

# The mobile Boolean model: an overview and further results

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## Abstract

This paper offers an overview of the mobile Boolean stochastic geometric model which is a time-dependent version of the ordinary Boolean model in a Euclidean space of dimension  $d$ . The main question asked is that of obtaining the law of the detection time of a fixed set. We give various ways of thinking about this which result into some general formulas. The formulas are solvable in some special cases, such the inertial and Brownian mobile Boolean models. In the latter case, we obtain some expressions for the distribution of the detection time of a ball, when the dimension  $d$  is odd and asymptotics when  $d$  is even. Finally, we pose some questions for future research.

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## 1 Introduction

Particles start from the points of a point process in  $\mathbb{R}^d$  and perform i.i.d. stochastic motions. Each particle carries a set (detection set) along (typically, a ball of fixed radius). The goal is to find the time until one of the particles detects a fixed set  $K$ . This is a paper that should have been written long ago. Its first version, [9] appeared in the proceedings of an obscure conference without many proofs. After actually working out the Brownian case, we discovered that Spitzer [8] had done most of the job for  $d = 2$  or  $3$ . A few years after that, we posed some open questions [10], some of which were resolved by Peres *et al.* [12]. The purpose of this paper is twofold. First, because we think that an overview of the topic is needed. Second, because there are several interesting questions that can be asked. This overview contains some new elements too. For example, it contains a method for computing the exact distribution of the detection time in odd dimensions and asymptotics of the distribution in even dimensions. We also compare what happens when we run the particles a Brownian motions vs. linear motions with random speeds. Roughly speaking, Brownian motions discover small objects faster than linear motions.

The paper is organized as follows. First, we give an overview of the general Boolean model. Then we pass on to the dynamic case. We then pose the detection problem and provide some general formulas involving capacity notions. The inertial Boolean model (particles move on straight lines with constant speed) is presented next. The Brownian Boolean model is then analyzed in detail. We conclude with some open problems. The first part

of the paper, i.e., up to and including Section 4, is largely an overview but with organized notation chosen so that the model can be explained in much greater generality than the “solvable” one (the Brownian Boolean model) Section 5 is new (inertial Boolean model), but quite simple. Section 6 deals with the Brownian Boolean model and explains the new formulas regarding the distributions of detection times and their expectations, separately in even and odd dimensions.

Throughout the paper we use the following notations:  $f(x) \sim g(x)$ , as  $x \rightarrow \infty$ , means  $f(x)/g(x) \rightarrow 1$ , as  $x \rightarrow \infty$ . Similarly, when  $x$  tends to another value. Also,  $f(x) \stackrel{\log}{\sim} g(x)$  means  $\log f(x) \sim \log g(x)$ . The closed unit ball in  $\mathbb{R}^d$  centered at the origin is denoted by  $\mathbb{B}$ . The Euclidean norm of  $x \in \mathbb{R}^d$  is denoted by  $|x|$  and the  $d$ -dimensional Lebesgue measure of a Borel set  $A \subset \mathbb{R}^d$  is denoted as  $\text{vol}(A)$ . In particular, we let  $\omega_d := \text{vol}(\mathbb{B})$ , keeping in mind that

$$\omega_d = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)}, \quad d \in \mathbb{Z}_+,$$

which gives

$$\omega_{2n} = \frac{\pi^n}{n!}, \quad \omega_{2n+1} = \frac{2^{n+1}\pi^n}{(2n+1)!!}, \quad n \in \mathbb{Z}_+,$$

where  $(2n+1)!! = 1 \cdot 3 \cdot 5 \cdots (2n+1)$ . When  $A, B$  are sets (perhaps random) in  $\mathbb{R}^d$ , then  $A \pm B$  is their Minkowski sum (difference). For example,  $A$  could be a singleton and  $B$  the trajectory of a stochastic process. The modified Bessel function of the second kind and of order  $\nu \in \mathbb{R}$  is denoted by  $K_\nu$ . It admits the integral representation

$$K_\nu(z) = \int_0^\infty \exp(-z \cosh t) \cosh(\nu t) dt, \quad \text{Re}(z) > 0.$$

Alternative formulas for  $K_\nu$  when  $\nu$  is a half-integer are in the appendix. We remark also that the term “Boolean model” is not standard. Some people refer to it as germ-grain model and allow the underlying point process to be general. In this paper, we reserve the term “Boolean model” for the case when the underlying point process is (homogeneous) Poisson and use “general Boolean model” for the case where the point process is more general.

## 2 A little background on the general Boolean model

The general Boolean model, also known as the germ-grain model [13, Ch. 4], is one the basic objects of study of stochastic geometry. Consider a random point process  $\Phi$  on  $\mathbb{R}^d$  (the “germs”) and a random compact set  $G$  (a “grain”). Conditionally on  $\Phi$ , let  $\{G_x, x \in \Phi\}$  be i.i.d. copies<sup>1</sup> of  $G$ . The *general Boolean model* is then defined by<sup>2</sup>

$$\Xi := \bigcup_{x \in \Phi} (x + G_x). \tag{1}$$

(If  $G$  is a deterministic set, then  $\Xi$  can also be expressed as  $\Xi = \Phi + G$ .) Alternatively, if  $\mathcal{K}$  is the collection of compact subsets of  $\mathbb{R}^d$ , we may consider the point process

$$\mathcal{N} := \{(x, G_x) : x \in \Phi\} \tag{2}$$

<sup>1</sup>We think of  $\Phi$  both as a random discrete set, and as a random point process. Thus  $\{x \in \mathbb{R}^d : x \in \Phi\}$  stands for the random set or for the support of the random measure. When  $B$  is a Borel subset of  $\mathbb{R}^d$  we use  $\Phi(B)$  to denote the value of the random measure at  $B$ , i.e., the number of points of the random set in  $B$ .

<sup>2</sup>When  $A, B \subset \mathbb{R}^d$ , we let  $A + B := \{a + b : a \in A, b \in B\}$  (Minkowski addition). Similarly,  $A - B := \{a - b : a \in A, b \in B\}$ . We let  $x + B := \{x\} + B$ .

as a discrete random subset of the product space  $\mathbb{R}^d \times \mathcal{K}$ . In fact, it is also a marked point process because to each  $x$  there is a unique  $G_x \in \mathcal{K}$  such that  $(x, G_x)$  is an element of the random set  $\mathcal{N}$ . Putting an appropriate probability measure on the set of marked point processes on  $\mathbb{R}^d \times \mathcal{K}$  gives another way of constructing a general Boolean model.

The *capacity functional* of the general Boolean model is defined as

$$\mathsf{T}_\Xi(K) := \mathbb{P}(K \cap \Xi \neq \emptyset),$$

for  $K$  a compact set.

