Asymptotics of Eigenpolynomials of Exactly-Solvable Operators

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Summary

The main topic of this doctoral thesis is asymptotic properties of zeros in polynomial families appearing as eigenfunctions to exactly-solvable differential operators (ES-operators). The study was initially inspired by a number of striking results from computer experiments performed by G. Masson and B. Shapiro for a more restrictive class of operators. Our research is also motivated by a classical question going back to S. Bochner on a general classification of differential operators possessing an infinite sequence of orthogonal eigenpolynomials. In general however, the sequence of eigenpolynomials of an ES-operator does not constitute an orthogonal system and it can therefore not be studied by means of the extensive theory known for such systems. Our study can thus be considered as the first step to a natural generalization of the asymptotic behaviour of the roots of classical orthogonal polynomials. Also, many special functions appear as solutions to certain second-order differential equations and hence solutions to exactly-solvable differential equations might be thought of as higher order special functions. Our main definition is as follows. An **exactly-solvable operator of order k** is a linear differential operator of the form

$$T = \sum_{j=1}^{k} Q_j(z) \frac{d^j}{dz^j}$$

where the Q_j are complex polynomials in one variable satisfying the condition $\deg Q_j \leq j$ with equality for at least one j. We are interested in the sequence of polynomials $\{p_n\}_{n=0}^{\infty}$ satisfying $T(p_n) = \lambda_n p_n$ where $\deg p_n = n$, and λ_n is the spectral parameter. One can show that for all sufficiently large integers n there exists a unique and monic p_n for any T as above. If $\deg Q_k = k$ for the leading term then T is called a **non-degenerate ES-operator**, whereas if $\deg Q_k < k$ we call T a **degenerate ES-operator**. The major difference between these two classes is that in the non-degenerate case the union of all roots of all p_n is contained in a compact set, as opposed to the degenerate case where the largest modulus of the roots of p_n tends to infinity when $n \to \infty$. Computer experiments indicate that roots of eigenpolynomials of ES-operators fill piecewise real analytic curves in the complex plane. One of the main technical tools in this thesis is the Cauchy transform of a probabil-

ity measure, which in the considered situation satisfies an algebraic equation - in the degenerate case however only after an appropriate scaling of the eigenpolynomials. Due to the connection between the (asymptotic) root measure and its Cauchy transform it is thus possible to obtain detailed information on the limiting zero distribution.

Some Remarks. Let us mention some interesting but still unexploited connections of our results to other fascinating fields:

1. The existence of a probability measure μ with compact support such that its Cauchy transform satisfies an algebraic equation implies that the Cauchy transform also satisfies a differential equation (of order at most k) expressed by a so called Fuchsian operator in the Weyl algebra $A_1(z, \partial)$. It is an open problem to determine the class of all Fuchsian operators which arise via the limiting root measure of eigenpolynomials associated to degenerate ES-operators.

2. In many situations considered in this thesis the support of the limiting root measure is a curvilinear tree in the complex plane which can be straightened out in a certain canonical holomorphic coordinate. The resulting usual tree has only a finite number of possible angles between its edges. This bears a strong resemblance with the notion of amoeba in tropical algebraic geometry, which is an extremely active research area. But the actual connection between our 'generalized' amoebas and the classical amoebas is unclear at the moment.

3. The support of the limiting root measure seems to be a part of the global Stokes line of the associated ordinary linear differential equation. But for orders $k > 2$ there is no satisfactory theory of Stokes lines, and the question about the exact interrelation between the latter support and Stokes lines is still widely open.

4. In the non-degenerate situation considered in this thesis (see paper I) it is only the leading coefficient of the differential equation which effects the limiting zero distribution. To change the situation in such a way that the whole symbol of the differential operator will be important one has to, instead of the usual spectral problem, consider its homogenized version where the polynomial coefficient Q_j is multiplied by λ_n^j where λ_n is the spectral parameter. This direction is at present pursued by J. Borcea, R. Bøgvad and B. Shapiro.

This thesis consists of five papers, all devoted to the study of zeros of eigenpolynomials of ES-operators.

Paper I: In this paper we study roots of polynomials arising as eigenfunctions to non-degenerate ES-operators. In this case the eigenvalue equation can be considered as a natural generalization to higher orders of the Gauss hypergeometric equation

$$(z^2-1)f''(z)+(az+b)f'(z)+cf(z)=0$$

to which (for certain choices of a, b and c) the classical orthogonal Jacobi polynomials appear as solutions. But for orders $k>2$ the sequence of eigenpolynomials is in general not an orthogonal system. The main result in this paper is a complete characterization of the asymptotic distribution of zeros of the eigenpolynomial of an arbitrary non-degenerate ES-operator. We thus settle several of the conjectures stated by G. Masson and B. Shapiro in [4]. Namely, for any such operator, we construct the probability measure μ_n (we call this the root measure) of its unique and monic eigenpolynomial p_n by placing a point mass of size $1/n$ at each zero of p_n. We then prove that μ_n converges (weakly) to a unique probability measure μ whose Cauchy transform satisfies the equation $C(z)^k=1/Q_k(z)$ for almost all $z\in\mathbb{C}$. Thus the asymptotic zero distribution of p_n depends only on the leading polynomial Q_k of T. Moreover, supp μ is a tree which contains all the zeros of Q_k and is contained in the convex hull of the zeros of Q_k. The proof of these properties of supp μ depends heavily on Lemma 3 in section 3 due to H. Rullgård.

Paper II: In this paper we adress a classical question going back to S. Bochner and H.L. Krall. A system of orthogonal polynomials which also appear as eigenfunctions of some finite-order differential operator is called a Bochner-Krall system and the corresponding spectral operator a Bochner-Krall operator. It is an open problem to classify all Bochner-Krall systems - a complete classification is only known for operators of order $k\leq 4$. Here we use the results obtained in paper I to settle a special case of a general conjecture describing the leading terms of all Bochner-Krall operators. Namely, in [4] it is conjectured that the leading coefficient of any such operator is a power of a polynomial of degree at most 2. Our main result in this paper is an affirmative answer to this conjecture for

Bochner- Krall operators of Nevai type - that is operators for which the system of eigenpolynomials is orthogonal with respect to a measure of the so called Nevai class.

Paper III: This paper is devoted to a study on the location of zeros of individual eigenpolynomials of a confluent hypergeometric operator, which is the simplest example of a degenerate ES-operator. The classical Laguerre polynomials appear as a special case, and some well-known results about these are recovered by yet another method and generalized. We pay particular attention to hyperbolicity properties of the eigenpolynomials of this operator.

Paper IV: In this paper we extend our previous study to asymptotic properties of zeros of eigenpolynomials for arbitrary degenerate ES-operators. We prove that (as opposed to the non-degenerate case) the zeros of the unique and monic nth degree eigenpolynomial p_n do not stay in a compact set as its degree tends to infinity. However, computer experiments indicate the existence of a limiting root measure supported on a tree in this case too, but that it has compact support only after an appropriate scaling. The main achievement in this paper is an explicit conjecture and partial results (lower bound) on the growth of the largest modulus of the roots of p_n. This can be seen as a generalization of the asymptotic behaviour of the maximal root for classical orthogonal polynomials appearing as eigenfunctions to degenerate ES-operators - our conjecture is confirmed by known results when reduced to such systems. Namely, let T be a degenerate ES-operator of order k and denote by j_0 the largest j for which $\deg Q_j = j$ in T (note that $j_0 < k$). Denote by r_n the largest modulus of the roots of p_n. Then

$$\lim_{n\to\infty} \frac{r_n}{n^d} = c_T \quad \text{where} \quad d := \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j} \right)$$

and where c_T is a positive constant. Based on this conjecture we introduce the properly scaled polynomial $q_n(z) = p_n(n^d z)$ for which the zeros are (conjecturally) contained in a compact set when $n \to \infty$. In fact, our natural heuristic arguments (see section 3) show that d as above is the only possible choice which results in a nice algebraic equation satisfied by the Cauchy transform of the asymptotic root measure of $q_n(z)$. From this equation we can conclude which terms of T that are relevant for the limiting zero distribution of its eigenpolynomials. Numerical evidence clearly illustrates that distinct operators whose scaled eigenpolynomials satisfy the same Cauchy transform equation when $n \to \infty$ yield identical zero distributions. It is still an open problem to describe its support explicitly.

Finally, as in the non-degenerate case, operators of the type we consider here occur in the theory of Bochner-Krall systems. By comparing known results on orthogonal polynomials with results on the eigenpolynomials studied here, we believe it will be possible to gain new insight into the nature of Bochner-Krall systems.

Paper V: In this paper we extend earlier results from paper IV by establishing a lower bound for the largest modulus of the roots of the eigenpolynomial of an arbitrary degenerate ES-operator. We thus confirm the lower-bound part of our main conjecture as stated in paper IV.

Acknowledgements

First I would like to thank my dear PhD advisor Professor Boris Shapiro for introducing me to this very fascinating and fruitful problem. Without him this thesis would never have come into being. One of his first comments a few weeks after I had entered the Department of Mathematics at Stockholm University was: *"You resemble an electron - you are everywhere at the same time!"*, and I guess he meant that my existence was like an overlap of wavefunctions that never would collapse - and he was right. The probablity for him to collapse as an effect of my intense nature was much higher (he mentioned this several times) but in time we developed a fruitful collaboration. I thank him for his professional guidance, his patience, kindness and support during my work.

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The list of people I am indepted to could be made much longer. I would like to thank all employees at the Department of Mathematics at Stockholm University. You have all been a source of inspiration to me. I am also deeply grateful to the staff at the Department of Mathematics at Lund University where I as a young student discovered the beauty of mathematics.

Finally I want to thank my family - my parents, grandparents, brother with family, and the most important person in my life: my daughter Nadja - for their love and support.

My interest in natural science began with astronomy which was my main occupation as a teenager. Over time my focus passed to physics until I finally realized one day that it's all mathematics! Years of research in the field of mathematics has deeply affected my understanding of Nature. Nowadays, whenever I look up to the sky I can't stop wondering if any of the constellations of stars I studied as a youngster isn't actually the distribution of zeros of a polynomial arising as an eigenfunction to an operator I haven't yet considered.

I dedicate this thesis to my mother Anni and to my grandparents Slavka and Nikola who are the reason I came this far and became the person I am today. You have been the main source of inspiration in my life - thank you for believing in me!

Tanja Bergkvist, February 2007

PAPER I

On Polynomial Eigenfunctions for a Class of Differential Operators

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Abstract

In this paper we study the asymptotic properties of zeros of polynomials arising as eigenfunctions of a certain class of operators (non-degenerate exactly-solvable operators). We prove that when the degree of the unique and monic eigenpolynomial tends to infinity, its zeros are distributed according to a certain probability measure which is compactly supported on a tree and which depends only on the leading term of the operator.

1 Introduction

Jacobi polynomials are solutions of the differential equation

$$(z^2-1)f''(z)+(az+b)f'(z)+cf(z)=0, \tag{1}$$

where a, b, c are constants satisfying $a > b$, $a+b > 0$ and $c = n(1-a-n)$ for some nonnegative integer n. It is a classical fact that the zeros of the Jacobi polynomials lie in the interval $[-1, 1]$, and that their density in this interval is proportional to $1/\sqrt{1-|z|^2}$ in the limit when the degree n tends to infinity. The usual proof of this statement involves the observation that, for fixed a and b, the Jacobi polynomials constitute an orthogonal system of polynomials with respect to a certain weight function on the interval $[-1, 1]$. The desired conclusion then follows from the general theory of orthogonal systems of polynomials.

The following appears to be a natural generalization of the differential equation (1). Let $k \geq 2$ be an integer, and let $Q_0, \ldots, Q_k$ be polynomials in one complex variable satisfying $\deg Q_j \leq j$ with equality when $j = k$. Moreover, we make a normalization by assuming that Q_k is monic. Consider the differential operator

$$T_Q(f) = \sum_{j=0}^{k} Q_j f^{(j)} \tag{2}$$

where $f^{(j)}$ denotes the jth derivative of f. Operators of this type appear for example in the theory of Bochner-Krall systems of orthogonal polynomials, see [3]. This operator was studied by G. Masson and B. Shapiro in [4]. Particular attention was given the more special operators $T'(f) = Q_k f^{(k)}$ and $T''(f) = (d/dz)^k (f(z)Q_k(z))$. These are indeed special cases of (2) obtained by taking $Q_j = 0$ or $Q_j = \binom{k}{j} Q_k^{(k-j)}$ respectively, for $j = 0, \ldots, k-1$. The following result, which shows that T_Q has plenty of polynomial eigenfunctions, was proved for the operators T' and T'' in [4]:

Theorem 1 *For all sufficiently large integers n there exists a unique constant λ_n and a unique monic polynomial p_n of degree n which satisfy*

$$T_Q(p_n) = \lambda_n p_n. \tag{3}$$

Moreover, we have $\lambda_n / n(n-1)\ldots(n-k+1) \to 1$ when $n \to \infty$.

G. Masson and B. Shapiro made a number of striking conjectures, based on numerical evidence, about the zeros of the eigenpolynomials p_n. They also observed that when $k > 2$, the sequence p_n is in general not an orthogonal system of polynomials, so it cannot be studied by means of the extensive theory known for such systems.

The goal of this paper is to prove some of the conjectures in [4]. More precisely, we shall show that in the limit when $n \to \infty$, the zeros of p_n are distributed according to a certain probability measure. This probability measure depends only on the "leading polynomial" Q_k and may be independently characterized in the following way.

Theorem 2 *Let Q_k be a monic polynomial of degree k. Then there exists a unique probability measure μ_{Q_k} with compact support whose Cauchy transform*

$$C(z) = \int \frac{d\mu_{Q_k}(\zeta)}{z - \zeta} \tag{4}$$

satisfies $C(z)^k = 1/Q_k(z)$ for almost all $z \in \mathbf{C}$.

We record some properties of the measure μ_{Q_k} which will be encountered in the proof of Theorem 2. Let $\operatorname{supp} \mu$ denote the support of a measure μ. Also, let

$$\Psi(z) = \int Q_k(z)^{-1/k}\, dz$$

be a primitive function of $Q_k(z)^{-1/k}$. At this point, we think of Ψ as a locally defined function in any simply connected domain where Q_k does not vanish. The choice of a branch of $Q_k(z)^{1/k}$ and an integration constant is of no importance here. As need arises, specifications will be made concerning these choices.

Theorem 3 *Let Q_k and μ_{Q_k} be as in Theorem 2. Then* $\operatorname{supp}\mu_{Q_k}$ *is the union of finitely many smooth curve segments, and each of these curves is mapped to a straight line by the mapping Ψ. Moreover,* $\operatorname{supp}\mu_{Q_k}$ *contains all the zeros of Q_k, is contained in the convex hull of the zeros of Q_k and is connected and has connected complement.*

If p is a polynomial of degree n, we can construct a probability measure μ by placing a point mass of size $1/n$ at each zero of p. We will call μ the root measure of p. Our main result is

Theorem 4 *Let p_n be the monic degree n eigenpolynomial of the operator T_Q and let μ_n be the root measure of p_n. Then μ_n converges weakly to μ_{Q_k} when $n \to \infty$.*

To illustrate, we show the zeros of the eigenpolynomial p_{40} for the order 5 operator T_Q with $Q_5(z) = z(z-1+i)(z-5)(z-2-4i)(z-4-4i)$ and $Q_0 = \cdots = Q_4 = 0$. Large dots represent the zeros of Q_5 (which are, in this case, also zeros of p_n) and small dots represent (the remaining) zeros of p_{40}. It is remarkable how well the zeros of the eigenpolynomial line up along the curves predicted by our results. Notice also how these curves are straightened out by the mapping Ψ, see Figure 1.

It is not difficult to deduce various other features of the measure μ_{Q_k} from the properties given in Theorem 3 and the defining property (4). For example, a recipe for computing the angles between the different curve segments is conjectured in [4]. The correctness of the procedure follows easily from our results. We refrain from going into details, but the key observation is the following. Suppose z_0 is

Figure 1: Zeros of the polynomial Q_5 and the eigenpolynomial p_{40} (left) and the image of these zeros under a branch of the mapping Ψ.

a point on one of the curve segments in $\operatorname{supp} \mu_{Q_k}$ and let C_1 and C_2 be the limiting values of $C(z)$ as z approaches z_0 from different sides of the curve. Then C_1 and C_2 are kth roots of $1/Q_k(z_0)$, and their actual values are easily found if the combinatorics of $\operatorname{supp} \mu_{Q_k}$ are known (which was assumed in the recipe mentioned above). From the fact that $\pi\mu_{Q_k} = \partial C/\partial \bar{z} \geq 0$, it follows that the curve must be perpendicular to $\overline{C}_1 - \overline{C}_2$ at z_0. Using this observation where several curves meet, it is possible to deduce the angles between them. Notice also that the density of μ_{Q_k} at z_0 is proportional to $|C_1 - C_2|$.

It is particularly easy to compute μ_{Q_k} when Q_k has only real zeros. Namely, denote the zeros by $z_1, \ldots, z_k$ in increasing order. From Theorem 3 we know that $\operatorname{supp} \mu_{Q_k} = [z_1, z_k]$. A direct computation shows that on the subinterval $[z_j, z_{j+1}]$, the measure is given by

$$\mu_{Q_k} = \frac{1}{\pi} \frac{\partial C}{\partial \bar{z}} = \frac{2}{\pi |Q_k|^{1/k}} \sin\left(\frac{\pi j}{k}\right) dx$$

where dx denotes Lebesgue measure on the real line. This remains true even if Q_k has multiple zeros, except if all the zeros coincide. In this case, μ_{Q_k} reduces, of course, to a point mass at this multiple zero.

Let us finally discuss some possible applications and directions for further research. As we already mentioned, operators of the type we consider occur in the theory of Bockner-Krall orthogonal systems. More precisely, a Bochner-Krall system (BKS), is a sequence of polynomials p_n which are both eigenpolynomials of an operator T_Q (here one omits the assumption that $\deg Q_k = k$) and also orthogonal with respect to a suitable inner product. It is a long standing problem to classify all BKS. A great deal is known about the asymptotic distribution of zeros of orthogonal polynomials. By comparing such results with our results on the distribution of zeros of eigenpolynomials, we believe that it will be possible to gain new insight into the nature of BKS. To get the most out of this approach, however, it would be desirable to have generalizations of our results to the case $\deg Q_k < k$. Computer experiments performed by the first author indicate that a limiting measure exists in this case too, but that it may not have compact support.

This paper is organized as follows. In section 2 we compute the matrix for the operator T_Q with respect to the basis of monomials $1, z, z^2, \ldots$, and use this to prove Theorem 1 for an arbitrary operator of the type we consider. Section 3 contains a proof of the uniqueness part of Theorem 2. Along the way, we also prove essentially all the statements in Theorem 3. In section 4 we recall some basic facts on the weak topology of measures in the complex plane and on logarithmic potentials and Cauchy transforms. We also outline the connection of these concepts to root measures of polynomials and prove a general result on the relation between the zeros of a polynomial and those of its derivative. In the final section 5 we apply the ideas from the previous section to give a proof of Theorem 4. The existence part of Theorem 2 is also a consequence of this proof.

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2 Calculation of the matrix

Proof of Theorem 1. We now prove that Theorem 1 as stated in Introduction holds for any operator of the type we consider. Recall that the differential operator T_Q is defined by

$$T_Q = Q_k \frac{d^k}{dz^k} + Q_{k-1}\frac{d^{k-1}}{dz^{k-1}} + \cdots + Q_1 \frac{d}{dz} + Q_0$$

where the Q_m are polynomials such that $\deg Q_m \leq m$ for $m = 0, \ldots, k$ and $\deg Q_k = k$. Let $p_n(z) = \sum_{i=0}^{n} a_{n,i} z^i$ be a monic polynomial of degree n and let $Q_m(z) = \sum_{j=0}^{m} q_{m,j} z^j$. Using these notations we get

$$\begin{aligned} T_Q(p_n) &= \sum_{m=0}^{k} Q_m \cdot \frac{d^m}{dz^m} p_n = \sum_{m=0}^{k} \left[\sum_{j=0}^{m} q_{m,j} z^j \right] \cdot \left[\sum_{i \geq m}^{n} a_{n,i} \cdot \frac{i!}{(i-m)!} z^{i-m} \right] \\ &= \sum_{m=0}^{k} \sum_{s=0}^{n} \left[\sum_{\substack{s=j+i-m \\ m \leq i \leq n \\ 0 \leq j \leq m}} q_{m,j} \cdot a_{n,i} \cdot \frac{i!}{(i-m)!} \right] z^s \\ &= \sum_{s=0}^{n} \left[\sum_{m=0}^{k} \sum_{\substack{s=j+i-m \\ m \leq i \leq n \\ 0 \leq j \leq m}} q_{m,j} \cdot a_{n,i} \cdot \frac{i!}{(i-m)!} \right] z^s. \end{aligned}$$

Lemma 1 *If p_n is monic and $T_Q(p_n) = \lambda_n p_n$ then*

$$\lambda_n = \sum_{m=0}^{k} q_{m,m} \cdot \frac{n!}{(n-m)!}.$$

Proof. With p_n monic and $T_Q(p_n) = \lambda_n \cdot p_n = \lambda_n z^n + \lambda_n \cdot a_{n,n-1} z^{n-1} + \ldots + \lambda_n \cdot a_{n,0}$, finding the eigenvalue λ_n amounts to finding the coefficient at z^n in $T_Q(p_n)$. Note that $\deg \left(Q_m \frac{d^m}{dz^m} p_n\right) \leq m + n - m = n$ with equality if and only if $\deg Q_m = m$. Thus we can assume that $p_n = z^n$ since any lower degree terms of p_n will result in terms of degree lower than n in $T_Q(p_n)$. Thus

$$
\begin{aligned}
T_Q(z^n) &= \sum_{m=0}^{k} Q_m \cdot \frac{d^m}{dz^m} z^n = \sum_{m=0}^{k} Q_m \cdot \frac{n!}{(n-m)!} z^{n-m} \\
&= \sum_{m=0}^{k} \left[\left(\sum_{j=0}^{m} q_{m,j} z^j \right) \cdot \frac{n!}{(n-m)!} z^{n-m} \right] \\
&= \sum_{m=0}^{k} \left[\sum_{j=0}^{m} q_{m,j} \cdot \frac{n!}{(n-m)!} z^{j+n-m} \right].
\end{aligned}
$$

Setting $j = m$ we get

$$
\lambda_n z^n = \sum_{m=0}^{k} q_{m,m} \cdot \frac{n!}{(n-m)!} z^n \quad \Longrightarrow \quad \lambda_n = \sum_{m=0}^{k} q_{m,m} \cdot \frac{n!}{(n-m)!}.
$$

$\square$

Lemma 2 *For $n \geq 1$ the coefficient vector X of p_n with components $a_{n,0}, \ldots, a_{n,n-1}$ satisfies the linear system $MX = Y$, where Y is a vector and M is an upper triangular matrix, both with entries expressible in the coefficients $q_{m,j}$.*

Proof. The relation

$$
T_Q(p_n) = \lambda_n \cdot p_n
$$

is equivalent to

$$
\sum_{s=0}^{n} \left[\sum_{m=0}^{k} \sum_{\substack{s=j+i-m \\ m \leq i \leq n \\ 0 \leq j \leq m}} q_{m,j} \cdot a_{n,i} \cdot \frac{i!}{(i-m)!} \right] z^s = \lambda_n \sum_{s=0}^{n} a_{n,s} z^s.
$$

With $j = m + s - i$ the condition $j \leq m$ gives $i \geq s$ and the condition $j \geq 0$ results in $m \geq i - s$. Therefore the above system will be equivalent to

$$
\sum_{s=0}^{n} \left[\sum_{s \leq i \leq n} \quad \sum_{i-s \leq m \leq \min(i,k)} q_{m,m+s-i} \cdot \frac{i!}{(i-m)!} \cdot a_{n,i} - \lambda_n \cdot a_{n,s} \right] z^s = 0.
$$

Thus for each s we have

$$
\sum_{s \leq i \leq n} \quad \sum_{i-s \leq m \leq \min(i,k)} q_{m,m+s-i} \cdot \frac{i!}{(i-m)!} \cdot a_{n,i} - \lambda_n \cdot a_{n,s} = 0
$$

or, equivalently,

$$\sum_{s \le i \le n-1} \sum_{i-s \le m \le \min(i,k)} q_{m,m+s-i} \cdot \frac{i!}{(i-m)!} \cdot a_{n,i} - \lambda_n \cdot a_{n,s} =$$

$$= \sum_{n-s \le m \le \min(n,k)} q_{m,m+s-n} \cdot \frac{n!}{(n-m)!} \cdot a_{n,n}$$

where $a_{n,n} = 1$. The $n \times n$ matrix M is thus constructed for $0 \le s \le n-1$ and $0 \le i \le n-1$. The left-hand side of the above equation corresponds to the $(s+1)$st row in M multiplied by the column vector X, and the right-hand side represents the $(s+1)$st row in Y. Thus the entries of M are given by

$$M_{s+1,i+1} = \sum_{i-s \le m \le \min(i,k)} q_{m,m+s-i} \cdot \frac{i!}{(i-m)!} - \lambda_n \cdot \delta_{i,s} \qquad (5)$$

where δ denotes the Kronecker delta. The condition $i \ge s$ implies that M is upper triangular. $\square$

We can now prove Theorem 1. Using Lemma 1 we get

$$\begin{aligned} \frac{\lambda_n}{n(n-1)\dots(n-k+1)} &= \frac{\sum_{m=0}^{k} q_{m,m} \cdot \frac{n!}{(n-m)!}}{n(n-1)\dots(n-k+1)} \\ &= \frac{q_{0,0}\frac{n!}{n!} + q_{1,1}\frac{n!}{(n-1)!} + q_{2,2}\frac{n!}{(n-2)!} + \dots + q_{k-1,k-1}\frac{n!}{(n-k+1)!} + q_{k,k}\frac{n!}{(n-k)!}}{n(n-1)\dots(n-k+1)} \\ &= \frac{q_{0,0}}{n(n-1)\dots(n-k+1)} + \frac{q_{1,1}}{(n-1)\dots(n-k+1)} + \dots \\ &\quad \dots + \frac{q_{k-1,k-1}}{(n-k+1)} + q_{k,k}. \end{aligned}$$

Thus

$$\lim_{n\to\infty} \frac{\lambda_n}{n(n-1)\dots(n-k+1)} = q_{k,k} = 1,$$

and the first part of Theorem 1 is proved. To prove the uniqueness part, we show that the determinant of the matrix M constructed in Lemma 2 is nonzero for all sufficiently large n. Since the matrix is upper triangular its determinant equals the product of the diagonal elements. Thus it suffices to prove that for sufficiently large n every diagonal element is nonzero.

The diagonal element $M_{i+1,i+1}$ of M is obtained by letting $i = s$ in (5) and so

$$M_{i+1,i+1} = \sum_{0 \le m \le \min(i,k)} q_{m,m} \cdot \frac{i!}{(i-m)!} - \lambda_n$$

for $i = 0, \ldots, n-1$. But the last expression equals $\lambda_i - \lambda_n$. Indeed, if $i \ge k$ then

$$\sum_{0 \le m \le \min(i,k)} q_{m,m} \cdot \frac{i!}{(i-m)!} = \sum_{0 \le m \le k} q_{m,m} \cdot \frac{i!}{(i-m)!} = \lambda_i.$$

If $i < k$ then this is again true since by definition $i!/(i-m)! = 0$ for $i < m \le k$. Thus we have to show that $\lambda_i - \lambda_n \ne 0 \quad \forall i < n$ as $n \to \infty$. For small values of i ($i < k$) we have $\lambda_i < \infty$ but $\lambda_n \to \infty$ as $n \to \infty$, implying $\lambda_i - \lambda_n \ne 0$. For larger values of i $(0 < m < k \le i)$ we get

$$\begin{aligned}
\lambda_n - \lambda_i &= \sum_{m=0}^{k} q_{m,m} \frac{n!}{(n-m)!} - \sum_{m=0}^{k} q_{m,m} \frac{i!}{(i-m)!} \\
&= \sum_{m=0}^{k} q_{m,m} \left[\frac{n!}{(n-m)!} - \frac{i!}{(i-m)!} \right].
\end{aligned}$$

Dividing the last expression by $\frac{n!}{(n-k)!} - \frac{i!}{(i-k)!}$ we obtain

$$\frac{\lambda_n - \lambda_i}{\frac{n!}{(n-k)!} - \frac{i!}{(i-k)!}} = q_{k,k} + \sum_{m=1}^{k-1} q_{m,m} \frac{\frac{n!}{(n-m)!} - \frac{i!}{(i-m)!}}{\frac{n!}{(n-k)!} - \frac{i!}{(i-k)!}}.$$

which tends to $q_{k,k} = 1$ as $n \to \infty$, since for each $m \le k-1$ we get

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{n!}{(n-m)!} - \frac{i!}{(i-m)!}}{\frac{n!}{(n-k)!} - \frac{i!}{(i-k)!}} &= \lim_{n\to\infty} \frac{\frac{i!}{(n-m)!}\left(\frac{n!}{i!} - \frac{(n-m)!}{(i-m)!}\right)}{\frac{i!}{(n-k)!}\left(\frac{n!}{i!} - \frac{(n-k)!}{(i-k)!}\right)} \\
&= \lim_{n\to\infty} \frac{(n-k)!}{(n-m)!} \cdot \frac{\left(\frac{n!}{i!} - \frac{(n-m)!}{(i-m)!}\right)}{\left(\frac{n!}{i!} - \frac{(n-k)!}{(i-k)!}\right)} \\
&= 0.
\end{aligned}$$

Therefore $\lambda_n - \lambda_i \ne 0$. Thus, as $n \to \infty$, every diagonal element of M becomes nonzero so M is invertible and the system$MX = Y$ has a unique solution. $\square$

3 Probability measures whose Cauchy transform satisfies an algebraic equation

In this section we will prove the uniqueness part of Theorem 2 and show that the measure μ_{Q_k}, if it exists, has the properties stated in Theorem 3. The proof relies heavily on the following lemma.

Lemma 3 *Let $A \subset \mathbf{C}$ be a finite set, $U \subset \mathbf{C}$ a convex domain and $\chi : U \to A$ a measurable function such that $\partial\chi/\partial\bar{z} \geq 0$ (in the sense of distributions). Let $a \in A$, $z_0 \in U$ and assume that $\chi^{-1}(a) \cap \{|z - z_0| < r\}$ has positive Lebesgue measure for every $r > 0$. Then $\chi(z) = a$ almost everywhere in $U \cap (z_0 + \Gamma_a)$ where*

$$\Gamma_a = \{z \in \mathbf{C}; \mathrm{Re}(az) \geq \mathrm{Re}(bz), \forall b \in A\}. \tag{6}$$

Note that if $\chi^{-1}(a) \cap \{|z - z_0| < r\}$ has positive Lebesgue measure for every $a \in A$ and all $r > 0$, then χ is determined completely (outside a set of measure 0) since the cones Γ_a cover the whole complex plane.

Proof. Let χ_a denote the characteristic function of the set $\chi^{-1}(a)$. We will show that if $z_1, z_2 \in U$ with $z_2 - z_1 \in \Gamma_a$, and ϕ is a positive test function such that $z_1 + \operatorname{supp}\phi$ and $z_2 + \operatorname{supp}\phi$ are both contained in U, then

$$(\phi * \chi_a)(z_1) \leq (\phi * \chi_a)(z_2). \tag{7}$$

The desired conclusion follows from this. Indeed, let ϕ_j be a sequence of positive test functions such that $\operatorname{supp}\phi_j \to 0$ and $\int \phi_j \, d\lambda = 1$, where λ denotes planar Lebesgue measure. We know then that $\phi_j * \chi_a$ converges in L^1_{loc} to χ_a. Hence, for any $\epsilon, r > 0$ we can find for all sufficiently large j a point z_1 with $|z_1 - z_0| < r$ such that $(\phi_j * \chi_a)(z_1) > 1 - \epsilon$. It follows from (7) that $(\phi_j * \chi_a)(z_2) > 1 - \epsilon$ and hence

$$|(\phi_j * \chi)(z_2) - a| = \left| \int \phi_j(z_2 - \zeta)(\chi(\zeta) - a) \, d\lambda(\zeta) \right| < \epsilon \max_{b \in A} |b - a|$$

for all $z_2 \in z_1 + \Gamma_a$. Letting ϵ and r tend to 0 and $j \to \infty$ it follows that $\chi(z) = \lim_{j\to\infty}(\phi_j * \chi)(z) = a$ for almost all z in $z_0 + \Gamma_a$.

We now prove the inequality (7). Without loss of generality we may assume that $z_2 - z_1 > 0$ and that $a = 0$, for the general case can be reduced to this situation by replacing χ with the function $e^{i\theta}(\chi(e^{i\theta}z)-a)$ where $\theta = \arg(z_2-z_1)$. The assumption that $z_2-z_1 \in \Gamma_a$ then implies that A is contained in the closed left half plane $\{\mathrm{Re}\, z \le 0\}$.

