Summaries of the Lectures of the Dimitsana Summer School on Stochastic Analysis and Optimization in Finance July 9 - July 21, 2001

1 F. Delbaen: Introduction to Mathematical Finance

We introduce the basic principles of elementary mathematical finance:
1) no transaction costs
2) full information for everybody
3) no restrictions on selling and buying.

Information is described using a filtered probability space

$$\left(\Omega, (\mathcal{F}_t)_{0\leq t}, P\right), \ \mathcal{F}_\infty = \bigvee_{0\leq t} \mathcal{F}_t .$$

We suppose the usual assumptions to be satisfied ($\mathcal{F}_0$ contains all null sets of $\mathcal{F}_\infty$, and $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_t$ for all t). Prices are described by càdlàg processes adapted to the filtration $(\mathcal{F}_t)_{0\leq t}$. For these lectures we suppose that prices are locally bounded (or even continuous in most cases). By cleverly extending filtrations one can show that discrete time can be embedded in this continuous time setting. The time horizon is ∞. But this is more a matter of notation. Prices are denoted by $\left(\bar{S}^0, \ldots, \bar{S}^d\right)$, i.e. there are $d+1$ assets. Locally bounded means there exists a sequence $(T_n)_{n\geq 1}$ of stopping times so that

$$\sup_{t\leq T_n} \left\|\bar{S}_t\right\| \leq K_n \quad \text{and} \quad T_n \nearrow \infty .$$

The asset 0 plays a special role. It is the bank account. We suppose $\bar{S}^0$ to be continuous, of finite variation and, most importantly, $\bar{S}^0_t > 0$ almost surely. In many cases

$$\bar{S}^0_t = \exp(rt) \quad \text{or} \quad \bar{S}^0_t = \exp\left(\int_0^t r_u du\right),$$

where r ($(r_u)_{0\leq u}$) is the interest rate (interest rate process). We suppose $\bar{S}^0_0 = 1$ for simplicity. Strategies are described using predictable processes. Elementary strategies are of the form:

$$H = \sum_{k=0}^{N} f_k 1_{(T_k, T_{k+1}]},$$

where $0 \leq T_0 \leq T_1 \leq \dots T_{N+1} < \infty$ are stopping times. $f_k : \Omega \to \mathbb{R}^d$ are $\mathcal{F}_{T_k}$-measurable, and f_k denotes the number of stocks $(1 \to d)$ kept during the interval $(T_k, T_{k+1}]$. To exclude obvious money making strategies we conclude that

(a) All transactions are financed through the bank account. Therefore f_0 is not defined. $\to$ self-financing.

(b) Transactions at times T_i are done at new prices $\bar{S}_{T_i}$ and not at their left limits. This is the reason why we take $(T_k, T_{k+1}]$ and not $[T_k, T_{k+1})$.

It turns out that the discounted prices are better to make the calculations.

2 M. Yor: Introduction to Stochastic Calculus

-) After defining "fair", "unfair" and "extra-fair" processes, as the processes (X_t) adapted to a filtration $(\mathcal{F}_t)$, which satisfy respectively:

$$\forall t > s \,,\, \mathrm{E}\,[X_t \,|\, \mathcal{F}_s] = X_s \,;\, \mathrm{E}\,[X_t \,|\, \mathcal{F}_s] \leq X_s \,;\, \mathrm{E}\,[X_t \,|\, \mathcal{F}_s] \geq X_s \,,$$

I gave their usual terminology: martingale, supermartingale and submartingale, following Doob (1953) and relations between these processes and (resp.) harmonic, superharmonic and subharmonic functions.

-) I then recalled the (additive) Doob-Meyer decomposition theorem.

-) The vector space of processes discussed above forms the space of $(\mathcal{F}_t, P)$ semimartingales. Stochastic calculus may be considered as a means / search(?) of understanding how the nature (martingale, etc.) of these processes is transformed under changes of probability, filtration, "time calendar", space (i.e. $(X_t) \to (f(X_t))$, which is solved via the famous Itô formula)

-) I introduced Brownian motion from a series expansion:

$$B(f) = \sum (f, e_n) G_n \,,$$

in terms of $\mathcal{N}(0,1)$ independent variables (G_n), then showed it has quadratic variation equal to t, hence infinite variation. I then constructed stochastic integrals $\left(\int_0^t \varphi(u,\omega) dB_u\right)$ for predictable processes φ which satisfy:

$$\mathrm{E}\left[\int_0^t \varphi^2(u) du\right] < \infty$$

and finished with a presentation of Lévy's characterization of BM.

Note: I intend in the final version of this lecture to give some historical elements about the development of the notion of semimartingale.

3 T. Konstantopoulos: Notions of Stochastic Models

We give several arguments for the necessity of stochastic modelling (randomness in physics, complexity of various systems, convenience and computation tools, etc.) and contrast it to deterministic modelling. We explain the passage from microscopic to macroscopic description and, as a simple example, present an alternative construction of Brownian motion via Donsker's theorem (here, we explain in some detail, the notion of weak convergence on $C[0,\infty)$, scaling properties of Brownian motion, and the Radon-Nikodym theorem). The "simplest" stochastic models can be seen to possess the Markov property which "lifts" the semigroup property from the dynamical system level to the level of probability kernels. A stochastic version of a simple deterministic system ($\dot{x} = \lambda - \mu x$) is discussed, and Poisson processes are introduced. The latter processes are used to construct the stochastic version via the integral equation

$$X_t = X_0 + N_t - M_{\int_0^t X_s ds} .$$

The related equation

$$X_t = X_0 + N_t - c \int_0^t X_s dM_s \quad (0 < c < 1)$$

is also discussed.