The capacity functional is a fundamental object in the theory of random sets [11]. If  $X$  is a random locally compact subset of  $\mathbb{R}^d$  then

$$\mathsf{T}_X(K) := \mathbb{P}(K \cap X \neq \emptyset)$$

is defined on compact sets  $K$ . It is submodular, i.e.,  $\mathsf{T}_X(K_1 \cup K_2) \leq \mathsf{T}_X(K_1) + \mathsf{T}_X(K_2) - \mathsf{T}_X(K_1 \cap K_2)$  and upper semicontinuous, i.e.,  $\mathsf{T}_X(K_n) \downarrow \mathsf{T}_X(K)$  whenever  $K_n$  is a decreasing sequence of compact sets with  $\bigcap_n K_n = K$ . An example of a capacity functional is the one given above. Another example is obtained by considering a Brownian motion  $\xi := \{\xi(t), t \geq 0\}$  in  $\mathbb{R}^d$  started from some  $\xi(0) \in \mathbb{R}^d$ . Letting

$$\xi^{(t)} := \{\xi(s), 0 \leq s \leq t\}$$

be the initial segment of  $\xi$  up to time  $t$ , we have  $\mathsf{T}_{\xi^{(t)}}(K) = \mathbb{P}(\xi^{(t)} \cap K \neq \emptyset)$ . If we set

$$T_K := \inf\{t \geq 0 : \xi(t) \in K\},$$

we obtain

$$\mathsf{T}_{\xi^{(t)}}(K) = \mathbb{P}(T_K \leq t).$$

Thus,  $t \mapsto \mathsf{T}_{\xi^{(t)}}(K)$  is the distribution function of the random variable  $T_K$ .

Going back to the general Boolean model, we observe that if (the law of)  $\Phi$  is invariant under translations and ergodic, and if  $\text{vol}$  denotes the Lebesgue measure on  $\mathbb{R}^d$ , then  $C_\Xi(\{0\})$  is the a.s. (and in  $L^1$ ) limit of  $\text{vol}(\Xi \cap [-h, h]^d)/h^d$ , as  $h \uparrow \infty$  and is known as the *volume fraction* of  $\Xi$ .

An important particular case is when  $\Phi$  is a *spatially homogeneous Poisson process* with intensity  $\lambda$ . In this case,  $\Xi$  is referred to as a Boolean model<sup>3</sup> [14]. The Boolean model has several computational advantages. For example, owing to the thinning property of a Poisson process, the set of points  $x \in \Phi$  such that  $x + G_x$  intersects a given closed set  $A$ ,

$$\Phi_A := \{x \in \Phi : A \cap (x + G_x) \neq \emptyset\},$$

forms an inhomogeneous Poisson process in  $\mathbb{R}^d$  with intensity measure

$$\mathbb{E}\Phi_A(dx) = \lambda_A(x) dx = \lambda \mathbb{P}(A \cap (x + G) \neq \emptyset) dx.$$

Indeed, conditionally on  $\Phi$ , the random variables  $\{\mathbf{1}(A \cap (x + G_x) \neq \emptyset), x \in \Phi\}$  are independent and  $\mathbf{1}(A \cap (x + G_x) \neq \emptyset) \stackrel{(d)}{=} \mathbf{1}(A \cap (x + G) \neq \emptyset)$ , for all  $x \in \Phi$ . Since  $A \cap (x + G) \neq \emptyset$  is equivalent to  $(A - x) \cap G \neq \emptyset$ , we have

$$\lambda_A(x) = \lambda \mathsf{T}_G(A - x).$$

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<sup>3</sup>The terminology is highly nonstandard. Some authors use the term ‘‘Boolean model’’ or ‘‘germ-grain model’’ for what we called ‘‘general Boolean model’’. We decided to reserve the term ‘‘Boolean model’’ for the case where  $\Phi$  is a Poisson process.

Notice also that if  $\Xi$  is a Boolean model then the marked point process  $\mathcal{N}$  introduced in (2) is a Poisson process on  $\mathbb{R}^d \times \mathcal{K}$  with intensity measure  $\lambda dx \mathbb{P}(G \in dg)$  on the space  $\mathbb{R}^d \times \mathcal{K}$ . Conversely, given a finite measure  $\mu(dg)$  on  $\mathcal{K}$ , we can construct a Poisson process  $\mathcal{N}$  on  $\mathbb{R}^d \times \mathcal{K}$  with intensity measure  $dx \mu(dg)$ . The projection of  $\mathcal{N}$  on  $\mathbb{R}^d$  is the set of germs and the projection on  $\mathcal{K}$  is the set of grains.

In applications of stochastic geometry, it is sometimes the case that the grains depend on a parameter  $t$  (time) and increase with  $t$ . The general mobile Boolean model has a time parameter  $t$  which principally affects the locations of the germs.

### 3 The general mobile Boolean model

#### 3.1 Definitions

In its most general form, the general mobile Boolean model is defined in terms of a germ point process  $\Phi$  on  $\mathbb{R}^d$  and of a time-dependent grain process  $G = \{G(t), t \geq 0\}$ . The latter is assumed to be a stochastic process with values in  $\mathcal{K}$  and continuous sample paths, when  $\mathcal{K}$  is equipped with the topology induced by the Hausdorff metric. Conditionally on  $\Phi$ , let  $\{G_x, x \in \Phi\}$  be i.i.d. copies of  $G$  and define, for each  $t \geq 0$ , the general Boolean model

$$\Xi(t) = \bigcup_{x \in \Phi} (x + G_x(t)).$$

The *general mobile Boolean model* is the process  $\Xi = \{\Xi(t), t \geq 0\}$ . Physically, we think of particles located at points  $x \in \Phi$  at time 0, each having a grain  $G_x(0)$  “around it”. At time  $t$ , point  $x$  moves to a new position and, as a result, the grain around it becomes  $x + G_x(t)$ . Notice that the set

$$\mathcal{W}(t) := \bigcup_{0 \leq s \leq t} \Xi(s)$$

represents everything covered by the general Boolean model up to time  $t$  and is itself too a general Boolean model with grains the points of  $\Phi$  and germs being i.i.d. copies of

$$G^{(t)} := \bigcup_{0 \leq s \leq t} G(s).$$

Indeed,

$$\mathcal{W}(t) = \bigcup_{x \in \Phi} (x + \bigcup_{0 \leq s \leq t} G_x(s)) = \bigcup_{x \in \Phi} (x + G_x^{(t)}). \quad (3)$$

Note that  $\mathcal{W}(t)$  is increasing in  $t$ .

#### 3.2 The canonical form

The canonical way of introducing the distribution of the grain process  $\{G(t), t \geq 0\}$  is by considering a specific point  $h(G(0))$  of  $G(0)$  as its “center” (where, formally,  $h$  is a measurable function from  $\mathcal{K}$  into  $\mathbb{R}^d$ ). Then

$$\xi(t) := h(G(t)), \quad t \geq 0,$$

describes the motion of the center. We may thus describe the law of  $\{G(t), t \geq 0\}$ , in two steps. First by specifying the law of  $\{\xi(t), t \geq 0\}$  and then, conditionally on  $\{\xi(t), t \geq 0\}$ , by specifying the law of

$$D(t) := G(t) - \xi(t), \quad t \geq 0.$$

(In the simplest case, we may assume that  $\{D(t), t \geq 0\}$  is independent of  $\{\xi(t), t \geq 0\}$ .) We can then write

$$\Xi(t) = \bigcup_{x \in \Phi} (x + \xi_x(t) + D_x(t)),$$

where  $\xi_x(t) = h(G_x(t))$ ,  $D_x(t) = G_x(t) - \xi_x(t)$ . The two representations for  $\Xi(t)$  are absolutely equivalent. Notice, however, that if we let

$$\Phi(t) := \{x + \xi(t) : x \in \Phi\}$$

we can write, with a slight abuse of notation as regards the indices of  $D_x(t)$ ,

$$\Xi(t) = \bigcup_{x \in \Phi(t)} (x + D_x(t)),$$

and think of  $\Xi(t)$  as a general Boolean model with germs the points of  $x \in \Phi(t)$  and grains  $D_x(t)$ .

