 \quad \text{as } \varepsilon \downarrow 0$$

Since X^1+Y^1 has the same jumps as X,

$$Z = X - (X^1+Y^1)$$

is a.s. continuous, with continuous ~~mean~~ and covariance, and independent increments. Hence Z is Gaussian.

Since $|\Delta Z_t| \cdot |\Delta X^1_t + \Delta Y^{\varepsilon,1}_t| = 0$,

we have that Z, $X^1 + Y^{\varepsilon,1}$ are independent $\forall \varepsilon > 0$

Hence Z, X^1+Y^1 are independent

9) Lévy – Itô decomposition

Let (X_t) be II process in $\mathbb{R}^d$ (cont. in prob.)
Then there exists a Poisson random measure η on $\mathbb{R}_+ \times \mathbb{R}_0^d$ such that

$$\int |x|^2 \wedge 1 \quad E\eta(ds,dx) < \infty ,$$

and a Gaussian process (Z_t) with independent increments, continuous mean EZ_t and continuous covariance function $E Z_s \cdot Z_t$, such that

$$X_t = Z_t + \int_{\substack{0 \le s \le t \\ |x| \le 1}} x \left[\eta(ds,dx) - E\eta(ds,dx) \right] + \int_{\substack{0 \le s \le t \\ |x| > 1}} x \, \eta(ds,dx) \quad (*)$$

10) Lévy – Itô decomposition for Lévy processes

Let (X_t) be Lévy in $\mathbb{R}^d$
The additional assumption of stationarity of increments (viz, translation invariance) implies that :

1. $EZ_t = at$, $a \in \mathbb{R}^d$
2. $EZ_s \cdot Z_t = \sigma \cdot (s \wedge t)$, $\sigma \in \mathbb{R}^{d \times d}$, positive semidefinite matrix
3. $E\eta(ds,dx) = ds \cdot \nu(dx)$ $(= \text{Lebesgue} \times \nu)$

   where ν is a σ-finite measure on $\mathbb{R}_0^d$ with
   $$\int_{\mathbb{R}_0^d} (|x|^2 \wedge 1) \, \nu(dx) < \infty$$

11) Lévy-Khinchine (-Kolmogorov) formula

Let (X_t) be Lévy in $\mathbb{R}^d$. Then:

$$Ee^{i\theta X_t} = e^{t\psi(\theta)}$$

$$\psi(\theta) = i\,\theta\cdot b - \tfrac{1}{2}\,\theta\cdot\sigma\cdot\theta + \int_{\mathbb{R}^d_0} \left\{e^{i\theta\cdot x} - 1 - i\,\theta\cdot x\ 1(|x|\le 1)\right\} \nu(dx) \qquad (2)$$

Proof: Take characteristic function of ⊛.

12) Lévy-Khichine formula for infinitely divisible distribution

Let μ be infinitely divisible probability measure in $\mathbb{R}^d$.

Hence $\mu = \mu_n^{*n}$ for each n

It can be shown that $\mu_n \Rightarrow \delta_0$ (weak convergence to point mass at o). Let $\hat\mu(\theta) = \int e^{i\theta\cdot x}\mu(dx)$, $\hat\mu_n(\theta) = \int e^{i\theta\cdot x}\mu_n(dx)$

Hence $\hat\mu_n(\theta) \to 1$ uniformly on each $-\varepsilon \le \theta \le \varepsilon$

Hence $\hat\mu(\theta) \ne 0$

Hence $\psi(\theta) = \log\hat\mu(\theta)$ is well-defined $: \mathbb{R} \to \mathbb{C}$

and we can choose a continuous branch of it

So $\hat\mu(\theta) = e^{\psi(\theta)}$

Similarly, $\hat\mu_n(\theta) = e^{\psi_n(\theta)}$

But $\psi(\theta) = n\,\psi_n(\theta)$. So $\psi_n(\theta) = \frac{1}{n}\psi(\theta)$

Since $\hat\mu_n(\theta) = e^{\psi_n(\theta)} = e^{\frac{1}{n}\psi(\theta)}$ is a characteristic function, so is any integer power $\hat\mu_n(\theta)^m = e^{\frac{m}{n}\psi(\theta)}$

By Bôchner's theorem, $e^{t\psi(\theta)}$ is a characteristic function for each $t \ge 0$. Let μ_t be the probability measure corresponding to $e^{t\psi(\theta)}$

Since $e^{t\psi(\theta)} = e^{(t-t_k)\psi(\theta)} e^{(t_k-t_{k-1})\psi(\theta)} \cdots e^{(t_2-t_1)\psi(\theta)} e^{t_1\psi(\theta)}$

we have ~~that~~ we can define random variables

$$(X_t,\ t \geq 0)$$

with compatible finite-dimensional distributions.

By Carathéodory–Kolmogorov extension, and the fact that $e^{t\psi(\theta)} \xrightarrow[t\to 0]{} 1$ implies continuity in probability, it follows that $(X_t, t\geq 0)$ is a Lévy process (i.e. we can choose a suitable càdlàg version). Hence ② holds.

Take $t=1$ in ② to obtain that any infinitely divisible distribution μ has characteristic function

$$\hat{\mu}(\theta) = e^{\psi(\theta)} \quad \text{with} \quad \psi(\theta) \text{ as in ②}.$$

REFERENCES

Kingman (1993) Poisson Processes

Lévy (1937) Théorie de l'Addition des Variables Aléatoires

Chow & Teicher (1978) Probability Theory

Kallenberg (1997) Foundations of Modern Probability

Applebaum (2004) Lévy Processes and Stochastic Calculus

Bertoin

Sato