4 F. Delbaen: Basic Models in Finance

Using the elementary strategy H, the final (discounted) value of the portfolio is given by

$$V_0 + (H \cdot S)_\infty = V_0 + \sum_{k=0}^{N} f_k \left(S_{T_{k+1}} - S_{T_k} \right) ,$$

where scalar product is taken and

$$S_t : \Omega \to \mathbb{R}^d$$

is defined by

$$S_t^j = \frac{\bar{S}_t^j}{\bar{S}_t^0} \quad \text{(discounting!)} .$$

V_0 is the initial investment at time 0. The standard example is

$$S_t = S_0 \exp\left((\mu - r)t + \sigma B_t - \frac{1}{2}\sigma^2 t \right) ,$$

where B is a Brownian motion or more generally a Lévy process.
We call a strategy H admissible if H is predictable, if $(H \cdot S)$ can be defined as a stochastic integral, if $(H \cdot S)_\infty$ exists as a limit $\lim_{t\to\infty} (H \cdot S)_t$ almost surely and if there exists a real constant $a \geq 0$ such that $(H \cdot S) \geq -a$ as a process.

$$\mathcal{K} = \{(H \cdot S)_\infty : H \text{ admissible}\}$$

is only a cone. The no-arbitrage assumption reads $\mathcal{K} \cap L_+^0 = \{0\}$.

5 T. Konstantopoulos

The basic theory of martingales, first in discrete time, is presented. Emphasis is given given on
(i) optional stopping theorems and
(ii) convergence theorems.
The upcrossing inequality is proved, and the basic convergence theorem is explained in terms of gambling ($\Leftrightarrow$ impossibility of making money in a fair game of chance, using a predictable strategy). We also discuss "discrete-time stochastic integrals"

$$(H \cdot X)_n = \sum_{k=1}^{n} H_k \left(X_k - X_{k-1} \right) ,$$

where $H_k \in \mathcal{F}_{k-1} \, \forall k$. We present several inequalities and prove the basic maximal inequality

$$P \left[\max_{0 \leq n \leq N} X_n \geq \lambda \right] \leq \frac{\mathrm{E}\left[|X| \right]}{\lambda}$$

for submartingales. We pass on to continuous time (semi-)martingales and construct X_{t+} and X_{t-}. We discuss the role of completeness of the filtration. Also, we discuss the existence of a càdlàg modification under the usual assumptions.
Examples given:
Random walk and exponential martingales ($\rightarrow$ Chernoff's bound), Brownian motion and exponential martingales ($\rightarrow$ exponential inequality), Brownian motion stopped at $T_a \wedge T_b$.
We also explained the role of uniform integrability for extending a martingale from $(X_n, n \in \mathbb{N})$ to $(X_n, n \in \mathbb{N} \cup \{\infty\})$. Finally, we presented the theorem that characterizes a martingale via the constancy of its expectation over all bounded stopping times.

6 M. Yor: Brownian Motion

-) I discussed several basic properties of Brownian motion, such as: scaling, time inversion, time reversal, Markov property, martingale property, ...

-) I gave several presentations of the Brownian bridge, i.e. $(B_u, u \leq t)$ conditioned to be at 0 at time t. In particular, I showed, using time-inversion, that if

$$g_t = \sup \{ s \leq t \, : \, B_s = 0 \} ,$$

then

$$\left(\frac{B_{u g_t}}{\sqrt{g_t}}, u \leq 1 \right)$$

is a standard Brownian bridge.

-) I also gave the semimartingale decomposition of the Brownian bridge

$$(b_u, u \leq 1)$$

as:

$$b_u = \beta_u - \int_0^u ds \frac{b_s}{(1-s)} .$$

References
-) Lévy (1939): Sur certains processus stochastiques homogènes. *Comp. Math.*
-) Jeulin, Yor (1979): Inégalité de Hardy, Semimartingales, et faux-amis. *Sém. Proba.* XIII, *Lecture Notes in Mathematics* 721.

7 T. Konstantopoulos: Stochastic Integrals and Brownian Motion

We first explained some previously discussed points:
(i) the construction of a Brownian bridge via sampling without replacement;
(ii) its relation to BES(3);
(iii) the uniform integrability of

$$\{\mathrm{E}[X \mid \mathcal{G}], \mathcal{G} \subset \mathcal{F}\} \quad \text{when} \quad \mathrm{E}[|X|] < \infty .$$

We then introduced local martingales

$$\left(\text{example:} \quad \frac{1}{\mathrm{BES}(3)} \quad \text{is a local martingale but not a martingale}\right) .$$

We defined the predictable and optional σ-algebras on $\mathbb{R}_+ \times \Omega$. We defined quadratic variation and showed how it can be constructed for a continuous bounded martingale. We proved its uniqueness, explained why $\langle X^T \rangle = \langle X \rangle^T$, for stopping times T, and extended it to local martingales. We discussed in detail the covariation (bracket) process $\langle X, Y \rangle$ and some of its properties. We started stochastic integration, as usual, by first integrating simple predictable processes, and showed their martingale property. We concluded the lecture by reminding the audience of the definition of $\int f(t) dB_t$ ($f \in L^2$, deterministic, B Brownian motion) via an $L^2(\mathbb{R}) \leftrightarrow L^2(\Omega)$ isometry and promised that the idea will be generalized and used for the definition of more general stochastic integrals $\int H_t dX_t$.

8 F. Delbaen: Theorems of Finance

The fundamental theorem of asset pricing (FTAP) is stated using the (NFLVR)-condition, i.e.

(a) No arbitrage
(b) $\mathcal{K}_1$ is bounded in probability,
where $\mathcal{K}_1$ is the set of outcomes of 1-admissible strategies.
We discuss the equivalent formulation
if H^n are admissible $f^n = (H^n \cdot S)_\infty \geq -\varepsilon_n, \varepsilon_n \searrow 0$, then $f^n \xrightarrow{P} 0$.
We discuss weaker concepts that lead to the assumption that S must be a semimartingale in order to avoid weak forms of (NFLVR). $(H \cdot S)_\infty$ exists as otherwise we could profit from the oscillation using "buy low - sell high" strategies (compare to the proof of the limit theorem for martingales).