For any $\epsilon > 0$, let $\tilde{\chi}_\epsilon = \log(\chi - \epsilon)$ where we have chosen a branch of the logarithm function which is continuous in the left half plane. Let ψ be a positive test function and note that $\partial(\psi * \chi)/\partial\bar{z} \ge 0$ and $\mathrm{Re}\, \psi * \chi \le 0$. It follows that

$$\mathrm{Re}\, \frac{\partial}{\partial \bar{z}} \log(\psi * \chi - \epsilon) = \mathrm{Re}\left(\frac{1}{\psi * \chi - \epsilon} \cdot \frac{\partial(\psi * \chi)}{\partial \bar{z}} \right) \le 0.$$

When $\mathrm{supp}\, \psi \to 0$ with $\int \psi \, d\lambda = 1$, we have that $\log(\psi * \chi - \epsilon) \to \tilde{\chi}_\epsilon$ in L^1_{loc}, and hence as a distribution. By passing to the limit it follows that

$$\mathrm{Re}\, \frac{\partial \tilde{\chi}_\epsilon}{\partial \bar{z}} \le 0.$$

If we write $\tilde{\chi}_\epsilon = \sigma_\epsilon + i\tau_\epsilon$, this means that

$$\frac{\partial \sigma_\epsilon}{\partial x} \le \frac{\partial \tau_\epsilon}{\partial y}. \tag{8}$$

Fix a positive test function ϕ such that $z_j + \mathrm{supp}\, \phi \subset U$ for $j = 1, 2$ and consider the function $(\phi * \sigma_\epsilon)(z_1 + \xi)$ of the real variable ξ. It follows from (8) and the fact that τ_ϵ is uniformly bounded for all ϵ that

$$\begin{aligned} \frac{\partial}{\partial \xi}(\phi * \sigma_\epsilon)(z_1 + \xi) &= \int \frac{\partial \phi}{\partial x}(z_1 + \xi - \zeta)\sigma_\epsilon(\zeta)\, d\lambda(\zeta) \\ &\le \int \frac{\partial \phi}{\partial y}(z_1 + \xi - \zeta)\tau_\epsilon(\zeta)\, d\lambda(\zeta) \\ &\le M \end{aligned}$$

where the constant M does not depend on ϵ. In particular,

$$(\phi * \sigma_\epsilon)(z_2) - (\phi * \sigma_\epsilon)(z_1) \le M|z_2 - z_1|. \tag{9}$$

On the other hand it is clear that

$$(\phi * \sigma_\epsilon)(z) = \log \epsilon \cdot (\phi * \chi_a)(z) + O(1). \tag{10}$$

Now (7) follows from (9) and (10) when $\epsilon \to 0$. $\square$

We deduce two corollaries of Lemma 3.

Corollary 1 *Let $U \subset \mathbf{C}$ be a convex domain and $A \subset \mathbf{C}$ a finite set. If v is a subharmonic function defined in U such that $2\partial v/\partial z \in A$ almost everywhere, then v is convex.*

Recall that a subharmonic function can locally be written as the sum of a harmonic function and a logarithmic potential. It follows that the distribution $\partial v/\partial z$ is represented by a locally integrable function. The condition $2\partial v/\partial z \in A$ should be interpreted by saying that $2\partial v/\partial z$ is represented by a measurable function with values in A.

Proof. Let $\chi = 2\partial v/\partial z$. Since v is subharmonic, $\partial\chi/\partial\bar{z} \geq 0$. Take any point $z_0 \in U$ and let A_0 be the set of all $a \in A$ such that $\chi^{-1}(a)$ has positive measure in every neighbourhood of z_0. Let U_0 be a convex neighbourhood of z_0 such that $\chi(z) \in A_0$ almost everywhere in U_0. By Lemma 3, $\chi(z) = a$ almost everywhere in $U_0 \cap (z_0 + \Gamma_a)$ where Γ_a is defined by (6) but with A_0 in place of A. This implies that $v(z) = v(z_0) + \operatorname{Re} a(z - z_0)$ for all $z \in U_0 \cap (z_0 + \Gamma_a)$, so that

$$v(z) = v(z_0) + \max_{a \in A_0} \operatorname{Re} a(z - z_0), \quad z \in U_0.$$

We have shown that in a neighbourhood of z_0, v is the maximum of certain linear functions, hence it is convex there. Since z_0 was arbitrary, it follows that v is convex. $\square$

Corollary 2 *Let $A \subset \mathbf{C}$ be a finite set, $U \subset \mathbf{C}$ a convex domain and let $\chi : U \to A$ be a measurable function. Then $\partial\chi/\partial\bar{z} \geq 0$ if and only if there exist real numbers c_a (possibly equal to $-\infty$) such that $\chi(z) = a$ almost everywhere in G_a where*

$$G_a = \{z \in U; c_a + \operatorname{Re}(az) \geq c_b + \operatorname{Re}(bz), \forall b \in A\}.$$

Proof. Suppose c_a are real numbers such that $\chi(z) = a$ almost everywhere in G_a. Let $v(z) = \max_{a \in A} (c_a + \operatorname{Re}(az))$. Then v is subharmonic and $\chi = 2\partial v/\partial z$, hence

$$\frac{\partial\chi}{\partial\bar{z}} = 2\frac{\partial^2 v}{\partial z \partial\bar{z}} \geq 0.$$

Suppose conversely that $\partial\chi/\partial\bar{z} \geq 0$. Since $\partial\chi/\partial\bar{z}$ is real, there exists a real valued function v defined in U with $2\partial v/\partial z = \chi$. It follows from Corollary 1 that v is convex. Moreover, we see from the proof that

$$v(z) = \max_{a \in A} (c_a + \mathrm{Re}(az))$$

where

$$c_a = \inf_{z \in U} (v(z) - \mathrm{Re}(az)).$$

If we define G_a using these constants c_a it follows that $v(z) = c_a + \mathrm{Re}(az)$ for $z \in G_a$, hence $\chi(z) = 2\partial v/\partial z = a$ in G_a. $\square$

Fix a monic polynomial Q_k of degree k and suppose that μ is a compactly supported probability measure whose Cauchy transform $C(z)$ satisfies

$$C(z)^k = 1/Q_k(z). \tag{11}$$

We will first show that μ has the properties asserted in Theorem 3, except that $\operatorname{supp} \mu_{Q_k}$ is contained in the convex hull of the zeros of Q_k, which will be proved in section 5.

Lemma 4 *If the Cauchy transform of μ satisfies* (11)*, then the support of μ is the union of finitely many smooth curve segments. These curves are mapped to lines by Ψ.*

Proof. It is sufficient to prove that $\operatorname{supp} \mu$ has these properties in a neighbourhood of any given point z_0. Assume first that $Q_k(z_0) \neq 0$. Choose a branch of $Q_k(z)^{-1/k}$ defined in a simply connected neighbourhood of z_0 and let Ψ be a primitive function of $Q_k(z)^{-1/k}$. Let U be a convex neighbourhood of $\Psi(z_0)$ so small that Ψ maps a neighbourhood of z_0 bijectively onto U. By (11) we can write $C(z) = \chi(\Psi(z))Q_k(z)^{-1/k}$ for $z \in \Psi^{-1}(U)$, where χ has values in the set of kth roots of unity. If we write $w = \Psi(z)$, then

$$\begin{aligned} \pi\mu &= \frac{\partial C}{\partial \bar{z}} = \frac{\partial \chi(\Psi(z))}{\partial \bar{z}} \cdot Q_k^{-1/k} = \Psi^* \left(\frac{\partial \chi}{\partial \bar{w}} \right) \cdot \overline{\frac{\partial \Psi}{\partial z}} \cdot Q_k^{-1/k} \\ &= \Psi^* \left(\frac{\partial \chi}{\partial \bar{w}} \right) \cdot |Q_k|^{-2/k} \end{aligned}$$

where Ψ^* denotes the pullback of distributions in U by Ψ. Since μ is positive, it follows that

$$\frac{\partial \chi}{\partial \bar{w}} \geq 0.$$

By Corollary 2, U is the union of sets G_a whose boundaries are finite unions of line segments, such that χ is constant in each G_a. It follows that $\operatorname{supp}\mu \cap \Psi^{-1}(U) = \Psi^{-1}(\operatorname{supp}\partial\chi/\partial\bar{z})$ is the union of finitely many curve segments which are mapped to straight lines by Ψ.

If z_0 is a zero of Q_k, we take a disc D centered at z_0 wich does not contain any other zeros of Q_k. If γ is any ray emanating at z_0, we can define single valued branches of $Q(z)^{-1/k}$ and Ψ in $D \setminus \gamma$. Notice that Ψ is continuous up to z_0. Let U be any half disc centered at $\Psi(z_0)$ and contained in $\Psi(D \setminus \gamma)$. It follows as in the first part of the proof that $\operatorname{supp}\mu$ has the required properties in $\Psi^{-1}(U)$. By varying γ and U, we see that the same holds in a full neighbourhood of z_0. $\square$

Hence $\operatorname{supp}\mu$ can be thought of as a graph whose edges are smooth curve segments connecting certain vertices. The statement that $\operatorname{supp}\mu$ is connected and has connected complement then means that it is a connected graph without cycles, that is a tree. Recall that a connected graph is a tree precisely if the number of vertices exceeds the number of edges by exactly one.

Lemma 5 *If the Cauchy transform of μ satisfies* (11)*, then the support of μ is a tree.*

Proof. We will first prove that $\operatorname{supp}\mu$ is connected. To do this we will show that if U is a bounded domain which is connected and simply connected, and the boundary of U does not intersect $\operatorname{supp}\mu$, then either $\operatorname{supp}\mu \subset U$ or $\operatorname{supp}\mu \subset \mathbf{C} \setminus U$. From this it easily follows that $\operatorname{supp}\mu$ is connected. Now it is clear that all the zeros of Q_k are either contained in U or in the complement of U, since $C(z)$ defines a continuous branch of $Q_k(z)^{-1/k}$ along ∂U. Observe also that

$$\frac{1}{2\pi i}\int_{\partial U} C(z)\,dz = \frac{1}{2\pi i}\int_{\mathbf{C}}\int_{\partial U}\frac{dz}{z-\zeta}\,d\mu(\zeta) = \int_U d\mu(\zeta). \qquad (12)$$

Now if all the zeros of Q_k are contained in the complement of U, there is an analytic continuation of $C(z)$ across U, hence the left hand side of (12) vanishes. It follows that $\operatorname{supp}\mu \subset \mathbf{C} \setminus U$. If on the other hand, all the zeros of Q_k are contained in U, then $C(z)$ has an analytic continuation in $\mathbf{C} \setminus U$ which is asymptotically equal to a/z for some kth root of unity a when $z \to \infty$. Thus the left hand side of (12) is equal to a. Since the right hand side is positive, a must be 1, which means that all the mass of μ is in U. Hence we have proved that $\operatorname{supp} \mu$ is connected.

Now let E be the set of all curve segments in $\operatorname{supp} \mu$ and let V be the set of vertices which are endpoints of the edges in E. We may assume that V contains all the zeros of Q_k. To every pair $e \in E$, $v \in V$ such that v is an endpoint of e, we assign a number $\nu(e, v)$ by the following rule. Let γ be a small loop winding once around v in the clockwise direction, and let $\nu(e, v)$ be the jump of $(2\pi i)^{-1} \log C(z)$ when z crosses e moving along γ. This number, which is defined modulo $\mathbf{Z}$, will be uniquely determined if we require that $0 < \nu(e, v) < 1$. Assume now that v is not a zero of Q_k and let $e_1, \ldots, e_r$ be the curves in E having v as one endpoint. (If some curve has both its endpoints in v, it will be counted twice.) Select a branch of $Q_k(z)^{1/k}$ near v and observe that by Lemma 3 and the proof of Lemma 4, $Q_k(z)^{1/k} C(z)$ is a kth root of unity, which moves once around the unit circle in the counterclockwise direction as z moves along γ. It follows that $\nu(e_1, v) + \cdots + \nu(e_r, v) = 1$. If instead v is a zero of Q_k of multiplicity m, a slight modification of the argument shows that $\nu(e_1, v) + \cdots + \nu(e_r, v) = 1 - m/k$. On the other hand, it is clear that $\nu(e, v_1) + \nu(e, v_2) = 1$ where v_1, v_2 are the endpoints of $e \in E$. Hence the sum of all the $\nu(e, v)$ is equal both to $\sharp V - 1$ and to $\sharp E$. Since $\operatorname{supp} \mu$ is a connected graph, this implies that it is a tree. $\square$

We are now ready to prove the uniqueness part of Theorem 2. This is done by means of the following two lemmas.

Lemma 6 *Suppose the Cauchy transform of μ satisfies* (11) *and let u be the logarithmic potential of μ. If Ψ^{-1} is a (locally defined) inverse of a primitive function of $Q_k(z)^{-1/k}$, then $u \circ \Psi^{-1}$ is convex.*

Proof. Let χ be as in the proof of Lemma 4. Since $2\partial u/\partial z = C(z)$ we have

$$\begin{aligned} 2\frac{\partial}{\partial w} u(\Psi^{-1}(w)) &= 2\frac{\partial u}{\partial z}(\Psi^{-1}(w)) \cdot Q_k(\Psi^{-1}(w))^{1/k} \\ &= C(\Psi^{-1}(w)) \cdot Q_k(\Psi^{-1}(w))^{1/k} \\ &= \chi(w). \end{aligned}$$

It follows from Corollary 1, that $u \circ \Psi^{-1}$ is convex. $\square$

Lemma 7 *Let μ be a measure whose Cauchy transform satisfies* (11), *let $\Omega = \mathbf{C} \setminus \operatorname{supp} \mu$ and let $\Psi(z)$ be defined in Ω by*

$$\Psi(z) = \int \log(z - \zeta)\, d\mu(\zeta).$$

Then Ψ is a multivalued function mapping Ω onto a domain $H = \{w; \operatorname{Re} w > h(\operatorname{Im} w)\}$ where h is a continuous function, and $\Psi^{-1} : H \to \Omega$ is a single valued function.

Proof. It is clear that Ψ is a holomorphic function in Ω defined up to multiples of $2\pi i$ and that $\Psi'(z) = C(z)$. Let γ be a curve segment of $\operatorname{supp} \mu$ and let U be a one-sided neighbourhood of γ in Ω on which Ψ has a single valued branch. Now the restriction of Ψ to U has an analytic continuation across γ, and by Lemma 4, Ψ maps γ to a line segment. Moreover, since in the notation of the proof of Lemma 4, $\chi = 1$ in $\Psi(U)$ and $\operatorname{Re} \chi \leq 1$ everywhere, it follows that $\Psi(\gamma)$ is not horizontal and that $\Psi(U)$ is on the right hand side of $\Psi(\gamma)$. Putting the segments $\Psi(\gamma)$ together as U moves around $\operatorname{supp} \mu$, we obtain a broken line of the form $\{\operatorname{Re} w = h(\operatorname{Im} w)\}$ bounding a domain $H = \{\operatorname{Re} w > h(\operatorname{Im} w)\}$. It is clear that Ψ maps Ω into H and the boundary of Ω to the boundary of H. Now $\psi(z) = \exp(-\Psi(z))$ is a single valued proper mapping from $\Omega \cup \{\infty\}$ to $D = \{\zeta; \log|\zeta| < -h(-\arg \zeta)\}$ which does not vanish in Ω and has a simple zero at ∞. It follows that $\psi : \Omega \cup \{\infty\} \to D$ is a bijection, hence $\Psi^{-1}(w) = \psi^{-1}(e^{-w})$ is a single valued holomorphic mapping. $\square$

Corollary 3 *If μ_1 and μ_2 are two probability measures whose Cauchy transforms satisfy* (11), *then $\mu_1 = \mu_2$.*

Proof. Let Ψ be defined as in Lemma 7 with μ_1 in place of μ, and let u_1 and u_2 be the logarithmic potentials of μ_1 and μ_2. Then $u_1(\Psi^{-1}(w)) = \operatorname{Re} w$ for all $w \in H$ and $u_2(\Psi^{-1}(w)) = \operatorname{Re} w$ when $\operatorname{Re} w$ is sufficiently large. Since $u_2 \circ \Psi^{-1}$ is convex by Lemma 6, it follows that $u_2(\Psi^{-1}(w)) \geq \operatorname{Re} w$ for all $w \in H$, hence $u_1(z) \leq u_2(z)$ for almost all z. Similarly, $u_2(z) \leq u_1(z)$ for almost all z, and it follows that $\mu_1 = \Delta u_1/2\pi = \Delta u_2/2\pi = \mu_2$. $\square$

4 Root measures and the Cauchy transform

In this section we describe the basic connections between root measures and the Cauchy transform which will be used to prove Theorem 4.

Let μ_n be a sequence of measures in the complex plane. The sequence is said to converge weakly to a measure μ if

$$\int \phi(z)\, d\mu_n(z) \to \int \phi(z)\, d\mu(z)$$

for every continuous function ϕ with compact support. If in addition there exists a compact set K such that $\operatorname{supp} \mu_n \subset K$ for every n, we will say that μ_n converges weakly with compact support to μ and write $\mu_n \to \mu$ (w.c.s.).

If $K \subset \mathbf{C}$ is a compact set and $M(K)$ denotes the space of all probability measures with support in K, equipped with the weak topology, it is known that $M(K)$ is a sequentially compact Hausdorff space. We will use this fact to select a convergent subsequence from a sequence of measures as a first step in the proof of Theorem 4.

If ϕ is a locally integrable function and μ is a compactly supported measure, the convolution

$$(\phi * \mu)(z) = \int \phi(z-\zeta)\, d\mu(\zeta)$$

is a locally integrable function defined almost everywhere in the complex plane. If $\mu_n \to \mu$ (w.c.s.), it is easy to show that $\phi * \mu_n \to \phi * \mu$ in L^1_{loc}.

We will be particularly interested in the cases where $\phi(z) = \log|z|$ or $\phi(z) = 1/z$. Convolution with these functions defines the logarithmic potential

$$u(z) = \int \log|z-\zeta|\, d\mu(\zeta)$$

and the Cauchy transform

$$C(z) = \int \frac{d\mu(\zeta)}{z-\zeta}$$

of μ. It is well known that the measure μ can be reconstructed from either u or C by the formula

$$\mu = \frac{1}{2\pi} \cdot \Delta u = \frac{1}{\pi} \cdot \frac{\partial C}{\partial \bar{z}}$$

where $\Delta = (\partial/\partial x)^2 + (\partial/\partial y)^2$ is the Laplace operator and $\partial/\partial \bar{z} = (\partial/\partial x + i\partial/\partial y)/2$. These identities should be understood in the sense of distribution theory.

Let p be a polynomial of degree n and let μ be the root measure of p, as defined in the introduction. If p is monic, the logarithmic potential of μ is given by

$$\frac{1}{n} \log|p(z)| = \int \log|z-\zeta|\, d\mu(\zeta), \tag{13}$$

and for any p, the Cauchy transform of μ is

$$\frac{p'(z)}{np(z)} = \int \frac{d\mu(\zeta)}{z-\zeta}. \tag{14}$$

These two identities, which can easily be verified, are among the main ingredients in the proof of Theorem 4. We will here use them to prove a general lemma which will be needed later.

Lemma 8 *Let p_m be a sequence of polynomials, such that $n_m := \deg p_m \to \infty$ and let μ_m and μ'_m be the root measures of p_m and p'_m respectively. If $\mu_m \to \mu$, $\mu'_m \to \mu'$ (w.c.s.) and u and u' are the logarithmic potentials of μ and μ', then $u' \leq u$ with equality in the unbounded component of $\mathbf{C} \setminus \operatorname{supp} \mu$.*

Proof. Assume with no loss of generality that the p_m are monic. Let K be a compact set containing the zeros of every p_m. By (13) we then have

$$u(z) = \lim_{m\to\infty} \frac{1}{n_m} \log|p_m(z)|$$

and

$$u'(z) = \lim_{m\to\infty} \frac{1}{n_m - 1} \log \left| \frac{p'_m(z)}{n_m} \right| = \lim_{m\to\infty} \frac{1}{n_m} \log \left| \frac{p'_m(z)}{n_m} \right|$$

with convergence in L^1_{loc}. Hence by (14),

$$\begin{aligned} u'(z) - u(z) &= \lim_{m\to\infty} \frac{1}{n_m} \log \left| \frac{p'_m(z)}{n_m p_m(z)} \right| \\ &= \lim_{m\to\infty} \frac{1}{n_m} \log \left| \int \frac{d\mu_m(\zeta)}{z-\zeta} \right|. \end{aligned} \tag{15}$$

Now, if ϕ is a positive test function it follows that

$$\begin{aligned} \int \phi(z)(u'(z) - u(z))\, d\lambda(z) &= \lim_{m\to\infty} \frac{1}{n_m} \int \phi(z) \log \left| \int \frac{d\mu_m(\zeta)}{z-\zeta} \right| d\lambda(z) \\ &\leq \lim_{m\to\infty} \frac{1}{n_m} \int \phi(z) \int \frac{d\mu_m(\zeta)}{|z-\zeta|}\, d\lambda(z) \\ &= \lim_{m\to\infty} \frac{1}{n_m} \iint \frac{\phi(z)\, d\lambda(z)}{|z-\zeta|}\, d\mu_m(\zeta) \end{aligned} \tag{16}$$

where λ denotes Lebesgue measure in the complex plane. Since $1/|z|$ is locally integrable, the function $\int \phi(z)|z-\zeta|^{-1}\, d\lambda(z)$ is continuous, and hence bounded by a constant M for all z in K. Since $\operatorname{supp} \mu_m \subset K$, the last expression in (16) is bounded by M/n_m, hence the limit when $m \to \infty$ is 0. This proves that $u' \leq u$.

In the complement of $\operatorname{supp} \mu$, u is harmonic and u' is subharmonic, hence $u' - u$ is a negative subharmonic function. Moreover, in the complement of K, $p'_m/(n_m p_m)$ converges uniformly on compact sets to the Cauchy transform $C(z)$ of μ. Since $C(z)$ is a nonconstant holomorphic function in the unbounded component of $\mathbf{C} \setminus K$, it follows from (15) that $u' - u = 0$ there. By the maximum principle for subharmonic functions it follows then that $u' - u = 0$ in the unbounded component of $\mathbf{C} \setminus \operatorname{supp} \mu$. The proof is complete. □

5 Root measures of eigenpolynomials

We now turn to the proof of Theorem 4. The plan is to show that μ_n converges to a measure whose Cauchy transform satisfies (11). This will prove Theorem 4 and the existence part of Theorem 2. Let μ_n be the root measure of p_n as in the statement of Theorem 4. Also let $\mu_n^{(i)}$ be the root measure of the ith derivative $p_n^{(i)}$. We begin by showing that there is a compact set K containing the supports of all the measures $\mu_n^{(i)}$.

Lemma 9 *Let $Q_0, ..., Q_k$ be fixed and let p_n be an eigenpolynomial of degree n of the operator T_Q. Then there exists a compact set K such that all the zeros of every $p_n^{(i)}$ lie in K for every n and every $i \geq 0$. If $Q_0 = \ldots = Q_{k-1} = 0$, K may be taken as the convex hull of the zeros of Q_k.*

Proof. The case with $Q_0 = \cdots = Q_{k-1} = 0$ was treated in [4]. In the general case it suffices to check the roots of p_n, since by Gauss-Lucas' theorem the roots of any derivative $p_n^{(i)}$ are contained in the convex hull of the roots of p_n. Moreover it suffices to show that there exists a compact set containing the zeros of p_n for large values of n, since for any finite value of n there are finitely many roots of the polynomial p_n, and these are clearly contained in some compact set.

Let z be a root of p_n. Then

$$T_Q(p_n)(z) = \sum_{i=0}^{k} Q_i(z) \cdot p_n^{(i)}(z) = \lambda_n \cdot p_n(z) = 0$$

or, equivalently,

$$Q_k(z) \cdot p_n^{(k)}(z) + Q_{k-1}(z) \cdot p_n^{(k-1)}(z) + \ldots + Q_1(z) \cdot p_n^{(1)}(z) = 0. \quad (17)$$

We will show that for sufficiently large choices of $|z|$ and n this equation cannot not hold. It is possible to find some r_0 and some n_0 such that if $|z| \geq r_0$ and $n > n_0$, then z cannot be a root of p_n. Using formula (14) we have

$$\frac{p_n^{(i+1)}(z)}{(n-i) \cdot p_n^{(i)}(z)} = \int \frac{d\mu_n^{(i)}(\zeta)}{z - \zeta} =: b_i.$$

Thus

$$p_n^{(k-1)}(z) = \frac{p_n^{(k)}(z)}{(n-k+1) \cdot b_{k-1}},$$

$$p_n^{(k-2)}(z) = \frac{p_n^{(k-1)}(z)}{(n-k+2)\cdot b_{k-2}} = \frac{p_n^{(k)}(z)}{(n-k+1)(n-k+2)\cdot b_{k-1}\cdot b_{k-2}},$$

and so on. Generally we have

$$p_n^{(i)}(z) = \frac{p_n^{(k)}(z)}{(n-k+1)\dots(n-i)\cdot\prod_{j=i}^{k-1} b_j}.$$

Now assume that z is the root of p_n with the largest modulus and let $|z| = r$. With ζ being a root of some $p_n^{(i)}$ we have $|\zeta| \le |z|$ by Gauss-Lucas' theorem. We will estimate $b_i = \int \frac{d\mu_n^{(i)}(\zeta)}{z-\zeta}$ so that $|b_i| \ge 1/2r \quad \forall i \le k$. We have

$$\frac{1}{z-\zeta} = \frac{1}{z}\cdot\frac{1}{1-\zeta/z} = \frac{1}{z}\cdot\frac{1}{1-\theta}$$

and $|\theta| = |\zeta/z| \le 1$.
With $w = 1/(1-\theta)$ we obtain

$$|w-1| = \left|\frac{1}{1-\theta} - \frac{(1-\theta)}{(1-\theta)}\right| = \frac{|\theta|}{|1-\theta|} = |\theta||w| \le |w|$$
$$\Leftrightarrow$$
$$|w-1| \le |w|$$
$$\Leftrightarrow$$
$$\mathrm{Re}(w) \ge \frac{1}{2}.$$

Using this result we get

$$\begin{aligned} |b_i| &= \left|\int \frac{d\mu_n^{(i)}(\zeta)}{z-\zeta}\right| = \frac{1}{r}\left|\int \frac{d\mu_n^{(i)}(\zeta)}{1-\theta}\right| = \\ &= \frac{1}{r}\left|\int w\,d\mu_n^{(i)}(\zeta)\right| \ge \frac{1}{r}\left|\int \mathrm{Re}(w)\,d\mu_n^{(i)}(\zeta)\right| \ge \\ &\ge \frac{1}{2r}\int d\mu_n^{(i)}(\zeta) = \frac{1}{2r}. \end{aligned}$$

Now we choose r_0 in such a way that $|Q_k(w)| \ge r^k/2$ as $|w| \ge r_0$ and then a constant C such that $|Q_i(w)| \le C\cdot r^i$ for every $i = 1,\dots,k-1$. Finally we choose n_0 such that $\frac{C\cdot 2^{k-i+1}}{(n-i)\dots(n-k+1)} < \frac{1}{k-1}$ as $n > n_0$ for every $i = 1, \dots, k-1$. Then, as $|z| = r \geq r_0$ and $n > n_0$, we get

$$\begin{aligned}
\left| \frac{Q_i(z) \cdot p_n^{(i)}(z)}{Q_k(z) \cdot p_n^{(k)}(z)} \right| &= \frac{|Q_i(z)|}{|Q_k(z)|} \cdot \frac{(n-k)!}{(n-i)!} \cdot \frac{1}{\prod_{j=i}^{k-1} |b_j|} \leq \\
&\leq \frac{|Q_i(z)|}{|Q_k(z)|} \cdot \frac{(n-k)!}{(n-i)!} \cdot 2^{k-i} \cdot r^{k-i} \leq \\
&\leq \frac{C \cdot r^i}{r^k/2} \cdot \frac{(n-k)!}{(n-i)!} \cdot 2^{k-i} \cdot r^{k-i} = \\
&= \frac{C \cdot 2^{k-i+1}}{(n-i)\dots(n-k+1)} < \frac{1}{k-1}.
\end{aligned}$$

Dividing (17) by $Q_k(z) \cdot p_n^{(k)}(z)$ we obtain

$$1 + \sum_{i=1}^{k-1} \frac{Q_i(z) \cdot p_n^{(i)}(z)}{Q_k(z) \cdot p_n^{(k)}(z)} = 0,$$

but with $r \geq r_0$ and $n > n_0$ we get

$$\left| \sum_{i=1}^{k-1} \frac{Q_i(z) \cdot p_n^{(i)}(z)}{Q_k(z) \cdot p_n^{(k)}(z)} \right| \leq \sum_{i=1}^{k-1} \left| \frac{Q_i(z) \cdot p_n^{(i)}(z)}{Q_k(z) \cdot p_n^{(k)}(z)} \right| < \sum_{i=1}^{k-1} \frac{1}{k-1} = 1$$

and so (17) cannot be fulfilled with such choices of r and n. □

Assume that N is a subsequence of the natural numbers such that

$$\mu^{(j)} = \lim_{n \to \infty, n \in N} \mu_n^{(j)} \tag{18}$$

exists for $j = 0, \dots, k$. The following lemma shows that the Cauchy transform of $\mu = \mu^{(0)}$ satisfies (11).

Lemma 10 *The measures $\mu^{(j)}$ are all equal and the Cauchy transform $C(z)$ of this common limit satisfies $C(z)^k = 1/Q_k(z)$ for almost every z.*

Proof. By (14) we have that

$$\frac{p_n^{(j+1)}(z)}{(n-j)p_n^{(j)}(z)} \to \int \frac{d\mu^{(j)}(\zeta)}{z - \zeta} \tag{19}$$

with convergence in L^1_{loc}, and by passing to a subsequence once again we can assume that we have pointwise convergence almost everywhere. From the relation $T_Q p_n = \lambda_n p_n$ it follows that

$$Q_k \frac{p_n^{(k)}}{n\ldots(n-k+1)p_n} = \frac{\lambda_n}{n\ldots(n-k+1)} - \sum_{l=0}^{k-1} \frac{Q_l}{(n-l)\ldots(n-k+1)} \prod_{j=0}^{l-1} \frac{p_n^{(j+1)}}{(n-j)p_n^{(j)}}. \qquad (20)$$

Now $\lambda_n/n\ldots(n-k+1) \to 1$ by Theorem 1, while the sum converges pointwise to 0 almost everywhere by virtue of the factors $(n-l)\ldots(n-k+1)$ in the denominators. It follows that

$$\frac{p_n^{(k)}(z)}{n\ldots(n-k+1)p_n(z)} \to \frac{1}{Q_k(z)} \qquad (21)$$

when $n \to \infty$ through the sequence N for almost every z. If $u^{(j)}$ denotes the logarithmic potential of $\mu^{(j)}$, then it follows from (13) and (21) that

$$u^{(k)} - u^{(0)} = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{p_n^{(k)}}{n\ldots(n-k+1)p_n} \right| = -\lim_{n\to\infty} \frac{1}{n} \log |Q_k| = 0.$$

On the other hand we have from Lemma 8 that $u^{(0)} \geq u^{(1)} \geq \cdots \geq u^{(k)}$, hence the potentials $u^{(j)}$ are all equal, and it follows that $\mu^{(j)} = \Delta u^{(j)}/2\pi$ are all equal. Finally we have from (19) and (21) that

$$C(z)^k = \lim_{n\to\infty} \prod_{j=0}^{k-1} \frac{p_n^{(j+1)}(z)}{(n-j)p_n^{(j)}(z)} = \lim_{n\to\infty} \frac{p_n^{(k)}(z)}{n\ldots(n-k+1)p_n(z)} = \frac{1}{Q_k(z)}$$

for almost every z. This completes the proof. □

Corollary 4 *There exists a unique measure μ_{Q_k} satisfying the requirements in Theorem 2. The sequence μ_n converges weakly to μ_{Q_k}. Moreover,* $\operatorname{supp} \mu_{Q_k}$ *is contained in the convex hull of the zeros of Q_k.*

Proof. By Theorem 1, the operator T_Q has an eigenpolynomial p_n of degree n for all sufficiently large n. By Lemma 9, there exists a compact set K such that $\operatorname{supp} \mu_n^{(j)} \subset K$ for all n. By compactness, there exists a subsequence N such that the limit (18) exists for $j = 0, \ldots, k$. By Lemma 10, $\mu_{Q_k} = \mu^{(0)}$ has the required properties, so existence is proved. Uniqueness was established in section 3. Since we may take $Q_0 = \ldots = Q_{k-1} = 0$, and in this case $\operatorname{supp} \mu_n^{(j)} \subset K$ where K is the convex hull of the zeros of Q_k by Lemma 9, it follows that $\operatorname{supp} \mu_{Q_k}$ is also contained in K.

Assume that μ_n does not converge to μ_{Q_k} Then we can find a subsequence N' of the natural numbers such that μ_n stays away from a fixed neighbourhood of μ_{Q_k} in the weak topology, for all $n \in N'$. Again by compactness, we can find a subsequence N of N' such that the limit (18) exists for $j = 0, \ldots, k$. By Lemma 10 and the uniqueness of μ_{Q_k}, it follows that $\mu^{(0)} = \mu_{Q_k}$, contradicting the assumption that μ_n stays away from μ_{Q_k} for all n in N' and hence all n in N. The proof is complete. $\square$

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PAPER II

On Bochner-Krall Orthogonal Polynomial Systems

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Abstract

In this paper we address the classical question going back to S. Bochner and H. L. Krall to describe all systems $\{p_n(x)\}_{n=0}^{\infty}$ of orthogonal polynomials (OPS) which are the eigenfunctions of some finite order differential operator. Such systems of orthogonal polynomials are called *Bochner-Krall OPS* (or BKS for short) and their spectral differential operators are accordingly called *Bochner-Krall operators* (or BK-operators for short). We show that the leading coefficient of a Nevai type BK-operator is of the form $((x-a)(x-b))^{N/2}$. This settles the special case of the general conjecture 7.3. of [4] describing the leading terms of all BK-operators.