In the simplest case,  $\Phi$  is a homogeneous Poisson process with intensity  $\lambda$ , and  $\{D(t), t \geq 0\}$  is independent of  $\{\xi(t), t \geq 0\}$ . Then  $\Phi(t)$  is again a homogeneous Poisson process with intensity  $\lambda$ . Thus, if  $\Xi(0)$  is a Boolean model and the trajectory of the center is chosen independently of  $D$ , then, for each  $t > 0$ ,  $\Xi(t)$  is also a Boolean model identical in distribution to  $\Xi(0)$ .

Note: Without further ado we shall, henceforth, define a mobile Boolean model to be one for which the germ point process  $\Phi$  is homogeneous Poisson and the independence between  $\xi(\cdot) = h(G(\cdot))$  and  $D(\cdot)$  holds. Dropping the adjective homogeneous gives an inhomogeneous mobile Boolean model. An inhomogeneous mobile Boolean model is arguably a good model for a sensor network. Sensing devices are initially located at the points  $x$  of the inhomogeneous Poisson process  $\Phi$  and move according to independent random motions  $x + \xi_x(t)$  (assuming  $\xi_x(0) = 0$ ). The set  $D_x(t)$  represents the part of space which can be sensed by the sensor at time  $t$ . If the problem is to discover an unknown target, then inhomogeneity in  $\Phi$  allows for the possibility of incorporating prior information about the location of the target. Randomness in  $D_x(t)$  may model the different sensing abilities of the devices. And time-dependence in  $D_x(t)$  allows for modeling of loss (or gain) of energy of the device. Finally, randomness in the trajectories is natural too.

## 4 The detection problem

We now consider the problem of finding the distribution of the first time that a general mobile Boolean model will detect a fixed compact set  $K$ . The expressions (4) and (6) below concern, respectively, a general mobile Boolean model and an inhomogeneous mobile Boolean model. The expressions (8), and (9) concern both a homogeneous mobile Boolean model. The last one relates the distribution of the detection time of  $K$  by to the distribution of the first hitting time of a set by the process  $x + \xi(t)$  representing the random motion of a sensing device initially located at the point  $x$ .

## 4.1 Detection time for a general mobile Boolean model

Let  $K$  be a compact subset of  $\mathbb{R}^d$  and let

$$S_K := \inf\{t \geq 0 : K \cap \mathcal{W}(t) \neq \emptyset\}.$$

This is called the *detection time of the set  $K$* . We are interested in deriving information about the law of  $K$ . Under natural assumptions on the law of  $\Phi$  (e.g., if  $\Phi$  is a Poisson process), and if the grain  $G$  has nonempty interior with positive probability, then, due to compactness, the probability that  $K$  is contained in  $\mathcal{W}(0)$  is positive. Hence  $\mathbb{P}(S_K = 0)$  is typically (e.g., under the previous assumptions) positive. By the monotonicity of  $\mathcal{W}(t)$ ,

$$\mathbb{P}(S_K \leq t) = \mathbb{P}(K \cap \mathcal{W}(t) \neq \emptyset) = \mathsf{T}_{\mathcal{W}(t)}(K), \quad (4)$$

and, by the expression (3) for  $\mathcal{W}(t)$ ,

$$\begin{aligned} \mathsf{T}_{\mathcal{W}(t)}(K) &= \mathbb{P}\left(\bigcup_{x \in \Phi} K \cap (x + G_x^{(t)}) \neq \emptyset\right) \\ &= \mathbb{P}(\exists x \in \Phi \ K \cap (x + G_x^{(t)}) \neq \emptyset). \end{aligned}$$

So, if we consider the point process

$$\Phi_K := \{x \in \Phi : K \cap (x + G_x^{(t)}) \neq \emptyset\}, \quad (5)$$

we have

$$\mathsf{T}_{\mathcal{W}(t)}(K) = \mathbb{P}(\Phi_K \neq \emptyset),$$

which is the probability that for some point  $x$  of  $\Phi$  the set  $x + G_x^{(t)}$  intersects  $K$ . This point process depends on the general Boolean model in a rather complicated way.

## 4.2 Detection time for a (possibly inhomogeneous) mobile Boolean model

Things become simple in the case of an inhomogeneous mobile Boolean model where  $\Phi$  is a Poisson process with intensity measure  $\lambda(dx)$ . In this case, arguing as earlier,  $\Phi_K$  is also a Poisson process with intensity measure

$$\mathbb{P}(G^{(t)} \cap (K - x) \neq \emptyset) \lambda(dx) = \mathsf{T}_{G^{(t)}}(K - x) \lambda(dx),$$

and thus,

$$\mathbb{P}(S_K \leq t) = \mathsf{T}_{\mathcal{W}(t)}(K) = 1 - \exp\left(-\int_{\mathbb{R}^d} \mathsf{T}_{G^{(t)}}(K - x) \lambda(dx)\right). \quad (6)$$

Since  $(G^{(t)} + D) \cap (K - x) \neq \emptyset$  iff  $x \in K - D - \xi(s)$  for some  $s \leq t$ , we have, by Fubini's theorem,

$$\mathbb{P}(S_K > t) = \exp\left(-\mathbb{E} \lambda\left(\bigcup_{0 \leq s \leq t} [K - D - \xi(s)]\right)\right) \quad (7)$$

If  $\Phi$  is a spatially homogeneous Poisson process with

$$\lambda(dx) = \lambda \cdot dx,$$

then

$$\mathbb{P}(S_K > t) = \exp\left(-\lambda \mathbb{E} \operatorname{vol}\left(\bigcup_{0 \leq s \leq t} [\xi(s) + D - K]\right)\right). \quad (8)$$

We point out that the sets involved in this union are formed by translating the set  $D - K = \{x - y : x \in D, y \in K\}$  by vectors  $\xi(s)$ . Put it otherwise, a particle performing motion  $\xi$  carries a neighborhood with shape  $D - K$ . The set swept by the particle up to time  $t$  can be called the  $(D - K)$ -sausage of  $\xi$  up to time  $t$ . The term *Wiener sausage* is reserved for the case when  $\xi$  is a Brownian motion.