Theorem: *Under (NFLVR) there exists a $Q \sim P$ such that S is a local martingale under Q. (The converse also holds and is much easier.)*

We discuss the relation between (a) and (b) in (NFLVR) without giving details.

9 T. Konstantopoulos: Stochastic Integration

We explain the isometry between the Banach space of bounded continuous martingales X with norm

$$\sqrt{\sup_t \mathrm{E}\left[X_t^2\right]}$$

and the space of predictable processes H with norm

$$\sqrt{\mathrm{E}\left[\int_0^\infty H_s^2 d\left\langle X\right\rangle_s\right]}.$$

This isometry, together with the fact that simple predictable processes are dense in the second space, allows for our first extension of the stochastic integral. We then prove various properties, including

$$\langle H \cdot X, K \cdot Y\rangle = HK \cdot \langle X, Y\rangle$$

and the Kunita-Watanabe inequality

$$\int |H_s K_s|\, d\left|\langle X, Y\rangle\right|_s \leq \sqrt{\int H_s^2 d\left\langle X\right\rangle_s \int K_s^2 d\left\langle Y\right\rangle_s}.$$

Finally, we extend the integral $H \cdot X = \int_0^\infty H_s dX_s$ to local martingales X and predictable processes H such that $\int_0^t H_s^2 d\left\langle X\right\rangle_s < \infty$ $\forall t$, a.s. We then show how to approximate $(H \cdot X)_t$ by using "Riemann-Itô" sums (when H is continuous) of the form

$$\sum_i H_{t_i^n}\left(X_{t_{i+1}^n} - X_{t_i}^n\right).$$

We discuss the Stratonovitch integral and compute $\int_0^t X_s dX_s$, using Riemann-Itô sums.

10 T. Konstantopoulos: Itô's Formula

We prove Itô's formula

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle X\rangle_s$$

for a continuous local martingale X, and $f \in C^2$, using Riemann-Itô sums approximation and Taylor's expansion for f. We introduce semimartingales, their bracket processes, and discuss the basic properties

$$\langle H\cdot X, K\cdot Y\rangle = HK\cdot\langle X,Y\rangle \quad \text{and} \quad H\cdot(K\cdot X) = (HK)\cdot X\,.$$

We prove Itô's formula for $\mathbb{R}^d$-valued semimartingales, by first proving the product formula

$$X_tY_t = X_0Y_0 + (X\cdot Y)_t + (Y\cdot X)_t + \langle X,Y\rangle_t\ ,$$

and then by approximating by polynomials. We discuss several applications:

1) Exit problem for Brownian motion in $\mathbb{R}^d$: $f(x) = \mathrm{E}_x\left[f(B_\tau)\right]$
2) Laplace's and the heat equation
3) Recurrence properties for Brownian motion in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^d$ $(d \geq 3)$
4) The Dol'eans-Dade martingale

Finally, we define the concept of an $\mathcal{F}_t$-Brownian motion and prove Lévy's characterization theorem for the Brownian motion in $\mathbb{R}^d$.

11 F. Delbaen: Option Pricing

In a complete market (i.e. where $\forall g \in L^\infty\ \exists g_0 \in \mathbb{R}, H$ so that $(H\cdot S)$ is bounded and $g = g_0 + (H\cdot S)_\infty$) we can price every contingent claim by $\mathrm{E}_Q\left[g\right] = g_0$, where Q is the unique local martingale measure for S. Of course, we suppose first $g \in L^\infty$, then $g \geq 0$.
Standard example: Samuelson's model $\Rightarrow$ Black-Scholes formula for option:

$$C(S_0,T,\sigma,r,K) = \mathrm{E}_Q\left[\left(S_T - K\mathrm{e}^{-rT}\right)^+\right]$$

$$= \mathrm{E}_Q\left[\left(S_0\exp\left\{\sigma\sqrt{T}N - \frac{1}{2}\sigma^2T\right\} - K\mathrm{e}^{-rT}\right)^+\right] = S_0\Phi(d_1) - K\mathrm{e}^{-rT}\Phi(d_2)\,.$$

By differentiation under the expectation (allowed!) we get without effort the Greeks. We discuss that

$$\mathrm{E}_Q\left[\left(S_T - K\mathrm{e}^{-rT}\right)^+ \mid \mathcal{F}_t\right] = f(S_t,t)$$

is a martingale and get, using Itô, the differential equation. Going back ("undiscount"), we get the differential equation for $C(\bar{S}_t, T-t,\sigma,r,K)$ (Exercise).

12 M. Yor: Stochastic Differential Equations

-) As a comment on the preceding lecture no. 11, I discussed briefly the convex set of probability measures:

$$\mathcal{M} = \{P \text{ on } C(\mathbb{R}_+, \mathbb{R}) \,|\, \text{under } P,\ (X_t(\omega) = \omega(t), t \geq 0) \text{ is a local martingale}\} \,.$$

and proved that the extremal points of $\mathcal{M}$ are those for which the "replication property" holds

-) Then I discussed Girsanov's theorem, which plays an important role in the study of SDE's.

-) My discussion of SDE's:

(i) Pathwise solutions and solutions in law were defined.

(ii) Concerning pathwise solutions, I started with the very standard fact that if $\sigma, b : \mathbb{R} \to \mathbb{R}$, are Lipschitz functions, then:

$$X_t = x + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) ds \,,\ t \geq 0 \,, \tag{12.1}$$

has a unique strong solution, obtained by (e.g.) Picard's iteration procedure.

(iii) Pathwise Uniqueness implies that the solution is strong (i.e. adapted with respect to the filtration of (B_t)), and there is uniqueness in law. This is the famous Yamada-Watanabe theorem.

(iv) It is quite remarkable that we can go quite further than the "standard" Lipschitz condition to solve (12.1), i.e. it suffices that σ be Hölder($\frac{1}{2}$), and b be bounded Borel.