1 Summary

Consider a sequence of polynomials $\{p_n\}_{n=0}^{\infty}$ in a variable x, where $\deg p_n = n$. This sequence is orthogonal with respect to a measure μ if $\int p_n(x)p_m(x)d\mu(x)$ is nonzero precisely when $n = m$. We are here concerned with polynomials orthogonal with respect to a measure of the so-called Nevai class, see [11] and below. Furthermore we say that $\{p_n\}$ is a sequence of eigenpolynomials if there exists a differential operator $\mathfrak{d} = \sum_{k=1}^{N} a_k(x)(d/dx)^k$ where a_k are polynomials in x. Finally, $\{p_n\}$ is called a Bochner-Krall system if it is both orthogonal and a system of eigenpolynomials. If it is orthogonal with respect to a measure of Nevai class, we say that it is a Bochner-Krall system of Nevai type.

It is an open problem to classify all Bochner-Krall systems. In [4] it is conjectured that the leading coefficient a_N for any Bochner-Krall system is a power of a polynomial of degree at most 2. Our main result is an affirmative answer to this conjecture for Bochner-Krall systems of Nevai type.

Main Theorem. *Let* $\{p_n\}$ *be a compact type* BKS *orthogonal with respect to a measure* μ *and with differential operator* $\mathfrak{d}$*. If* μ *is of the Nevai class and the convex hull of* $\operatorname{supp}\mu$ *is the interval* $[a,b]$*, then* N *is even and* $a_N(x)$ *is a constant multiple of* $((x-a)(x-b))^{N/2}$.

2 Introduction

Let $P_{\mathbb{R}}$ and $P_{\mathbb{C}}$ denote the spaces of all real and complex polynomials, respectively, in a variable x. By a real (or complex) *polynomial system* we will mean a sequence $\{p_n\}_{n=0}^{\infty}$ of polynomials in $P_{\mathbb{R}}$ (or $P_{\mathbb{C}}$) such that $\deg p_n = n$. By an orthogonal polynomial system (OPS) one understands a real polynomial system $\{p_n\}$ such that $\langle p_n, p_m\rangle$ is nonzero precisely when $n = m$, where $\langle\,,\,\rangle$ is some reasonable inner product on the linear space $P_{\mathbb{R}}$. If an orthogonal polynomial system for a given inner product exists, the p_n are unique, up to multiplication by scalars.

Orthogonal polynomial systems have been studied in various degrees of generality. Classically, one has considered inner products of the form

$$\langle p, q\rangle = \sigma(p \cdot q)$$

where σ is a moment functional, that is a linear functional on $P_{\mathbb{R}}$. It is known that all moment functionals can be represented by an integral

$$\sigma(p) = \int p(x)\, d\mu(x) \qquad (1)$$

where μ is a (possibly signed) Borel measure on the real line. The most complete theories have been obtained in the case where μ is positive with compact support, and moreover belongs to the so-called Nevai class. If the density of μ is a function ρ and $\log\rho$ is integrable on the smallest interval containing $\operatorname{supp}\mu$, then μ is of the Nevai class, but this condition is not necessary. See [11] for the precise definition of the Nevai class, denoted there by $M(a,b)$. In what follows we will mainly be concerned with orthogonal polynomial systems of this particular kind, which we will call orthogonal systems of *Nevai type*. Recently, there has been interest in more general inner products called Sobolev, which are of the form

$$\langle p, q\rangle = \sum_{k=0}^{M} \sigma_k(p^{(k)} \cdot q^{(k)})$$

where the σ_k are moment functionals. For the basics of the classical theory of orthogonal polynomials see e.g. [12] and [3].

Consider now a differential operator

$$\mathfrak{d} = \sum_{k=1}^{N} a_k(x) \frac{d^k}{dx^k} \tag{2}$$

where the coefficients $a_k(x)$ are polynomials in $P_{\mathbb{C}}$. We are interested in eigenpolynomials of this operator, that is polynomials $p \in P_{\mathbb{C}}$ satisfying $\mathfrak{d}p = \lambda p$ for some constant λ. Already S. Bochner observed that the operator $\mathfrak{d}$ has infinitely many linearly independent eigenpolynomials if and only if $\deg a_k \leq k$, with equality for at least one k. In this case there is precisely one monic degree n eigenpolynomial p_n for all sufficiently large n. For generic a_k, the same is true for every $n \geq 0$. If a complex polynomial system consists of eigenpolynomials for an operator of the form (2), we will call it a *system of eigenpolynomials*.

A Bochner-Krall system (BKS for short) is defined to be a real polynomial system which is both orthogonal (with respect to some inner product $\langle\, ,\, \rangle$) and a system of eigenpolynomials (for some differential operator $\mathfrak{d}$). In this case $\langle\, ,\, \rangle$ is called a Bochner-Krall inner product, and $\mathfrak{d}$ is called a Bochner-Krall operator. If a BKS is an orthogonal system of Nevai type, we will call it for short a *Nevai type BKS*. The results we report in this note are valid for all Nevai type BKS.

It is an open problem to classify all Bochner-Krall systems. A complete classification is only known for Bochner-Krall operators $\mathfrak{d}$ with $N \leq 4$. The corresponding BKS are various classical systems such as the Jacobi-type, the Laguerre-type, the Legendre-type, the Bessel and the Hermite polynomials, see [4], Th. 3.1. In general, it is not even known which differential operators are Bochner-Krall operators for some BKS. In [4] it is conjectured that the leading coefficient a_N of a Bochner-Krall operator is a power of a polynomial of degree at most 2. Our main result is an affirmative answer to this conjecture for Nevai type BKS.

Our results are obtained by studying the asymptotic distribution of zeros of a polynomial system. To make this more precise, let $\{p_n\}$ be a polynomial system and for fixed $n \geq 1$, let $\alpha_1, \ldots, \alpha_n$ denote the (real or complex) zeros of p_n. Let $\nu_n = \frac{1}{n}\sum_{i=1}^{n} \delta(x - \alpha_i)$ be the probability measure in the complex plane with point masses at these zeros. We call the measures ν_n the *root measures* of the polynomial system $\{p_n\}$. If the sequence of root measures ν_n converges weakly to a measure ν when $n \to \infty$, we say that ν is the asymptotic distribution of zeros of the polynomial system.

The following results, which characterize the asymptotic distribution of zeros for Nevai type OPS and for systems of eigenpolynomials respectively, are crucial to our treatment.

Suppose that a polynomial system $\{p_n\}$ is orthogonal with respect to a positive measure μ, and that the convex hull of $\operatorname{supp}\mu$ is a compact interval $[a, b]$. It is well known (see [3]) that the zeros of every p_n are contained in the interval $[a, b]$. The following is a more precise result on the distribution of zeros for orthogonal systems of Nevai type.

Theorem A. See [11], Th. 3, p. 50. *Let the polynomial system $\{p_n\}$ be orthogonal with respect to a measure μ of Nevai class on $\mathbb{R}$, and let the convex hull of* $\operatorname{supp}\mu$ *be $[a, b]$. Then the asymptotic distribution of zeros of $\{p_n\}$ is an absolutely continuous measure ν which depends only on $[a, b]$. The support of ν is precisely $[a, b]$ and its density in this interval is given by*

$$\rho(x) = \frac{1}{\pi\sqrt{(b-x)(x-a)}}.$$

Next we describe the asymptotic distribution of zeros for a system of eigenpolynomials.

Theorem B. See [1] Th. 2 and 4. *Let $\{p_n\}$ be a system of eigenpolynomials for an operator $\mathfrak{d}$ with a_N monic of degree N. Then the asymptotic distribution of zeros of $\{p_n\}$ is a probability measure ν with the following properties:*

a) ν has compact support;
b) its Cauchy transform $C(x) = \int \frac{d\nu(\zeta)}{x-\zeta}$ satisfies the equation $C(x)^N = 1/a_N(x)$ for almost all $x \in \mathbb{C}$.
These properties determine ν uniquely.

Note that the limiting measure ν is independent of all terms in (2) except the leading term $a_N(x)\frac{d^N}{dx^N}$.

To derive from these two results a statement about Nevai type BKS, we will need the following

Proposition 1. *Let $\{p_n\}$ be a system of eigenpolynomials for a differential operator $\mathfrak{d}$, and assume that all the zeros of p_n are real. Then there exists a compact set containing all the zeros of every p_n if and only if $\deg a_N = N$.*

Now it is easy to derive the following

Main Theorem. *Let $\{p_n\}$ be a Nevai type BKS, orthogonal with respect to a measure μ and with differential operator $\mathfrak{d}$. If μ is of the Nevai class and the convex hull of $\operatorname{supp}\mu$ is the interval $[a, b]$, then N is even and $a_N(x)$ is a constant multiple of $((x-a)(x-b))^{N/2}$.*

References and acknowledgements. There exists a really vast literature devoted to the classification problem for OPS. Classification of BKS has also attracted considerable attention, see e.g. [4] with its 100 references and [7],[8], [9] and references therein. The authors are happy to be able to contribute to this both classical and active area. We are grateful to Dr.M. Shapiro and Professor H. Shapiro for a number of discussions on the topic. The third author wants to acknowledge the hospitality of Mathematisches Forschungsinstitut Oberwolfach in September 2001 in whose peaceful and serene atmosphere he found some information on BKS and realized that the results of [10] and [1] are applicable to the BKS-classification problem. We should mention that when a preliminary version of this note was shown to some experts in this field, Professors Kwon and Lee sent us their paper [6] in preparation containing several results very much in the same spirit as in the present note. We also want to thank Prof. G. J. Yoon for clarifying to us the significance of the Nevai class.

3 Proofs

We need to prove Proposition 1 and the main theorem (as its corollary). Since the proof of Theorem B, in the situation where we will need it, follows easily along the same lines, we will include such a proof for the convenience of the reader.

Consider a polynomial system $\{p_n\}$ with the associated root measures ν_n. Assume that the supports of the measures ν_n are all contained in the same compact set, and that $\nu_n \to \nu$ in the weak topology. Let $C_n(x)$ be the Cauchy transform of ν_n and note that

$$C_n(x) = \int \frac{d\nu_n(\zeta)}{x-\zeta} = \frac{p_n'(x)}{np_n(x)}.$$

If $C(x)$ denotes the Cauchy transform of ν, it follows that

$$\frac{p_n'(x)}{np_n(x)} \to C(x)$$

for almost every $x \in \mathbb{C}$.

Suppose now that $\{p_n\}$ is a system of eigenpolynomials for an operator $\mathfrak{d}$ and that $\operatorname{supp} \nu_n$ are all contained in the same compact subset of the real line. Then there exists at least a subsequence of the ν_n converging weakly to some measure ν. Moreover, if we let $\nu_n^{(j)}$ denote the root measure of the jth derivative of p_n, then it follows from Rolle's theorem that (a subsequence of) $\nu_n^{(j)}$ converges weakly to ν for every $j > 0$. In particular,

$$\frac{p_n^{(j+1)}(x)}{(n-j)p_n^{(j)}(x)} \to C(x)$$

for almost every $x \in \mathbb{C}$, where $C(x)$ is the Cauchy transform of ν. If we divide both sides of the differential equation $\mathfrak{d}p_n = \lambda_n p_n$ by $n(n-1)\dots(n-N+1)p_n$ we obtain

$$a_N(x) \prod_{j=0}^{N-1} \frac{p_n^{(j+1)}(x)}{(n-j)p_n^{(j)}(x)} + \frac{a_{N-1}(x)}{n-N+1} \prod_{j=0}^{N-2} \frac{p_n^{(j+1)}(x)}{(n-j)p_n^{(j)}(x)} + \dots =$$

$$= \frac{\lambda_n}{n(n-1)\dots(n-N+1)}.$$

When $n \to \infty$ all the terms on the left hand side but the first tend to zero, and so

$$a_N(x)C(x)^N = \lim_{n\to\infty} \frac{\lambda_n}{n(n-1)\dots(n-N+1)}$$

for almost all x. But it can be seen (see [1]) that $\lambda_n = \sum_{k=1}^{N} c_k n(n-1)\dots(n-k+1)$ where c_k is the coefficient at x^k in $a_k(x)$. In particular, if $\deg a_N < N$ i.e. $c_N = 0$, then $\lambda_n/n(n-1)\dots(n-N+1) \to 0$ when $n \to \infty$, and it follows that $C(x) = 0$ for almost all x. This implies that $\nu = 0$, a contradiction. This argument proves one of the implications in Proposition 1. For the other implication we refer to [1], Lemma 9. Moreover, if a_N is monic of degree N, then $\lambda_n/n(n-1)\dots(n-N+1) \to 1$, and it follows that $C(x)^N = 1/a_N(x)$.

Suppose now that $\{p_n(x)\}$ is a Nevai type BKS as in the main theorem. By the remark preceding Theorem A, the zeros of every $p_n(x)$ are contained in the interval $[a, b]$. It follows from Proposition 1 that $\deg a_N(x) = N$, and we might as well assume that $a_N(x)$ is monic. Hence the Cauchy transform $C(x)$ of the limit $\nu = \lim_{n\to\infty} \nu_n$ satisfies $C(x)^N = 1/a_N(x)$. On the other hand, a direct computation of the Cauchy transform, using the expression for ν in Theorem A, gives $C(x)^2 = 1/(x-a)(x-b)$. Comparing these results yields $a_N(x) = ((x-a)(x-b))^{N/2}$.

4 Final remarks

Problem 1. The major problem in the context of this paper is whether every BKS, orthogonal with respect to a positive mesure with compact support, is a Jacobi-type OPS, compare [4], Conjecture 7.3. For the constant leading coefficient the analogous fact was proved in [9].

Problem 2. Is there an analogue of Theorem A on the asymptotic zero distribution for a signed measure μ with compact support? What is the situation for a probability measure with a noncompact support as well as for Sobolev orthogonal polynomial systems. (There exists a literature on this topic.)

Problem 3. Generalize the results of [1] to operators with $\deg a_N < N$. Preliminary computer experiments show that similar results on the asymptotic distribution of zeros would hold for all operators (2).

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PAPER III

On Generalized Laguerre Polynomials with Real and Complex Parameter

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submitted

Abstract

In this paper we consider families of polynomials arising as eigenfunctions to the confluent hypergeometric operator $T = Q_1(z)\frac{d}{dz} + Q_2(z)\frac{d^2}{dz^2}$ where the polynomial coefficients Q_1 and Q_2 are linear. We study the location and properties of zeros of individual eigenpolynomials. The classical Laguerre polynomials appear as a special case and some well-known results about these are recovered and generalized.

1 Introduction

The confluent hypergeometric operator studied in this paper is a special case of a wider class of operators which we are interested in. Namely, consider the differential operator

$$T_Q = \sum_{j=1}^{k} Q_j(z)\frac{d^j}{dz^j}$$

where the Q_j are polynomials in one complex variable satisfying the condition $\deg Q_j \leq j$ with equality for at least one j. In [3] we studied the eigenvalue problem

$$T_Q(p_n) = \lambda_n p_n$$

where T_Q is an operator of the above kind of order k and where in particular $\deg Q_k = k$ (we call this the *non-degenerate case*). We proved that for such an operator there exists a unique monic eigenpolynomial p_n of degree n for all sufficiently large integers n. The main topic in [3] was asymptotic properties of the zeros of p_n. Our main result was that when the degree n tends to infinity, the zeros of p_n are distributed according to a certain probability measure which is *compactly supported on a tree* and which depends *only* on the leading polynomial Q_k. Moreover, we proved that the zeros of p_n are all contained in the convex hull of the zeros of Q_k.

An operator of the above type of order k but with the condition $\deg Q_k < k$ for the leading term is referred to as a *degenerate operator*. In this paper we restrict our study to properties of zeros of eigenpolynomials of the the simplest degenerate operator, namely the confluent hypergeometric operator[1]

$$T = Q_1(z)\frac{d}{dz} + Q_2(z)\frac{d^2}{dz^2}$$

where $\deg Q_1 = \deg Q_2 = 1$. With $Q_1(z) = \alpha z + \beta$ and $Q_2(z) = \gamma z + \delta$ where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $\alpha, \gamma \neq 0$, one can show (see Lemma 2 in Section 2) that by an appropriate affine transformation of z any such operator can be rewritten as

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right],$$

where $\delta, \kappa \in \mathbb{C}$. In what follows T will denote this operator. The corresponding eigenvalue equation then becomes

$$zp_n^{''}(z) + (z+\kappa)p_n^{'}(z) = np_n(z), \tag{1}$$

since $\lambda_n = n$. One can prove, using a similar method as in [3], that there exists a *unique and monic* eigenpolynomial p_n of degree n for *every* integer n in (1), see Lemma 1 in Section 2.

In this paper we study the location of zeros of individual eigenpolynomials of T. In the sequel we will extend this study to asymptotic properties of zeros of eigenpolynomials of arbitrary degenerate operators.

The *Laguerre polynomials* appear as solutions to the Kummer hypergeometric equation

$$zy^{''}(z) + (\alpha + 1 - z)y^{'}(z) + ny(z) = 0$$

[1]Various familiar functions of mathematical analysis such as Hermite polynomials, Laguerre polynomials, Whittaker functions, Bessel functions and cylinder functions, are confluent hypergeometric functions, that is solutions to confluent hypergeometric equations.

when $\alpha \in \mathbb{R}$, $\alpha > -1$ and $n \in \mathbb{Z}$.[2] Making the transformation $z \to -z$ it is easy to see that this equation corresponds to our eigenvalue equation (1) when $\kappa \in \mathbb{R}$ and $\kappa > 0$. Thus the classical Laguerre polynomials appear normalized[3] as solutions to (1). One of the most important properties of the Laguerre polynomials is that they constitute an orthogonal system with respect to the weight function $e^{-x}x^{\alpha}$ on the interval $[0, \infty)$. It is well-known that the Laguerre polynomials are *hyperbolic* - that is all roots are real - and that the roots of two consecutive Laguerre polynomials p_n and p_{n+1} are *interlacing*. For other choices of the complex parameter α in Kummer's equation the sequence $\{p_n\}$ is in general *not* an orthogonal system and it can therefore not be studied by means of the theory known for such systems.

One of the results in this paper is the characterization of the exact choices on α for which T has hyperbolic eigenpolynomials and also for which α two consecutive eigenpolynomials have interlacing roots. It turns out that these properties are not restricted to the Laguerre polynomials solely. Our study can therefore be considered as a generalization of the properties of zeros of Laguerre polynomials to any family of polynomials appearing as eigenfunctions of the operator T. We also recover some well-known results (Theorems 3 and 4) by another method.

In what follows p_n denotes the nth degree unique and monic eigenpolynomial of T. These are our results:

Theorem 1. *The following two conditions are equivalent:*
(i) *there exists a real affine transformation $z \to az + b$ such that our operator can be written*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta \in \mathbb{C}$, $\kappa \in \mathbb{R}$ and $\kappa > -1$,
(ii) *p_n is hyperbolic for all n.*

[2]Observe that this equation has a degree n polynomial solution if and only if n is an integer. Without the condition that n is an integer we obtain the *Laguerre functions*.

[3]The nth degree Laguerre polynomial becomes monic when multiplied by $n!(-1)^n$.

Remark. Each p_n is actually *strictly* hyperbolic here, that is all roots are real *and* simple, see Corollary 2. Note that (i)⇒(ii) for $\kappa > 0$ also follows from the general theory of orthogonal polynomial systems, since then the p_n are normalized Laguerre polynomials.

Theorem $1^{'}$**.** *The following two conditions are equivalent:*
(i)$'$ *there exists a real affine transformation $z \to az + b$ such that our operator can be written*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta \in \mathbb{C}$, $\kappa \in \mathbb{R}$ and $\kappa > -1$ or $\kappa = -1, -2, -3, \ldots, -(n-1)$,
(ii)$'$ *p_n is hyperbolic.*

Remark. Thus if κ is a negative integer then all p_n such that $n > |\kappa|$ are hyperbolic. Note that when the degree n tends to infinity, p_n is hyperbolic for all negative integer values of κ.

The above results imply the following corollaries:

Corollary 1. *The following two conditions are equivalent:*
(i) *there exists a complex affine transformation $z \to \alpha z + \beta$ such that our operator can be written*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta \in \mathbb{C}$, $\kappa \in \mathbb{R}$ and $\kappa > -1$,
(ii) *the roots of p_n lie on a straight line in $\mathbb{C}$ for all n.*

Corollary $1^{'}$**.** *The following two conditions are equivalent:*
(i)$'$ *there exists a complex affine transformation $z \to \alpha z + \beta$ such that our operator can be written*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta \in \mathbb{C}$, $\kappa \in \mathbb{R}$ and $\kappa > -1$ or $\kappa = -1, -2, -3, \ldots, -(n-1)$,
(ii)$'$ *the roots of p_n lie on a straight line in $\mathbb{C}$.*

Remark. Thus if κ is a negative integer, then the roots of every p_n such that $n > |\kappa|$ lie on a straight line in $\mathbb{C}$.

Theorem 2. *Let*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta, \kappa \in \mathbb{C}$. *Then all roots of* p_n *are simple, unless* $\kappa = -1, -2, \ldots, -(n-1)$.

Combining Theorem 1 (hyperbolicity) and Theorem 2 (simplicity) we obtain the following

Corollary 2. *The eigenpolynomials of* T *are strictly hyperbolic (all roots are real and simple) for all* n *if and only if there exists a real affine transformation* $z \to az + b$ *such that our operator can be written*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta \in \mathbb{C}$, $\kappa \in \mathbb{R}$ *and* $\kappa > -1$.

Corollary 3. *Let*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta \in \mathbb{C}$ *and* $\kappa = 0, -1, -2, \ldots, -(n-1)$. *Then the eigenpolynomial* p_n *has* $(n+\kappa)$ *distinct roots, all of which are simple except the root at the origin which has multiplicity* $(1-\kappa)$. *Note that for* $\kappa = 0$ *all roots are simple.*

Moreover, it is possible to count the exact number of real roots of p_n. Namely,

Theorem 3. *Let*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta \in \mathbb{C}$, $\kappa \in \mathbb{R}$ *and* $\kappa < -(n-1)$. *Then* p_n *has no real roots if* n *is even, and* p_n *has precisely one real root if* n *is odd.*

Theorem 4. *Let*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta \in \mathbb{C}$, $\kappa \in \mathbb{R}$ and $-(n-1) < \kappa < -1$ such that κ is not an integer. Let $[\kappa]$ denote the integer part of κ. Then the number of real roots of p_n equals

$$\begin{cases} n+[\kappa]+1, & \text{if } [\kappa] \text{ is odd} \\ n+[\kappa], & \text{if } [\kappa] \text{ is even.} \end{cases}$$

It is a classical fact that the roots of any two consecutive Laguerre polynomials interlace along the real axis. These polynomials arise (normalized and after letting $z \to -z$) as eigenfunctions to our operator T when $\kappa > 0$. Here we extend this result and prove that the interlacing property also holds for polynomials arising as eigenfunctions of T when $\kappa = 0, -1, -2, \ldots, -(n-1)$. We have the following

Theorem 5. *Assume that our operator, after some complex affine transformation, can be written*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta \in \mathbb{C}, \kappa \in \mathbb{R}$. Then the roots of any two consecutive eigenpolynomials p_n and p_{n+1} are interlacing if $\kappa = 0, -1, -2, \ldots, -(n-1)$.

Remark. If the eigenpolynomials are hyperbolic then the meaning of this is obvious, while if they are not hyperbolic the roots interlace along a straight line in the complex plane (see Corollary $1'$).

Recent results on zero asymptotics. When α is arbitrary and real the polynomial solutions to Kummer's equation are referred to as *generalized Laguerre polynomials*. Some properties of the zeros when $\alpha \leq -1$ have been studied in [18]. In [23] similar results and several others are derived by considering the Laguerre polynomials as a limiting case of the Jacobi polynomials. In this paper we recover some of these results using yet another method. The asymptotic zero distribution for the generalized Laguerre polynomials (and several others) with real and degree dependent parameter α_n ($\alpha_n/n \to \infty$) have been studied in [6] using a continued fraction technique, and the same results are derived in [11] via a differential equation approach. It is known that when $\alpha \leq -1$ the zeros accumulate along certain interesting contours in the complex plane. More recent results on this can be found in [14] where a Riemann-Hilbert formulation for the Laguerre polynomials together with the steepest descent method (introduced in [6]) is used to obtain asymptotic properties of the zeros. The asymptotic location of the zeros depends on $A = \lim_{n\to\infty} -\frac{\alpha_n}{n} > 0$, and the results show a great sensitivity of the zeros to α_n's proximity to the integers. For $A > 1$ the contour is an open arc. For $0 < A < 1$ the contour consists of a closed loop together with an interval on the positive real axis. In the intermediate case $A = 1$ the contour is a simple closed contour. The case $A > 1$ is well-understood (see [21]), and uniform asymptotics for the Laguerre polynomials as $A > 1$ were obtained more recently, see [9], [15] and [26]. For fixed n interesting results can be found in [7] and [8].

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2 Proofs

We start with the following preliminary result:

Lemma 1. *Let*

$$T_Q = \sum_{j=0}^{k} Q_j(z) \frac{d^j}{dz^j}$$

be a linear differential operator where the polynomial coefficients satisfy $\deg Q_0 = 0$*,* $\deg Q_j = j$ *for exactly one* $j \in [1, k]$*, and* $\deg Q_m < m$ *if* $m \neq 0, j$*. Then* T_Q *has a unique and monic eigen-*

polynomial p_n of degree n for every integer n. Also, using the notation $Q_m = \sum_{j=0}^{m} q_{m,j} z^j$, we have

$$\lim_{n\to\infty} \frac{\lambda_n}{n(n-1)\dots(n-j+1)} = q_{j,j}$$

where λ_n is the eigenvalue.

Proof. In [3] we proved that for any operator T_Q as above but with the weaker restriction $\deg Q_j \leq j$ for all $j \in [0, n]$, the eigenvalue equation can be expressed as follows:

For $n \geq 1$ the coefficient vector X of $p_n = (a_{n,0}, a_{n,1}, \dots, a_{n,n-1})$ satisfies the linear system $MX = Y$, where Y is a vector and M is an upper triangular matrix, both with entries expressible in the coefficients $q_{m,j}$ of the Q_j.

We then used this to prove that there exists a unique monic eigenpolynomial of degree n for all sufficiently large integers n. Here we use the same method to prove that for the operator in Lemma 1 (of which the operator studied in this paper is a special case) we actually obtain a unique and monic eigenpolynomial of *every* degree n.

If we compute the matrix M with respect to the basis of monomials $1, z, z^2, \dots$, a diagonal element $M_{i+1,i+1}$ of M at the position $(i+1, i+1)$ (where $0 \leq i \leq n-1$) is given by

$$M_{i+1,i+1} = \sum_{0 \leq m \leq min(i,k)} q_{m,m} \cdot \frac{i!}{(i-m)!} - \lambda_n$$

where

$$\lambda_n = \sum_{m=0}^{k} q_{m,m} \cdot \frac{n!}{(n-m)!},$$

where we have used the notation $Q_m = \sum_{j=0}^{m} q_{m,j} z^j$. For the operator T_Q in Lemma 1 we have $\deg Q_m < m$ if $m \neq 0, j$, and so $q_{m,m} = 0$ for all $m \neq 0, j$. Inserting this in the expression for λ_n we obtain

$$\lambda_n = \sum_{m=0}^{k} q_{m,m} \cdot \frac{n!}{(n-m)!} = q_{0,0} + q_{j,j} \cdot \frac{n!}{(n-j)!} =$$
$$= q_{0,0} + q_{j,j} \cdot n(n-1)\dots(n-j+1),$$

and thus

$$\lim_{n\to\infty} \frac{\lambda_n}{n(n-1)\dots(n-j+1)} =$$
$$\lim_{n\to\infty} \left(\frac{q_{0,0}}{n(n-1)\dots(n-j+1)} + q_{j,j} \right) = q_{j,j}.$$

To prove the uniqueness of p_n we calculate the determinant of the matrix M and since it is upper triangular this equals the product of the diagonal elements. Thus, if we prove that all diagonal elements are nonzero for every n, then M is invertible for every n and the system $MX = Y$ has a unique solution for every n and we are done. Inserting $q_{m,m} = 0$ for $m \neq 0, j$ we get

$$\begin{aligned} M_{i+1,i+1} &= \sum_{0\leq m\leq min(i,k)} q_{m,m} \cdot \frac{i!}{(i-m)!} - \lambda_n = \\ &= \sum_{0\leq m\leq min(i,k)} q_{m,m} \cdot \frac{i!}{(i-m)!} - \left(q_{0,0} + q_{j,j} \cdot \frac{n!}{(n-j)!} \right) = \\ &= q_{j,j} \cdot \left(\frac{i!}{(i-j)!} - \frac{n!}{(n-j)!} \right) \neq 0 \end{aligned}$$

where $q_{j,j} \neq 0$ since $\deg_{Q_j} = j$ and $i < n$. Note that for $i < j$ one sets $i!/(i-j)! = 0$. □

Remark. The operator

$$T = Q_1(z)\frac{d}{dz} + Q_2(z)\frac{d^2}{dz^2}$$

which we are interested in here is a special case of the operator T_Q in Lemma 1.

Lemma 2. *Any operator*

$$T = (\alpha z + \gamma)\frac{d}{dz} + (\beta z + \delta)\frac{d^2}{dz^2}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *can be written*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

for some $\delta, \kappa \in \mathbb{C}$.

Proof. Dividing $T = (\alpha z + \gamma)\frac{d}{dz} + (\beta z + \delta)\frac{d^2}{dz^2}$ by β we obtain

$$T^* = T/\beta = \left(\frac{\alpha}{\beta}z + \frac{\gamma}{\beta}\right)\frac{d}{dz} + \left(z + \frac{\delta}{\beta}\right)\frac{d^2}{dz^2}$$

and making the translation $\tilde{z} = z + \frac{\delta}{\beta}$ we have

$$\begin{aligned}\tilde{T}^* &= \left(\frac{\alpha}{\beta}\left(\tilde{z} - \frac{\delta}{\beta}\right) + \frac{\gamma}{\beta}\right)\frac{d}{d\tilde{z}} + \left(\tilde{z} - \frac{\delta}{\beta} + \frac{\delta}{\beta}\right)\frac{d^2}{d\tilde{z}^2} = \\ &= \left(\frac{\alpha}{\beta}\tilde{z} - \frac{\alpha\delta}{\beta^2} + \frac{\gamma}{\beta}\right)\frac{d}{d\tilde{z}} + \tilde{z}\frac{d^2}{d\tilde{z}^2}.\end{aligned}$$

Finally with $\bar{z} = \frac{\alpha}{\beta}\tilde{z} \Leftrightarrow \tilde{z} = \frac{\beta}{\alpha}\bar{z}$ we have $d\bar{z}/d\tilde{z} = \alpha/\beta$ and so

$$\begin{cases} \frac{d}{d\tilde{z}} = \frac{\alpha}{\beta}\frac{d}{d\bar{z}} \\ \frac{d^2}{d\tilde{z}^2} = \frac{d}{d\bar{z}}\left(\frac{d}{d\tilde{z}}\right)\frac{d\bar{z}}{d\tilde{z}} = \frac{d}{d\bar{z}}\left(\frac{\alpha}{\beta}\frac{d}{d\bar{z}}\right)\frac{\alpha}{\beta} = \frac{\alpha^2}{\beta^2}\frac{d^2}{d\bar{z}^2} \end{cases}$$

and we get

$$\begin{aligned}\bar{\tilde{T}}^* &= \left(\frac{\alpha}{\beta}\tilde{z} - \frac{\alpha\delta}{\beta^2} + \frac{\gamma}{\beta}\right)\frac{d}{d\tilde{z}} + \tilde{z}\frac{d^2}{d\tilde{z}^2} = \\ &= \left(\frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha}\bar{z} - \frac{\alpha\delta}{\beta^2} + \frac{\gamma}{\beta}\right)\frac{\alpha}{\beta}\frac{d}{d\bar{z}} + \frac{\beta}{\alpha}\bar{z}\cdot\frac{\alpha^2}{\beta^2}\frac{d^2}{d\bar{z}^2} = \\ &= \frac{\alpha}{\beta}\left[\left(\bar{z} - \frac{\alpha\delta}{\beta^2} + \frac{\gamma}{\beta}\right)\frac{d}{d\bar{z}} + \bar{z}\frac{d^2}{d\bar{z}^2}\right] = \delta\left[(\bar{z} + \kappa)\frac{d}{d\bar{z}} + \bar{z}\frac{d^2}{d\bar{z}^2}\right]\end{aligned}$$

where $\delta = \frac{\alpha}{\beta}$ and $\kappa = -\frac{\alpha\delta}{\beta^2} + \frac{\gamma}{\beta}$. □

Note that if $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ then $\delta, \kappa \in \mathbb{R}$.

We will now study hyperbolicity of the eigenpolynomials of T in detail. Note that performing the transformations in Lemma 2 above with all coefficients real does not affect hyperbolicity of the polynomial eigenfunctions.