### 4.3 Detection time for a homogeneous mobile Boolean model in its canonical form

Suppose now that

$$G(t) = \xi(t) + D,$$

where  $D$  is a fixed deterministic compact set, for instance a closed ball, and  $\{\xi(t), t \geq 0\}$  a random process with continuous sample paths and  $\xi(0) = 0$ ; see Section 3.2. Then

$$G^{(t)} = \bigcup_{0 \leq s \leq t} (\xi(s) + D) = \xi^{(t)} + D,$$

and

$$G^{(t)} \cap (K - x) \neq \emptyset \iff (\xi^{(t)} + D) \cap (K - x) \neq \emptyset \iff (x + \xi^{(t)}) \cap (K - D) \neq \emptyset.$$

Therefore, if we define the first hitting time

$$T_B^x := \inf\{t \geq 0 : x + \xi(s) \in B\}$$

of a closed set  $B$  by the process  $x + \xi(\cdot)$ , we have

$$\mathbb{T}_{G^{(t)}}(K - x) = \mathbb{P}(T_{K-D}^x \leq t).$$

Using (6) and (8) we obtain

$$\mathbb{P}(S_K > t) = \exp\left(-\lambda \int_{\mathbb{R}^d} \mathbb{P}(T_{K-D}^x \leq t) dx\right). \quad (9)$$

In particular, if  $T_{K-D}^x$  has density  $f_{K-D}^x(t)$  then the hazard rate  $h_K(t)$  of  $S_K$ , defined by

$$\mathbb{P}(S_K \leq t + \delta \mid S_K > t) = h_K(t)\delta + o(\delta), \quad \text{as } \delta \downarrow 0,$$

exists and is given by

$$h_K(t) = \lambda \int_{\mathbb{R}^d} f_{K-D}^x(t) dx.$$

Computing the distribution of  $S_K$  exactly may be hard, but asymptotics may be possible, via knowledge of the Laplace transform of  $T_{K-D}^x$  and Tauberian theorems.

#### 4.4 The isotropic case

Suppose that the process  $\{\xi(t), t \geq 0\}$ , with  $\xi(0) = 0$ , is isotropic, i.e., that if  $Q$  is a proper rotation of  $\mathbb{R}^d$  then  $\{Q\xi(t), t \geq 0\}$  has the same law as  $\{\xi(t), t \geq 0\}$ . Assuming further that

$$D = r\mathbb{B}, \quad K = r_0\mathbb{B}, \quad (10)$$

where  $\mathbb{B} = \{x \in \mathbb{R}^d : |x| \leq 1\}$  is the unit ball of radius 1 centered at the origin, then, clearly, the integral in (9) can be simplified. Indeed, with  $e_1 = (1, 0, \dots, 0)$ ,

$$T_{K-D}^x = \inf\{t \geq 0 : |x + \xi(t)| \leq r + r_0\} \stackrel{(d)}{=} \inf\{t \geq 0 : |x|e_1 + \xi(t)| \leq r + r_0\},$$

so, letting

$$T_{r+r_0}^\rho := \inf\{t \geq 0 : |\rho e_1 + \xi(t)| \leq r + r_0\} \quad (11)$$

be the first hitting time of  $r + r_0$  by the radial process  $|\rho e_1 + \xi(t)|$ , we have

$$\begin{aligned} \int_{\mathbb{R}^d} \mathbb{P}(T_{K-D}^x \leq t) dx &= \sigma_{d-1} \int_0^\infty \mathbb{P}(T_{r+r_0}^\rho \leq t) \rho^{d-1} d\rho \\ &= \omega_d (r + r_0)^d + \sigma_{d-1} \int_{r+r_0}^\infty \mathbb{P}(T_{r+r_0}^\rho \leq t) \rho^{d-1} d\rho, \end{aligned}$$

where  $\omega_d = \pi^{d/2}/\Gamma(1 + d/2)$  is the  $d$ -dimensional Lebesgue measure of the unit ball in  $\mathbb{R}^d$  and  $\sigma_{d-1} = d\omega_d$ . Hence

$$\mathbb{P}(S_K > t) = e^{-\lambda \omega_d (r+r_0)^d} \exp\left(-\lambda \sigma_{d-1} \int_{r+r_0}^\infty \rho^{d-1} \mathbb{P}(T_{r+r_0}^\rho \leq t) d\rho\right). \quad (12)$$

Although the integral has been reduced from a  $d$ -dimensional one to 1-dimensional, finding the distribution of (11) is still a  $d$ -dimensional problem.

## 5 The inertial Boolean model

We take  $\xi$  to be a linear stochastic motion. Let  $v$  be a random variable in  $\mathbb{R}^d$  and let the motion of a typical particle be

$$\xi(t) := tv, \quad t \geq 0.$$

The random set covered by time  $t$  is

$$\mathcal{W}(t) = \bigcup_{x \in \Phi} (x + \{vs, 0 \leq s \leq t\} + K).$$

Letting  $\xi^{(t)} := \{\xi(s), 0 \leq s \leq t\}$ , and  $G^{(t)} := \xi^{(t)} + D$ , we have

$$\begin{aligned} \mathbb{T}_{G^{(t)}}(K - x) &= \mathbb{P}((\xi^{(t)} + D) \cap (K - x) \neq \emptyset) \\ &= \mathbb{P}(\{x + sv, 0 \leq s \leq t\} \text{ intersects } K - D), \end{aligned} \quad (13)$$

which is laborious (but not impossible) to compute explicitly (but we don't need the explicit formula). Assume that the particles are initially placed at the points of a Poisson process



with intensity measure  $\lambda(dx)$ . Then, from (7), the distribution of the detection time of  $K$  is

$$\mathbb{P}(S_K > t) = \exp\left(-\mathbb{E}\lambda\left(\bigcup_{0 \leq s \leq t} [K - D - sv]\right)\right).$$

Take now  $K = r\mathbb{B}$ ,  $D = r_0\mathbb{B}$ , with  $\mathbb{B}$  the unit ball in  $\mathbb{R}^d$ . Then  $K - D = (r + r_0)\mathbb{B}$ . Let

$$R := r + r_0.$$

Assuming further that  $\lambda$  is isotropic (invariant under rotations), so that

$$\lambda(dx) = d\theta |x|^{d-1} \mu(d|x|),$$

where  $d\theta$  is the natural spherical measure on the boundary of the unit ball and  $\mu$  a measure on  $\mathbb{R}_+$ , we have

$$\lambda\left(\bigcup_{0 \leq s \leq t} (K - D - sv)\right) = \frac{1}{2}\lambda(R\mathbb{B}) + \frac{1}{2}\lambda(R\mathbb{B} + tve_1) + \lambda(C_{R,tv}),$$

where  $C_{R,tv}$  is a cylinder with height  $tv$ . Specifically,  $C_{R,tv}$  contains the  $x \in \mathbb{R}^d$  such that

$$\begin{aligned} x \cdot v &\leq t|v|^2, \\ |v|^2 x - (x \cdot v)v &\in |v|^2 \in R\mathbb{B}. \end{aligned}$$

Even when the law of  $v$  is assumed isotropic, the integrals can be quite hard to compute exactly unless we further assume invariance under translations for  $\lambda$ , i.e., take now  $\lambda$  to be a multiple of the Lebesgue measure:

$$\lambda(dx) = \lambda \cdot dx.$$

In this case, things are very simple:

$$\lambda\left(\bigcup_{0 \leq s \leq t} (K - D - sv)\right) = \lambda \cdot R^d \omega_d + \lambda \cdot R^{d-1} \sigma_{d-1} t |v|. \quad (14)$$

Therefore, if  $K - D = R\mathbb{B}$ , and if the original location of particles is a homogeneous Poisson process with intensity  $\lambda$  then, assuming that  $\mathbb{E}|v| < \infty$ ,

$$\mathbb{P}(S_K > t) = \exp(-\lambda R^d \omega_d) \exp(-\lambda R^{d-1} \sigma_{d-1} (\mathbb{E}|v|) t). \quad (15)$$