(v) I then gave a number of examples, including the Ornstein-Uhlenbeck process, the "bang bang" process, and finally a proof that

$$\sinh(B_t) \overset{\text{(law)}}{=} \int_0^t \exp(B_s)\, dC_s \,, \quad \text{for fixed } t \,,$$

where B and C denote two independent Brownian motions.

<u>References:</u>

-) Any book on stochastic integration (KS, RY, RW, ...) contains these discussions on SDE's.

-) Øksendal, Kloeden-Platen, Gand present a number of explicitly solvable SDE's.

-) A nice discussion of diffusions is in: Varadhan: Handbook of Statistics, Vol. 19 (2001).

13 F. Delbaen: Stochastic Differential Equation-s in Finance

We discuss the American call option and show that because of the submartingale property of $(S_t - Ke^{-rt})^+$ early exercise is not interesting. The problem of the

American put is then treated. $(Ke^{-rt} - S_t)^+$ is not a submartingale.

We discuss the price of the perpetual American put and show how to get the solution. The smooth fit condition is obtained using an argument based on local time.

Time constraints did not permit to introduce alternative models based on s-tochastic differential equations.

It can be shown that

$$dS_t = \mu(S_t)dt + \sigma(S_t)dB_t$$

yields equivalent martingale measures if "tractable" conditions on μ and σ are fulfilled (not treated in the lecture).

14 M. Yor: Relations between SDE and PDE

-) Following my lecture no. 12, I started with the discussion of the Markov property of (X_t), solution of (12.1), under uniqueness conditions.

-) Then, followed naturally a discussion of the semigroup (P_t) associated with (X_t), as well as the infinitesimal generator A. One obtains the Kolmogorov equation:

$$P_t f(x) = f(x) + \int_0^t ds P_s(Af)(x) \, ,$$

for functions $f \in D(A)$.

-) The classical Doob h-transform is then discussed, as well as computation of the transform of the infinitesimal generator A under even more general trans-forms, i.e.

$$P^{\varphi}_{x|\mathcal{F}_t} \equiv \left(\frac{\varphi(X_t)}{\varphi(x)}\right) \exp\left(-\int_0^t ds \left(\frac{A\varphi}{\varphi}\right)(X_s)\right) \cdot P_{x|\mathcal{F}_t} \, .$$

-) Finally, I discussed the Feynman-Kac formula for Brownian motion

References:

Over the years, I found out that both Durret and Karatzas-Shreve have a nice discussion of the relations between SDE and PDE. But, again, almost every book on stochastic differential equations will offer such discussions.
Should probably look at A. Friedmann
Stroock's book (Camb. Univ. Press): Probability from an analyst's point of view.

15 F. Delbaen: Advanced Alternative modelling with Brownian motion

devoted to the discussion of the American put and the details to calculate the exercise boundary for the perpetual put. see remark for lecture no. 13 morning.

16 T. Konstantopoulos: Optimization and Control in Finance

Concepts of optimal stochastic control, decision-making in a random environment. First simple application: the secretary problem. The principle of dynamic programming is based on

$$\max_{u,v} [g(u) + h(u,v)] = \max_u \left[g(u) + \max_v h(u,v) \right] , \; g, h \geq 0 \, .$$

Dynamic programming equation (recursion) for discrete-time deterministic recursions. Introduction to Markov chains and the dynamic programming equation for a discounted reward. Application: gambling (maximize log(final fortune) when playing a (p,q)-coin). Controlled stochastic differential equations:

$$dX_t = b(t, X_t, U_t)dt + \sigma(t, X_t, U_t)dB_t \, .$$

Letting T be the exit time from a domain $G \subset \mathbb{R}_+ \times \mathbb{R}^d$,

$$J^U(s,x) := \mathrm{E}_{s,x} \left[\int_s^T F(t, X_t, U_t)dt + K(T, X_T) 1_{\{T<\infty\}} \right] ,$$

we derive for the value function $\Phi(y) = \sup_y J^U(y)$, the Hamilton-Jacobi-Bellman equation

$$\sup_U \left\{ F(y,U) + (A^U \Phi)(y) \right\} = 0 \, , \; y \in G \, ,$$

and discuss its meaning and applicability. We solve the optimal portfolio selection problem, as a simple application.

17 F. Delbaen: Complete and Incomplete Markets

A model is incomplete $\Leftrightarrow \mathcal{M}^e$ (the set of equivalent local martingale measures) contains more than one element.

We discuss the relation (for $g \geq 0$)

$$\sup_{Q \in \mathcal{M}^e} \mathrm{E}_Q [g] = \inf \{ \alpha \mid \exists f \in \mathcal{K} \, , \, \alpha + f \geq g \} \, .$$

In case this expression is finite, inf = min whereas the sup is not necessarily a max, see further.

We ask the students to find out the relation with linear programming. Primal and dual program! Take Ω finite to avoid surprises from functional analysis. We introduce in various stages the following concepts and results (Ansel-Stricker and Delbaen-Schachermayer):

$f \in \mathcal{K}$ is called maximal if $g \in \mathcal{K}$, $g \geq f$ implies $g = f$.

Let $f \in \mathcal{K}$. Equivalent are:

(a) f is maximal.

(b) $\exists Q \in \mathcal{M}^e \; \exists H$ admissible with $f = (H \cdot S)_\infty$ and $(H \cdot S)$ is a Q-martingale which is uniformly integrable.

(c) $\exists Q \in \mathcal{M}^e \; \mathrm{E}_Q[f] = 0$.

We remark that this does not imply $\mathrm{E}_{Q'}[f] = 0 \; \forall Q' \in \mathcal{M}^e$.

From this we deduce that $g \geq 0$ can be hedged $\Leftrightarrow \exists Q \in \mathcal{M}^e$

$$\mathrm{E}_Q[g] = \sup_{Q' \in \mathcal{M}} \mathrm{E}_{Q'}[g] \; .$$

hedged means $\exists g_0 \in \mathbb{R} \; \exists H \; \exists Q \in \mathcal{M}^e$ with $g = g_0 + (H \cdot S)_\infty$ and $(H \cdot S)$ is a Q-uniformly integrable martingale or $g = g_0 + f$ with f maximal in $\mathcal{K}$.