Proof of Theorems 1 **and** $1'$ **and their corollaries.** We first need the following well-known corollary (see [2]):

Corollary of Sturm's Theorem. *All roots of a monic and real polynomial are real if and only if the nonzero polynomials in its Sturm sequence have positive leading coefficients.*

Here the Sturm sequence is defined as follows. Let $p = p_0$ be a given real polynomial. Define $p_1 = p^{'}$ (the derivative of p) and choose the p_i to satisfy

$$\begin{aligned} p_0 &= p_1 q_1 - p_2, \quad \deg p_2 < \deg p_1 \\ p_1 &= p_2 q_2 - p_3, \quad \deg p_3 < \deg p_2 \\ p_2 &= p_3 q_3 - p_4, \quad \deg p_4 < \deg p_3 \\ &\vdots \end{aligned}$$

where the q_i are polynomials, and so on until a zero remainder is reached. That is, for each $i \geq 2$, p_i is the negative of the remainder when p_{i-2} is divided by p_{i-1}. Then the sequence $(p_0, p_1, p_2, \ldots)$ is called the *Sturm sequence* of the polynomial p.

We now calculate the Sturm sequence for a monic and real degree n eigenpolynomial $p = p_n$ of the operator $T = \delta\big[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\big]$, where $\delta \in \mathbb{C}$ and $\kappa \in \mathbb{R}$. Note that p is real if $\kappa \in \mathbb{R}$ and any two operators differing by a complex constant have identical polynomial eigenfunctions. Since our eigenpolynomials by assumption are monic, the first two elements in the Sturm sequence, p and $p^{'}$, clearly have positive leading coefficients, namely 1 and n. Define $R(i) = p_{i+1}$ in the Sturm sequence above. Then $R(1)$ is the negative of the remainder when p is divided by $p^{'}$. With $\deg p = n$ we have $\deg R(i) = n - i - 1$. The last element in the Sturm sequence (if it has not already stopped) will be the constant $R(n-1)$. We now claim that for every n and every $i \geq 1$ we have

$$\begin{cases} R(i) = A \cdot \sum_{j=0}^{n-i-1} \binom{n-i-1}{j} \frac{(\kappa+n-i-1)!}{(\kappa+j)!} z^j & \text{if } i \text{ is } \textit{odd} \\ R(i) = B \cdot \sum_{j=0}^{n-i-1} \binom{n-i-1}{j} \frac{(\kappa+n-i-1)!}{(\kappa+j)!} z^j & \text{if } i \text{ is } \textit{even} \end{cases} \tag{2}$$

where

$$\begin{cases} A = (n-1)(\kappa+n-1)(n-3)(\kappa+n-3)\ldots(n-i)(\kappa+n-i), \\ B = n(n-2)(\kappa+n-2)(n-4)(\kappa+n-4)\ldots(n-i)(\kappa+n-i). \end{cases}$$

It is obvious that with $\kappa = 0$ the leading coefficients of all the $R(i)$ are positive and p will be hyperbolic. For $\kappa \in \mathbb{R}$ and $\kappa \neq 0$ we have the following conditions for the leading coefficients $R(i)_{lc}$ of the $R(i)$ to be positive:

$$\begin{cases} R(1)_{lc} > 0 \quad \Rightarrow \quad \kappa > 1 - n \\ R(2)_{lc} > 0 \quad \Rightarrow \quad \kappa > 2 - n \\ R(3)_{lc} > 0 \quad \Rightarrow \quad \kappa > 3 - n \\ \vdots \\ R(i)_{lc} > 0 \quad \Rightarrow \quad \kappa > i - n \\ \vdots \\ R(n-1)_{lc} > 0 \quad \Rightarrow \quad \kappa > -1. \end{cases}$$

These conditions together yield $\kappa > -1$. Note that if some factor $(\kappa + n - j) = 0$, then not only the leading coefficient is zero, but the whole polynomial $R(i)$ is zero. So for $\kappa = j - n$ with $j \in [1, n-1]$ we also get hyperbolic p_n since the Sturm sequence by definition stops when a zero remainder is reached, and thus the leading coefficients of the previous components of the Sturm sequence are positive. So by the corollary of Sturm's Theorem, p_n is hyperbolic for *all* n *if and only if* $\kappa > -1$, and p_n is hyperbolic for a *particular* n *if and only if* $\kappa > -1$ *or* $\kappa = -1, -2, \ldots, -(n-1)$. One can prove by induction that the Sturm sequence polynomials are of the form claimed in (2) (see Appendix) . Moreover, it is obvious that if the roots of p_n lie on a straight line they can be transformed to the real axis by some complex affine transformation, and thus T must be on the form claimed by Theorem 1 or $1'$, and so Corollaries 1 and $1'$ follow.$\square$

To prove Theorem 2 we need the following

Lemma 3. *Let* $p_n = \sum_{j=0}^{n} a_{n,j} z^j$ *be the nth degree monic polynomial eigenfunction of* $T = \delta\big[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\big]$ *where* $\delta, \kappa \in \mathbb{C}$. *Note that* T *and* δT *have identical eigenpolynomials. Then the coefficients* $a_{n,j}$ *of* p_n *are given by*

$$a_{n,j} = \binom{n}{j} \frac{(\kappa + n - 1)!}{(\kappa + j - 1)!}, \quad \forall j \in [0, n].$$

Proof. Inserting $p_n = \sum_{j=0}^{n} a_{n,j} z^j$ in $(z+\kappa)p_n' + zp_n'' = np_n$ we have

$$(z+\kappa)\sum_{j=1}^{n} j a_{n,j} z^{j-1} + z\sum_{j=2}^{n} j(j-1)a_{n,j}z^{j-2} = n\sum_{j=0}^{n} a_{n,j}z^j$$

$$\Leftrightarrow$$

$$\sum_{j=1}^{n} j a_{n,j} z^j + \sum_{j=1}^{n} \kappa j a_{n,j} z^{j-1} + \sum_{j=2}^{n} j(j-1)a_{n,j}z^{j-1} = \sum_{j=0}^{n} n a_{n,j} z^j$$

$$\Leftrightarrow$$

$$\sum_{j=1}^{n} j a_{n,j} z^j + \sum_{j=0}^{n-1} \kappa(j+1)a_{n,j+1}z^j + \sum_{j=1}^{n-1} j(j+1)a_{n,j+1}z^j = \sum_{j=0}^{n} n a_{n,j} z^j.$$

Comparing coefficients we obtain

$$j a_{n,j} + \kappa(j+1)a_{n,j+1} + j(j+1)a_{n,j+1} = n a_{n,j}$$

$$\Leftrightarrow$$

$$a_{n,j} = \frac{(j+1)(\kappa+j)}{(n-j)} \cdot a_{n,j+1}$$

Applying this iteratively and using $a_{n,n} = 1$ (by monicity of p_n) we arrive at

$$a_{n,j} = \binom{n}{j}\frac{(\kappa+n-1)!}{(\kappa+j-1)!}, \quad \forall j \in [0,n].$$

□

Proof of Theorem 2. Let $\alpha \neq 0$ be a root of p_n that is not simple. Then, by repeatedly differentiating our eigenvalue equation $zp_n'' + (z+\kappa)p_n' = np_n$ and inserting $z = \alpha$, we get $p_n^{(j)}(\alpha) = 0$ $\forall j$, which means the multiplicity of α is infinite, which is absurd. Thus, for all $\kappa \in \mathbb{C}$, any non-zero root α of p_n is simple[4]. Next we prove that if $\kappa \neq -1, -2, \ldots, -(n-1)$ and if $\alpha = 0$ is a root of p_n then it must be simple too. Let $\alpha = 0$ be a root of p_n of multiplicity m and write $p_n(z) = z^m q(z)$ where $\alpha = 0$ is not a root of $q(z)$. Then $p_n'(z) = mz^{m-1}q(z) + z^m q'(z)$ and

[4]This also follows from the uniqueness theorem for a second order differential equation.

$p_n^{''}(z) = m(m-1)z^{m-2}q(z) + mz^{m-1}q^{'}(z) + mz^{m-1}q^{'}(z) + z^m q^{''}(z)$. Inserting this in our eigenvalue equation we obtain

$$\begin{aligned}
\lambda_n p_n(z) =& z p_n^{''}(z) + (z+\kappa)p_n^{'}(z) \\
&\Leftrightarrow \\
z^{m-1}[\lambda_n z q(z)] =& m(m-1)z^{m-1}q(z) + mz^m q^{'}(z) \\
&+ mz^m q^{'}(z) + z^{m+1}q^{''}(z) + mz^m q(z) + z^{m+1}q^{'}(z) \\
&+ \kappa m z^{m-1} q(z) + \kappa z^m q^{'}(z) \\
=& z^{m-1}[m(m-1)q(z) + mzq^{'}(z) + mzq^{'}(z) \\
&+ z^2 q^{''}(z) + mzq(z) + z^2 q^{'}(z) + \kappa m q(z) + \kappa z q^{'}(z)].
\end{aligned}$$

Equating the expressions in the brackets and setting $z = 0$ we arrive at the relation $m(m-1)q(0) + \kappa m q(0) = 0 \Leftrightarrow m(m-1+\kappa) = 0$. Thus $m = 0$ or $m = 1-\kappa$ for the multiplicity m of the root $\alpha = 0$. But if $m = 0$ then $\alpha = 0$ is not a root of p_n whence all roots of p_n are simple and we are done. If $\kappa = 0$ then $m = 0$ or $m = 1$ (it will soon be proved that the latter is true, see below). If $\kappa \neq 0, -1, -2\ldots, -(n-1)$ then either $m = 0$ and we are done, or $m = 1-\kappa$. Since m is the multiplicity of the root it must be a non-negative integer, and therefore $m = 1-\kappa$ is impossible unless $\kappa = 0, -1, -2, \ldots, -(n-1)$. Thus $\alpha = 0$ is not a root of p_n if $\kappa > -1$ and $\kappa \neq 0$. Also, $m = 1-\kappa$ is absurd if $\kappa \notin \mathbb{Z}$, and thus $m = 0$ for $\kappa \notin \mathbb{Z}$. Now consider the case $\kappa \in \mathbb{Z}$ with $\kappa \leq -n$. Then either $m = 0$ or $m = 1-\kappa$. By Lemma 3 the constant term $a_{n,0}$ of p_n equals

$$a_{n,0} = \frac{(\kappa - 1 + n)!}{(\kappa - 1)!} = (\kappa-1+n)(\kappa-2+n)(\kappa-3+n)\ldots(\kappa+2)(\kappa+1)\kappa.$$

and this cannot be zero if $\kappa \in \mathbb{Z}$ and $\kappa \leq -n$ - hence there is no zero at the origin ($m = 0$). Finally we prove that for $\kappa = -1, -2, \ldots, -(n-1)$ the multiplicity of the root $\alpha = 0$ is $m = 1-\kappa > 1$ and so in this case not all roots of p_n are simple. Recall that $m(m-1+\kappa) = 0$, so if $m \neq 0$ then $m = 1-\kappa$ and we are done. Thus we have to prove that we do have a root at the origin for $\kappa = 0, -1, -2, \ldots, -(n-1)$. But this is only possible if $a_{n,0} = 0$, and this is indeed the case if $\kappa = 0, -1, -2, \ldots, -(n-1)$ and we can conclude that all roots of p_n are simple for all $\kappa \in \mathbb{C} \setminus \{-1, -2, \ldots, -(n-1)\}$. $\square$

Using Lemma 3 we also obtain the following

Proposition 1. *Let $p_n(\kappa, z)$ denote the* n*th degree monic eigenpolynomial of $T = \delta\big[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\big]$ where $\delta, \kappa \in \mathbb{C}$. Then, using the explicit representation of p_n in Lemma 3, we obtain the identity*

$$p_n^{(m)}(\kappa, z) = \frac{n!}{(n-m)!} p_{n-m}(\kappa+m, z), \quad n = 0, 1, \ldots; m = 1, 2, \ldots$$

and the recurrence formula

$$p_n(\kappa, z) = (z + 2n + \kappa - 2)p_{n-1}(\kappa, z) - (n-1)(n+\kappa-2)p_{n-2}(\kappa, z),$$

where $p_0(\kappa, z) = 1$ and $p_1(\kappa, z) = z + \kappa$.

Proof of Corollary 3. By Theorem 2 all nonzero roots of p_n are simple, and from the proof of Theorem 2 we know that for $\kappa = 0, -1, -2, \ldots, -(n-1)$ the multiplicity of the root at the origin is $m = 1 - \kappa$. We have a total of n roots of p_n and thus there are $n - (1-\kappa) + 1 = n + \kappa$ distinct roots. $\square$

As stated in **Theorems 3 and 4**, it is possible to count the exact number of real roots of p_n if $\kappa \in \mathbb{R}$ in T. We use *Sturm's Theorem* to count the number of real roots in any interval:[5]

Sturm's Theorem. *Let $(p_0(t), p_1(t), p_2(t), \ldots)$ be the Sturm sequence of a polynomial $p(t)$ (as defined in the proof of Theorems 1 and $1'$). Let $u < v$ be real numbers. Assume that U is the number of sign changes in the sequence $(p_0(u), p_1(u), p_2(u), \ldots)$ and let V be the number of sign changes in the sequence $(p_0(v), p_1(v), p_2(v), \ldots)$. Then the number of real roots of $p(t)$ between u and v (with each multiple root counted exactly once) is exactly $U - V$.*

Remark. Combining Sturm's Theorem with Theorem 2 and its Corollary 3 it is possible to recover Theorems 1 and $1'$ in the direction $\Rightarrow$. Namely, we get $(i) \Rightarrow (ii)$ if $\kappa > -1$ and p_n is the nth degree monic polynomial eigenfunction of T, since then the Sturm sequence of p_n has $(n+1)$ nonzero elements, all with positive leading coefficients. With $u = -\infty$ and $v = \infty$ we then have

[5] see [2].

$U = n$ and $V = 0$, and therefore the number of real roots of p_n is $U - V = n$, so p_n is hyperbolic (Theorem 1). And similarly $(i)' \Rightarrow (ii)'$ for $\kappa = -1, -2, \ldots, -(n-1)$, since the Sturm sequence stops as soon as the zero remainder is reached, and here it has $(n + \kappa + 1)$ nonzero elements, all with positive leading coefficients. Therefore, with $u = -\infty$ and $v = \infty$, we have $U = n+\kappa$ and $V = 0$. By Corollary 3 all roots of p_n are simple except the root at the origin which has multiplicity $1-\kappa$. Thus, counted with multiplicity, p_n has $U - V + (-\kappa) = n$ real roots and is therefore hyperbolic (Theorem $1'$) .

We already know that if $\kappa \neq -1, -2, \ldots, -(n-1)$, then all roots of p_n are simple and no element in the Sturm sequence of p_n is identically zero. The leading coefficients of the elements of the Sturm sequence are (see the proof of Theorems 1 and $1'$) given by

$$\begin{cases} p_{lc} = 1 \\ p'_{lc} = n \\ R(1)_{lc} = (n-1)(\kappa + n - 1) \\ R(2)_{lc} = n(n-2)(\kappa + n - 2) \\ R(3)_{lc} = (n-1)(\kappa + n - 1)(n-3)(\kappa + n - 3) \\ R(4)_{lc} = n(n-2)(\kappa + n - 2)(n-4)(\kappa + n - 4) \\ R(3)_{lc} = (n-1)(\kappa + n - 1)(n-3)(\kappa + n - 3)(n-5)(\kappa + n - 5) \\ R(4)_{lc} = n(n-2)(\kappa + n - 2)(n-4)(\kappa + n - 4)(n-6)(\kappa + n - 6) \\ \vdots \\ R(n-1)_{lc} = \ldots \end{cases}$$

We now use Sturm's Theorem to prove Theorems 3 and 4:

Proof of Theorem 3. Let p_n be the monic degree n eigenpolynomial of T where $\kappa < -(n-1)$, i.e. $\kappa + n - 1 < 0$ and therefore $\kappa + n - j < 0$ for every $j \geq 1$. Thus we get the following for the leading coefficients of the Sturm sequence elements:

$$\begin{cases} p_{lc} = 1 > 0 \\ p'_{lc} = n > 0 \\ R(1)_{lc} < 0 \\ R(2)_{lc} < 0 \\ R(3)_{lc} > 0 \\ R(4)_{lc} > 0 \\ R(5)_{lc} < 0 \\ R(6)_{lc} < 0 \\ \vdots \end{cases}$$

This pattern continues up to the last element $R(n-1)$ of the sequence. Inserting $v = \infty$ in the Sturm sequence of p_n we find that there is a sign change at every $R(i)$ where i is odd. Therefore the number of sign changes V in this sequence equals the number of $R(i)$ where i is odd. Thus:

$$V = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Inserting $u = -\infty$ in the Sturm sequence we find that there is a sign change between the first two elements in the sequence and then at every $R(i)$ where i is even. and hence the number of sign changes U equals 1+ [the number of $R(i)$ where i is even]. Thus:

$$U = \begin{cases} \frac{n-2}{2} + 1 = \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} + 1 = \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

By Theorem 2 all roots of p_n are simple and thus the number of real roots of p_n equals $U - V = \begin{cases} 0 & \text{if } n \text{ is even.} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$ $\square$

Proof of Theorem 4. Let p_n be the monic eigenpolynomial of T where $\kappa \in \mathbb{R}$ and $j - n < \kappa < j - n + 1$ for $j \in [1, n-2]$. Then $(\kappa + n - j) > 0$ and $(\kappa + n - j - 1) < 0$ and $[\kappa] = j - n$. Again we consider the leading coefficients in the Sturm sequence of p_n. Clearly $p_{lc} = 1 > 0$, $p'_{lc} = n > 0$ and $R(i)_{lc} > 0$ $\forall i \in [1, j]$. For the remaining leading coefficients we have

$$\begin{cases} R(j+1)_{lc} < 0 \\ R(j+2)_{lc} < 0 \\ R(j+3)_{lc} > 0 \\ R(j+4)_{lc} > 0 \\ R(j+5)_{lc} < 0 \\ R(j+6)_{lc} < 0 \\ \qquad\vdots \end{cases}$$

and this pattern continues up to the last element $R(n-1)$ in the sequence. Consider the sequence we obtain by inserting $v = \infty$ in this Sturm sequence. We have sign changes at every $R(j+l)$ where l is odd. Our last element is $R(n-1) = R(j+(n-j-1))$. Also note that if $n-j-1 = n-n-[\kappa]-1 = -[\kappa]-1$ is even then $[\kappa]$ is odd, and if $n-j-1$ is odd then $[\kappa]$ is even. Thus the number of sign changes V in this sequence is

$$V = \begin{cases} \frac{n-j-1}{2} & \text{if } [\kappa] \text{ is odd,} \\ \frac{n-j}{2} & \text{if } [\kappa] \text{ is even.} \end{cases}$$

Now insert $u = -\infty$ in the Sturm sequence. The number of sign changes from the first element p in the sequence till the element $R(j)$ is $(1+j)$. For the remaining $n-j-1$ elements of this sequence we have a change of sign at every $R(j+l)$ where l is even. Thus the number of sign changes is $(n-j-1)/2$ if $(n-j-1)$ is even $\Leftrightarrow$ $[\kappa]$ is odd, and the number of sign changes is $(n-j-2)/2$ if $(n-j-1)$ is odd $\Leftrightarrow$ $[\kappa]$ is even. Thus for the total number of sign changes U in this sequence we get

$$U = \begin{cases} (1+j) + \frac{n-j-1}{2} = \frac{n+j+1}{2} & \text{if } [\kappa] \text{ is odd} \\ (1+j) + \frac{n-j-2}{2} = \frac{n+j}{2} & \text{if } [\kappa] \text{ is even.} \end{cases}$$

Therefore the number of real roots $U - V$ of p_n, counted with multiplicity, is precisely

$$U - V = \begin{cases} \frac{n+j+1}{2} - \frac{n-j-1}{2} = j+1 = n+[\kappa]+1 & \text{if } [\kappa] \text{ is odd} \\ \frac{n+j}{2} - \frac{n-j}{2} = j = n+[\kappa] & \text{if } [\kappa] \text{ is even.} \end{cases}$$

since all roots of p_n are simple in this case by Theorem 2. $\square$

Proof of Theorem 5. The proof of the interlacing property consists of a sequence of five lemmas. Lemmas 4 and 8 are well-known. Lemmas 4,5 and 6 are used in the proof of Lemma 7, which is proved using an idea due to S. Shadrin presented in [20]. The five lemmas are the following:

Lemma 4. *If R_n and R_{n+1} are strictly hyperbolic polynomials of degrees n and $n+1$ respectively, then $R_n + \epsilon R_{n+1}$ is hyperbolic for any sufficiently small ϵ.*

Lemma 5. *Let p_n and p_{n+1} be two polynomial eigenfunctions of the operator $T = (z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}$ with $\kappa = 0, -1, -2, \ldots, -(n-1)$. Then $p_n + \epsilon p_{n+1}$ is hyperbolic for any sufficiently small ϵ.*

Proof of Lemma 5. From Corollary 3 we know that p_n and p_{n+1} have all their roots simple except for the root at the origin which for both polynomials has multiplicity $1-\kappa$. Thus we can write $p_n + \epsilon p_{n+1} = z^{1-\kappa}(R_{n+\kappa-1} + \epsilon R_{n+\kappa})$, where $R_{n+\kappa-1}$ and $R_{n+\kappa}$ are strictly hyperbolic polynomials of degrees $n+\kappa-1$ and $n+\kappa$ respectively. By Lemma 4, $R_{n+\kappa-1} + \epsilon R_{n+\kappa}$ is hyperbolic for any sufficiently small ϵ, and then clearly $z^{1-\kappa}(R_{n+\kappa-1} + \epsilon R_{n+\kappa}) = p_n + \epsilon p_{n+1}$ is also hyperbolic for any sufficiently small ϵ. $\square$

Lemma 6. *Let $T = \kappa + (z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}$ with $\kappa = 0, -1, -2, \ldots, -(n-1)$, and let p_n and p_{n+1} be two consecutive eigenpolynomials of T. Then letting T act on any linear combination $\alpha p_n + \beta p_{n+1}$ with $\alpha, \beta \in \mathbb{R}$ that is hyperbolic (i.e. has all its roots real) results in a hyperbolic polynomial.*

Proof of Lemma 6. Note that the operators $T = \kappa + (z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}$ and $T = (z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}$ have identical eigenpolynomials. Let $f = \alpha p_n + \beta p_{n+1}$ be a hyperbolic linear combination with real coefficients of two consecutive eigenpolynomials of T. Then f' is a hyperbolic polynomial by Gauss-Lucas Theorem. By Rolle's Theorem f and f' have interlacing roots and so by the well-known Lemma 8 below, $(f+f')$ is a hyperbolic polynomial. By Corollary 3 both p_n and p_{n+1} have a root at the origin of multiplicity $1-\kappa$. Thus $f = \alpha p_n + \beta p_{n+1}$ has a root at the origin of multiplicity at least $1-\kappa$, and f' has a root at the origin of multiplicity at least $-\kappa$. Thus the polynomial

$(f+f^{'})$ has a root at the origin of multiplicity at least $(-\kappa)$ and we can write $(f+f^{'}) = z^{-\kappa}g$ for some hyperbolic polynomial g. Now $z^{\kappa}(f+f^{'}) = g$ is a hyperbolic polynomial. But $D[z^{\kappa}(f+f^{'})] = \kappa z^{\kappa-1}(f+f^{'})+z^{\kappa}(f^{'}+f^{''}) = z^{\kappa-1}[\kappa f+(z+\kappa)f^{'}+zf^{''}] = z^{\kappa-1}T(f)$ where $T(f) = \kappa f + (z+\kappa)f^{'} + zf^{''}$. By Gauss-Lucas Theorem one has that $D[z^{\kappa}(f+f^{'})]$ is a hyperbolic polynomial and therefore $T(f) = z^{1-k}D[z^{\kappa}(f+f^{'})]$ is a hyperbolic polynomial. □

Lemma 7. *Let $T = \kappa + (z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}$. Any linear combination $\alpha p_n + \beta p_{n+1}$ with real coefficients of two consecutive eigenpolynomials of T with $\kappa = 0, -1, -2, \dots, -(n-1)$ is a hyperbolic polynomial.*

Proof of Lemma 7. Applying to $\alpha p_n + \beta p_{n+1}$ some high power T^{-N} of the inverse operator one gets

$$T^{-N}(\alpha p_n + \beta p_{n+1}) = \frac{\alpha}{\lambda_n^N}p_n + \frac{\beta}{\lambda_{n+1}^N}p_{n+1} =$$

$$= \frac{\alpha}{\lambda_n^N}(p_n + \epsilon p_{n+1}),$$

where ϵ is arbitrarily small for the appropriate choice of N (since $0 < \lambda_n < \lambda_{n+1}$). Thus, by Lemma 5, the polynomial $T^{-N}(\alpha p_n + \beta p_{n+1})$ is hyperbolic for sufficiently large N. Assume that $\alpha p_n + \beta p_{n+1}$ is non-hyperbolic and take the largest N_0 for which $R_{N_0} = T^{-N_0}(\alpha p_n + \beta p_{n+1})$ is still non-hyperbolic. Then $R_{N_0} = T(R_{N_0+1})$ where $R_{N_0+1} = T^{-N_0-1}(\alpha p_n + \beta p_{n+1})$. Note that R_{N_0+1} is hyperbolic and that if $\kappa = 0, -1, -2, \dots, -(n-1)$ then letting T act on any hyperbolic linear combination $\alpha p_n + \beta p_{n+1}$ with real coefficients results in a hyperbolic polynomial by Lemma 6. Contradiction. □

Lemma 8 [classical]. If R_n and R_{n+1} are any real polynomials of degrees n and $n+1$, respectively, then saying that every linear combination $\alpha R_n + \beta R_{n+1}$ with real coefficients is hyperbolic is equivalent to saying that
(i) both R_n and R_{n+1} are hyperbolic, and
(ii) their roots are interlacing.

We now **prove Theorem 5**. Consider the operator $T = (z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}$ where $\kappa = 0, -1, -2, \dots, -(n-1)$, and let p_n and p_{n+1} be two consecutive eigenpolynomials of T. Recall that δT and T

have identical eigenpolynomials and that by Corollary $1'$ the roots in this case lie on straight lines in the complex plane. By Lemma 5 the linear combination $p_n + \epsilon p_{n+1}$ is hyperbolic for any sufficiently small ϵ. Using Lemmas 5 and 6 we can therefore apply Lemma 7 which says that any linear combination $\alpha p_n + \beta p_{n+1}$ with real coefficients α and β is a hyperbolic polynomial. By Lemma 8 this implies that the roots of p_n and p_{n+1} are interlacing and we are done. $\square$

Remark. Note that we can recover the interlacing property for the normalized Laguerre polynomials using the same proof as in Theorem 5 but with a small modification of Lemma 6. Namely, if $\kappa > 0$, then the application of T to *any* hyperbolic polynomial results in a hyperbolic polynomial. For if f is a hyperbolic polynomial, then f' is hyperbolic by Gauss-Lucas Theorem, f and f' have interlacing roots by Rolle's Theorem, and by the well-known Lemma 8 the linear combination $(f + f')$, and therefore $z^\kappa(f + f')$, is a hyperbolic polynomial. Finally $D[z^\kappa(f + f')] = z^{\kappa-1}T(f)$ is hyperbolic by Gauss-Lucas Theorem.

When suitably scaled, it is possible to find a limiting expansion for p_n when $n \to \infty$ that is closely related to a Bessel function. Because of the scaling however, the convergence to the Bessel function only gives information about the asymptotic behaviour of p_n in an infinitesimal neighbourhood of the origin. Although other methods must be used to get information elsewhere, it is interesting that on the infinitesimal scale our eigenpolynomials mimic the global behaviour of this particular Bessel function. We have the following proposition, where $J_{\kappa-1}$ denotes the Bessel function of the first kind of order $(\kappa - 1)$:

Proposition 2. *Let $p_n(\kappa, z)$ denote the unique and monic eigenpolynomial of the operator*

$$T = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta, \kappa \in \mathbb{C}$ and κ is not a negative integer. We then have the limit formula

$$\lim_{n\to\infty} \frac{n^{1-\kappa}}{n!} p_n(\kappa, z/n) = (-z)^{(1-\kappa)/2} J_{\kappa-1}(2i\sqrt{z}),$$

where the convergence holds for all $z \in \mathbb{C}$ and uniformly on compact z-sets.

Remark. The *Bessel function of the first kind of order κ* is defined by the series

$$J_\kappa(z) \equiv \sum_{\nu=0}^{\infty} \frac{(-1)^\nu (z/2)^{\kappa+2\nu}}{\nu! \Gamma(\kappa+\nu+1)}$$

where $z, \kappa \in \mathbb{C}$ and $|z| < \infty$. Clearly $z^{-\kappa} J_\kappa(z)$ is an entire analytic function for all $z \in \mathbb{C}$ if κ is not a negative integer. This Bessel function is a solution to *Bessel's equation* [6] *of order κ*, which is the second-order linear differential equation

$$z^2 \frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - \kappa^2)y = 0.$$

From now on we adopt the notational convention $\Gamma(n+\kappa) = (n+\kappa-1)!$ for $\kappa \in \mathbb{C}$, where Γ is the Gamma function. In order to prove Proposition 2, we will need the following technical

Lemma 9.

$$\lim_{n\to\infty} \binom{n+\kappa-1}{n-\nu} n^{1-\kappa-\nu} = \frac{1}{\Gamma(\kappa+\nu)}$$

where $n, \nu \in \mathbb{R}$ and $\kappa \in \mathbb{C} \setminus \{-1, -2, \ldots\}$.

Proof of Lemma 9. Using the following well-known asymptotic formula:

Corollary of the Stirling formula. [7]

$$\lim_{n\to\infty} \frac{\Gamma(n+\alpha)}{\Gamma(n)} n^{-\alpha} = 1$$

where $\alpha \in \mathbb{C}$ and $n \in \mathbb{R}$,

[6] Bessel's equation is encountered in the study of boundary value problems in potential theory for cylindrical domains. The solutions to Bessel's equation are referred to as *cylinder functions*, of which the Bessel functions are a special kind.

[7] see e.g.[25]

we get

$$\begin{aligned}
\lim_{n\to\infty}\binom{n+\kappa-1}{n-\nu}n^{1-\kappa-\nu} &= \frac{1}{\Gamma(\kappa+\nu)}\lim_{n\to\infty}\frac{\Gamma(n+\kappa)}{\Gamma(n-\nu+1)}n^{1-\kappa-\nu} \\
&= \frac{1}{\Gamma(\kappa+\nu)}\lim_{n\to\infty}\frac{\Gamma(n+\kappa)}{\Gamma(n)}n^{-\kappa}\lim_{n\to\infty}\frac{\Gamma(n)}{\Gamma(n-\nu+1)}n^{1-\nu} \\
&= \frac{1}{\Gamma(\kappa+\nu)}.
\end{aligned}$$

Proof of Proposition 2. By Lemma 3 our eigenpolynomials have the following explicit representation:

$$p_n(\kappa,z) = \sum_{\nu=0}^{n}\binom{n}{\nu}\frac{(\kappa+n-1)!}{(\kappa+\nu-1)!}z^\nu = \sum_{\nu=0}^{n}\binom{n+\kappa-1}{n-\nu}\frac{n!}{\nu!}z^\nu$$

where $\kappa\in\mathbb{C}$.

Thus, with the scaling $z\to z/n$ and using Lemma 9, we get

$$\begin{aligned}
\lim_{n\to\infty}\frac{n^{1-\kappa}}{n!}p_n(\kappa,z/n) &= \lim_{n\to\infty}\sum_{\nu=0}^{n}\binom{n+\kappa-1}{n-\nu}n^{1-\kappa}\frac{1}{\nu!}\left(\frac{z}{n}\right)^\nu \\
&= \lim_{n\to\infty}\sum_{\nu=0}^{n}\binom{n+\kappa-1}{n-\nu}n^{1-\kappa-\nu}\frac{z^\nu}{\nu!} \\
&= \sum_{\nu=0}^{\infty}\frac{z^\nu}{\Gamma(\kappa+\nu)\nu!} = (-z)^{(1-\kappa)/2}J_{\kappa-1}(2i\sqrt{z}).
\end{aligned}$$

□

Appendix: Proof of (2) in Section 2.