This gives

$$\mathbb{E}S_K = \frac{e^{-\lambda \omega_d R^d}}{\lambda d \omega_d (\mathbb{E}|v|) R^{d-1}} = \frac{1}{\lambda d \omega_d (\mathbb{E}|v|)} \frac{1}{R^{d-1}} - \frac{1}{d} R + o(R), \quad \text{as } R \downarrow 0. \quad (16)$$

However, if  $\mathbb{E}|v| = \infty$ , then the Poisson process (5) has intensity measure proportional to (13) which integrates, with respect to the  $d$ -dimensional Lebesgue measure, to the expectation of (14) and this is infinity if  $t > 0$ . Therefore,

$$\text{if } \mathbb{E}|v| = \infty \text{ then } S_K = 0, \text{ a.s.}$$

A more elaborate problem is the computation of the law of

$$\inf\{t \geq 0 : r_0\mathbb{B} \cap \mathcal{W}(t) \neq \emptyset, a \notin \mathcal{W}(t)\}$$

where

$$\mathcal{W}(t) = \bigcup_{x \in \Phi} (x + \{vs, 0 \leq s \leq t\} + r\mathbb{B}),$$

which is the set covered up to time  $t$  by the inertial Boolean model when the particles carry balls of radii  $r$  each. In other words, the problem is that of finding information about the first time that the particles will detect a fixed ball of radius  $r_0$  before anyone of them goes close to some point  $a$  (the enemy). This problem will be addressed in the future.

## 6 The Brownian Boolean model

We now specialize further and take  $\xi$  to be a standard Brownian motion in  $\mathbb{R}^d$ . In other words,  $\xi(0) = 0$  and  $\xi = (\xi_1, \dots, \xi_d)$ , where the  $\xi_i, i = 1, \dots, d$  are i.i.d. standard Brownian motions in  $\mathbb{R}$ . Let  $D$  and  $K$  be balls with radii  $r$  and  $r_0$ , respectively, as in (10), and let  $S$  be the detection time of  $K$ . The distribution of  $S$  depends on  $r_0$  and  $r$  through their sum, so we write

$$R = r_0 + r$$

for convenience. Let

$$\varrho(t) := |\rho e_1 + \xi(t)|.$$

Then  $\varrho$  is standard Bessel process of dimension  $d$  started at  $\varrho(0) = \rho$ , denoted as  $\text{BES}^d(\rho)$  by Revuz and Yor [15]. It is a strong Markov process (in fact, a Feller diffusion) satisfying the Itô equation

$$\varrho(t) = \varrho(0) + \beta(t) + \frac{d-1}{2} \int_0^t \frac{1}{\varrho(s)} ds,$$

where  $\beta$  is a standard Brownian motion in  $\mathbb{R}$ . Letting  $T_R^\rho$  be the first hitting time of the closed ball of radius  $R$  centered at the origin by the Bessel process started at  $\rho$ , we have, from (12),

$$\mathbb{P}(S > t) = \exp\left(-\lambda \sigma_{d-1} \int_0^\infty \rho^{d-1} \mathbb{P}(T_R^\rho \leq t) d\rho\right) =: e^{-\lambda V^R(t)}. \quad (17)$$

The infinitesimal generator  $\mathcal{A}$  of  $\varrho$  is the radial part of  $\frac{1}{2}\Delta$ , where  $\Delta$  is the Laplacian on  $\mathbb{R}^d$ :

$$\mathcal{A}f(\rho) = \frac{1}{2} \frac{1}{\rho^{d-1}} \frac{\partial}{\partial \rho} \left( \rho^{d-1} \frac{\partial f}{\partial \rho} \right) = \frac{1}{2} f''(\rho) + \frac{d-1}{2\rho} f'(\rho),$$

acting on  $C^2$  functions. Therefore, for  $s > 0$ , the function

$$L(\rho) := \mathbb{E}[e^{-sT_R^\rho}]$$

satisfies

$$\mathcal{A}L = sLa$$

i.e., the ODE

$$\frac{1}{2}L''(\rho) + \frac{d-1}{2\rho}L'(\rho) = sL(\rho), \quad R < \rho < \infty, \quad (18)$$

with boundary conditions

$$L(R+) = 1, \quad \lim_{\rho \rightarrow \infty} L(\rho) = 0. \quad (19)$$

Change variables using

$$L(\rho) = \rho^b \tilde{L}(a\rho), \quad (20)$$

for appropriate constants  $a > 0, b \in \mathbb{R}$ . The ODE reduces to

$$a^2 \rho^2 \tilde{L}''(a\rho) + (d-1+2b)a\rho \tilde{L}'(a\rho) + (b^2 - 2b + bd - 2s\rho^2) \tilde{L}(a\rho) = 0.$$

Choosing

$$b = \frac{d}{2} - 1$$

gives

$$a^2 \rho^2 \tilde{L}''(a\rho) + a\rho \tilde{L}'(a\rho) - (b^2 + 2s\rho^2) \tilde{L}(a\rho) = 0.$$

Letting

$$a = \sqrt{2s} \quad (21)$$

(and letting  $x := a\rho$ ), we obtain the following ODE

$$x^2 \tilde{L}''(x) + x \tilde{L}'(x) - (b^2 + x^2) \tilde{L}(x) = 0. \quad (22)$$

We recognize this as the modified Bessel ODE [5, Sec. 3.7] the fundamental solutions of which are the modified Bessel functions  $I_{\pm b}$  and  $K_b$ . The standard Bessel ODE differs from (22) by a change of sign in the last term. The fundamental solutions of the standard Bessel ODE are the Bessel functions of first and second kind  $J_{\pm b}, N_b$ , whose series representations are easily obtained from the ODE; see equations (3.82), (3.83) and (3.85) in [5]. The modified Bessel functions  $I_{\pm b}$  and  $K_b$  (of first and second kind, respectively) are related to  $I_{\pm b}$  and  $K_b$  via

$$\begin{aligned} I_{\pm b}(x) &= i^{-b} J_{\pm b}(ix) \\ K_b(x) &= \frac{\pi}{2} i^b [i J_b(ix) - N_b(ix)], \end{aligned}$$

and are real-valued, despite appearances; see (3.100), (3.101) and (3.86) in [5]. Since  $I_{\pm b}$  explodes as  $x \rightarrow \infty$ , we are left with only one choice for (22):

$$\tilde{L}(x) = C K_b(x),$$

where  $C$  is a constant. In terms of the original function, i.e., using the change of variables (21), (20),

$$L(\rho) = C \rho^{-b} K(\rho \sqrt{2s}).$$

The boundary conditions (19) determine  $C$ :

$$C = R^b / K_b(R\sqrt{2s}).$$

So the solution to the ODE (18) with boundary conditions (19) is given by

$$L(\rho) = \mathbb{E}[e^{-sT_R^\rho}] = \frac{\rho^{-b} K_b(\rho \sqrt{2s})}{R^{-b} K_b(R\sqrt{2s})}, \quad \rho \geq R. \quad (23)$$