The latter is a definition which does not use a specific measure and is therefore conceptually better.

18 M. Yor: Poisson and Lévy processes

-) I presented the Poisson process in detail, insisting in particular on the construction of associated martingales, which finally yield:

(a) Watanabe's characterization of the Poisson process;

(b) the property that the "Poisson Market" is complete, i.e. every square integrable martingale may be written as

$$c + \int_0^t m_s \left(dN_s - c ds \right) \; ,$$

for some suitable predictable $(m_s, s \geq 0)$.

-) This then led me to the construction of Poisson random measures with given intensity measures, which I compared to Gaussian measures.

-) Poisson point processes are obtained from Poisson random measures on $E \times \mathbb{R}_+$ with intensity $\nu(de)dt$.

-) This allows to construct Lévy processes.

References

This presentation is close to Chapters 0 and 1 of Bertoin's book: Lévy processes.

19 P. Cheridito: Gaussian Random Vectors

Definition: *We call a d-dimensional random vector X Gaussian if*

$$\mathrm{E}\left[\mathrm{e}^{\mathrm{i}u^T X}\right] = \exp\left(-\frac{1}{2}u^T C u + \mathrm{i}u^T m\right), \, u \in \mathbb{R}^d,$$

for some $m \in \mathbb{R}^d$ and a symmetric, positive semi-definite $d \times d$-matrix C.

Proposition: *Let $(X_1, \ldots, X_l, X_{l+1}, \ldots, X_d)$ be a Gaussian random vector.*

a) *If*

$$\mathrm{Cov}\,(X_j, X_k) = 0$$

for all $j = 1, \ldots, l$ and $k = l+1, \ldots, d$, then $(X_1, \ldots, X_l)$ and $(X_{l+1}, \ldots, X_d)$ are independent.

b)

$$\mathrm{E}\left[X_d \,|X_1, \ldots, X_{d-1}\right] = \sum_{j=1}^{d-1} a_j X_j \,,$$

for constants $a_1, \ldots, a_{d_1}$.

Proposition: *Let X be a d-dimensional Gaussian random vector and A an $m \times d$-matrix. Then AX is an m-dimensional Gaussian vector.*

Theorem: *Let $\{X^n\}_{n=0}^{\infty}$ be a sequence of d-dimensional Gaussian random vectors that converges to a d-dimensional random vector X in probability. Then X is Gaussian and the convergence is also in L^2.*

References
-) Fernique (1995): *Fonctions aléatoire gaussiennes, vecteurs aléatoire gaussiens.* Sherbrooke Univ.
-) Ibragimov and Rozanov (1978): *Gaussian Random Processes.* Springer-Verlag.
-) Neveu (1968): *Processus aléatoires gaussiens.* Presses Univ. Montréal.

20 F. Delbaen: Minimum Variance and Markowitz Theory

This concept is different from the previous lectures in the sense that the probability P is important. We make the following assumption:

$$(*) \quad \exists Q_0 \in \mathcal{M}^e \quad \text{such that} \quad \frac{dQ_0}{dP} \in L^2 \,.$$

The space $\mathcal{K}^2$ is the L^2 closure of elements f obtained by $(H \cdot S)_\infty$ where H is elementary and $(H \cdot S)$ bounded. Kabanov-Stricker showed that under $(*)$,

$$\left\{Q \,|\, \frac{dQ}{dP} \in L^2\right\}$$

is $L^1(P)$-dense in $\mathcal{M}^a$.

NFLVR $\Rightarrow 1 \notin \mathcal{K}^2$ and let q be the orthogonal projection of 1 on $\mathcal{K}^2 \Rightarrow 1 - q \perp \mathcal{K}^2$. We get after some calculation

$$\mathrm{E}_P[f] = \mathrm{E}_P[q]\,\sigma(f)\rho(f,q)\frac{1}{\sigma(q)}\,,$$

where σ denotes standard deviation and ρ correlation. This relation is the market line relation in mean-variance theory of Markowitz.

Exercise: rewrite this relation in undiscounted terms.

We discuss the generality of the argument: No utility function (only no-arbitrage) and no assumption of normal distributions.

We end the discussion with the following theorems:

(a) Schweizer: if S is continuous , then $q \leq 1$ a.s.

(b) Delbaen-Schachermayer: $q < 1$ a.s. if S is continuous.

the guess

$$\frac{1-q}{1-\mathrm{E}_P[q]}$$

as the equivalent minimal variance martingale measure.

21 M. Yor: Basic Theory of Lévy Processes

a) I "read" from the Lévy-Khintchine formula the decomposition of a generic Lévy process $(X_t, t \geq 0)$ taking values in $\mathbb{R}$:

$$X_t = at + \sqrt{q}B_t + \tilde{X}_t^{(>1)} + \tilde{X}_t^{(\leq 1)}\,,$$

where $a \in \mathbb{R}$, $q > 0$, $(B_t, t \geq 0)$ is a Brownian motion,

$$\tilde{X}_t^{(>1)} = \sum_{s\leq t} \Delta X_s 1_{\{|\Delta X_s|>1\}}\,,$$

$$\tilde{X}_t^{(\leq 1)} = \lim_{\varepsilon \searrow 0} \left\{ \sum_{s\leq t} \Delta X_s 1_{\{\varepsilon \leq |\Delta X_s| \leq 1\}} - t\int \nu(dx) x 1_{\{\varepsilon \leq |x| \leq 1\}} \right\}.$$

b) I illustrated the LK formula with a number of examples: -) Compound Poisson process (whose Lévy measures are finite), -) Gamma process $(\Gamma_t, t \geq 0)$, -) Difference of two Gamma processes: $\Gamma_t - \Gamma'_t \overset{(d)}{=} \sqrt{2}\beta_{\Gamma_t}$, which led me to the discussion of Bochner subordination:

$$Z_t = X_{Y_t}\,,$$

where (X_u) is a Lévy process, independent of the subordinator $(Y_t, t \geq 0)$, which is an increasing Lévy process.