Note that we have adopted the notational convention $\Gamma(n+\kappa) = (n+\kappa-1)!$ for $\kappa \in \mathbb{C}$, where Γ denotes the Gamma function. I start by calculating $R(1)$ and $R(2)$ and so the hypothesis (actually there are two hypotheses, one for even i and one for odd i) is true for one case of even i and one case of odd i. With the nth degree eigenpolynomial $p_n = \sum_{j=0}^{n} a_{n,j} z^j$ we have by Lemma 3 that

$$a_{n,j} = \binom{n}{j}\frac{(\kappa+n-1)!}{(\kappa+j-1)!} \quad \Rightarrow \quad p_n = \sum_{j=0}^{n}\binom{n}{j}\frac{(\kappa+n-1)!}{(\kappa+j-1)!}z^j.$$

Calculation of $R(1) =$ [the negative of the remainder when the eigenpolynomial p_n is divided by $p_n^{'}$]:

$$\frac{z}{n} + \frac{(n-1+\kappa)}{n}$$

$$\sum_{j=1}^{n} j\binom{n}{j}\frac{(\kappa+n-1)!}{(\kappa+j-1)!}z^{j-1} \;\Big)\; \sum_{j=0}^{n}\binom{n}{j}\frac{(\kappa+n-1)!}{(\kappa+j-1)!}z^j$$

$$-\Big[\sum_{j=1}^{n}\frac{j}{n}\binom{n}{j}\frac{(\kappa+n-1)!}{(\kappa+j-1)!}z^j\Big]$$

$$= \sum_{j=0}^{n-1}\binom{n}{j}\frac{(\kappa+n-1)!}{(\kappa+j-1)!}\Big[1-\frac{j}{n}\Big]z^j$$

$$-\Big[\sum_{j=0}^{n-1}\frac{j+1}{n}\binom{n}{j+1}(\kappa+n-1)\frac{(\kappa+n-1)!}{(\kappa+j)!}z^j\Big]$$

$$= \sum_{j=0}^{n-2}\Big[\binom{n}{j}\frac{(\kappa+n-1)!}{(\kappa+j-1)!}\Big(1-\frac{j}{n}\Big) - \frac{(j+1)}{n}\binom{n}{j+1}(\kappa+n-1)\frac{(\kappa+n-1)!}{(\kappa+j)!}\Big]z^j$$

and it remains to prove that the negative of this remainder equals

$$R(1) = (n-1)(\kappa+n-1)\sum_{j=0}^{n-2}\binom{n-2}{j}\frac{(\kappa+n-2)!}{(\kappa+j)!}z^j.$$

Developing the coefficient in front of z^j in our remainder we obtain

$$\binom{n}{j}\frac{(\kappa+n-1)!}{(\kappa+j-1)!}\left(1-\frac{j}{n}\right)-\frac{(j+1)}{n}\binom{n}{j+1}(\kappa+n-1)\frac{(\kappa+n-1)!}{(\kappa+j)!}$$
$$=\frac{n!}{(n-j)!j!}\frac{(\kappa+n-1)!}{(\kappa+j-1)!}-\frac{n!}{(n-j)!j!}\frac{j}{n}\frac{(\kappa+n-1)!}{(\kappa+j-1)!}-\frac{(j+1)}{n}\frac{n!(\kappa+n-1)}{(j+1)!(n-j-1)!}\frac{(\kappa+n-1)!}{(\kappa+j)!}$$
$$=\frac{n(n-1)(n-2)!}{(n-j-2)!(n-j-1)(n-j)j!}\frac{(\kappa+n-2)!(\kappa+n-1)(\kappa+j)}{(\kappa+j)!}$$
$$-\frac{(n-1)(n-2)!(\kappa+n-2)!(\kappa+n-1)j(\kappa+j)}{(n-j-2)!(n-j-1)(n-j)(\kappa+j)!j!}-\frac{(n-1)(n-2)!(\kappa+n-1)^2(\kappa+n-2)!}{j!(n-j-2)!(n-j-1)(\kappa+j)!}$$
$$=(n-1)(\kappa+n-1)\frac{(n-2)!}{j!(n-j-2)!}\frac{(\kappa+n-2)!}{(\kappa+j)!}\Bigg[\frac{n(\kappa+j)}{(n-j-1)(n-j)}-\frac{j(\kappa+j)}{(n-j-1)(n-j)}$$
$$-\frac{(\kappa+n-1)(n-j)}{(n-j-1)(n-j)}\Bigg]$$
$$=(n-1)(\kappa+n-1)\binom{n-2}{j}\frac{(\kappa+n-2)!}{(\kappa+j)!}\left[\frac{n\kappa+nj-j\kappa-j^2-\kappa n+\kappa j-n^2+nj+n-j}{n^2-nj-nj+j^2-n+j}\right]$$
$$=-(n-1)(\kappa+n-1)\binom{n-2}{j}\frac{(\kappa+n-2)!}{(\kappa+j)!},$$

and we are done.

Calculation of $R(2) =$ [the negative of the remainder when p_n' is divided by $R(1)$]:

$$\frac{nz}{(n-1)(\kappa+n-1)}+\frac{n(2n-3+\kappa)}{(n-1)(\kappa+n-1)}$$

$$\sum_{j=0}^{n-2}\binom{n-2}{j}\frac{(\kappa+n-2)!}{(\kappa+j)!}(n-1)(\kappa+n-1)z^j \;\Big|\; \sum_{j=0}^{n-1}(j+1)\binom{n}{j+1}\frac{(\kappa+n-1)!}{(\kappa+j)!}z^j$$

$$-\left[\sum_{j=1}^{n-1}n\binom{n-2}{j-1}\frac{(\kappa+n-2)!}{(\kappa+j-1)!}z^j\right]$$

$$=\sum_{j=0}^{n-2}\left[(j+1)\binom{n}{j+1}\frac{(\kappa+n-1)!}{(\kappa+j)!}-n\binom{n-2}{j-1}\frac{(\kappa+n-2)!}{(\kappa+j-1)!}\right]z^j$$

$$-\left[\sum_{j=0}^{n-2}\binom{n-2}{j}\frac{(\kappa+n-2)!}{(\kappa+j)!}n(2n-3+\kappa)z^j\right]$$

$$=\sum_{j=0}^{n-3}\left[\frac{(\kappa+n-2)!}{(\kappa+j-1)!}\left((j+1)\binom{n}{j+1}\frac{(\kappa+n-1)!}{(\kappa+j)!}-n\binom{n-2}{j-1}\right)-n(2n-3+\kappa)\binom{n-2}{j}\frac{(\kappa+n-2)!}{(\kappa+j)!}\right]z^j$$

and it remains to prove that the negative of this remainder equals

$$R(2)=n(n-2)(\kappa+n-2)\sum_{j=0}^{n-3}\binom{n-3}{j}\frac{(\kappa+n-3)!}{(\kappa+j)!}z^j.$$

Developing the coefficient in front of z^j in our remainder we have

$$\frac{(\kappa+n-2)!}{(\kappa+j-1)!}(j+1)\binom{n}{j+1}\frac{(\kappa+n-1)}{(\kappa+j)} - \frac{(\kappa+n-2)!}{(\kappa+j-1)!}n\binom{n-2}{j-1} - \frac{(\kappa+n-2)!}{(\kappa+j)!}\binom{n-2}{j}n(2n-3+\kappa)$$
$$= \frac{(\kappa+n-2)!}{(\kappa+j-1)!}\frac{n!}{j!(n-j-1)!}\frac{(\kappa+n-1)}{(\kappa+j)} - \frac{(\kappa+n-2)!}{(\kappa+j-1)!}\frac{n(n-2)!}{(j-1)!(n-j-1)!}$$
$$- \frac{(\kappa+n-2)!}{(\kappa+j)!}\frac{(n-2)!}{(j!(n-j-2)!}n(2n-3+\kappa)$$
$$= \frac{(\kappa+n-3)!(\kappa+n-2)(n-3)!(n-2)(n-1)n(\kappa+n-1)}{(\kappa+j)!j!(n-j-3)!(n-j-2)(n-j-1)}$$
$$- \frac{(\kappa+n-3)!(\kappa+n-2)n(n-2)(n-3)!j(\kappa+j)}{j!(n-j-3)!(n-j-2)(n-j-1)(\kappa+j)!}$$
$$- \frac{(\kappa+n-3)!(\kappa+n-2)(n-2)(n-3)!n(2n-3+\kappa)}{(\kappa+j)!j!(n-j-2)(n-j-3)!}$$
$$= \frac{(\kappa+n-3)!(n-3)!}{(\kappa+j)!j!(n-j-3)!}n(n-2)(\kappa+n-2)\left[\frac{(n-1)(\kappa+n-1)}{(n-j-2)(n-j-1)}\right.$$
$$\left.- \frac{j(\kappa+j)}{(n-j-2)(n-j-1)} - \frac{(2n-3+\kappa)(n-j-1)}{(n-j-2)(n-j-1)}\right]$$
$$= n(n-2)(\kappa+n-2)\binom{n-3}{j}\frac{(\kappa+n-3)!}{(\kappa+j)!}\left[\frac{-n^2+nj+n+nj-j^2-j+2n-2j-2}{n^2-nj-n-nj+j^2+j-2n+2j+2}\right]$$
$$= -n(n-2)(\kappa+n-2)\binom{n-3}{j}\frac{(\kappa+n-3)!}{(\kappa+j)!},$$

and we are done.

To prove the induction hypotheses we divide $R(i)$ by $R(i+1)$ to obtain $R(i+2)$. Here it is assumed that i is odd. The proof with even i differs only in small details from this proof and is therefore omitted here. For simplicity we use the notations

$$\begin{cases} A = (n-1)(\kappa+n-1)(n-3)(\kappa+n-3)\dots(n-i)(\kappa+n-i), \\ B = n(n-2)(\kappa+n-2)(n-4)(\kappa+n-4)\dots(n-i)(\kappa+n-i). \end{cases}$$

Dividing $R(i)$ by $R(i+1)$:

$$
\begin{array}{l|l}
\multicolumn{2}{r}{\frac{A}{B}z+\frac{A}{B}(2n-2i-3+\kappa)} \\
\hline
B\sum\limits_{j=0}^{n-i-2}\binom{n-i-2}{j}\frac{(\kappa+n-i-2)!}{(\kappa+j)!}z^j & A\sum\limits_{j=0}^{n-i-1}\binom{n-i-1}{j}\frac{(\kappa+n-i-1)!}{(\kappa+j)!}z^j \\
\cline{1-1}
 & -\left[A\sum\limits_{j=1}^{n-i-1}\binom{n-i-2}{j-1}\frac{(\kappa+n-i-2)!}{(\kappa+j-1)!}z^j\right] \\
\cline{2-2}
 & =A\sum\limits_{j=0}^{n-i-2}\left[\binom{n-i-1}{j}\frac{(\kappa+n-i-1)!}{(\kappa+j)!}-\binom{n-i-2}{j-1}\frac{(\kappa+n-i-2)!}{(\kappa+j-1)!}\right]z^j \\
 & -\left[A\sum\limits_{j=0}^{n-i-2}(2n{-}2i{-}3{+}\kappa)\binom{n-i-2}{j}\frac{(\kappa+n-i-2)!}{(\kappa+j)!}z^j\right] \\
\hline
\end{array}
$$

$$
=A\sum_{j=0}^{n-i-3}\left[\binom{n-i-1}{j}\frac{(\kappa+n-i-1)!}{(\kappa+j)!}-\binom{n-i-2}{j-1}\frac{(\kappa+n-i-2)!}{(\kappa+j-1)!}-(2n-2i-3+\kappa)\binom{n-i-2}{j}\frac{(\kappa+n-i-2)!}{(\kappa+j)!}\right]z^j
$$

and it remains to prove that the negative of this remainder equals the excpected (by hypothesis)

$$
R(i{+}2)=A(n{-}i{-}2)(\kappa{+}n{-}i{-}2)\sum_{j=0}^{n-i-3}\binom{n-i-3}{j}\frac{(\kappa+n-i-3)!}{(\kappa+j)!}z^j,
$$

i.e. we have to prove the following equality:

$$
\begin{aligned}
&\binom{n-i-1}{j}\frac{(\kappa+n-i-1)!}{(\kappa+j)!}-\binom{n-i-2}{j-1}\frac{(\kappa+n-i-2)!}{(\kappa+j-1)!}\\
&-(2n-2i-3+\kappa)\binom{n-i-2}{j}\frac{(\kappa+n-i-2)!}{(\kappa+j)!}=\\
&=-(n-i-2)(\kappa+n-i-2)\binom{n-i-3}{j}\frac{(\kappa+n-i-3)!}{(\kappa+j)!}.
\end{aligned}
$$

But

$$\binom{n-i-1}{j}\frac{(\kappa+n-i-1)!}{(\kappa+j)!}-\binom{n-i-2}{j-1}\frac{(\kappa+n-i-2)!}{(\kappa+j-1)!}-(2n-2i-3+\kappa)\binom{n-i-2}{j}\frac{(\kappa+n-i-2)!}{(\kappa+j)!}$$

$$=\frac{(n-i-1)!}{j!(n-i-j-1)!}\frac{(\kappa+n-i-1)!}{(\kappa+j)!}-\frac{(n-i-2)!}{(j-1)!(n-i-j-1)!}\frac{(\kappa+n-i-2)!}{(\kappa+j-1)!}$$

$$-(2n-2i-3+\kappa)\frac{(n-i-2)!}{j!(n-i-j-2)!}\frac{(\kappa+n-i-2)!}{(\kappa+j)!}$$

$$=\frac{(n-i-3)!(n-i-2)(n-i-1)(\kappa+n-i-3)!(\kappa+n-i-2)(\kappa+n-i-1)}{j!(n-i-j-1)(n-i-j-2)(n-i-j-3)!(\kappa+j)!}$$

$$-\frac{(n-i-3)!(n-i-2)j(\kappa+n-i-3)!(\kappa+n-i-2)(c+j)}{j!(n-i-j-3)!(n-i-j-2)(n-i-j-1)(\kappa+j)!}$$

$$-(2n-2i-3+\kappa)\frac{(n-i-3)!(n-i-2)(\kappa+n-i-3)!(\kappa+n-i-2)}{j!(n-i-j-3)!(n-i-j-2)(\kappa+j)!}$$

$$=(n-i-2)(\kappa+n-i-2)\frac{(n-i-3)!}{j!(n-i-j-3)!}\frac{(\kappa+n-i-3)!}{(\kappa+j)!}$$

$$\cdot\left[\frac{(n-i-1)(\kappa+n-i-1)-j(\kappa+j)-(2n-2i-3+\kappa)(n-i-j-1)}{(n-i-j-1)(n-i-j-2)}\right]$$

$$=(n-i-2)(\kappa+n-i-2)\frac{(n-i-3)!}{j!(n-i-j-3)!}\frac{(\kappa+n-i-3)!}{(\kappa+j)!}(-1)$$

$$=-(n-i-2)(\kappa+n-i-2)\binom{n-i-3}{j}\frac{(\kappa+n-i-3)!}{(\kappa+j)!}.$$

□

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PAPER IV

On Asymptotics of Polynomial Eigenfunctions for Degenerate Exactly-Solvable Differential Operators

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Abstract

In this paper we study roots of eigenpolynomials of degenerate exactly-solvable differential operators, i.e. $T = \sum_{j=1}^{k} Q_j D^j$ with polynomial coefficients Q_j in one complex variable satisfying the condition $\deg Q_j \le j$ with equality for at least one j, and in particular $\deg Q_k < k$. We show that the root with the largest modulus of the nth degree unique and monic eigenpolynomial p_n of T tends to infinity when $n \to \infty$, as opposed to the non-degenerate case ($\deg Q_k = k$), which we have treated previously in [2]. Our main result in this paper is an explicit conjecture and partial results on the growth of the largest modulus of the roots of p_n. Based on this conjecture we deduce the algebraic equation satisfied by the Cauchy transform of the asymptotic root measure of the appropriately scaled eigenpolynomials, for which the union of all roots is conjecturally contained in a compact set.

1 Introduction

In this paper we study asymptotic properties of roots in certain families of eigenpolynomials. Namely, consider a linear differential operator

$$T = \sum_{j=1}^{k} Q_j D^j,$$

where $D = d/dz$ and the Q_j are complex polynomials in a single variable z satisfying the condition $\deg Q_j \le j$ for all j, and $\deg Q_k < k$ for the leading term. Such operators will be referred to as *degenerate exactly-solvable operators*, see Definition 1 below.

In this paper we study polynomial eigenfunctions of such operators, that is polynomials satisfying

$$T(p_n) = \lambda_n p_n \tag{1}$$

for some value of the spectral parameter λ_n, where n is a positive integer and $\deg p_n = n$.

The basic motivation for this study comes from two sources: 1) a classical question going back to S. Bochner, and 2) the *generalized Bochner problem*, which we describe below.

1) In 1929 Bochner asked about the classification of differential equations (1) having an infinite sequence of *orthogonal* polynomial solutions, see [15]. Such a system of polynomials $\{p_n\}_{n=0}^{\infty}$ which are both eigenpolynomials of some finite order differential operator *and* orthogonal with respect to some suitable inner product, are referred to as *Bochner-Krall orthogonal polynomial systems* (BKS), and the corresponding operators are called *Bochner-Krall operators.* It is an open problem to classify all BKS - a complete classification is only known for Bochner-Krall operators of order $k \leq 4$, and the corresponding BKS are various classical systems such as the Jacobi type, the Laguerre type, the Legendre type and the Bessel and Hermite polynomials, see [7].

2) The problem of a general classisfication of linear differential operators for which the eigenvalue problem (1) has a certain number of eigenfunctions in the form of a finite-order polynomial in some variables, is referred to as *the generalized Bochner problem*, see [19] and [20]. In the former paper a classification of operators possessing infinitely many finite-dimensional subspaces with a basis in polynomials is presented, and in the latter paper a general method has been formulated for generating eigenvalue problems for linear differential operators in one and several variables possessing polynomial solutions.

Notice that for the operators considered here the sequence of eigenpolynomials is in general not an orthogonal system and it can therefore not be studied by means of the extensive theory known for such systems.

Definition 1. We call a linear differential operator T of the kth order *exactly-solvable* if it preserves the infinite flag $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_n \subset \cdots$, where $\mathcal{P}_n$ is the linear space of all polynomials of degree less than or equal to n.[1] Or, equivalently, the problem (1) has an infinite sequence of polynomial eigenfunctions *if and only if* the operator T is *exactly-solvable*, see [21].

Notice that any exactly-solvable operator is of the form $T = \sum_{j=1}^{k} Q_j D^j$. They split into two major classes: *non-degenerate* and *degenerate*, where in the former case $\deg Q_k = k$, and in the latter case $\deg Q_k < k$ for the leading term. The major difference between these two classes is that in the non-degenerate case the union of all roots of all eigenpolynomials of T is contained in a compact set (see [2]), contrary to the degenerate case, which we prove in this paper.

The importance of studying eigenpolynomials of exactly-solvable operators is among other things motivated by numerous examples coming from classical orthogonal polynomials. Our study can be considered as a natural generalization of the behaviour of the maximal root for classical orthogonal polynomial families such as the Laguerre and Hermite polynomials, which appear as solutions to the eigenvalueproblem (1) for certain choices on the polynomial coefficients Q_j for a second-order degenerate exactly-solvable operator; the Laguerre polynomials appear as solutions to the differential equation $zy''(z) + (1-z)y'(z) + ny(z) = 0$, and the Hermite polynomials are solutions to the differential equation $y''(z) - 2zy'(z) + 2ny(z) = 0$ where n is a nonnegative integer. Recent studies and interesting results on the asymptotic zero behaviour for these polynomials and the corresponding generalized polynomials can be found in e.g [6], [9],[13], [17], [18], [11], [12] and references therein. In [11] one can find bounds on the spacing of zeros of certain functions belonging to the Laguerre-Polya class satisfying a second order differential equation, and as a corollary new sharp inequalities on the extreme zeros of the Hermite, Laguerre and Jacobi polynomials are established.

[1] Correspondingly, a linear differential operator of the kth order is called *quasi-exactly-solvable* if it preserves the space $\mathcal{P}_n$ for some fixed n.

Let us briefly recall our previous results. In [2] we treated the asymptotic zero distribution for polynomial families appearing as solutions to (1) where T is an arbitrary non-degenerate exactly-solvable operator. This seems to be a natural generalization to higher orders of the Gauss hypergeometric equation. As a special case, the Jacobi polynomials appear as solutions to $(z^2-1)y''(z)+(az+b)y'(z)+cy(z)=0$, where a,b and c are constants satisfying $a>b, a+b>0$ and $c=n(1-a-n)$ for some nonnegative integer n. It is a classical fact that the zeros of the Jacobi polynomials lie in the interval $[-1,1]$ and that their density in this interval is proportional to $1/\sqrt{1-|z|^2}$ when the degree n tends to infinity, which follows from the general theory of orthogonal polynomial systems. However, for higher-order operators of this kind, the sequence of eigenpolynomials is in general not an orthogonal system. In [2] we proved that when $n\to\infty$, the roots of the nth degree eigenpolynomial p_n for a non-degenerate exactly-solvable operator are distributed according to a certain probability measure which has compact support and which depends *only* on the leading polynomial Q_k. Namely,

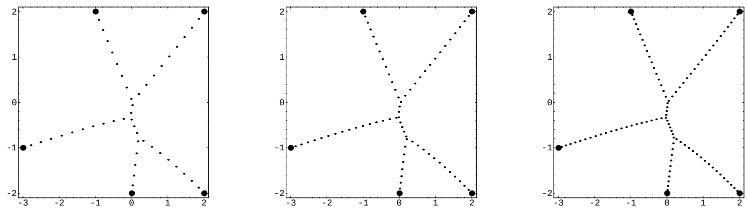
Theorem A. *Let Q_k be a monic polynomial of degree k. Then there exists a unique probability measure μ_{Q_k} with compact support whose Cauchy transform $C(z)=\int\frac{d\mu_{Q_k}(\zeta)}{z-\zeta}$ satisfies $C(z)^k=1/Q_k(z)$ for almost all $z\in\mathbb{C}$.*

Theorem B. *Let Q_k and μ_{Q_k} be as in Theorem A. Then* supp *μ_{Q_k} is the union of finitely many smooth curve segments, and each of these curves is mapped to a straight line by the locally defined mapping $\Psi(z)=\int Q_k(z)^{-1/k}dz$. Moreover,* supp *$\mu_{Q_k}$ contains all the zeros of Q_k, is contained in the convex hull of the zeros of Q_k and is connected and has connected complement.*

If p_n is a polynomial of degree n we construct the probability measure (root measure) μ_n by placing the point mass of size $\frac{1}{n}$ at each zero of p_n. The following is our main result from [2]:

Theorem C. *Let p_n be the monic degree n eigenpolynomial of a non-degenerate exactly-solvable operator, and let μ_n be the root measure of p_n. Then μ_n converges weakly to μ_{Q_k} when $n\to\infty$.*

To illustrate, we show the zeros of some polynomial eigenfunctions for the non-degenerate exactly-solvable operator $T = Q_5 D^5$, where $Q_5 = (z-2+2i)(z+1-2i)(z+3+i)(z+2i)(z-2i-2)$. In the pictures below large dots represent the zeros of Q_5 and small dots represent the zeros of the eigenpolynomials p_{50}, p_{75} and p_{100} respectively.



n=50 n=75 n=100.

As a consequence of the above results we were able to prove a special case of a general conjecture describing the leading terms of all Bochner-Krall operators, see [3].

In the present paper we partially extend the above results to the case of degenerate exactly-solvable operators. Numerical evidence shows that the roots of the nth degree eigenpolynomial are distributed on a tree in this case too, but that the limiting root measure is compactly supported only after an appropraite scaling of the roots. In what follows we will wlog assume that p_n is monic.

We start with the following preliminary result:

Lemma 1. *Let $T = \sum_{j=1}^{k} Q_j D^j$ be a degenerate exactly-solvable operator of order k. Then, for all sufficiently large integers n, there exists a unique constant λ_n and a unique monic polynomial p_n of degree n which satisfy $T(p_n) = \lambda_n p_n$. If* $\deg Q_j = j$ *for* precisely one *value $j < k$, then there exists a unique constant λ_n and a unique monic polynomial p_n of degree n which satisfy $T(p_n) = \lambda_n p_n$ for* every *integer $n = 1, 2, \ldots$.*

In what follows we denote by r_n the largest modulus of all roots of the unique and monic nth degree eigenpolynomial p_n of T, i.e.

$$r_n = \max\{|z| : p_n(z) = 0\}.$$

These are our main results:

Theorem 1.[2] *Let T be a degenerate exactly-solvable operator of order k. Then $r_n \to \infty$ when $n \to \infty$.*

Next we establish a lower bound for r_n when $n \to \infty$.

Theorem 2. *Let $T = \sum_{j=1}^{k} Q_j D^j = \sum_{j=1}^{k} \left(\sum_{i=0}^{\deg Q_j} \alpha_{j,i} z^i\right) D^j$ be a degenerate exactly-solvable operator of order k. Then for any $\gamma < b$ we have*

$$\lim_{n\to\infty} \frac{r_n}{n^\gamma} = \infty,$$

where

$$b := \overset{+}{\min_{j\in[1,k-1]}} \left(\frac{k-j}{k-j+\deg Q_j - \deg Q_k} \right),$$

and where the notation $\min^+$ *means that the minimum is taken only over positive terms* $(k-j+\deg Q_j - \deg Q_k)$.

Corollary 1. *Let $T = \sum_{j=1}^{k} Q_j D^j$ be a degenerate exactly-solvable operator of order k such that* $\deg Q_j \le j_0$ *for all $j > j_0$, and in particular* $\deg Q_k = j_0$*, where j_0 is the largest j such that* $\deg Q_j = j$*. Then* $\lim_{n\to\infty} \frac{r_n}{n^\gamma} = \infty$ *for any $\gamma < 1$.*

Corollary 2. *Let $T = \sum_{j=1}^{k} Q_j D^j$ be a degenerate exactly-solvable operator of order k such that* $\deg Q_j = 0$ *for all $j > j_0$, where j_0 is the largest j for which* $\deg Q_j = j$*. Then* $\lim_{n\to\infty} \frac{r_n}{n^\gamma} = \infty$ *for any $\gamma < \frac{k-j_0}{k}$.*

In fact our extensive numerical experiments and natural heuristic arguments (see Section 3) support the following conjecture:

[2]This theorem is joint work with H. Rullgård.

Main Conjecture. *Let* $T = \sum_{j=1}^{k} Q_j D^j$ *be a degenerate exactly-solvable operator of order* k *and denote by* j_0 *the largest* j *for which* $\deg Q_j = j$*. Then*

$$\lim_{n\to\infty} \frac{r_n}{n^d} = c_T,$$

where $c_T > 0$ *is a positive constant and*

$$d := \max_{j\in[j_0+1,k]} \left(\frac{j - j_0}{j - \deg Q_j}\right).$$

Remark. Note that Main Conjecture implies Theorem 2 since $b \leq d$.

The next two theorems support the above conjecture:

Theorem 3. *Let* T *be a degenerate exactly-solvable operator of order* k *consisting of precisely two terms:* $T = Q_{j_0}D^{j_0} + Q_k D^k$*. Then there exists a positive constant* c *such that*

$$\lim_{n\to\infty} \inf \frac{r_n}{n^d} \geq c$$

where $d := \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right) = \frac{k-j_0}{k-\deg Q_k}$.

This result can be generalized to operators consisting of any number of terms, but with certain conditions on the degree of the polynomial coefficients Q_j when $j > j_0$, where j_0 is the largest j for which $\deg Q_j = j$. Namely,

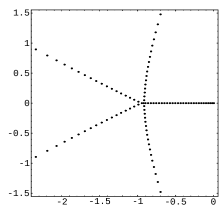
Theorem 4. *Let* T *be a degenerate exactly-solvable operator of order* k*. Denote by* j_0 *the largest* j *such that* $\deg Q_j = j$ *and let* $(j - \deg Q_j) \geq (k - \deg Q_k)$ *for every* $j > j_0$*. Then there exists a positive constant* $c > 0$ *such that*

$$\lim_{n\to\infty} \inf \frac{r_n}{n^d} \geq c$$

where $d := \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right) = \frac{k-j_0}{k-\deg Q_k}$.

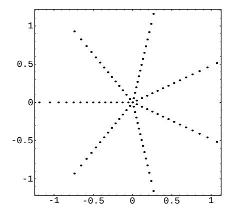
Computer experiments indicate that roots of eigenpolynomials scaled according to the Main Conjecture fill certain interesting curves in $\mathbb{C}$. To illustrate this phenomenon let us present some typical pictures. Below p_n denotes the nth degree unique and monic eigenpolynomial of the given operator, and $q_n(z) = p_n(n^d z)$ denotes the corresponding appropriately scaled polynomial, where d is as in Main Conjecture, and for which the union of all roots is (conjecturally) contained in a compact set.

Fig.1:



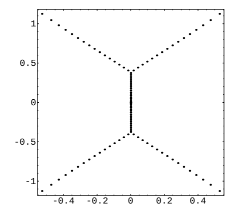
roots of
$q_{100}(z) = p_{100}(100z)$

Fig.2:



roots of
$q_{100}(z) = p_{100}(100z)$

Fig.3:



roots of
$q_{100}(z) = p_{100}(100z)$

Fig.1: $T_1 = zD + zD^2 + zD^3 + zD^4 + zD^5$.
Fig.2: $T_2 = z^2D^2 + D^7$.
Fig.3: $T_3 = z^3D^3 + z^2D^4 + zD^5$.

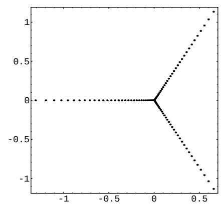
In Section 3 we will derive the (conjectural) algebraic equation satisfied by the Cauchy transform $C(z)$ of the asymptotic root measure of the scaled eigenpolynomial q_n for an arbitrary degenerate exactly-solvable operator. From this equation one can obtain detailed information about the above curves and also conclude which terms of the operator that are relevant for the asymptotic zero distribution of its eigenpolynomials.[3] Namely, with d and j_0 as in Main Conjecture and assuming Q_{j_0} is monic, we have

$$z^{j_0} C^{j_0}(z) + \sum_{j \in A} \alpha_{j, \deg Q_j} z^{\deg Q_j} C^j(z) = 1,$$

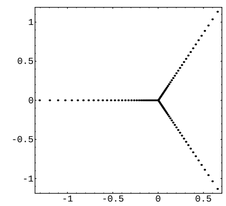
where $A = \{j : (j - j_0)/(j - \deg Q_j) = d\}$ and $\alpha_{j,\deg Q_j}$ is the leading coefficient of Q_j. For details see Section 3.

[3]As was mentioned earlier, in the non-degenerate case which we have treated previously, the asymptotic zero distribution of the eigenpolynomials depends *only* on the leading coefficient Q_k. For the operators considered here however, the situation is more complicated.

Numerical evidence clearly illustrates that distinct operators whose scaled eigenpolynomials satisfy the same Cauchy transform equation when $n \to \infty$, will yield identical asymptotic zero distributions. Below we show one such example. For further details see Section 4.3.



T_4, roots of
$q_{100}(z) = p_{100}(100^{2/3}z)$



$\tilde{T}_4$, roots of
$q_{100}(z) = p_{100}(100^{2/3}z)$

where $T_4 = z^3D^3 + z^2D^5$ and $\widetilde{T}_4 = z^2D^2 + z^3D^3 + zD^4 + z^2D^5 + D^6$.

Let us finally mention some possible applications of our results and directions for further research. As was mentioned earlier, operators of the type we consider occur in the theory of Bochner-Krall orthogonal systems. A great deal is known about the asymptotic zero distribution of orthogonal polynomials. By comparing such known results with results on the asymptotic zero distribution of the eigenpolynomials considered here, we believe it will be possible to gain new insight into the nature of BKS.

This paper is structured as follows. In Section 2 we give the proofs of the lemma, the theorems and the corollaries stated in this section. In Section 3 we explain how we arrived at Main Conjecture and how we obtain as its corollary the algebraic Cauchy transform equation. In Section 4 we display numerical evidence supporting Main Conjecture and its corollary. In Section 5 (Appendix) we give the detailed calculations which led to the corollary of Main Conjecture, and we also prove that for a class of operators of the type we consider, the conjectured upper bound for r_n implies the conjectured lower bound. Finally, in Section 6, we discuss some open problems and directions for further research.

Acknowledgements. I am sincerely greatful to my PhD advisor Professor B. Shapiro for introducing me to this very fascinating problem and for his constant support during my work. I would also like to thank Professor J-E. Björk for stimulating discussions on the topic. I am also indepted to anonymous referees for their interesting suggestions which contributed to the improvement of this paper. My research was supported by Stockholm University.

2 Proofs

Proof of Lemma 1. In [2] we proved that for any exactly-solvable operator T, the eigenvalue problem $T(p_n) = \lambda_n p_n$ can be written as a linear system $MX = Y$, where X is the coefficient vector of the monic nth degree eigenpolynomial p_n with components $a_{n,0}, a_{n,1}, a_{n,2}, \ldots, a_{n,n-1}$, and Y is a vector and M is an upper triangular $n \times n$ matrix, both with entries expressible in the coefficients of T. With $T = \sum_{j=1}^{k} Q_j D^j$, $Q_j = \sum_{i=0}^{\deg Q_j} \alpha_{j,i} z^i$, and $p_n(z) = \sum_{i=0}^{n} a_{n,i} z^i$, the eigenvalue λ_n is given by

$$\lambda_n = \sum_{j=1}^{k} \alpha_{j,j} \frac{n!}{(n-j)!},$$

and the diagonal elements of the matrix M are given by

$$M_{i+1,i+1} = \sum_{1 \le j \le \min(i,k)} \alpha_{j,j} \frac{i!}{(i-j)!} - \lambda_n = \sum_{j=1}^{k} \alpha_{j,j} \left[\frac{i!}{(i-j)!} - \frac{n!}{(n-j)!} \right]$$

for $i = 0, 1, \ldots, n-1$. The last equality follows since $i!/(i-j)! = 0$ for $i < j \le k$ by definition (see Lemma 2 in [2]). In order to prove that p_n is unique we only need to check that the determinant of M is nonzero, which implies that M is invertible, whence the system $MX = Y$ has a unique solution. Notice that M is upper triangular and thus its determinant equals the product of its diagonal elements. We therefore prove that every diagonal element $M_{i+1,i+1}$ ($i \in [0, n-1]$) is nonzero for all sufficiently large integers n for an arbitrary T as above, as well as for *every* n if $\deg Q_j = j$ for *exactly* one j.

From the expression

$$-M_{i+1,i+1} = \sum_{j=1}^{k} \alpha_{j,j} \left[\frac{n!}{(n-j)!} - \frac{i!}{(i-j)!} \right]$$

it is clear that $M_{i+1,i+1} \neq 0$ for every $i \in [0, n-1]$ and *every* n if $\alpha_{j,j} \neq 0$ for *exactly* one j, that is if $\deg Q_j = j$ for precisely one j, and thus we have proved the second part of Lemma 1.