Compare now (17) with (8). Since  $\bigcup_{0 \leq s \leq t} (\xi(s) + D - K) = \bigcup_{0 \leq s \leq t} (\xi(s) + R\mathbb{B})$ , the quantity in the exponent in (17) is the expected volume of the Wiener sausage

$$W^R(t) := \bigcup_{0 \leq s \leq t} (\xi(s) + R\mathbb{B});$$

see comments at the end of §4.2. We have

$$V^R(t) := \mathbb{E} \operatorname{vol} W^R(t) = \int_{\mathbb{R}^d} \mathbb{P}(T_R^x \leq t) dx = \omega_d R^d + \sigma_{d-1} \int_R^\infty \rho^{d-1} \mathbb{P}(T_R^\rho \leq t) d\rho.$$

Via (23), we have an expression for the Laplace transform of the the expected volume of the Wiener sausage:

$$\begin{aligned} \widehat{V}^R(s) &:= \int_0^\infty e^{-st} V^R(t) dt = \frac{\omega_d R^d}{s} + \frac{\sigma_{d-1}}{s} \int_R^\infty (\mathbb{E} e^{-sT_R^\rho}) \rho^{d-1} d\rho \\ &= \frac{\omega_d R^d}{s} + \frac{\sigma_{d-1}}{s} \int_R^\infty \frac{R^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(\rho\sqrt{2s})}{\rho^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(R\sqrt{2s})} \rho^{d-1} d\rho \\ &= \frac{\omega_d R^d}{s} + \frac{\sigma_{d-1} R^{\frac{d}{2}-1}}{s K_{\frac{d}{2}-1}(R\sqrt{2s})} \int_R^\infty K_{\frac{d}{2}-1}(\rho\sqrt{2s}) \rho^{d/2} d\rho. \end{aligned} \quad (24)$$

We now use a couple of facts about the functions  $K_b$ ; see [1]. First, we have the recursion formula

$$K_{b+1}(x) - K_{b-1}(x) = \frac{2b}{x} K_b(x).$$

Second, we have the derivative

$$K_b'(x) = \frac{b}{x} K_b(x) - K_{b+1}(x).$$

Combining these we get

$$\frac{d}{dx} (K_b(x) x^b) = -K_{b-1}(x) x^b,$$

and so

$$\int_x^\infty K_{\frac{d}{2}-1}(y) y^{\frac{d}{2}} dy = K_{\frac{d}{2}}(x) x^{\frac{d}{2}},$$

and, for  $\lambda > 0$ ,

$$\int_x^\infty K_{\frac{d}{2}-1}(\lambda y) y^{\frac{d}{2}} dy = \frac{1}{\lambda} K_{\frac{d}{2}}(\lambda x) x^{\frac{d}{2}}.$$

The last integral in (24) evaluates to

$$\frac{R^{\frac{d}{2}}}{\sqrt{2s}} K_{\frac{d}{2}}(R\sqrt{2s}),$$

and so

$$\widehat{V}_d^R(s) = \frac{\omega_d R^d}{s} + \frac{\sigma_{d-1} R^{d-1}}{\sqrt{2s^3}} \frac{K_{\frac{d}{2}}(R\sqrt{2s})}{K_{\frac{d}{2}-1}(R\sqrt{2s})},$$

where we added a subscript  $d$  to indicate dependence on the dimension. We can save some space by observing that, due to Brownian scaling,

$$V_d^R(t) = R^d V_d^1(t/R^2), \quad (25)$$

$$\widehat{V}_d^R(s) = R^{d+2} \widehat{V}_d^1(R^2 s).$$

and so it is only

$$\widehat{V}_d^1(s) = \frac{\omega_d}{s} + \frac{\sigma_{d-1}}{\sqrt{2s^3}} \frac{K_{\frac{d}{2}}(\sqrt{2s})}{K_{\frac{d}{2}-1}(\sqrt{2s})} \quad (26)$$

we should be looking for. Since  $K_b = K_{-b}$ , the case  $d = 1$  is trivial:

$$\widehat{V}_1^1(s) = \frac{2}{s} + \frac{2}{\sqrt{2s^3}}. \quad (27)$$

Inverting this Laplace transform gives

$$V_1^1(t) = 2 + \sqrt{\frac{8t}{\pi}}.$$

By the scaling relation (25),

$$V_1^R(t) = 2R + \sqrt{\frac{8t}{\pi}},$$

that is, the expected change of volume (=length) from its initial value does not depend on  $R$ .

## 6.1 Odd dimensions

Consider now the case where

$$d = 2n + 1, \quad n = 0, 1, \dots$$

We will produce an algorithm for computing  $\widehat{V}_{2n+1}^1(s)$  recursively, and carry out its first few steps. The modified Bessel functions of half-integer order have a simple form:

$$K_{n+\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}} y_n(1/x), \quad n = 0, 1, \dots,$$

where  $y_n(x)$  is a polynomial of degree  $n$  with integer coefficients:

$$y_n(x) := \sum_{k=0}^n \frac{(n+k)!}{(n-k)!k!} \left(\frac{x}{2}\right)^k, \quad n = 0, 1, \dots,$$

known as the Bessel polynomial of degree  $n$ ; see [6, formula (3)]. Note, in particular, that

$$y_n(x) = (2n-1)!!x^n + (2n-1)!!x^{n-1} + \dots + \frac{n(n+1)}{2}x + 1,$$

i.e., the coefficients of the two highest powers are equal to the double factorial

$$(2n-1)!! = \frac{(2n)!}{n!2^n} = (2n-1)(2n-3)(2n-5) \dots 3 \cdot 1.$$

See Appendix A for a table of the first few Bessel polynomials and their corresponding Bessel functions. Consequently,

$$\widehat{V}_{2n+1}^1(s) = \frac{\omega_{2n+1}}{s} + \frac{\omega_{2n+1}}{s} \frac{2n+1}{\sqrt{2s}} \frac{y_n(1/\sqrt{2s})}{y_{n-1}(1/\sqrt{2s})}, \quad n = 0, 1, \dots, \quad (28)$$

where

$$y_{-1}(x) := 1,$$

as follows by comparison to (27). We thus have

$$\begin{aligned}\frac{1}{\omega_3} \widehat{V}_3^1(s) &= \frac{1}{s} + \frac{3}{s} \frac{\sqrt{2s+1}}{2s} \\ \frac{1}{\omega_5} \widehat{V}_5^1(s) &= \frac{1}{s} + \frac{5}{s} \frac{(2s)^{3/2} + 6s + 3\sqrt{2s}}{(2s)^{3/2} [\sqrt{2s+1}]} \\ \frac{1}{\omega_7} \widehat{V}_7^1(s) &= \frac{1}{s} + \frac{7}{s} \frac{(2s)^{3/2} + 12s + 15\sqrt{2s} + 15}{\sqrt{2s} [(2s)^{3/2} + 6s + 3\sqrt{2s}]} \\ \frac{1}{\omega_9} \widehat{V}_9^1(s) &= \frac{1}{s} + \frac{9}{s} \frac{(2s)^{5/2} + 40s^2 + 45(2s)^{3/2} + 210s + 105\sqrt{2s}}{(2s)^{3/2} [(2s)^{3/2} + 12s + 15\sqrt{2s} + 15]}.\end{aligned}$$