c) In particular, the (increasing) stable process $\left(\tau_t^{(\alpha)}, t \geq 0\right)$, $(0 < \alpha \leq 1)$, among which the stable (1/2) process, obtained from hitting times of levels of one-dimensional Brownian motion allow to construct, by taking $X_u \equiv \beta_u$, a Brownian motion independent of $\tau^{(\alpha)}$, all symmetric stable processes:

$$Z_t^{(2\alpha)} = \beta_{\tau_t^{(\alpha)}}\,,\ t \geq 0\,, \quad (0 < 2\alpha \leq 2)\,.$$

d) Introducing the local time for Brownian motion, and its inverse (τ_t), I showed how to obtain many subordinators from:

$$\left(\int_0^{\tau_t} ds f(B_s)\,,\ t \geq 0\right)\,.$$

22 P. Cheridito: Gaussian Systems and Fractional Brownian Motion (fBm)

Definition: *Let Λ be a set. $(X_\lambda)_{\lambda\in\Lambda} : \Omega \to \mathbb{R}^\Lambda$ is a Gaussian system if and only if for all $d \in \mathbb{N}$ and $\{\lambda_1, \ldots, \lambda_d\} \subset \Lambda$, $(X_{\lambda_1}, \ldots, X_{\lambda_d})$ is a Gaussian random vector. We call a Gaussian system $(X_\lambda)_{\lambda\in\Lambda}$ a Gaussian process if $\Lambda = \mathbb{N}, \mathbb{Z}, \mathbb{R}_+$ or $\mathbb{R}$.*

Theorem:
a) *Let $(X_\lambda)_{\lambda\in\Lambda}$ be a Gaussian system. Then*

$$\Gamma_{\lambda\mu}^X = \mathrm{Cov}\left(X_\lambda, X_\mu\right)\,,\ \lambda, \mu \in \Lambda\,,$$

is symmetric and positive semi-definite.
b) *For all $(m_\lambda)_{\lambda\in\Lambda} \in \mathbb{R}^\Lambda$ and all symmetric, positive semi-definite $(\Gamma_{\lambda\nu})_{\lambda,\mu\in\Lambda} \in \mathbb{R}^{\Lambda^2}$, there exists a Gaussian system $(X_\lambda)_{\lambda\in\Lambda}$ with*

$$\mathrm{E}\left[X_\lambda\right] = m_\lambda\,,\ \lambda \in \Lambda\,,$$

and

$$\mathrm{Cov}\left(X_\lambda, X_\mu\right) = \Gamma_{\lambda\mu}\,,\ \lambda, \mu \in \Lambda\,.$$

Definition: *A fBm with Hurst paramter $H \in (0,1]$, is a continuous centred Gaussian process $\left(B_t^H\right)_{t\in\mathbb{R}}$ with*

$$\mathrm{Cov}\left(B_t^H, B_s^H\right) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)\,,\ t, s \in \mathbb{R}\,.$$

For $H = 1$, B^H can be represented as

$$B_t^1 = t\xi\,,\ t \in \mathbb{R}\,,$$

where ξ is a standard normal random variable. For $H \in (0,1)$ Mandelbrot and Van Ness gave the following representation:

$$B_t^H = c_H \int_{-\infty}^{t} \left[(t-u)^{H-\frac{1}{2}} - 1_{\{u \leq 0\}}(-u)^{H-\frac{1}{2}} \right] dW_u \,,$$

where $(W_t)_{t \in \mathbb{R}}$ is a two-sided Brownian motion and c_H a normalizing constant.

References:

Gaussian processes:
-) Hida and Hitsuda (1993): *Gaussian processes*, AMS

fBm:
-) Kolmogorov (1940): Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum. *Doklady*, 26.
-) Mandelbrot and Van Ness (1968): Fractional Brownian motions, fractional noises and applications. *SIAM Review*, 10.
-) Decreusefond Üstünel (1999): Stochastic analysis of the fractional Brownian motion. *Potential Anal.* 10.
-) Lin (1995). Stochastic analysis of fractional Brownian motion. *Stochastics and Stochastics Reports* 55.

23 F. Delbaen: Coherent Risk Measures

We introduce coherent risk measures and compare them with VaR (Value at Risk). In credit risk situations we show how unsuitable and dangerous the application of VaR can be. Different examples of coherent risk measures are given using the characterisation of coherent risk measures.

References:

-) http://www.math.ethz.ch/~delbaen
-) lecture notes of Scuola Normale di Pisa (see the Friday July 20 summary for the theorems presented)

24 T. Konstantopoulos: Fluctuation Theory

Let X be a Lévy process (we exclude the compound Poisson case).

$$S_t = \sup_{0 \leq u \leq t} X_u \vee 0 \,, \; Z_t = S_t - X_t \,:$$

(Strong) Markov property, and Skorohod reflection. Assuming that 0 is regular for Z, we let L be the local time of Z and L^{-1} its right-continuous inverse, which is seen to be a subordinator. Moreover, the ladder process

$$\left(L^{-1}(x) \,, \, S_{L^{-1}(x)} \equiv H(x) \right)_{x \geq 0}$$

is Lévy in $\mathbb{R}_+^2$.