Now assume that $\deg Q_j = j$ for more than one j and denote by j_0 the largest such j. Then $\alpha_{j_0,j_0} \neq 0$ and we have

$$\begin{aligned}
- \quad & M_{i+1,i+1} = \sum_{j=1}^{j_0} \alpha_{j,j} \left[\frac{n!}{(n-j)!} - \frac{i!}{(i-j)!} \right] \\
= \quad & \frac{n!}{(n-j_0)!} \left[\alpha_{j_0,j_0} \left(1 - \frac{i!/(i-j_0)!}{n!/(n-j_0)!} \right) + \sum_{1 \leq j < j_0} \alpha_{j,j} \frac{(n-j_0)!}{(n-j)!} \right. \\
- \quad & \left. \sum_{1 \leq j < j_0} \frac{(n-j_0)! i!}{n!(i-j)!} \right].
\end{aligned}$$

The last two sums in the brackets on the right-hand side of the above equality tend to zero when $n \to \infty$, since $j_0 > j$ and $i \leq n-1$. Thus for all sufficiently large n we get

$$-M_{i+1,i+1} = \frac{n!}{(n-j_0)!} \left[\alpha_{j_0,j_0} \left(1 - \frac{i!/(i-j_0)!}{n!/(n-j_0)!} \right) \right] \neq 0$$

for every $i \in [0, n-1]$, and we have proved the first part of Lemma 1. $\square$

To prove Theorem 1 we need the following. If p_n is a polynomial of degree n we construct the probability measure μ_n by placing a point mass of size $\frac{1}{n}$ at each zero of p_n. We call μ_n the *root measure* of p_n. By definition, for any polynomial p_n of degree n, the Cauchy transform $C_{n,j}$ of the root measure $\mu_n^{(j)}$ for the j*th* derivative $p_n^{(j)}$ is defined by

$$C_{n,j}(z) := \frac{p_n^{(j+1)}(z)}{(n-j)p_n^{(j)}(z)} = \int \frac{d\mu_n^{(j)}(\zeta)}{z - \zeta},$$

for $j = 0, 1, \ldots, n-1$, and it is well-known that the measure $\mu_n^{(j)}$ can be reconstructed from $C_{n,j}$ by the formula $\mu_n^{(j)} = \frac{1}{\pi} \cdot \frac{\partial C_{n,j}}{\partial \bar{z}}$ where $\partial/\partial\bar{z} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$.

Proof of Theorem 1. Let $T = \sum_{j=1}^{k} Q_j D^j$ and denote by j_0 the largest j for which $\deg Q_j = j$. Note that since T is degenerate we have $j_0 < k$. Thus, using the above notation, we have

$$\begin{aligned}\frac{p_n^{(j)}(z)}{p_n(z)} &= C_{n,0}(z)C_{n,1}(z)\cdots C_{n,j-1}(z)\cdot n(n-1)\cdots(n-j+1)\\ &= \frac{n!}{(n-j)!}\prod_{m=0}^{j-1} C_{n,m}(z).\end{aligned}$$

With the notation $Q_j(z) = \sum_{i=0}^{\deg Q_j} \alpha_{j,i} z^i$ we have $\lambda_n = \sum_{j=1}^{j_0} \alpha_{j,j} \frac{n!}{(n-j)!}$. Now dividing the equation $T(p_n) = \lambda_n p_n$ by p_n we obtain

$$Q_k(z)\frac{p_n^{(k)}(z)}{p_n(z)} + Q_{k-1}(z)\frac{p_n^{(k-1)}(z)}{p_n(z)} + \ldots + Q_1(z)\frac{p_n'(z)}{p_n(z)} = \sum_{j=1}^{j_0} \alpha_{j,j}\frac{n!}{(n-j)!}$$

$$\Leftrightarrow$$

$$\begin{aligned}&Q_k(z)\frac{n!}{(n-k)!}\prod_{m=0}^{k-1} C_{n,m}(z) + Q_{k-1}(z)\frac{n!}{(n-k+1)!}\prod_{m=0}^{k-2} C_{n,m}(z) + \ldots\\ &\ldots + Q_1(z)\frac{n!}{(n-1)!}C_{n,0}(z) = \sum_{j=1}^{j_0}\alpha_{j,j}\frac{n!}{(n-j)!}.\end{aligned} \tag{2}$$

Dividing both sides of this equation by $\frac{n!}{(n-k)!}$ we get

$$\begin{aligned}&Q_k(z)\prod_{m=0}^{k-1} C_{n,m}(z)\Bigg[1 + \frac{(n-k)!}{(n-k+1)!}\frac{1}{C_{n,k-1}(z)}\frac{Q_{k-1}(z)}{Q_k(z)} +\\ &\frac{(n-k)!}{(n-k+2)!}\frac{1}{C_{n,k-1}(z)C_{n,k-2}(z)}\frac{Q_{k-2}(z)}{Q_k(z)} + \ldots\\ &\ldots + \frac{(n-k)!}{(n-1)!}\frac{1}{\prod_{m=1}^{k-1} C_{n,m}(z)}\frac{Q_1(z)}{Q_k(z)}\Bigg] = \sum_{j=1}^{j_0}\alpha_{j,j}\frac{(n-k)!}{(n-j)!}.\end{aligned} \tag{3}$$

Now assume that all zeros of all p_n are uniformly bounded. Then we can take a subsequence $\{p_{n_i}\}$ such that all the corresponding root measures μ_{n_i} are weakly convergent to a compactly supported probability measure. Then all Cauchy transforms $C_{n_i,m}$ will be uniformly convergent to a non-vanishing function outside some large disc, which in particular contains all the roots of $Q_k(z)$. Since $j_0 < k$, the right-hand side of (3) tends to zero when $n \to \infty$. On the other hand, in the left-hand side of (3), all terms in the bracket except for the constant term 1 tend to zero when $n \to \infty$, and thus the limit of the left-hand side equals $\lim_{n\to\infty} Q_k(z) \prod_{m=0}^{k-1} C_{n,m}(z) = K \neq 0$, and we obtain a contradiction when $n \to \infty$. $\square$

In order to prove Theorem 2 we need the following two lemmas, where Lemma 2 is used to prove Lemma 3.

Lemma 2. *Let z_n be a root of p_n with the largest modulus r_n. Then, for any complex number z_0 such that $|z_0| = r_0 \geq r_n$, we have $|C_{n,j}(z_0)| \geq \frac{1}{2r_0}$ for all $j \geq 0$.*

Proof. Recall that $C_{n,j}(z) := \int \frac{d\mu_n^{(j)}(\zeta)}{z-\zeta} = \frac{p_n^{(j+1)}(z)}{(n-j)p_n^{(j)}(z)}$. With ζ being some root of $p_n^{(j)}(z)$ we have $|\zeta| \leq |z_0|$ by Gauss-Lucas theorem. Thus $\frac{1}{z_0-\zeta} = \frac{1}{z_0} \cdot \frac{1}{1-\zeta/z_0} = \frac{1}{z_0} \cdot \frac{1}{1-\theta}$ where $|\theta| = |\zeta/z_0| \leq 1$. With $w = \frac{1}{1-\theta}$ we obtain

$$|w-1| = \frac{|\theta|}{|1-\theta|} = |\theta||w| \leq |w| \Leftrightarrow |w-1| \leq |w| \Rightarrow Re(w) \geq 1/2,$$

and thus

$$\begin{aligned} |C_{n,j}(z_0)| &= \left| \int \frac{d\mu_n^{(j)}(\zeta)}{z_0-\zeta} \right| = \frac{1}{r_0} \left| \int \frac{d\mu_n^{(j)}(\zeta)}{1-\theta} \right| = \frac{1}{r_0} \left| \int w \, d\mu_n^{(j)}(\zeta) \right| \\ &\geq \frac{1}{r_0} \left| \int Re(w) d\mu_n^{(j)}(\zeta) \right| \geq \frac{1}{2r_0} \int d\mu_n^{(j)}(\zeta) = \frac{1}{2r_0}. \end{aligned}$$

$\square$

Lemma 3. *Let $T = \sum_{j=1}^{k} Q_j D^j = \sum_{j=1}^{k} \left(\sum_{i=0}^{\deg Q_j} \alpha_{j,i} z^i \right) D^j$ be a degenerate exactly-solvable operator of order k. Wlog we assume that Q_k is monic, i.e. $\alpha_{k,\deg Q_k} = 1$. Let z_n be a root of p_n with the largest modulus r_n. Then the following inequality holds:*

$$1 \leq \sum_{j=1}^{k-1} \sum_{i=0}^{\deg Q_j} |\alpha_{j,i}| 2^{k-j} \frac{r_n^{k-j-\deg Q_k+i}}{(n-k+1)^{k-j}} + \sum_{0 \leq i < \deg Q_k} \frac{|\alpha_{k,i}|}{r_n^{\deg Q_k - i}}. \quad (4)$$

Proof. From $C_{n,j}(z) = \frac{p_n^{(j+1)}(z)}{(n-j)p^{(j)}(z)}$ we get

$$p^{(j)}(z) = \frac{p_n^{(k)}(z)}{(n-k+1)(n-k+2)\cdots(n-j)\prod_{m=j}^{k-1} C_{n,m}(z)} \quad (5)$$

for all $j < k$. Inserting z_n in the eigenvalue equation $Tp_n(z) = \lambda_n p_n(z)$ we obtain

$$\sum_{j=1}^{k-1} \left(\sum_{i=0}^{\deg Q_j} \alpha_{j,i} z_n^i \right) p_n^{(j)}(z_n) + \left(\sum_{i=0}^{\deg Q_k} \alpha_{k,i} z_n^i \right) p_n^{(k)}(z_n) = \lambda_n p_n(z_n) = 0,$$

and after division by $z_n^{\deg Q_k} p_n^{(k)}(z_n)$ we obtain

$$\sum_{j=1}^{k-1} \left(\sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{1}{z_n^{\deg Q_k - i}} \right) \frac{p_n^{(j)}(z_n)}{p_n^{(k)}(z_n)} + \sum_{0 \leq i < \deg Q_k} \alpha_{k,i} \frac{1}{z_n^{\deg Q_k - i}} + 1 = 0.$$

Thus, applying (5) and Lemma 2, we obtain

$$\begin{aligned}
1 &= \left| \sum_{j=1}^{k-1} \left(\sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{1}{z_n^{\deg Q_k - i}} \right) \frac{p_n^{(j)}(z_n)}{p_n^{(k)}(z_n)} + \sum_{0 \leq i < \deg Q_k} \alpha_{k,i} \frac{1}{z_n^{\deg Q_k - i}} \right| \\
&\leq \sum_{j=1}^{k-1} \left| \sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{1}{z_n^{\deg Q_k - i}} \right| \frac{|p_n^{(j)}(z_n)|}{|p_n^{(k)}(z_n)|} + \sum_{0 \leq i < \deg Q_k} \frac{|\alpha_{k,i}|}{r_n^{\deg Q_k - i}} \\
&\leq \sum_{j=1}^{k-1} \sum_{i=0}^{\deg Q_j} \frac{|\alpha_{j,i}|}{r_n^{\deg Q_k - i}} \frac{1}{(n-k+1)\cdots(n-j)\prod_{m=j}^{k-1} |C_{n,m}(z_n)|} \\
&+ \sum_{0 \leq i < \deg Q_k} \frac{|\alpha_{k,i}|}{r_n^{\deg Q_k - i}} \\
&\leq \sum_{j=1}^{k-1} \sum_{i=0}^{\deg Q_j} \frac{|\alpha_{j,i}|}{r_n^{\deg Q_k - i}} \frac{(2r_n)^{k-j}}{(n-k+1)^{k-j}} + \sum_{0 \leq i < \deg Q_k} \frac{|\alpha_{k,i}|}{r_n^{\deg Q_k - i}} \\
&= \sum_{j=1}^{k-1} \sum_{i=0}^{\deg Q_j} |\alpha_{j,i}| 2^{k-j} \frac{r_n^{k-j-\deg Q_k+i}}{(n-k+1)^{k-j}} + \sum_{0 \leq i < \deg Q_k} \frac{|\alpha_{k,i}|}{r_n^{\deg Q_k - i}}.
\end{aligned}$$

$\square$

The proof of Theorem 2 follows from Theorem 1 and Lemma 3.

Proof of Theorem 2. Applying Theorem 1, we see that the last sum on the right-hand side of inequality (4) in Lemma 3 tends to zero when $n \to \infty$. Now consider the double sum on the right-hand side of (4). If, for given i and j, the exponent $(k-j-\deg Q_k+i)$ of r_n is negative or zero, the corresponding term tends to zero when $n \to \infty$ by Theorem 1. We now consider the remaining terms in the double sum, namely those for which the exponent $(k-j-\deg Q_k+i)$ of r_n is positive. If $r_n \le c_0(n-k+1)^\gamma$ where $c_0 > 0$ and $\gamma < \frac{k-j}{k-j+i-\deg Q_k}$ for given $j \in [1, k-1]$ and given $i \in [0, \deg Q_j]$, then the corresponding term

$$\frac{r_n^{k-j+i-\deg Q_k}}{(n-k+1)^{k-j}} = \left(\frac{r_n}{(n-k+1)^{\frac{k-j}{k-j+i-\deg Q_k}}}\right)^{k-j+i-\deg Q_k}$$

in the double sum tends to zero when $n \to \infty$. Thus assume that $r_n \le c_0(n-k+1)^\gamma$ where c_0 is a positive constant and $\gamma < b$ where

$$b = \overset{+}{\min_{\substack{j\in[1,k-1]\\ i\in[0,j]}}} \frac{k-j}{k-j+i-\deg Q_k} = \overset{+}{\min_{j\in[1,k-1]}} \frac{k-j}{k-j+\deg Q_j-\deg Q_k},$$

and where the notation $\min^+$ means that we only take the minimum over positive terms $(k-j+i-\deg Q_k)$ and $(k-j+\deg Q_j-\deg Q_k)$.[4] Then every term in the double sum tends to zero when $n \to \infty$, and we obtain a contradiction to (4) when $n \to \infty$. Thus for all sufficiently large integers n we must have $r_n > c_0(n-k+1)^\gamma$ for all $\gamma < b$, and hence $\liminf_{n\to\infty} \frac{r_n}{n^\gamma} > c_0$ for any $\gamma < b$. But for any such γ we can form $\gamma' = \frac{\gamma+b}{2}$ for which $\gamma' < b$ and $\gamma < \gamma'$, and thus $\lim_{n\to\infty} \frac{r_n}{n^\gamma} = \infty$ for all $\gamma < b$. $\square$

[4]On the left-hand side in the expression for b above we take the minimum over $i \in [0, \deg Q_j]$, so we can put $i = \deg Q_j$ in this expression. Thus with $b = \min^+_{j\in[1,k-1]} \frac{k-j}{k-j+\deg Q_j-\deg Q_k}$ we get that $\gamma < \frac{k-j}{k-j+i-\deg Q_k}$ for *every* $j \in [1, k-1]$ and *every* $i \in [0, \deg Q_j]$. Then if $r_n \le c_0(n-k+1)^\gamma$ and $\gamma < b$, every term with positive exponent $(k-j+i-\deg Q_k)$ will tend to zero when $n \to \infty$.

Proof of Corollary 1. For this class of operators we have

$$\begin{aligned} b &:= \overset{+}{\min_{j\in[1,k-1]}} \frac{k-j}{k-j+\deg Q_j - \deg Q_k} \\ &= \overset{+}{\min_{j\in[1,k-1]}} \frac{k-j}{k-j+\deg Q_j - j_0} = \frac{k-j_0}{k-j_0} = 1, \end{aligned}$$

and the proof is complete applying Theorem 2. $\square$

Proof of Corollary 2. For this class of operators we have

$$\begin{aligned} b &:= \overset{+}{\min_{j\in[1,k-1]}} \left(\frac{k-j}{k-j+\deg Q_j - \deg Q_k}\right) \\ &= \overset{+}{\min_{j\in[1,k-1]}} \left(\frac{k-j}{k-j+\deg Q_j}\right) = \min_{j\in[1,j_0]} \frac{k-j}{k} = \frac{k-j_0}{k} \end{aligned}$$

where the third equality follows from choosing any j for which $\deg Q_j = j$, and the minimum is then attained for $j = j_0$ (note that for $j > j_0$ we get $(k-j)/(k-j+\deg Q_j) = 1 > (k-j_0)/k$), and the proof is complete applying Theorem 2. $\square$

Remark. Note that for the class of operators considered in Corollary 1 the Main Conjecture claims that $\lim_{n\to\infty} \frac{r_n}{n} = c_T$ for some $c_T > 0$, since $d := \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right) = \frac{k-j_0}{k-j_0} = 1$ (the maximum is attained by choosing any $j > j_0$ such that $\deg Q_j = j_0$, e.g. $j = k$), and for the class of operators considered in Corollary 2 the Main Conjecture claims that $\lim_{n\to\infty} \frac{r_n}{n^{(k-j_0)/k}} = c_T$ for some $c_T > 0$, since

$$d := \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right) = \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j}\right) = \frac{k-j_0}{k}.$$

Remark. For a class of operators containing the operators considered in Corollaries 1 and 2 we can actually prove that the conjectured upper bound $\lim_{n\to\infty} \sup \frac{r_n}{n^d} \le c_1$ implies the conjectured lower bound $\lim_{n\to\infty} \inf \frac{r_n}{n^d} \ge c_0$ where $c_1 \ge c_0 > 0$, see Theorem 5 in Section 5.2.

Proof of Theorem 3. Clearly $\deg Q_{j_0} = j_0$ since there exists at least one such $j < k$. Set

$$T = Q_{j_0} D^{j_0} + Q_k D^k = \sum_{i=0}^{j_0} \alpha_{j_0,i} z^i D^{j_0} + \sum_{i=0}^{\deg Q_k} \alpha_{k,i} z^i D^k,$$

where $\alpha_{j_0,j_0} \neq 0$, and where we wlog assume that Q_k is monic. From inequality (4) in Lemma 3 we have

$$\begin{aligned} 1 &\leq \sum_{i=0}^{j_0} |\alpha_{j_0,i}| 2^{k-j_0} \frac{r_n^{i-\deg Q_k + k - j_0}}{(n-k+1)^{k-j_0}} + \sum_{0 \leq i < \deg Q_k} |\alpha_{k,i}| \frac{1}{r_n^{\deg Q_k - i}} \\ &\leq \sum_{i=0}^{j_0} |\alpha_{j_0,i}| 2^{k-j_0} \frac{r_n^{i-\deg Q_k + k - j_0}}{(n-k+1)^{k-j_0}} + \epsilon, \end{aligned}$$

where we choose n so large that $\epsilon < 1$ (this is possible since $\epsilon \to 0$ when $n \to \infty$ due to Theorem 1). Thus for sufficiently large n

$$\begin{aligned} c_0 &\leq \sum_{i=0}^{j_0} |\alpha_{j_0,i}| 2^{k-j_0} \frac{r_n^{i-\deg Q_k + k - j_0}}{(n-k+1)^{k-j_0}} \\ &\leq \sum_{i=0}^{j_0} |\alpha_{j_0,i}| 2^{k-j_0} \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j_0}} \\ &= K \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j_0}}, \end{aligned}$$

where $1 - \epsilon = c_0 \to 1$ when $n \to \infty$, and $K > 0$ since $\alpha_{j_0,j_0} \neq 0$ (the second inequality follows since $i \leq j_0$). Thus

$$r_n \geq \left(\frac{c_0}{K}\right)^{1/(k-\deg Q_k)} (n-k+1)^{\frac{k-j_0}{k-\deg Q_k}}$$

for sufficiently large integers n, and hence there exists a positive constant $c = (1/K)^{1/(k-\deg Q_k)}$ such that

$$\lim_{n\to\infty} \inf \frac{r_n}{n^{\left(\frac{k-j_0}{k-\deg Q_k}\right)}} \geq c.$$

Finally, it is clear that for this two-term operator

$$d := \max_{j \in [j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j} \right) = \frac{k-j_0}{k-\deg Q_k},$$

and we are done. $\square$

Remark. If, in Theorem 3, Q_k is a monomial (i.e. $Q_k = z^{\deg Q_k}$), then there exists a positive constant c such that $r_n \geq c(n-k+1)^d$ for *every* n, where $d := \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right) = \frac{k-j_0}{k-\deg Q_k}$. This is easily seen from the calculations in the proof of Theorem 3 since $\sum_{0\leq i<\deg Q_k}|\alpha_{k,i}|\frac{1}{r_n^{\deg Q_k - i}}$ on the right-hand side of (4) vanishes, and therefore $1 \leq K\frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j_0}}$ for *every* n. From the second part of Lemma 1 we know that for this class of operators there exists a unique eigenpolynomial p_n for every n, and the conclusion follows.

Proof of Theorem 4. For this class of operators $(j-\deg Q_j) \geq (k-\deg Q_k)$ for every $j > j_0$ and thus

$$d := \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right) = \frac{k-j_0}{k-\deg Q_k}.$$

Assuming that Q_k is monic we have the inequality

$$1 \leq \sum_{j=1}^{k-1}\sum_{i=0}^{\deg Q_j}|\alpha_{j,i}|2^{k-j}\frac{r_n^{k-j+i-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{0\leq i<\deg Q_k}\frac{|\alpha_{k,i}|}{r_n^{\deg Q_k - i}} \tag{6}$$

by Lemma 3. The last sum here tends to zero when $n \to \infty$ by Theorem 1. Considering the double sum on the right-hand side of (6) we see that for every j we have (since $i \leq \deg Q_j$) that

$$\begin{aligned}
& \sum_{i=0}^{\deg Q_j}|\alpha_{j,i}|2^{k-j}\frac{r_n^{k-j+i-\deg Q_k}}{(n-k+1)^{k-j}} \\
= & \sum_{i=0}^{\deg Q_j}|\alpha_{j,i}|2^{k-j}\frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}}r_n^{i-\deg Q_j} \\
= & \frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}}\left(2^{k-j}|\alpha_{j,\deg Q_j}| + \sum_{i<\deg Q_j}2^{k-j}|\alpha_{j,i}|r_n^{i-\deg Q_j}\right) \\
= & K_{j,n}\frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}},
\end{aligned} \tag{7}$$

where

$$K_{j,n} = 2^{k-j}|\alpha_{j,\deg Q_j}| + \sum_{i<\deg Q_j}2^{k-j}|\alpha_{j,i}|r_n^{i-\deg Q_j},$$

where $K_{j,n} > 0$ since $\alpha_{j,\deg Q_j} \neq 0$. Also, $K_{j,n} < \infty$ when $n \to \infty$ since $(i - \deg Q_j) < 0$ for the exponent of r_n, and then using Theorem 1 (note that $K_{j,n} \to 2^{k-j}|\alpha_{j,\deg Q_j}|$ when $n \to \infty$). With the decomposition
$A = \{j : \deg Q_j = j\}$,
$B = \{j : \deg Q_j < j \quad \textit{and} \quad (k - j + \deg Q_j - \deg Q_k) > 0\}$,
$C = \{j : \deg Q_j < j \quad \textit{and} \quad (k - j + \deg Q_j - \deg Q_k) \leq 0\}$,
and using (7) we see that inequality (6) is equivalent to:

$$\begin{aligned} 1 \quad &\leq \quad \sum_{j=1}^{k-1}\sum_{i=0}^{j}|\alpha_{j,i}|2^{k-j}\frac{r_n^{k-j+i-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{0\leq i<\deg Q_k}\frac{|\alpha_{k,i}|}{r_n^{\deg Q_k - i}} \\ &= \quad \sum_{j\in A}K_{j,n}\frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{j\in B}K_{j,n}\frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} \\ &+ \quad \sum_{j\in C}K_{j,n}\frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{0\leq i<\deg Q_k}\frac{|\alpha_{k,i}|}{r_n^{\deg Q_k-i}}. \end{aligned}$$

The last two sums on the right-hand side of this inequality both tend to zero when $n \to \infty$, the last one due to Theorem 1, and the sum over C since $(j - \deg Q_j) \geq (k - \deg Q_k) \Leftrightarrow (k - j + \deg Q_j - \deg Q_k) \leq 0$ for every $j \in C$ by assumption, and then applying Theorem 1. Therefore, when $n \to \infty$, we get the inequality

$$c_0 \leq \sum_{j\in A}K_{j,n}\frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{j\in B}K_{j,n}\frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} \tag{8}$$

where

$$c_0 = 1 - \sum_{j\in C}K_{j,n}\frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} - \sum_{0\leq i<\deg Q_k}\frac{|\alpha_{k,i}|}{r_n^{\deg Q_k-i}}.$$

Note that $c_0 \to 1$ when $n \to \infty$.

Now assume that B is empty. This corresponds to an operator such that $(j - \deg Q_j) \geq (k - \deg Q_k)$ for every j for which $\deg Q_j < j$. Then inequality (8) above becomes

$$
\begin{aligned}
c_0 &\leq \sum_{j\in A} K_{j,n}\frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j}} \\
&= \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j_0}}\left(K_{j_0,n} + \sum_{j\in A\setminus\{j_0\}} K_{j,n}\frac{1}{(n-k+1)^{j_0-j}}\right) \\
&\leq K_A\frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j_0}} \qquad (9)
\end{aligned}
$$

where K_A is a positive constant which is finite when $n \to \infty$, since $j_0 - j > 0$ for every $j \in A\setminus\{j_0\}$ (recall that j_0 is the largest element in A by definition). Thus for all sufficiently large integers n we have

$$
r_n \geq \left(\frac{c_0}{K_A}\right)^{1/(k-\deg Q_k)}(n-k+1)^{\frac{k-j_0}{k-\deg Q_k}},
$$

and therefore there exists a positive constant $c = (1/K_A)^{1/(k-\deg Q_k)}$ such that

$$
\lim_{n\to\infty}\inf \frac{r_n}{n^{\left(\frac{k-j_0}{k-\deg Q_k}\right)}} \geq c,
$$

and we are done.

Now assume that B is nonempty. Again inequality (8) holds, i.e.

$$
c_0 \leq \sum_{j\in A} K_{j,n}\frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{j\in B} K_{j,n}\frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}},
$$

where $c_0 \to 1$ when $n \to \infty$. From (9) we have

$$
\sum_{j\in A} K_{j,n}\frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j}} \leq K_A\frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j_0}}
$$

for the sum over A for large n and thus

$$
\begin{aligned}
c_0 &\leq \sum_{j\in A} K_{j,n}\frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{j\in B} K_{j,n}\frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} \\
&\leq K_A\frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j_0}} + \sum_{j\in B} K_{j,n}\frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} \\
&= \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j_0}}\Bigg(K_A + \sum_{j\in B} K_{j,n}\frac{r_n^{\deg Q_j-j}}{(n-k+1)^{j_0-j}}\Bigg) \\
&\leq K_{AB}\frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j_0}},
\end{aligned}
$$

where K_{AB} is a positive and finite constant when $n\to\infty$ (note that $K_{AB}\to K_A$ when $n\to\infty$, since $(\deg Q_j - j)<0$ and $j_0-j>0$ for every $j\in B$). Thus for all sufficiently large integers n we have

$$
r_n \geq \big(\frac{c_0}{K_{AB}}\big)^{1/(k-\deg Q_k)}(n-k+1)^{\frac{k-j_0}{k-\deg Q_k}},
$$

so there exists a positive constant $c=(1/K_{AB})^{1/(k-\deg Q_k)}$ such that

$$
\lim_{n\to\infty}\inf\frac{r_n}{n^{\left(\frac{k-j_0}{k-\deg Q_k}\right)}} \geq c.
$$

□

3 Main Conjecture and its Corollary

In this section we explain how we arrived at Main Conjecture (see Section 1) and obtain as a corollary of our method the (conjectural) algebraic equation satisfied by the Cauchy transform of the asymptotic root measure of the properly scaled eigenpolynomials.

How did we arrive at Main Conjecture?

Let $T=\sum_{j=1}^k Q_jD^j=\sum_{j=1}^k\big(\sum_{i=0}^{\deg Q_j}\alpha_{j,i}z^i\big)D^j$ be an arbitrary degenerate exactly-solvable operator of order k and denote by j_0 the largest j for which $\deg Q_j=j$. Wlog we assume that Q_{j_0} is monic, i.e. $\alpha_{j_0,j_0} = 1$. Consider the *scaled eigenpolynomial* $q_n(z) = p_n(n^d z)$, where $p_n(z)$ is the unique and monic nth degree eigenpolynomial of T, and d is some real number. The goal is now to obtain a well-defined algebraic equation for the Cauchy transform of the root measure μ_n of the scaled eigenpolynomial q_n when $n \to \infty$, and as we will see in the process of doing this, we are forced to choose d as in Main Conjecture.[5]

Basic assumption. When performing our calculations we assume that the root measures $\mu_n^{(0)}, \mu_n^{(1)}, \mu_n^{(2)} \dots, \mu_n^{(k-1)}$ of the *scaled eigenpolynomial* $q_n(z)$ and its derivatives up to the kth order exist when $n \to \infty$ and that they are are all weakly convergent to the *same* asymptotic root measure μ.[6] Thus the corresponding Cauchy transforms are all asymptotically identical, and we define $C(z) := \lim_{n\to\infty} C_{n,j}(z)$ for all $j \in [0, k-1]$, where $C(z)$ is the Cauchy transform of μ and is considered for z's away from the support of μ. Computer experiments strongly indicate that this assumption is true - for details see Section 4.2.

From the definition of the Cauchy transform we obtain

$$\begin{aligned}
\prod_{i=0}^{j-1} C_{n,i}(z) &= \prod_{i=0}^{j-1} \frac{q_n^{(i+1)}(z)}{(n-j)q_n^{(i)}(z)} \\
&= \frac{q_n^{(1)}(z)}{nq_n(z)} \cdot \frac{q_n^{(2)}(z)}{(n-1)q^{(1)}(z)} \cdot \frac{q_n^{(3)}(z)}{(n-2)q_n^{(2)}(z)} \cdots \\
&\quad \cdots \frac{q_n^{(j-1)}(z)}{(n-j+2)q_n^{(j-2)}(z)} \cdot \frac{q_n^{(j)}(z)}{(n-j+1)q_n^{(j-1)}(z)} \\
&= \frac{q_n^{(j)}(z)}{n(n-1)\cdots(n-j+1)q_n(z)},
\end{aligned}$$

[5] It is already well-known that for the Laguerre polynomials, which appear as eigenpolynomials for a second order exactly-solvable operator, the largest root grows as n when $n \to \infty$ and thus $d = 1$ in this case, which is consistent with Main Conjecture.