These Laplace transforms can, in principle, be inverted by using partial fraction expansion and the fact that (see Erdélyi *et al.* [3, Ch. 7, p. 233])

$$\frac{1}{\sqrt{s} + \beta} = \int_0^\infty e^{-st} \left[ \frac{1}{\sqrt{\pi t}} - \beta e^{\beta^2 t} \operatorname{erfc}(\beta\sqrt{t}) \right] dt,$$

where

$$\operatorname{erfc}(t) := 1 - \operatorname{erf}(t), \quad \operatorname{erf}(t) := \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-u^2} du.$$

For example, writing

$$\frac{1}{\omega_3} \widehat{V}_3^1(s) = \frac{1}{s} + \frac{3}{\sqrt{2s^3}} + \frac{3}{2s^2},$$

we obtain

$$\frac{1}{\omega_3} V_3^1(t) = 1 + \frac{6}{\sqrt{\pi}} \sqrt{t} + \frac{3}{2} t.$$

Expanding  $\widehat{V}_5^1(s)$ , we obtain

$$\frac{1}{\omega_5} \widehat{V}_5^1(s) = \frac{1}{s} + \frac{10}{2s} \left[ \frac{1}{\sqrt{2s+1}} + \frac{3}{\sqrt{2s}(\sqrt{2s+1})} + \frac{3}{2s(\sqrt{2s+1})} \right].$$

Since

$$\begin{aligned}\frac{1}{s(\sqrt{s+1})} &= \int_0^\infty e^{-st} [1 - e^t \operatorname{erfc}(\sqrt{t})] dt \\ \frac{1}{s\sqrt{s}(\sqrt{s+1})} &= \frac{1}{s\sqrt{s}} - \frac{1}{s(\sqrt{s+1})} = \int_0^\infty e^{-st} \left[ 2\sqrt{\frac{t}{\pi}} - 1 + e^t \operatorname{erfc}(\sqrt{t}) \right] dt \\ \frac{1}{s^2(\sqrt{s+1})} &= \int_0^\infty e^{-st} \left[ 1 + t - 2\sqrt{\frac{t}{\pi}} - e^t \operatorname{erfc}(\sqrt{t}) \right] dt,\end{aligned}$$

letting  $g_1(t), g_2(t), g_3(t)$  be the functions in the square brackets of the last three lines, we have

$$\frac{1}{\omega_5} V_5^1(t) = 1 + 10 \left[ \frac{1}{2} g_1(t/2) + \frac{3}{2} g_2(t/2) + \frac{3}{2} g_3(t/2) \right] = 6 - 5e^{t/2} \operatorname{erfc}(\sqrt{t/2}) + \frac{15}{2} t.$$

Using the scaling relation (25), we can now obtain  $V_d^R(t)$ , for  $d = 1, 3, 5$  and, therefore, the distribution of the detection time via (17).

$$d = 1: \quad \mathbb{P}(S > t) = \exp(-2\lambda R - 4\lambda\sqrt{t/\pi}). \quad (29)$$

$$d = 3: \quad \mathbb{P}(S > t) = \exp\left(-\frac{4\pi\lambda}{3}R^3 - 8\sqrt{\pi t}\lambda R^2 - 2\pi\lambda t R\right). \quad (30)$$

$$d = 5: \quad \mathbb{P}(S > t) = \exp\left(-\frac{16\pi^2}{5}R^2 + \frac{8R^5}{3}e^{t/2R^2} \operatorname{erfc}(R^{-1}\sqrt{t/2}) - 4\pi^2 R^3 t\right) \quad (31)$$

Notice that the exponent is not a polynomial in  $R$ , as is apparent for the  $d = 5$  case.<sup>4</sup>

A more efficient way of doing the above is by means of the recursion formula

$$y_n(x) = (2n - 1)xy_{n-1}(x) + y_{n-2}(x), \quad n = 1, 2, \dots,$$

with initial conditions  $y_{-1}(x) = y_0(x) = 1$ . See [6, §7]. Let

$$H_n(s) := \frac{\sqrt{2s}}{2n+1} \left( s \frac{\widehat{V}_{2n+1}^1(s)}{\omega_{2n+1}} - 1 \right).$$

From (28) we have

$$H_n(s) = \frac{y_n(1/\sqrt{2s})}{y_{n-1}(1/\sqrt{2s})},$$

and so, from the recursion formula for Bessel polynomials,

$$H_n(s) = \frac{2n-1}{\sqrt{2s}} + \frac{1}{H_{n-1}(s)}, \quad n = 1, 2, \dots,$$

where  $H_0(s) = 1$ . This gives a recursive way for computing  $\widehat{V}_{2n+1}^1(s)$ .

Let

$$a_n(s) := \frac{2n-1}{\sqrt{2s}}.$$

A ‘‘closed’’ formula can also be obtained:

$$\frac{\widehat{V}_{2n+1}^1(s)}{\omega_{2n+1}} = \frac{1}{s} + \frac{a_{n+1}(s)}{s} \left\{ a_n(s) + \frac{1}{a_{n-1}(s) + \frac{1}{a_{n-2}(s) + \frac{1}{\ddots + \frac{1}{a_1(s)+1}}} \right\} \quad (32)$$

$$= \frac{1}{s} + \frac{2n+1}{2s^2} \left\{ (2n-1) + \frac{\sqrt{2s}}{(2n-3) + \frac{\sqrt{2s}}{(2n-5) + \frac{\sqrt{2s}}{\ddots + \frac{\sqrt{2s}}{1+\sqrt{2s}}}}} \right\} \quad (33)$$

<sup>4</sup>Answering a question posed by Günter Last to me a few years ago.

## 6.2 Large time asymptotics in all dimensions

Expression (26) allows us to find logarithmic asymptotics for  $\mathbb{P}(S > t)$ , as  $t \rightarrow \infty$ , in any dimension  $d$ . We repeat the expression here:

$$\frac{1}{\omega_d} \widehat{V}_d^1(s) = \frac{1}{s} + \frac{d}{\sqrt{2s^3}} \frac{K_{\frac{d}{2}}(\sqrt{2s})}{K_{\frac{d}{2}-1}(\sqrt{2s})} =: \frac{1}{s} + \widehat{g}_d(s).$$

The following asymptotics are known for Bessel functions [1]. For any  $b > 0$ , as  $z \rightarrow 0$ ,

$$K_b(z) \sim 2^{b-1} \Gamma(b) z^{-b}.$$

Therefore, for  $d \geq 3$ , we have

$$\widehat{g}_d(s) \sim \frac{d(d-2)}{2s^2}, \quad s \rightarrow 0.$$

Hence

$$g_d(t) \sim \frac{d(d-2)}{2} t, \quad t \rightarrow \infty.$$

By the scaling equation (25) and expression (17), we obtain

$$\begin{aligned} \mathbb{P}(S > t) &= \exp(-\lambda \omega_d R^d - \lambda \omega_d R^d g_d(t/R^2)). \\ &\stackrel{\log}{\sim} \exp\left(-\lambda \omega_d \frac{d(d-2)}{2} R^{d-2} t\right), \quad d \geq 3. \end{aligned}$$

The case  $d = 2$  has to be treated differently as it requires the behavior of  $K_0$  near zero which is different:

$$K_0(z) \sim \log(1/z), \quad z \rightarrow 0.$$

Hence

$$\widehat{g}_2(s) = \frac{2}{\sqrt{2s^3}} \frac{K_1(\sqrt{2s})}{K_0(\sqrt{2s})} \sim \frac{2}{\sqrt{2s^3}} \frac{1}{\sqrt{2s} \log \frac{1}{\sqrt{2s}}} = \frac{2}{s^2 \log(1/s)}$$