Problem: Compute

$$\mathrm{E}\left[\exp\left(-\alpha L^{-1}(x) - \beta S_{L^{-1}(x)}\right)\right] \equiv \exp\left(-xK(\alpha,\beta)\right), \ \alpha > 0, \beta > 0 .$$

We discuss excursion theory, and the compensation formula

$$\mathrm{E}\left[\sum_g F_g(\varepsilon_g)\right] = \mathrm{E}\left[\int_0^\infty dL(t) \int_{\mathcal{E}} F_t(\varepsilon) n(d\varepsilon)\right]$$

and use it to derive the law of $\left(G_\tau, S_(\tau)\right)$, where $\tau \sim \exp(q)$, independent of X, and G_τ = last zero of Z on $[0,\tau]$:

$$\mathrm{E}\left[\exp\left(-\alpha G_\tau - \beta S_\tau\right)\right] = \frac{K(q,0)}{K(\alpha+q,\beta)} .$$

We prove the Wiener-Hopf factorization

$$(\tau, X_\tau) = (G_\tau, S_\tau) \overset{\text{(indep.)}}{+} (\tau - G_\tau, X_\tau - S_\tau)$$

(infinitely divisible distributions), which leads to:

$$\mathrm{E}\left[\mathrm{e}^{\mathrm{i}\lambda X_\tau}\right] = \frac{q}{q+\Psi(\lambda)} = \Psi_q^+(\lambda)\Psi_q^-(\lambda) .$$

We obtain

$$K(\alpha,\beta) = K \exp\left(\int_0^\infty \int \left(\mathrm{e}^{-t} - \mathrm{e}^{-\alpha t - \beta x}\right) t^{-1} P\left[X_t \in dx\right] dt\right) ,$$

and the interesting formula

$$\mathrm{E}\left[\mathrm{e}^{-\lambda G_\tau}\right] = \exp\left(\int_0^\infty \left(\mathrm{e}^{-\lambda t} - 1\right) \mathrm{e}^{-qt} P\left[X_t \geq 0\right] dt\right)$$

(c.f. Sparre Andersen's formula for discrete time random walks). Finally, we derive the generalized arcsine law for $\frac{G_t}{t}$, when $P\left[X_t \geq 0\right]$ = constant.

25 P. Cheridito: Fractional Finance Models

We showed that fBm has the following properties:

1) The increments of B^H are independent if $H = \frac{1}{2}$, positively correlated if $H \in (\frac{1}{2}, 1]$, and negatively correlated if $H \in (0, \frac{1}{2})$.
2) The increments of B^H exhibit long-range dependence $\Leftrightarrow H \in (\frac{1}{2}, 1]$.
3) B^H is self-similar.

We presented the following models for the evolution of a stock price:

fractional Bachelier model:

$$S_t = S_0 + \nu t + \sigma B_t^H .$$

fractional Samuelson model:

$$S_t = S_0 \exp\left(\nu t + \sigma B_t^H\right) .$$

fractional stochastic volatility model:

$$\begin{aligned} dS_t &= S_t \left(\mu dt + \sigma_t dB_t\right) \\ d\sigma_t &= -\lambda \left(\sigma_t - b\right) dt + a_1 dB_t + a_2 dB_t^H \end{aligned}$$

We defined "FLVR", "arbitrage" and "strong arbitrage".

We showed that for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, B^H is not a semimartingale.

We introduced the notion of p-variation and showed that for all $p > \frac{1}{H}$, almost all paths of B^H are of bounded p-variation.

We recalled the definition of the Riemann-Stieltjes integral and presented the following

Theorem: *If $f : [a,b] \to \mathbb{R}$ is continuous and of bounded p-variation for some $p \in [1,2)$, $h \in C^1(\mathbb{R})$ and h' is locally Lipschitz, then*

$$\mathrm{RS} - \int_a^b h' \circ f df = h \circ f(b) - h \circ f(a) .$$

References:

Mathematical Finance:

-) Bachelier (1900): Théorie de la spéculation. *Ann. Sci. Ecole Norm. Sup.* 17.

-) Samuelson (1965): Rational theory of warrant pricing. *Ind. Manag. Rev.* Vol. 6, No. 2.

-) Delbaen and Schachermayer (1994): A general version of the fundamental theorem of asset pricing. *Math. Ann.* 300.

Finance with fBm:

-) Cutland, Kopp and Willinger (1995): Stock price returns and the Joseph effect: a fractional version of the Black-Scholes model. *Progress in Probability* 36.

26 P. Cheridito: Arbitrage in fBm Models

Theorem: *For all $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, the fractional Bachelier model and the fractional Samuelson model have FLVR in $\Theta^{\mathrm{S}}_{\mathrm{sf,adm}}(\mathcal{F}^{B^H})$ and strong arbitrage in $\Theta^{\mathrm{aS}}_{\mathrm{sf,adm}}(\mathcal{F}^{B^H})$.*

Theorem: *For all $H \in (0,1)$, the fractional Bachelier model and the fractional*

Samuelson model have no arbitrage in $\Theta^h_{\rm sf}(\mathcal{F}^{B^H})$.

References:
-) Rogers (1997): Arbitrage with fractional Brownian motion. *Mathematical Finance* 7.
-) Shiryaev (1998): On arbitrage and replication for fractal models. *Research Report no. 20*, MaPhySto.
-) Salopek (1998): Tolerance to arbitrage. *Stoch. Pro. Appl.* 76.
-) Cheridito (2000): Arbitrage in fractional Brownian motion models. *www.math.ethz.ch/~dito*

27 M. Yor: Brownian Excursions: Theory and Applications

a) Following lecture no. 24 on fluctuation theory, when specialized to Brownian motion, I first recalled and proved Lévy's theorem:

$$(M_t - B_t\,,\ M_t\,,\ t \geq 0) \overset{\text{(law)}}{=} (|B_t|\ ,\ L_t\,,\ t \geq 0)\ ,$$

where $M_t = \sup_{s \leq t} B_s$, and (L_t) is the local time of Brownian motion at 0, with the help of Skorohod's lemma. This result can be extended to $B^\mu_t \equiv B_t + \mu t$ on the LHS, while on the RHS (B_t) is replaced by the bang-bang process, which is the solution of

$$dX_t = d\beta_t - \mu \mathrm{sgn}(X_t) dt\,.$$

b) I proved Itô's representation theorem of Brownian excursions as a Poisson point process, as well as D. Williams representation of the characteristic Itô measure, either in terms of an integral (over lengths v) of the laws of the BES(3) bridges with lengths v, or in terms of an integral (over heights m) of the laws of two BES(3) processes put back to back as they reach level m.

c) I finally discussed the following results, obtained using excursion theory:

$$(M^*(b))^2 \overset{\text{(law)}}{=} T^{(3)}_{\frac{\pi}{2}}\ ,$$

and

$$(M^*(e))^2 \overset{\text{(law)}}{=} (M^*(b))^2 + \left(M^*(\tilde{b})\right)^2\ ,$$

where b is a standard Brownian bridge, $M^*(b) = \sup_{s \leq 1} |b(s)|$, e a standard excursion and $M^*(e) = \sup_{s \leq 1} |e(s)|$.