[6] Conjecturally supp μ is a tree, see Section 6 on Open Problems.

and thus the basic assumption implies

$$C^j(z) = \lim_{n\to\infty} \prod_{i=0}^{j-1} C_{n,i}(z) = \lim_{n\to\infty} \frac{q_n^{(j)}(z)}{n(n-1)\cdots(n-j+1)q_n(z)}. \quad (10)$$

In the above notation consider the eigenvalue equation $Tp_n(z) = \lambda_n p_n(z)$, where the eigenvalue λ_n is given by

$$\lambda_n = \sum_{j=1}^{k} \alpha_{j,j} \frac{n!}{(n-j)!} = \sum_{j=1}^{j_0} \alpha_{j,j} \frac{n!}{(n-j)!} = \sum_{j=1}^{j_0} \alpha_{j,j} n(n-1)\cdots(n-j+1).$$

Clearly this sum ends at j_0 since $\alpha_{j,j} = 0$ for all $j > j_0$ by definition of j_0 as the largest j for which $\deg Q_j = j$. We then have

$$Tp_n(z) = \lambda_n p_n(z)$$

$$\Leftrightarrow$$

$$\sum_{j=1}^{k} \left(\sum_{i=0}^{\deg Q_j} \alpha_{j,i} z^i \right) p_n^{(j)}(z) = \sum_{j=1}^{j_0} \alpha_{j,j} n(n-1)\cdots(n-j+1) p_n(z).$$

Substituting $z = n^d z$ in this equation we obtain

$$\sum_{j=1}^{k} \left(\sum_{i=0}^{\deg Q_j} \alpha_{j,i} n^{di} z^i \right) p_n^{(j)}(n^d z) = \sum_{j=1}^{j_0} \alpha_{j,j} n(n-1)\cdots(n-j+1) p_n(n^d z),$$

and with $q_n(z) = p_n(n^d z)$ we get

$$\sum_{j=1}^{k} \left(\sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n^{d(j-i)}} \right) q_n^{(j)}(z) = \sum_{j=1}^{j_0} \alpha_{j,j} n(n-1)\cdots(n-j+1) q_n(z).$$

Dividing this equation by $\frac{n!}{(n-j_0)!} q_n(z) = n(n-1)\cdots(n-j_0+1)q_n(z)$ we get

$$\begin{aligned} &\sum_{j=1}^{k} \left(\sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n^{d(j-i)}} \right) \frac{q_n^{(j)}(z)}{n(n-1)\cdots(n-j_0+1)q_n(z)} = \\ &= \sum_{j=1}^{j_0} \alpha_{j,j} \frac{n(n-1)\cdots(n-j+1)}{n(n-1)\cdots(n-j_0+1)}. \end{aligned} \quad (11)$$

Consider the right-hand side of (11). Since $j \le j_0$ all terms for which $j < j_0$ (if not already zero, which is the case if $\alpha_{j,j} = 0$, i.e. if $\deg Q_j < j$) tend to zero when $n \to \infty$, and therefore the limit of the right-hand side of (11) equals

$$\lim_{n\to\infty} \sum_{j=1}^{j_0} \alpha_{j,j} \frac{n(n-1)\cdots(n-j+1)}{n(n-1)\cdots(n-j_0+1)} = \alpha_{j_0,j_0} = 1,$$

since we assumed that Q_{j_0} is monic. Now consider the jth term in the sum on the left-hand side of (11). It equals

$$\begin{aligned}
&\sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n^{d(j-i)}} \cdot \frac{q_n^{(j)}(z)}{n(n-1)\cdots(n-j_0+1)q_n(z)} = \\
=\ &\sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n^{d(j-i)}} \cdot \frac{q_n^{(j)}(z)}{n(n-1)\cdots(n-j+1)q_n(z)} \cdot \frac{n\cdots(n-j+1)}{n\cdots(n-j_0+1)} \\
=\ &\sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n^{d(j-i)}} \cdot \prod_{i=0}^{j-1} C_{n,i}(z) \cdot \frac{n(n-1)\cdots(n-j+1)}{n(n-1)\cdots(n-j_0+1)} \\
=\ &\sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n^{d(j-i)+j_0-j}} \prod_{i=0}^{j-1} C_{n,i}(z) \frac{n\cdots(n-j+1)}{n^j} \frac{n^{j_0}}{n\cdots(n-j_0+1)}.
\end{aligned}$$

Taking the limit and using the basic assumption (10) we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n^{d(j-i)}} \cdot \frac{q_n^{(j)}(z)}{n(n-1)\cdots(n-j_0+1)q_n(z)} \\
=\ &\lim_{n\to\infty} \sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n^{d(j-i)+j_0-j}} C^j(z)
\end{aligned}$$

for the jth term and thus, taking the limit of the left-hand side of (11) we get

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{j=1}^{k} \left(\sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n^{d(j-i)}} \right) \frac{q_n^{(j)}(z)}{n(n-1)\cdots(n-j_0+1)q_n(z)} \\
=\ &\lim_{n\to\infty} \sum_{j=1}^{k} \left(\sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n^{d(j-i)+j_0-j}} \right) C^j(z).
\end{aligned}$$

Adding up, the following equation is satisfied by $C(z)$ for z's away from the support of μ:

$$\lim_{n\to\infty}\sum_{j=1}^{k}\Big(\sum_{i=0}^{\deg Q_j}\alpha_{j,i}\frac{z^i}{n^{d(j-i)+j_0-j}}\Big)C^j(z)=1. \tag{12}$$

In order to make (12) a well-defined algebraic equation, i.e. to avoid infinities in the denominator when $n\to\infty$, we must impose the following condition on the real number d in the exponent of n, namely

$$d(j-i)+j_0-j\geq 0 \quad\Leftrightarrow\quad d\geq\frac{j-j_0}{j-i}$$

for all $j\in[1,k]$ and all $i\in[0,\deg Q_j]$. Therefore we take $d=\max_{\substack{j\in[1,k]\\ i\in[0,\deg Q_j]}}\left(\frac{j-j_0}{j-i}\right)$, but this maximum is clearly obtained for the maximal value of i for any given j, so we may as well put $i=\deg Q_j$. Our condition then becomes $d=\max_{j\in[1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)$, and clearly the maximum is taken only over j for which $Q_j(z)$ is not identically zero. Finally we observe that since T is degenerate we have $j_0<k$ and thus we need only take this maximum over $j\in[j_0+1,k]$, since there always exists a positive value on d for any operator of the type we consider. Thus our condition becomes:

$$d=\max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right).$$

Corollary of Main Conjecture.

In the above notation (recall that $\alpha_{j_0,j_0}=1$ by monicity of Q_{j_0}), the following well-defined algebraic equation follows immediately when inserting d as defined in Main Conjecture into equation (12) and letting $n\to\infty$:

Corollary. *The Cauchy transform $C(z)$ of the asymptotic root measure μ of the scaled eigenpolynomial $q_n(z)=p_n(n^dz)$ of an exactly-solvable operator T as above with Q_{j_0} monic satisfies the following algebraic equation for almost all complex z in the usual Lebesgue measure on $\mathbb{C}$:*

$$z^{j_0}C^{j_0}(z)+\sum_{j\in A}\alpha_{j,\deg Q_j}z^{\deg Q_j}C^j(z)=1,$$

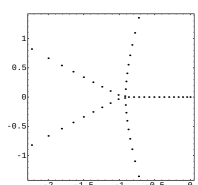
where A is the set consisting of all j for which the maximum $d := \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)$ is attained, i.e.
$A = \{j : (j - j_0)/(j - \deg Q_j) = d\}$.

For detailed calculations see Section 5.1.

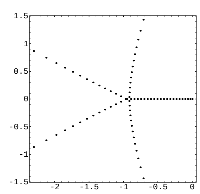
4 Numerical evidence

4.1 Evidence for Main Conjecture

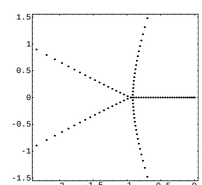
In the table on the last page of this section we present numerical evidence on the growth of $r_n = \max\{|z| : p_n(z) = 0\}$ which supports the choice of d in Main Conjecture. We have performed similar computer experiments for a large number of other degenerate exactly-solvable operators, and the results are in all cases consistent with Main Conjecture. Next we present some typical pictures on the zero distribution of the appropriately scaled eigenpolynomial $q_n(z) = p_n(n^d z)$ for some degenerate exactly-solvable operators, where p_n denotes the nth degree unique and monic polynomial eigenfunction of the given operator T. Conjecturally the zeros of $q_n(z)$ are contained in a compact set when $n \to \infty$.



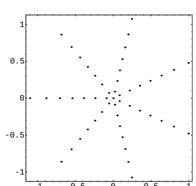
T_1, roots of $q_{50}(z) = p_{50}(50z)$



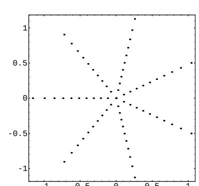
T_1, roots of $q_{75}(z) = p_{75}(75z)$



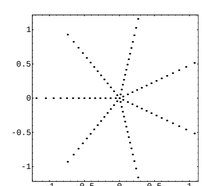
T_1, roots of $q_{100}(z) = p_{100}(100z)$



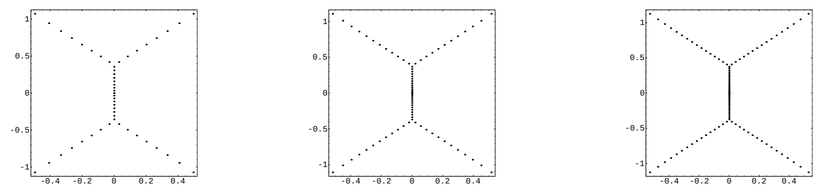
T_2, roots of $q_{50}(z) = p_{50}(50^{5/7}z)$



T_2, roots of $q_{75}(z) = p_{75}(75^{5/7}z)$



T_2, roots of $q_{100}(z) = p_{100}(100^{5/7}z)$



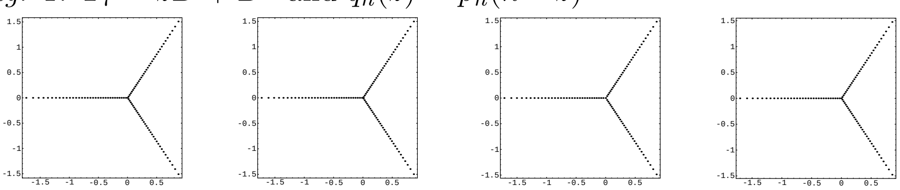
T_3, roots of $q_{50}(z) = p_{50}(50^{1/2}z)$ T_3, roots of $q_{75}(z) = p_{75}(75^{1/2}z)$ T_3, roots of $q_{100}(z) = p_{100}(100^{1/2}z)$

where $T_1 = zD + zD^2 + zD^3 + zD^4 + zD^5$, $T_2 = z^2D^2 + D^7$ and $T_3 = z^3D^3 + z^2D^4 + zD^5$.

4.2 On the basic assumption

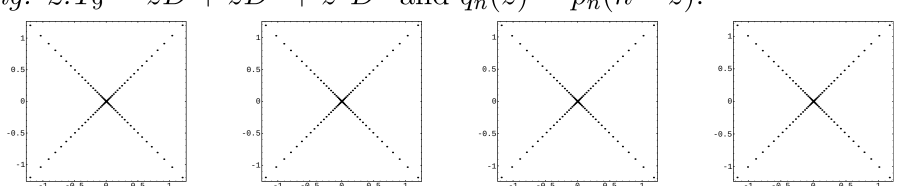
Below examples supporting the basic assumption, namely that the root measures of $q_n(z)$ and its derivatives up to the kth order exist when $n \to \infty$ and are all weakly convergent to the same measure μ.

Fig. 1: $T_7 = zD + D^3$ and $q_n(z) = p_n(n^{2/3}z)$

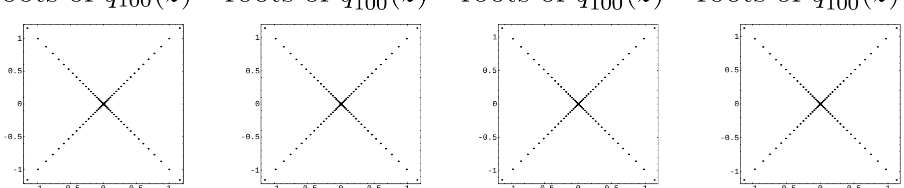


roots of $q_{100}(z)$ roots of $q'_{100}(z)$ roots of $q''_{100}(z)$ roots of $q'''_{100}(z)$

Fig. 2:$T_9 = zD + zD^4 + z^3D^7$ and $q_n(z) = p_n(n^{3/2}z)$.



roots of $q_{100}(z)$ roots of $q'_{100}(z)$ roots of $q''_{100}(z)$ roots of $q'''_{100}(z)$



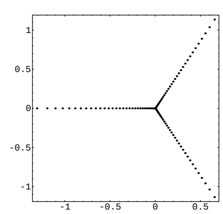
roots of $q^{(iv)}_{100}(z)$ roots of $q^{(v)}_{100}(z)$ roots of $q^{(vi)}_{100}(z)$ roots of $q^{(vii)}_{100}(z)$

4.3 On the Corollary of Main Conjecture

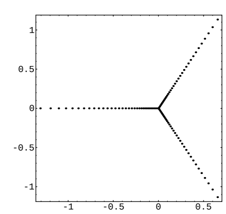
The algebraic equation in the corollary of Main Conjecture satisfied by the Cauchy transform of the asymptotic root measure of the scaled eigenpolynomial indicates that the asymptotic zero distribution depends only on the term $z^{j_0}D^{j_0}$ and the term(s) $\alpha_{j,\deg Q_j} z^{\deg Q_j} D^j$ for which $d = \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)$ is attained.[7] Thus any term Q_jD^j in T for which $j < j_0$ or such that $(j-j_0)/(j-\deg Q_j) < d$ is (conjecturally) irrelevant for the zero distribution when $n \to \infty$.

To illustrate this fact we now present some pictures of the zero distributions of the scaled eigenpolynomials for some *distinct* operators for which the Cauchy transforms of the corresponding scaled eigenpolynomials q_n satisfy the *same* equation when $n \to \infty$, namely the equation in the Corollary of Main Conjecture in Section 3.

As a first example consider the operator $T_4 = z^3D^3 + z^2D^5$. Here $d = \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right) = (5-3)/(5-2) = 2/3$, the corresponding scaled eigenpolynomial is $q_n(z) = p_n(n^{2/3}z)$, and we have $z^3C^3 + z^2C^5 = 1$ for the Cauchy transform of q_n when $n \to \infty$. Now consider the slightly modified operator $\widetilde{T}_4 = z^2D^2 + z^3D^3 + zD^4 + z^2D^5 + D^6$ and note that d is obtained again (only) for $j = 5$ (for $j = 4$ we have $(4-3)/(4-1) = 1/3 < 2/3$ and for $j = 6$ we have $(6-3)/(6-0) = 3/6 = 1/2 < 2/3$). We therefore obtain the same Cauchy transform equation as for T_4, and hence the terms z^2D^2, zD^4 and D^6 in $\widetilde{T}_4$ can be considered as irrelevant for the zero distribution for sufficiently large n. The pictures below clearly illustrate this.



T_4, roots of $q_{100}(z) = p_{100}(100^{2/3}z)$

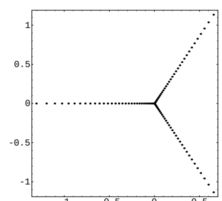


$\tilde{T}_4$, roots of $q_{100}(z) = p_{100}(100^{2/3}z)$

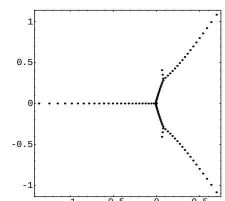
where $T_4 = z^3D^3 + z^2D^5$ and $\widetilde{T}_4 = z^2D^2 + z^3D^3 + zD^4 + z^2D^5 + D^6$.

[7]Recall that we normalized T by letting Q_{j_0} be monic.

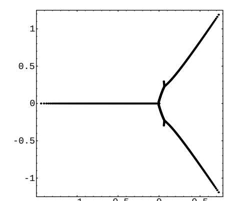
However, instead of D^6, we may add the more "disturbing" term zD^6 to T_4. Note that for the operator $\overline{T}_4 = z^2D^2 + z^3D^3 + zD^4 + z^2D^5 + zD^6$ for $j = 6$ we have $(6-3)/(6-1) = 3/5 = 0.6 < 2/3$. Adding any term Q_jD^j such that $(j-j_0)/(j-\deg Q_j) < d$ to a given operator, it is clear that the closer the value of $(j-j_0)/(j-\deg Q_j)$ is to d (in this case $2/3$), the more disturbing it is in the sense that it requires larger n for the corresponding zero distributions to coincide. See pictures below, where $q_n(z) = p_n(n^{2/3}z)$:



T_4, roots of $q_{100}(z)$



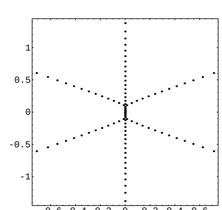
$\overline{T}_4$, roots of $q_{100}(z)$



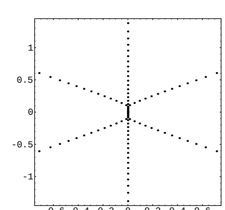
$\overline{T}_4$, roots of $q_{600}(z)$

Increasing n however, experiments indicate that the zero distributions of the scaled eigenpolynomials of T_4 and $\overline{T}_4$ coincide, as they (conjecturally) should.

As a second example, consider $T_5 = z^5D^5 + z^4D^6 + z^2D^8$ and $\widetilde{T}_5 = z^2D^2 + z^5D^5 + z^4D^6 + zD^7 + z^2D^8$, whose scaled eigenpolynomials $q_n(z) = p_n(n^{1/2}z)$ both satisfy the Cauchy transform equation $z^5C^5 + z^4C^6 + z^2C^8 = 1$ when $n \to \infty$. In the pictures below we see that the terms z^2D^2 and zD^7 of $\widetilde{T}_5$ seem to have no effect on the zero distribution for large n:

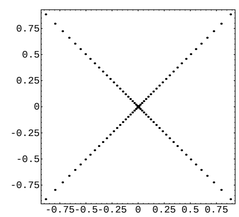


T_5, roots of $q_{100}(z)$

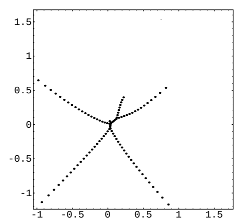


$\widetilde{T}_5$, roots of $q_{100}(z)$

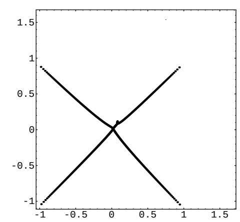
Finally note that for j_0 and for any j for which d is attained, it is only the highest degree term $\alpha_{j,\deg Q_j} z^{\deg Q_j}$ of Q_j that is involved in the Cauchy transform equation. Consider for example the following case, where adding lower degree terms to $\alpha_{j,\deg Q_j} z^{\deg Q_j}$ in the (relevant) Q_j seems to have no effect on the zero distribution for large n. Below $T_6 = z^3 D^3 + z^2 D^6$, and $\widetilde{T}_6 = [(1+13i)+(24i-3)z+11iz^2+z^3]D^3+[(22i-13)+(-9-14i)z+z^2]D^6$, and thus $q_n(z) = p_n(n^{3/4}z)$. Note the difference in scaling between the pictures:



T_6, roots of $q_{100}(z)$



$\widetilde{T}_6$, roots of $q_{100}(z)$



$\widetilde{T}_6$, roots of $q_{500}(z)$

Operator	n	r_n experimental	r_n conjectured
$T_1 = zD + zD^2 + zD^3 + zD^4 + zD^5$	50	$2.7 \cdot 50^{0.967595}$	$c_1 \cdot 50^1$
	100	$2.7 \cdot 100^{0.984180}$	$c_1 \cdot 100^1$
	200	$2.7 \cdot 200^{0.992557}$	$c_1 \cdot 200^1$
	250	$2.7 \cdot 250^{0.994272}$	$c_1 \cdot 250^1$
$T_2 = z^2D^2 + D^7$	50	$1.3 \cdot 50^{0.671977}$	$c_2 \cdot 50^{5/7}$
	100	$1.3 \cdot 100^{0.694847}$	$c_2 \cdot 100^{5/7}$
	200	$1.3 \cdot 200^{0.706226}$	$c_2 \cdot 200^{5/7}$
	300	$1.3 \cdot 300^{0.710085}$	$c_2 \cdot 300^{5/7}$
	400	$1.3 \cdot 400^{0.712043}$	$c_2 \cdot 400^{5/7}$
$T_3 = z^3D^3 + z^2D^4 + zD^5$	50	$4/3 \cdot 50^{0.469007}$	$c_3 \cdot 50^{1/2}$
	100	$4/3 \cdot 100^{0.484824}$	$c_3 \cdot 100^{1/2}$
	200	$4/3 \cdot 200^{0.492832}$	$c_3 \cdot 200^{1/2}$
	300	$4/3 \cdot 300^{0.495592}$	$c_3 \cdot 300^{1/2}$
	400	$4/3 \cdot 400^{0.497009}$	$c_3 \cdot 400^{1/2}$
$T_4 = z^3D^3 + z^2D^5$	50	$1.4 \cdot 50^{0.633226}$	$c_4 \cdot 50^{2/3}$
	100	$1.4 \cdot 100^{0.652141}$	$c_4 \cdot 100^{2/3}$
	200	$1.4 \cdot 200^{0.661412}$	$c_4 \cdot 200^{2/3}$
	300	$1.4 \cdot 300^{0.664511}$	$c_4 \cdot 300^{2/3}$
	400	$1.4 \cdot 400^{0.666066}$	$c_4 \cdot 400^{2/3}$
$\widetilde{T}_4 = z^2D^2 + z^3D^3 + zD^4 + z^2D^5 + D^6$	50	$1.4 \cdot 50^{0.632811}$	$\tilde{c}_4 \cdot 50^{2/3}$
	100	$1.4 \cdot 100^{0.651960}$	$\tilde{c}_4 \cdot 100^{2/3}$
	200	$1.4 \cdot 200^{0.661332}$	$\tilde{c}_4 \cdot 200^{2/3}$
	300	$1.4 \cdot 300^{0.664461}$	$\tilde{c}_4 \cdot 300^{2/3}$
	400	$1.4 \cdot 400^{0.666030}$	$\tilde{c}_4 \cdot 400^{2/3}$
$T_5 = z^5D^5 + z^4D^6 + z^2D^8$	50	$1.5 \cdot 50^{0.462995}$	$c_5 \cdot 50^{1/2}$
	100	$1.5 \cdot 100^{0.481684}$	$c_5 \cdot 100^{1/2}$
	200	$1.5 \cdot 200^{0.491066}$	$c_5 \cdot 200^{1/2}$
	300	$1.5 \cdot 300^{0.494304}$	$c_5 \cdot 300^{1/2}$
	400	$1.5 \cdot 400^{0.495971}$	$c_5 \cdot 400^{1/2}$
$\widetilde{T}_5 = z^2D^2 + z^5D^5 + z^4D^6 + zD^7 + z^2D^8$	50	$1.5 \cdot 50^{0.463391}$	$\tilde{c}_5 \cdot 50^{1/2}$
	100	$1.5 \cdot 100^{0.481837}$	$\tilde{c}_5 \cdot 100^{1/2}$
	200	$1.5 \cdot 200^{0.491129}$	$\tilde{c}_5 \cdot 200^{1/2}$
	300	$1.5 \cdot 300^{0.494342}$	$\tilde{c}_5 \cdot 300^{1/2}$
	400	$1.5 \cdot 400^{0.495998}$	$\tilde{c}_5 \cdot 400^{1/2}$
$T_6 = z^3D^3 + z^2D^6$	50	$1.4 \cdot 50^{0.702117}$	$c_6 \cdot 50^{3/4}$
	100	$1.4 \cdot 100^{0.725715}$	$c_6 \cdot 100^{3/4}$
	200	$1.4 \cdot 200^{0.737541}$	$c_6 \cdot 200^{3/4}$
	300	$1.4 \cdot 300^{0.741614}$	$c_6 \cdot 300^{3/4}$
	400	$1.4 \cdot 400^{0.743713}$	$c_6 \cdot 400^{3/4}$
$\widetilde{T}_6 = [(1+13i) + (24i-3)z + 11iz^2 + z^3]D^3$ $+[(22i-13) - (9+14i)z + z^2]D^6$	50	$1.4 \cdot 50^{0.769260}$	$\tilde{c}_6 \cdot 50^{3/4}$
	100	$1.4 \cdot 100^{0.760399}$	$\tilde{c}_6 \cdot 100^{3/4}$
	200	$1.4 \cdot 200^{0.756161}$	$\tilde{c}_6 \cdot 200^{3/4}$
	300	$1.4 \cdot 300^{0.754590}$	$\tilde{c}_6 \cdot 300^{3/4}$
	400	$1.4 \cdot 400^{0.753765}$	$\tilde{c}_6 \cdot 400^{3/4}$

5 Appendix

5.1 Arriving at the Corollary of Main Conjecture

The algebraic equation in the corollary of Main Conjecture follows immediately from inserting d as defined in Main Conjecture into equation (12) in Section 3 and letting $n \to \infty$. If we put d into (12) we namely get

$$\sum_{j=1}^{k} \left(\sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n^{\max_{j \in [j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j} \right)(j-i)+j_0-j}} \right) C^j(z) = 1. \quad (13)$$

Denote by $N_{j,i}$ the exponent of n in (13) for given j and i. Thus

$$N_{j,i} = \max_{j \in [j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j} \right)(j-i) + j_0 - j.$$

The terms in (13) for which this exponent is positive tend to zero as $n \to \infty$.

First we consider j for which $\deg Q_j = j$, and denote, as usual, by j_0 the largest such j. If $j = j_0$, then $i \leq \deg Q_{j_0} = j_0$ and thus for $j = j_0$ and $i = j_0$ we get

$$\begin{aligned} N_{j_0,j_0} &= \max_{j \in [j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j} \right)(j-i) + j_0 - j \\ &= \max_{j \in [j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j} \right)(j_0-j_0) + j_0 - j_0 = 0, \end{aligned}$$

and for $j = j_0$ and $i < j_0$ we have

$$\begin{aligned} N_{j_0,i} &= \max_{j \in [j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j} \right)(j-i) + j_0 - j \\ &> \max_{j \in [j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j} \right)(j_0-j_0) + j_0 - j_0 = 0. \end{aligned}$$

Thus $N_{j_0,j_0} = 0$ and $N_{j_0,i} > 0$ for $i < j_0$, and for the term corresponding to $j = j_0$ in (13) we get

$$\sum_{i=0}^{j_0} \alpha_{j_0,i} \frac{z^i}{n^{\max_{j \in [j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j} \right)(j_0-i)+j_0-j_0}} C^{j_0}(z) \to z^{j_0} C^{j_0}(z)$$

when $n \to \infty$ (recall that $\alpha_{j_0,j_0} = 1$).
Now let j be such that $\deg Q_j = j$ and $j < j_0$. Then $i \leq \deg Q_j = j$ and

$$\begin{aligned} N_{j,j} &= \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j \\ &= \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-j)+j_0-j = j_0-j>0, \end{aligned}$$

and for $i < j$ we get

$$\begin{aligned} N_{j,i} &= \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j \\ &> \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-j)+j_0-j = j_0-j>0, \end{aligned}$$

that is $N_{j,i} > 0$ for all $j < j_0$ such that $\deg Q_j = j$ and for all $i \leq j$. Thus for the corresponding terms in (13) we get

$$\sum_{j\in\{j<j_0:\deg Q_j=j\}} \sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n^{\max_{j\in[j_0+1,j]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j}} C^j(z) \to 0$$

when $n \to \infty$ for every $j < j_0$ for which $\deg Q_j = j$.

Now denote by j_m the j for which $d = \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)$ is attained. Note that there may be several distinct j for which this maximum is attained.[8] Then

$$\begin{aligned} N_{j_m,degQ_{j_m}} &= \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j \\ &= \left(\frac{j_m-j_0}{j_m-\deg Q_{j_m}}\right)(j_m-\deg Q_{j_m})+j_0-j_m \\ &= j_m-j_0+j_0-j_m=0, \end{aligned}$$

[8]Consider for example the Laplace type operator (that is with all polynomial coefficients Q_j linear) $T = zD + zD^2 + \ldots zD^k$. Here $j_0 = 1$ and the equation satisfied by the Cauchy transform of the asymptotic root measure of the scaled eigenpolynomial $q_n(z) = p_n(nz)$ is given by $zC(z) + zC^2(z) + \ldots zC^k(z) = 1$, since $d = \max_{j\in[2,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right) = 1$ is attained for *every* $j = 2, 3, \ldots k$.

and for $i < \deg Q_{j_m}$ we get

$$\begin{aligned} N_{j_m,i} &= \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j \\ &> \left(\frac{j_m-j_0}{j_m-\deg Q_{j_m}}\right)(j_m-\deg Q_{j_m})+j_0-j_m \\ &= j_m-j_0+j_0-j_m=0, \end{aligned}$$

i.e. $N_{j_m,\deg Q_{j_m}}=0$ and $N_{j_m,i}>0$ for $i<\deg Q_{j_m}$, and for the term corresponding to $j=j_m$ in (13) we get

$$\sum_{i=0}^{\deg Q_{j_m}} \alpha_{j_m,i}\frac{z^i}{n^{\max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j_m-i)+j_0-j_m}}C^{j_m}(z)\to$$

$$\to \alpha_{j_m,\deg Q_{j_m}} z^{\deg Q_{j_m}} C^{j_m}(z)$$

when $n\to\infty$. In case of several j for which d is attained, we put $A=\{j:(j-j_0)/(j-\deg Q_j)=d\}$, and for the corresponding terms in (13) we get

$$\sum_{j\in A}\sum_{i=0}^{\deg Q_j}\alpha_{j,i}\frac{z^i}{n^{\max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j}}C^j(z)\to$$

$$\to\sum_{j\in A}\alpha_{j,\deg Q_j}z^{\deg Q_j}C^j(z)$$

when $n\to\infty$. Now consider the remaining terms in (13), namely terms for which $j<j_0$ such that $\deg Q_j<j$, terms for which $j_0<j<j_m$, and terms for which $j_m<j\le k$ (clearly this last case does not exist if $j_m=k$).

We start with $j<j_0$ such that $\deg Q_j<j$. Then $i\le\deg Q_j<j$ and

$$\begin{aligned} N_{j,i} &= \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j \\ &> \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-j)+j_0-j=j_0-j>0, \end{aligned}$$

and for the corresponding terms in (13) we have

$$\sum_{j\in\{j<j_0:\deg Q_j<j\}} \sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n^{\max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j}} C^j(z) \to 0$$

when $n \to \infty$.

Now assume that $j_m < k$ and consider $j_m < j \le k$. Clearly $j_m > j_0$ since the maximum is taken over $j \in [j_0+1, k]$, and therefore $i \le \deg Q_j < j$ for $j_m < j \le k$. Also,

$$\max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right) = \left(\frac{j_m - j_0}{j_m - \deg Q_{j_m}}\right) > \left(\frac{j-j_0}{j-\deg Q_j}\right),$$

since the maximum is attained for j_m by assumption. Thus we get

$$\begin{aligned} N_{j,i} &= \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j \\ &= \left(\frac{j_m - j_0}{j_m - \deg Q_{j_m}}\right)(j-i)+j_0-j \\ &> \left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j \\ &\ge \left(\frac{j-j_0}{j-\deg Q_j}\right)(j-\deg Q_j)+j_0-j \\ &= j-j_0+j_0-j = 0, \end{aligned}$$

i.e. $N_{j,i} > 0$ for every $j_m < j \le k$ and every $i \le \deg Q_j$. For the corresponding terms in (13) we therefore get

$$\sum_{j_m<j\le k} \sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n^{\max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j}} C^j(z) \to 0$$

as $n \to \infty$.

Finally we consider $j_0 < j < j_m$. Note that this also covers the case $j_{m_1} < j < j_{m_2}$ where the maximum d is attained for both j_{m_1}

and j_{m_2}. Since $i \leq \deg Q_j < j$ we get

$$\begin{aligned}
N_{j,i} &= \max_{j \in [j_0+1,k]} \left(\frac{j - j_0}{j - \deg Q_j} \right) (j-i) + j_0 - j \\
&= \left(\frac{j_m - j_0}{j_m - \deg Q_{j_m}} \right) (j-i) + j_0 - j \\
&> \left(\frac{j - j_0}{j - \deg Q_j} \right) (j-i) + j_0 - j \\
&\geq \left(\frac{j - j_0}{j - \deg Q_j} \right) (j - \deg Q_j) + j_0 - j \\
&= j - j_0 + j_0 - j = 0,
\end{aligned}$$

i.e. $N_{j,i} > 0$ for every $j_0 < j < j_m$ and every $i \leq \deg Q_j$. Thus for the corresponding terms in (13) we get

$$\sum_{j_0 < j < j_m} \sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n^{\max_{j \in [j=+1,k]0} \left(\frac{j-j_0}{j-\deg Q_j} \right) (j-i) + j_0 - j}} C^j(z) \to 0$$

when $n \to \infty$.

Adding up these results we finally get the following equation by letting $n \to \infty$ in equation (13):

$$z^{j_0} C^{j_0}(z) + \sum_{j \in A} \alpha_{j,\deg Q_j} z^{\deg Q_j} C^j(z) = 1,$$

where j_0 is the largest j for which $\deg Q_j = j$, and A is the set consisting of all j for which the maximum $d = \max_{j \in [j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j} \right)$ is attained, i.e. $A = \{j : (j - j_0)/(j - \deg Q_j) = d\}$.

5.2 Theorem 5

Here we prove that for a class of operators containing the operators considered in Corollaries 1 and 2 the conjectured upper bound $\lim_{n\to\infty} \sup(r_n/n^d) \leq c_1$ implies the conjectured lower bound $\lim_{n\to\infty} \inf(r_n/n^d) \geq c_0$ for some constants $c_1 \geq c_0 > 0$ and where d is as in Main Conjecture. This follows automatically from inequality (4) in Lemma 3. We have the following

Theorem 5. *Let T be a degenerate exactly-solvable operator of order k which satisfies the condition*

$$b = \min_{j\in[1,k-1]}^{+} \left(\frac{k-j}{k-j+\deg Q_j - \deg Q_k} \right) = \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j} \right) = d$$

where the notation $\min^{+}$ *means that the minimum is taken only over positive values of* $(k-j+\deg Q_j - \deg Q_k)$*. Assume that the inequality* $r_n \le c_1(n-k+1)^d$ *holds for some positive constant* c_1 *for all sufficiently large* n*. Then there exists a positive constant* $c_0 \le c_1$ *such that* $r_n \ge c_0(n-k+1)^d$ *holds for all sufficiently large* n*. Thus*

$$\lim_{n\to\infty} \sup \frac{r_n}{n^d} \le c_1 \quad \Rightarrow \quad \lim_{n\to\infty} \inf \frac{r_n}{n^d} \ge c_0.$$

Proof. From inequality (4) in Lemma 3 we have

$$\begin{aligned} 1 \ &\le \ \sum_{j=1}^{k-1} \sum_{i=0}^{\deg Q_j} |\alpha_{j,i}| 2^{k-j} \frac{r_n^{k-j+i-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{0\le i<\deg Q_k} \frac{|\alpha_{k,i}|}{r_n^{\deg Q_k - i}} \\ &\le \ \sum_{j=1}^{k-1} K_j \frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{0\le i<i_k} \frac{|\alpha_{k,i}|}{r_n^{i_k-i}} \qquad (14) \end{aligned}$$

where the K_j are positive constants. The second sum on the right-hand side of (14) tends to zero when $n \to \infty$ due to Theorem 1. We now decompose the first sum on the right-hand side of (14) into three parts. Namely, let

$A = \{j : \frac{k-j}{k-j+\deg Q_j - \deg Q_k} = d\}$, and note that $(k-j+\deg Q_j - \deg Q_k) > 0$ here since $d > 0$.

$B = \{j : \frac{k-j}{k-j+\deg Q_j - \deg Q_k} > d\}$, and note that $(k-j+\deg Q_j - \deg Q_k) > 0$ here since $d > 0$.