Notice that

$$\widehat{g}_2(s) = s^{-2} \ell(1/s),$$

where the function  $\ell(z) = 2/\log(z)$  is slowly varying at infinity, viz.,  $\ell(\kappa z)/\ell(z) \rightarrow 1$ , as  $z \rightarrow \infty$ , for all  $\kappa > 0$ . Combining Karamata's Tauberian theorem with the monotone density theorem ( $g_2(t)$  is increasing function of  $t$ ) we conclude that

$$g_2(t) \sim t \ell(t) = \frac{2t}{\log t}, \quad \text{as } t \rightarrow \infty.$$

Arguing as before, this means that, as  $t \rightarrow \infty$ ,

$$\begin{aligned} \mathbb{P}(S > t) &= \exp(-\lambda \omega_2 R^2 - \lambda \omega_2 R^2 g_2(t/R^2)) \\ &\stackrel{\log}{\sim} \exp\left(-\frac{2\pi \lambda t}{\log t}\right). \end{aligned}$$

This agrees with the result of [8, Theorem 2].



### 6.3 Expectations

For  $d = 1$ , using  $\int_0^\infty \exp(-\sqrt{t}) dt = 2$ , we can compute the expectation of  $S$  explicitly by integrating (29):

$$\mathbb{E}S := \frac{\pi}{8} \frac{e^{-2\lambda R}}{\lambda^2}.$$

For  $d = 2$ , we have no explicit expression, but the earlier asymptotic expression for large  $t$  tells us that, as  $R \rightarrow 0$ ,  $\mathbb{E}S$  converges to a constant. This is a manifestation of the fact that Brownian motion is neighborhood recurrent in 2 dimensions.

For  $d = 3$ , we can use the integral  $\int_0^\infty \exp(-\sqrt{t} - t) dt = 1 + \frac{\pi e^{1/4}}{2}(\operatorname{erf}(1/2) - 1)$  to integrate (31):

$$\mathbb{E}S = \frac{e^{-\frac{4}{3}\pi\lambda R^3}}{2(\pi\lambda R)^{3/2}} \left[ \sqrt{\pi\lambda R} + 2\sqrt{2}\pi\lambda R^2 e^{8\lambda R^3} (\operatorname{erf}(2\sqrt{2}\lambda R^3/2) - 1) \right]$$

For higher dimensions, we can translate the previous asymptotics for  $t \rightarrow \infty$  into asymptotics for  $R \downarrow 0$  and obtain that

$$\mathbb{E}S \sim \frac{c_d}{R^{d-2}}, \text{ as } R \downarrow 0, \quad (34)$$

where  $c_d$  is a constant depending on  $d$  only. This estimate holds for all  $d \geq 2$ . Comparing (34) with (16) we find that in any dimension  $d \geq 2$ , it is better to make particles (sensors) perform Brownian motions rather than random straight lines with finite mean velocity if the goal is to detect a small object.<sup>5</sup>

## 7 Concluding remarks and open problems

In this paper, we reviewed the mobile Boolean model, focusing, in particular, in the inertial and Brownian cases. For the inertial case, we have an explicit expression (15) for the distribution of the detection time of in any dimension. For the Brownian case, we have explicit expressions (29), (30), (31) in dimensions  $d = 1, 3$  and 5, an algorithm for computing an explicit expression when  $d$  is odd (Section 6.1) and logarithmic asymptotics when  $d$  is even (Section 6.2). We worked with a target set  $K$  which is a ball. This enabled us to reduce the a multi-dimensional problem to one dimension. We also gave formulas or estimates for expectations.

For general compact set  $K$ , and dimension  $d = 3$ , Spitzer's [8] paper gives asymptotic estimates for the expected volume of a  $K$ -Wiener sausage, in terms of the Newtonian capacity  $\operatorname{cap}(K)$  of  $K$ , using entirely probabilistic methods:

$$\mathbb{E} \operatorname{vol}(\xi^{(t)} + K) = \operatorname{vol}(K) + \operatorname{cap}(K)t + 4(2\pi)^{-3/2} \operatorname{cap}(K)^2 \sqrt{t} + o(\sqrt{t}), \quad d = 3.$$

Translating this into a detection time probability estimate, and using the scaling relation, we have

$$P(S_{RK} > t) = \exp\left(\operatorname{vol}(K)R^3 + \operatorname{cap}(K)R^2t + 4(2\pi)^{-3/2} \operatorname{cap}(K)^2 R\sqrt{t} + o(R\sqrt{t})\right).$$

We next pose some open problems.

---

<sup>5</sup>This answers a question posed by Venkat Anantharam a few years ago, who, jokingly, commented that an engineer would never have sensors perform Brownian motions.

**Open problem 1.** We did not touch at all the coverage problem, i.e., the law of the random variable  $\inf\{t \geq 0 : K \subset \mathcal{W}(t)\}$ . For the Brownian Boolean model, and when the diameter of  $R$  tends to infinity, the problem has been solved in [12]. What are the corresponding asymptotics for the inertial cases?

**Open problem 2.** Let  $K_1$  and  $K_2$  be two sets (e.g., balls of radii  $r_1, r_2$ ), and let  $S_{K_1}, S_{K_2}$  be their detection times. Find the probability  $P(S_{K_1} < S_{K_2})$ .

**Open problem 3.** For the inertial Boolean model, find the distribution of the detection time of a ball before a fixed point is hit. (See remarks at the end of Section 5.)

**Open problem 4.** Investigate further the algorithm of Section 6.1 and, in particular, the “closed” formula (32)-(33).

**Open problem 5.** Let there be an independent space-time Poisson process  $\Psi$  in  $\mathbb{R}^d \times \mathbb{R}_+$  with fixed intensity. Interpret its points as “customers”. The Brownian Boolean model is a space-time serving mechanism clearing points whenever it meets them. Find necessary and sufficient stability conditions. This problem is related to a number of recent stability problems in queueing theory where the spatial dimension is just as important as the time dimension [4, 2, 7].

## A Modified Bessel functions of second kind of half-integer order and their corresponding Bessel polynomials

$$\begin{aligned}
 K_{n+\frac{1}{2}}(x) &= \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}} & y_n(x) &= \sum_{k=0}^n \frac{(n+k)!}{(n-k)!k!} \left(\frac{x}{2}\right)^k \\
 K_{1/2}(x) &= \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}} & y_0(x) &= 1 \\
 K_{3/2}(x) &= \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{x^{3/2}}(x+1) & y_1(x) &= 1+x \\
 K_{5/2}(x) &= \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{x^{5/2}}(x^2+3x+3) & y_2(x) &= 1+3x+3x^2 \\
 K_{7/2}(x) &= \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{x^{7/2}}(x^3+6x^2+15x+15) & y_3(x) &= 1+6x+15x^2+15x^3 \\
 K_{9/2}(x) &= \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{x^{9/2}}(x^4+10x^3+45x^2+105x+105) & y_4(x) &= 1+10x+45x^2+105x^3+105x^4.
 \end{aligned}$$

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