References:
-) Chung (1976)
-) Revuz and Yor, Chap. XII

28 P. Cheridito: Regularized fBm and Option Pricing

We showed that if $\varphi : \mathbb{R} \to \mathbb{R}$ is a measurable function that satisfies
(R1) $\varphi(0) = 0$ for $x < 0$.
(R2) $\int_{\mathbb{R}} [\varphi(t-u) - \varphi(-u)]^2 \, du < \infty$, $\forall t \in \mathbb{R}$.
(R3) $\varphi(t) = \varphi(0) + \int_0^t \psi(u) du$, $t \geq 0$, for some $\psi \in L^2(\mathbb{R}_+)$.
Then

$$X_t^{\varphi} = \int_{\mathbb{R}} [\varphi(t-u) - \varphi(-u)] \, dW_u \, , \; t \geq 0 \, ,$$

is a semimartingale. If $\varphi(0) = 0$, then X^{φ} is a FV-process. If $\varphi(0) \neq 0$, then, for all $T > 0$, the law of $(X_t^{\varphi})_{0 \leq t \leq T}$ is equivalent to Wiener measure.

We introduced the regularized fBm $R^{H,v,d}$ and discussed option prices in the model

$$\begin{aligned} S_t^0 &= 1 \, , \; t \in [0,T] \\ S_t &= S_0 \exp\left(\nu t + \sigma R_t^{H,v,d}\right) \, , \; t \in [0,T] \, . \end{aligned}$$

References:
-) Cheridito (2000): Regularized fractional Brownian motion and option pricing. *www.math.ethz.ch/~dito*
-) Cheridito (2001): Sensitivity of the Black-Scholes option price to the local path behaviour of the stochastic process modelling the underlying asset. *www.math.ethz.ch/~dito*

29 F. Delbaen: Capital Allocation and Risk Measures

We continue the study of the relation between VaR and coherent risk measures and show the minimality of TailVaR in the class of distribution invariant coherent risk measures dominating VaR. This result has been improved by Kusuoka.

The extension of coherent risk measures to the space L^0 of all random variables is discussed and necessary and sufficient conditions are given for this extension to take values in $\mathbb{R} \cup \{\infty\}$ (avoiding $-\infty$).

The capital allocation problem is dealt with through the help of the subgradient (more precisely, the weak * subgradient). Results of Aubin, Artzner-Ostroy, Billera-Heath are put in this context.

The extension of coherent risk measures to multi-period models was not treated.

30 M. Yor: Stochastic Calculus with Lévy processes

-) From the decomposition of a Lévy process presented in lecture no. 21, any Lévy process is a semimartingale, hence stochastic integration with respect to a semimartingale can be used for a Lévy process.

-) As an application, I discussed the integro-differential form of the infinitesimal generator of a Lévy process, as:

$$Af(x) = af'(x) + \frac{q}{2}f''(x) + \int \nu(dy)\left(f(x+y) - f(x) - f'(x)y1_{\{|y|\le 1\}}\right).$$

In the second half of the lecture, I discussed:

(i) the Escher transform for subordinators, and in particular the Gamma subordinator: for all $a > 0$,

$$\mathrm{E}\left[F\left(a\Gamma_u\,,\; u \le t\right)\right] = \mathrm{E}\left[F\left(\Gamma_u\,,\; u \le t\right) e_a\left(\Gamma_t\,,\; t\right)\right]$$

(exercise: find e_a!).

(ii) the Lamperti transform:

$$\exp\left(\xi_t\right) = X_{\int_0^t ds\, \exp(\xi_s)}\,,$$

where $(\xi_t\,,\; t \ge 0)$ is a Lévy process, and $(X_h\,,\; h \ge 0)$ the associated semi-stable Markov process (see Lamperti, TAMS (1972)).

(iii) the generalized Ornstein-Uhlenbeck process, associated to two (independent, for simplicity) Lévy processes (ξ_t, η_t), as:

$$U_t = \exp(\xi_t)\left(x + \int_0^t \exp(-\xi_{s_-})d\eta_t\right)$$

(a new Markov process).

Both (ii) and (iii) may be applied successfully to obtain the law of $\int_0^\infty ds\, \exp\left(\xi_s\right)$, when this variable is finite. These variables (perpetuities) are particularly important in Insurance. A well known particular case is Dufresne's result, which is:

$$\int_0^\infty ds\, \exp\left(2(B_s - \mu s)\right) \quad \text{is distributed as} \quad \frac{1}{2\Gamma_\mu} \quad (\mu > 0)\,.$$

Finally, I presented the result (obtained with J. Bertoin) that, for a subordinator (ξ_t), the perpetuity

$$I = \int_0^\infty ds\, \exp\left(-\xi_s\right)$$

satisfies

$$e \stackrel{\text{(law)}}{=} IR\,,$$

where on the LHS, e is exp(1), and on the RHS, I and R are independent, with

$$\mathrm{E}\left[I^k\right] = \frac{k!}{\Phi(1)\dots\Phi(k)} \quad , \quad \mathrm{E}\left[R^k\right] = \Phi(1)\dots\Phi(k)\,.$$

References:
Carmona, Petit, Yor in Barndorff-Nielsen/Mikosch/Resnick (2001)
also in: Revista Ibero-Amer. (1997).