$C = \{j : (k-j+\deg Q_j - \deg Q_k) \le 0\}$, and note that $j < k$ in (14).

Clearly due to the condition $b = d$ there are no terms for which $\frac{k-j}{k-j+\deg Q_j - \deg Q_k} < d$ and $(k-j+\deg Q_j - \deg Q_k) > 0$ both hold .

If $j \in A$ then

$$\frac{r_n^{k-j+\deg Q_k-\deg Q_k}}{(n-k+1)^{k-j}} = \left(\frac{r_n}{(n-k+1)^d}\right)^{k-j+\deg Q_j-\deg Q_k}$$

for the corresponding terms in the sum on the right-hand side of (14).

If $j \in B$ then $d(k-j+\deg Q_j - \deg Q_k) < (k-j)$, and this inequality together with the upper bound $r_n \le c_1(n-k+1)^d$ which we assume holds for all sufficiently large n, gives us

$$\frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} \le \frac{c_1(n-k+1)^{d(k-j+\deg Q_j-\deg Q_k)}}{(n-k+1)^{k-j}} \to 0$$

when $n \to \infty$ for the corresponding terms in (14).

If $j \in C$ then $(k-j+\deg Q_j - \deg Q_k) \le 0$ and we get

$$\frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} \to 0$$

when $n \to \infty$ for the corresponding terms in (14) due to Theorem 1. Note that if $(k-j+\deg Q_j-\deg Q_k) = 0$ the corresponding term tends to zero when $n \to \infty$ since $j < k$ in (14).

With this decomposition of the first sum on the right-hand side of the last inequality in (14) we can write

$$\begin{aligned}
1 &\le \sum_{j=1}^{k-1} K_j \frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{0\le i<i_k} \frac{|\alpha_{k,i}|}{r_n^{i_k-i}} \\
&\le \sum_{j\in A} K_j \left(\frac{r_n}{(n-k+1)^d}\right)^{k-j+\deg Q_j-\deg Q_k} \\
&+ \sum_{j\in B} K_j \frac{c_1(n-k+1)^{d(k-j+\deg Q_j-\deg Q_k)}}{(n-k+1)^{k-j}} \\
&+ \sum_{j\in C} K_j \frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{0\le i<i_k} \frac{|\alpha_{k,i}|}{r_n^{i_k-i}}
\end{aligned}$$

where the last three sums tend to zero when $n \to \infty$ by the above arguments (and the last one due to Theorem 1).
Thus for all sufficiently large n there exists a positive constant c' such that

$$c' \leq \sum_{j \in A} K_j \left(\frac{r_n}{(n-k+1)^d} \right)^{k-j+\deg Q_j - \deg Q_k} \tag{15}$$

where $c' \to 1$ when $n \to \infty$. If the set A contains precisely one element, then the sum in (15) consists of one single term, and we are done: there exists a positive constant $c_0 = (c'/K_j)^{1/(k-j+\deg Q_j - \deg Q_k)}$ such that $r_n \geq c_0(n-k+1)^d$ for all sufficiently large n, and thus $\lim_{n\to\infty} \inf(r_n/n^d) \geq c_0$.

But clearly, for some operators A will contain more than one element. If this is the case define $m := \min_{j\in A}(k-j+\deg Q_j - \deg Q_k)$ and denote by j_m the corresponding j for which this minimum is attained. Using the upper bound $r_n \leq c_1(n-k+1)^d$ we then get the following inequality from (15):

$$\begin{aligned}
c' \quad &\leq \quad \sum_{j\in A} K_j \left(\frac{r_n}{(n-k+1)^d} \right)^{k-j+\deg Q_j - \deg Q_k} = K_{j_m} \left(\frac{r_n}{(n-k+1)^d} \right)^m \\
&+ \quad \sum_{j \in A\setminus\{j_m\}} K_j \left(\frac{r_n}{(n-k+1)^d} \right)^m \left(\frac{r_n}{(n-k+1)^d} \right)^{k-j+\deg Q_j - \deg Q_k - m} \\
&\leq \quad K_{j_m} \left(\frac{r_n}{(n-k+1)^d} \right)^m \\
&+ \quad \sum_{j \in A\setminus\{j_m\}} K_j \left(\frac{r_n}{(n-k+1)^d} \right)^m c_1^{k-j+\deg Q_j - \deg Q_k - m} \\
&= \quad \left(\frac{r_n}{(n-k+1)^d} \right)^m \left(K_{j_m} + \sum_{j \in A\setminus\{j_m\}} K_j \cdot c_1^{k-j+\deg Q_j - \deg Q_k - m} \right) \\
&= \quad \left(\frac{r_n}{(n-k+1)^d} \right)^m \cdot K
\end{aligned}$$

where $K > 0$. Thus $r_n \geq \left(\frac{c'}{K}\right)^{1/m}(n-k+1)^d$ for all sufficiently large n, and therefore there exists a positive constant $c_0 = (1/K)^{1/m}$ (recall that $c' \to 1$ when $n \to \infty$) such that $\lim_{n\to\infty} \inf \frac{r_n}{n^d} \geq c_0$. $\square$

Remark. About the last inequality following from (15) in the above proof, consider for example the operator $T = zD + D^2 + zD^3 + zD^4$. In this case, by Lemma 3, we have

$$1 \leq \sum_{j=1}^{3} \frac{2^{4-j} r_n^{3-j+\deg Q_j}}{(n-3)^{4-j}} = 8\frac{r_n^3}{(n-3)^3} + 4\frac{r_n}{(n-3)^2} + 2\frac{r_n}{(n-3)},$$

where r_n is the largest modulus of all roots of the unique and monic eigenpolynomial of T. For this operator $d = 1$ and we see that $(4-j)/(3-j+\deg Q_j) = d$ for $j = 1$ and for $j = 3$. Now, assuming that $r_n \leq c_1(n-3)$ holds for some positive constant c_1 for large n, our inequality becomes

$$\begin{aligned} 1 &\leq 8\frac{r_n^3}{(n-3)^3} + 4\frac{r_n}{(n-3)^2} + 2\frac{r_n}{(n-3)} \\ &\leq 8\frac{r_n}{(n-3)} \cdot \frac{c_1^2(n-3)^2}{(n-3)^2} + 4\frac{c_1(n-3)}{(n-3)^2} + 2\frac{r_n}{(n-3)} \\ &= (8c_1^2 + 2)\frac{r_n}{(n-3)} + \frac{4c_1}{(n-3)} \end{aligned}$$

where the last term tends to zero as $n \to \infty$. Thus $r_n \geq c_0(n-3)$ for sufficiently large choices on n, where $c_0 = 1/(8c_1^2 + 2)$, and hence $\lim_{n\to\infty} \inf(r_n/n) \geq c_0$.

6 Open Problems

1. The main challenge is to obtain a complete proof of the Main Conjecture, see Introduction. This proof requires both the sharp upper and lower bounds of the largest root. The upper bound can apparently be obtained by a detailed study of the corresponding Riccati equation at ∞. If the Main Conjecture is settled then to achieve its corollary (the Cauchy transform equation) one can use a technique similar to that of [2] to prove the basic assumption, see Section 3.

2. As suggested by one of the referees, estimates similar to that of Main Conjecture can be formulated for the sequence of roots $z_{n,i}$ of p_n such that $\lim_{n\to\infty} \frac{r_{n,i}}{r_{n,n}} = \alpha$ where $0 < \alpha < 1$ and $|z_{n,i}| = r_{n,i}$.

3. In the case of orthogonal polynomials there is a number of results describing the growth of the largest modulus r_n of the roots as an expansion in powers of n, see e.g. [11] and [12] and references therein. In the present paper we conjectured the form of the leading term of r_n in our more general setup. As suggested by one of the referees, the question about the lower terms in the expansion of r_n is natural in our context as well.

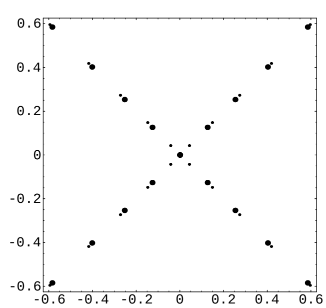
4. Operators of the type we consider occur in the theory of Bochner-Krall orthogonal systems, i.e. families of polynomials which are both eigenfunctions of some finite order differential operator and orthogonal with respect to some suitable inner product. A lot is known about the asymptotic zero distribution of orthogonal polynomials, and by comparing such known results with results on the asymptotic zero distribution of eigenpolynomials of degenerate exactly-solvable operators, we believe it will be possible to gain new insight into the nature of BKS. We have previously used our results from [2] to prove a special case of a general conjecture describing the leading terms of all Bochner-Krall operators, see [3]. Another problem relevant for BKS is to describe all exactly-solvable operators whose eigenpolynomials have real roots only.

5. Numerical evidence indicates that the roots of the scaled eigenpolynomials fill certain curves in the complex plane. The support of the limiting root measure μ seems to be a tree. This is the case for the non-degenerate exactly-solvable operators which we treated in [2], but then without such a scaling of the eigenpolynomials. By a tree we mean a connected compact subset Γ of $\mathbb{C}$ which consists of a finite union of analytis curves and where $\hat{\mathbb{C}} \setminus \Gamma$ is simply connected. The (conjectural) algebraic equation satisfied by the Cauchy transform contains a lot of information about μ, and it remains to describe its support explicitly.

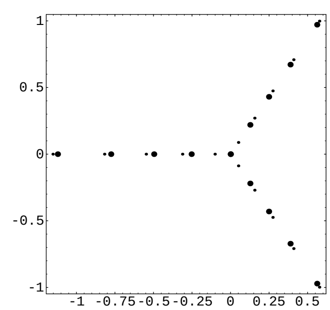
6. Conjecturally the support of the asymptotic zero distribution of the scaled eigenpolynomial q_n is the union of a finite number of analytic curves in the complex plane which we denote by Ξ_T, i.e. $\Xi_T = \text{supp } \mu$, where μ is the limiting root measure of q_n. Then the following conjecture seems to be quite plausible[9].

Conjecture 1. [Interlacing property] *For any family $\{q_n\}$ of appropriately scaled eigenpolynomials of a degenerate exactly-solvable operator, the zeros of any two consecutive polynomials q_{n+1} and q_n interlace along Ξ_T for all sufficiently large integers n.*

When defining the interlacing property some caution is required since the zeros of q_n do not lie exactly on Ξ_T. Thus identify some sufficiently small neighbourhood $N(\Xi_T)$ of Ξ_T with the normal bundle to Ξ_T by equipping $N(\Xi_T)$ with the projection onto Ξ_T along the fibres which are small curvilinear segments orthogonal to Ξ_T. We then say that two sets of points in $N(\Xi_T)$ interlace if their orthogonal projections on Ξ_T interlace in the usual sense. If Ξ_T has singularities one should first remove some sufficiently small neighbourhoods of these singularities and then proceed as above on the remaining part of Ξ_T. Conjecture 1 thus states that for any sufficiently small neighbourhood $N(\Xi_T)$ of Ξ_T there exists a number n_0 such that the interlacing property holds for the zeros of q_n and q_{n+1} for all $n \geq n_0$. We conclude this section by showing some pictures illustrating the interlacing property. Below, small dots represent the roots of q_{n+1} and large dots represent the roots of q_n for some fixed n.

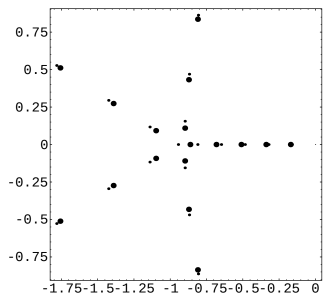


$T = z^2D^2 + z^3D^3 + zD^5$,
roots of q_{25} and q_{24}.

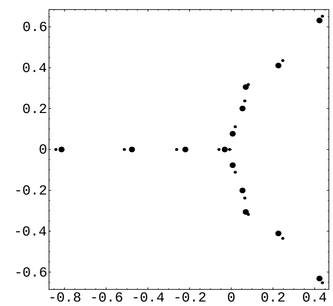


$T = zD + z^2D^2 + D^3$,
roots of q_{20} and q_{19}.

[9]The question concerning interlacing was raised by B. Shapiro. Also see [1].



$T = zD + zD^2 + zD^3 + zD^4 + zD^5$, roots of q_{23} and q_{22}.



$T = z^3D^3 + z^2D^5 + zD^6$, roots of q_{20} and q_{19}.

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PAPER V

On Roots of Eigenpolynomials for Degenerate Exactly-Solvable Differential Operators

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submitted

Abstract

In this paper we partially settle our conjecture from [1] on the roots of eigenpolynomials for degenerate exactly-solvable operators. Namely, for any such operator we establish a lower bound (which supports our conjecture) for the largest modulus of all roots of its unique and monic eigenpolynomial p_n as the degree n tends to infinity. The main theorem below thus extends earlier results obtained in [1] for a restrictive class of operators.

1 Introduction

We are interested in roots of eigenpolynomials satisfying certain linear differential equations. Namely, consider an operator

$$T = \sum_{j=1}^{k} Q_j D^j$$

where $D = d/dz$ and the Q_j are complex polynomials in one variable satisfying the condition $\deg Q_j \leq j$, with equality for at least one j, and in particular $\deg Q_k < k$ for the leading term. Such operators are referred to as *degenerate exactly-solvable operators*[1], see [1]. We are interested in eigenpolynomials of T, that is polynomials satisfying

$$T(p_n) = \lambda_n p_n \tag{1}$$

for some value of the spectral parameter λ_n, where n is a positive integer and $\deg p_n = n$. The importance of studying eigenpolynomials for these operators is among other things motivated by numerous

[1]Correspondingly, operators for which $\deg Q_k = k$ are called *non-degenerate exactly-solvable operators*. We have treated roots of eigenpolynomials for these operators in [2].

examples coming from classical orthogonal polynomials, such as the Laguerre and Hermite polynomials, which appear as solutions to (1) for certain choices on the polynomials Q_j when $k = 2$. Note however that for the operators considered here the sequence of eigenpolynomials $\{p_n\}$ is in general *not* an orthogonal system.

Let us briefly recall our previous results:

A. In [2] we considered eigenpolynomials of *non-degenerate exactly-solvable operators*, that is operators of the above type but with the condition $\deg Q_k = k$ for the leading term. We proved that when the degree n of the unique and monic eigenpolynomial p_n tends to infinity, the roots of p_n stay in a compact set in $\mathbb{C}$ and are distributed according to a certain probability measure which is supported by a tree and which depends only on the leading polynomial Q_k.

B. In [1] we studied eigenpolynomials of *degenerate exactly-solvable operators* ($\deg Q_k < k$). We proved that there exists a unique and monic eigenpolynomial p_n for all sufficiently large values on the degree n, and that the largest modulus of the roots of p_n tends to infinity when $n \to \infty$. We also presented an explicit conjecture and partial results on the growth of the largest root. Namely,

Conjecture (from [1]). *Let $T = \sum_{j=1}^{k} Q_j D^j$ be a degenerate exactly-solvable operator of order k and denote by j_0 the largest j for which $\deg Q_j = j$. Let $r_n = \max\{|\alpha| : p_n(\alpha) = 0\}$, where p_n is the unique and monic nth degree eigenpolynomial of T. Then*

$$\lim_{n\to\infty} \frac{r_n}{n^d} = c_0,$$

where $c_0 > 0$ is a positive constant and

$$d := \max_{j\in[j_0+1,k]} \left(\frac{j - j_0}{j - \deg Q_j} \right).$$

Extensive computer experiments listed in [1] confirm the existence of such a constant c_0. Now consider the scaled eigenpolynomial $q_n(z) = p_n(n^d z)$. We construct the probability measure μ_n by placing a point mass of size $1/n$ at each zero of q_n. Numerical evidence indicates that for each degenerate exactly-solvable operator

T, the sequence $\{\mu_n\}$ converges weakly to a probability measure μ_T which is (compactly) supported by a tree. In [1] we deduced the algebraic equation satisfied by the Cauchy transform of μ_T.[2] Namely, let

$$T = \sum_{j=1}^{k} Q_j(z) D^j = \sum_{j=1}^{k} \Big(\sum_{i=0}^{\deg Q_j} q_{j,i} z^i \Big) D^j$$

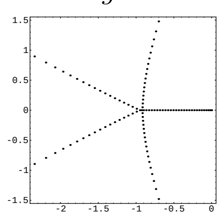
and denote by j_0 the largest j for which $\deg Q_j = j$. Assuming without loss of generality that Q_{j_0} is monic, i.e. $q_{j_0,j_0} = 1$, we have

$$z^{j_0} C^{j_0}(z) + \sum_{j \in A} q_{j,\deg Q_j} z^{\deg Q_j} C^j(z) = 1$$

where $C(z) = \int \frac{d\mu_T(\zeta)}{z-\zeta}$ is the Cauchy transform of μ_T and $A = \{j : (j - j_0)/(j - \deg Q_j) = d\}$, where d is defined in the conjecture.

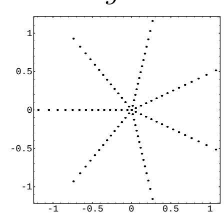
Below we present some typical pictures of the distribution of roots of the scaled eigenpolynomial $q_n(z) = p_n(n^d z)$ for some degenerate exactly-solvable operators.

Fig.1:



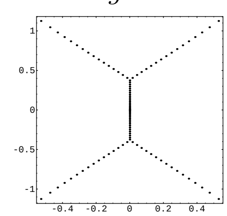
roots of
$q_{100}(z) = p_{100}(100z)$

Fig.2:



roots of
$q_{100}(z) = p_{100}(100z)$

Fig.3:



roots of
$q_{100}(z) = p_{100}(100z)$

Fig.1: $T_1 = zD + zD^2 + zD^3 + zD^4 + zD^5$.
Fig.2: $T_2 = z^2 D^2 + D^7$.
Fig.3: $T_3 = z^3 D^3 + z^2 D^4 + zD^5$.

In this paper we extend the results from [1] by establishing a lower bound for the largest modulus r_n of the roots of p_n for any degenerate exactly-solvable operator and which supports the above

[2]It remains to prove the existence of μ_T and to describe its support explicitly, see Open Problems and Conjectures.

conjecture.[3] This is our main result:

Main Theorem. *Let* $T = \sum_{j=1}^{k} Q_j D^j$ *be a degenerate exactly-solvable operator and denote by* j_0 *the largest* j *for which* $\deg Q_j = j$. *Let* p_n *be the unique and monic* n*th degree eigenpolynomial of* T *and* $r_n = \max\{|\alpha| : p_n(\alpha) = 0\}$. *Then there exists an integer* n_0 *and a positive constant* $s > 0$ *such that*

$$r_n \geq s \cdot n^d$$

for all $n > n_0$, *where*

$$d := \max_{j \in [j_0+1,k]} \Big(\frac{j - j_0}{j - \deg Q_j}\Big).$$

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2 Proofs

Lemma 1. *For any monic polynomial* $p(z)$ *of degree* $n \geq 2$ *for which all the zeros are contained in a disc of radius* $A \geq 1$, *there exists an integer* $n(j)$ *and an absolute constant* C_j *depending only on* j, *such that for every* $j \geq 1$ *and every* $n \geq n(j)$ *we have*

$$\frac{1}{C_j} \cdot \frac{n^j}{A^j} \leq \left|\left|\frac{p^{(j)}(z)}{p(z)}\right|\right|_{2A} \leq C_j \cdot \frac{n^j}{A^j} \tag{2}$$

where $p^{(j)}(z)$ *denotes the* j*th derivative of* $p(z)$, *and where we have used the maximum norm* $||p(z)||_{2A} = \max_{|z|=2A} |p(z)|$.

Remark. The right-hand side of the above inequality actually holds for all $n \geq 2$, whereas the left-hand side holds for all $n \geq n(j)$.

Proof. To obtain the inequality on the *right-hand side* we use the notation $p(z) = \prod_{i=1}^{n}(z-\alpha_i)$ where by assumption $|\alpha_i| \leq A$ for every complex root of $p(z)$. Then $p^{(j)}(z)$ is the sum of $n(n-1)\cdots(n-j+1)$

[3]It is still an open problem to prove the upper bound.

terms, each being the product of $(n-j)$ factors $(z-\alpha_i)$.[4] Thus $p^{(j)}(z)/p(z)$ is the sum of $n(n-1)\cdots(n-j+1)$ terms, each equal to 1 divided by a product consisting of $n-(n-j)=j$ factors $(z-\alpha_i)$. If $|z|=2A$ we get $|z-\alpha_i|\geq A \Rightarrow \frac{1}{|z-\alpha_i|}\leq \frac{1}{A}$, and thus

$$||\frac{p^{(j)}}{p}||_{2A} \leq \frac{n(n-1)\cdots(n-j+1)}{A^j} \leq C_j \cdot \frac{n^j}{A^j}.$$

Here we can choose $C_j=1$ for all j, but we refrain from doing this since we will need C_j large enough to obtain the constant $1/C_j$ in the left-hand side inequality. To prove the *left-hand side* inequality we will need inequalities (i)-(iv) below, where we need (i) to prove (ii), and we need (ii) and (iii) to prove (iv), from which the left-hand side inequality of this lemma follows.

For every $j\geq 1$ we have

(i) $\quad ||\frac{d}{dz}\big(\frac{p^{(j)}(z)}{p(z)}\big)||_{2A} \leq j\cdot\frac{n^j}{A^{j+1}}$.

For every $j\geq 1$ there exists a positive constant C'_j depending only on j, such that

(ii) $\quad \left|\left|\frac{p^{(j)}}{p}-\frac{(p')^j}{p^j}\right|\right|_{2A} \leq C'_j\cdot\frac{n^{j-1}}{A^j}$.

(iii) $\quad ||\frac{p'}{p}||_{2A} \geq \frac{n}{3A}$.

For every $j\geq 1$ there exists a positive constant C''_j and some integer $n(j)$ such that for all $n\geq n(j)$ we have

(iv) $\quad ||\frac{p^{(j)}}{p}||_{2A} \geq C''_j\cdot\frac{n^j}{A^j}$.

To prove (i), let $p(z)=\prod_{i=1}^n(z-\alpha_i)$, where $|\alpha_i|\leq A$ for each complex root α_i of $p(z)$. Then again $p^{(j)}(z)/p(z)$ is the sum of $n(n-1)\cdots(n-j+1)$ terms and each term equals 1 divided by a product consisting of j factors $(z-\alpha_i)$. Differentiating *each* such

[4] Differentiating $p(z)=\prod_{i=1}^n(z-\alpha_i)$ once yields $\binom{n}{1}=n$ terms each term being a product of $(n-1)$ factors $(z-\alpha_i)$, differentiating once again we obtain $n\binom{n-1}{1}=n(n-1)$ terms, each being the product of $(n-2)$ factors $(z-\alpha_i)$, etc.

term we obtain a sum of j terms each being on the form (-1) divided by a product consisting of $(j+1)$ factors $(z-\alpha_i)$.[5] Thus $\frac{d}{dz}\big(\frac{p^{(j)}(z)}{p(z)}\big)$ is a sum consisting of $j \cdot n(n-1)\cdots(n-j+1)$ terms, each on the form (-1) divided by $(j+1)$ factors $(z-\alpha_i)$. Using $\frac{1}{|z-\alpha_i|} \leq \frac{1}{A}$ for $|z| = 2A$ since $|\alpha_i| \leq A$ for all $i \in [1,n]$, we thus get

$$||\frac{d}{dz}(\frac{p^{(j)}(z)}{p(z)})||_{2A} \leq \frac{j \cdot n(n-1)\cdots(n-j+1)}{A^{j+1}} \leq j \cdot \frac{n^j}{A^{j+1}}.$$

To prove (ii) we use (i) and induction over j. The case $j=1$ is trivial since $\frac{p'}{p} - \frac{(p')^1}{p^1} = 0$. If we put $j=1$ in (i) we get $\big|\big|\frac{d}{dz}\big(\frac{p'}{p}\big)\big|\big|_{2A} \leq \frac{n}{A^2}$. But $\frac{d}{dz}\big(\frac{p'}{p}\big) = \frac{p^{(2)}}{p} - \frac{(p')^2}{p^2}$, and thus $\big|\big|\frac{p^{(2)}}{p} - \frac{(p')^2}{p^2}\big|\big| \leq \frac{n}{A^2}$, so (ii) holds for $j=2$. We now proceed by induction. Assume that (ii) holds for some $j = p \geq 2$, i.e. $\big|\big|\frac{p^{(p)}}{p} - \frac{(p')^p}{p^p}\big|\big|_{2A} \leq C'_p \cdot \frac{n^{p-1}}{A^p}$. Also note that with $j = p$ in (i) we have

$$||\frac{p^{(p+1)}}{p} - \frac{p^{(p)} \cdot p'}{p^2}||_{2A} = ||\frac{d}{dz}(\frac{p^{(p)}}{p})||_{2A} \leq p \cdot \frac{n^p}{A^{p+1}},$$

and also $||\frac{p'}{p}||_{2A} \leq \frac{n}{A}$ (from the right-hand side inequality of this lemma). Thus we have

$$\begin{aligned}
\left|\left|\frac{p^{(p+1)}}{p} - \frac{(p')^{p+1}}{p^{p+1}}\right|\right|_{2A} &= \left|\left|\frac{p^{(p+1)}}{p} - \frac{p^{(p)} \cdot p'}{p^2} + \frac{p^{(p)} \cdot p'}{p^2} - \frac{(p')^{p+1}}{p^{p+1}}\right|\right|_{2A} \\
&\leq \left|\left|\frac{p^{(p+1)}}{p} - \frac{p^{(p)} \cdot p'}{p^2}\right|\right|_{2A} \\
&+ \left|\left|\frac{p'}{p}\left(\frac{p^{(p)}}{p} - \frac{(p')^p}{p^p}\right)\right|\right|_{2A} \\
&\leq p \cdot \frac{n^p}{A^{p+1}} + \frac{n}{A} \cdot C'_p \cdot \frac{n^{p-1}}{A^p} \\
&= (p + C'_p) \cdot \frac{n^p}{A^{p+1}} = C'_{p+1} \cdot \frac{n^p}{A^{p+1}}.
\end{aligned}$$

[5] With $D = d/dz$ consider for example $D\frac{1}{\prod_{i=1}^{j}(z-\alpha_i)} = \frac{-1 \cdot D\prod_{i=1}^{j}(z-\alpha_i)}{\prod_{i=1}^{j}(z-\alpha_i)^2}$, which is a sum of j terms, each being on the form (-1) divided by a product consisting of $2j-(j-1) = (j+1)$ factors $(z-\alpha_i)$.

To prove (iii) observe that $\frac{p'(z)}{p(z)} = \sum_{i=1}^n \frac{1}{(z-\alpha_i)} = \sum_{i=1}^n \frac{1}{z} \cdot \frac{1}{1-\frac{\alpha_i}{z}}$. By assumption $|\alpha_i| \le A$ for all complex roots α_i of $p(z)$, so for $|z| = 2A$ we have $|\frac{\alpha_i}{z}| \le \frac{A}{2A} = \frac{1}{2}$ for all $i \in [1, n]$. Writing $w_i = \frac{1}{1-\frac{\alpha_i}{z}}$ we obtain

$$|w_i - 1| = \left| \frac{1}{1-\frac{\alpha_i}{z}} - \frac{1-\frac{\alpha_i}{z}}{1-\frac{\alpha_i}{z}} \right| = \frac{\left|\frac{\alpha_i}{z}\right|}{\left|1-\frac{\alpha_i}{z}\right|} \le \frac{1}{2}|w_i|,$$

which implies

$$Re\left(\frac{1}{1-\frac{\alpha_i}{z}}\right) = Re(w_i) \ge \frac{2}{3} \quad \forall i \in [1, n] \Rightarrow Re\left(\sum_{i=1}^n \frac{1}{1-\frac{\alpha_i}{z}}\right) \ge \frac{2n}{3}.$$

Thus

$$\begin{aligned}
\left|\left|\frac{p'(z)}{p(z)}\right|\right|_{2A} &= \max_{|z|=2A} \left|\frac{p'(z)}{p(z)}\right| = \max_{|z|=2A} \frac{1}{|z|} \cdot \left| \sum_{i=1}^n \frac{1}{1-\frac{\alpha_i}{z}} \right| \\
&\ge \frac{1}{2A} \cdot \left| \sum_{i=1}^n \frac{1}{1-\frac{\alpha_i}{z}} \right|_{2A} \ge \frac{1}{2A} \cdot Re\left(\sum_{i=1}^n \frac{1}{1-\frac{\alpha_i}{z}} \right) \\
&\ge \frac{n}{3A}.
\end{aligned}$$

To prove (iv) we note that from (iii) we obtain $\left|\left|\left(\frac{p'}{p}\right)^j\right|\right|_{2A} \ge \frac{n^j}{3^j A^j}$, and this together with (ii) yields

$$\begin{aligned}
\left|\left|\frac{p^{(j)}}{p}\right|\right|_{2A} &= \left|\left| \left(\frac{p'}{p}\right)^j + \frac{p^{(j)}}{p} - \left(\frac{p'}{p}\right)^j \right|\right|_{2A} \\
&\ge \left|\left| \left(\frac{p'}{p}\right)^j \right|\right|_{2A} - \left|\left| \frac{p^{(j)}}{p} - \left(\frac{p'}{p}\right)^j \right|\right|_{2A} \\
&\ge \frac{n^j}{3^j A^j} - C'_j \cdot \frac{n^{j-1}}{A^j} = \frac{n^j}{A^j}\left(\frac{1}{3^j} - \frac{C'_j}{n}\right) \ge C''_j \cdot \frac{n^j}{A^j},
\end{aligned}$$

where C''_j is a positive constant such that $C''_j \le \left(\frac{1}{3^j} - \frac{C'_j}{n}\right)$ for all $n \ge n(j)$. The left-hand side inequality in this lemma now follows from (iv) if we choose the constant C_j on right-hand side inequality so large that $\frac{1}{C_j} \le C''_j$. $\square$

To prove Main Theorem we will need the following lemma, which follows from Lemma 1:

Lemma 2. Let $0 < s < 1$ and $d > 0$ be real numbers. Let $p(z)$ be any monic polynomial of degree $n \geq 2$ such that all its zeros are contained in a disc of radius $A = s \cdot n^d$, and let $Q_j(z)$ be arbitrary polynomials. Then there exists some positive integer n_0 and positive constants K_j such that

$$\frac{1}{K_j} \cdot n^{d(\deg Q_j - j)+j} \cdot \frac{s^{\deg Q_j}}{s^j} \leq \left|\left| Q_j(z) \cdot \frac{p^{(j)}}{p} \right|\right|_{2sn^d} \leq K_j \cdot n^{d(\deg Q_j - j)+j} \cdot \frac{s^{\deg Q_j}}{s^j}$$

for every $j \geq 1$ and all $n \geq \max(n_0, n(j))$, where $n(j)$ is as in Lemma 1.

Proof. Let $Q_j(z) = \sum_{i=0}^{\deg Q_j} q_{j,i} z^i$. Then for $|z| = 2A >> 1$ we have

$$|Q(z)|_{2A} = |q_{j,\deg Q_j}| 2^{\deg Q_j} A^{\deg Q_j} \left(1 + O(\frac{1}{A})\right).$$

Since $A = s \cdot n^d$ there exists some integer n_0 such that $n \geq n_0 \Rightarrow A \geq A_0 >> 1$, and thus by Lemma 1 there exists a positive constant K_j such that the following inequality holds for all $n \geq \max(n(j), n_0)$ and all $j \geq 1$:

$$\frac{1}{K_j} \cdot \frac{n^j}{A^j} \cdot A^{\deg Q_j} \leq \left|\left| Q_j(z) \cdot \frac{p^{(j)}}{p} \right|\right|_{2A} \leq K_j \cdot \frac{n^j}{A^j} \cdot A^{\deg Q_j}.$$

Inserting $A = s \cdot n^d$ in this inequality we obtain

$$\frac{1}{K_j} \cdot \frac{n^j}{s^j n^{dj}} \cdot s^{\deg Q_j} n^{d \cdot \deg Q_j} \leq \left|\left| Q_j(z) \cdot \frac{p^{(j)}}{p} \right|\right|_{2sn^d} \leq K_j \cdot \frac{n^j}{s^j n^{dj}} \cdot s^{\deg Q_j} n^{d \cdot \deg Q_j}$$

$$\Leftrightarrow$$

$$\frac{1}{K_j} \cdot n^{d(\deg Q_j - j)+j} \cdot \frac{s^{\deg Q_j}}{s^j} \leq \left|\left| Q_j(z) \cdot \frac{p^{(j)}}{p} \right|\right|_{2sn^d} \leq K_j \cdot n^{d(\deg Q_j - j)+j} \cdot \frac{s^{\deg Q_j}}{s^j}$$

for every $j \geq 1$ and all $n \geq \max(n_0, n(j))$. $\square$

Proof of Main Theorem. Let $d = \max_{j \in [j_0+1,k]} \left(\frac{j - j_0}{j - \deg Q_j}\right)$ where

j_0 is the largest j for which $\deg Q_j = j$ in the degenerate exactly-solvable operator $T = \sum_{j=1}^{k} Q_j D^j$, where $Q_j(z) = \sum_{i=0}^{\deg Q_j} q_{j,i} z^i$. Let $p_n(z)$ be the nth degree unique and monic eigenpolynomial of T and denote by λ_n the corresponding eigenvalue. Then the eigenvalue equation can be written

$$\sum_{j=1}^{k} Q_j(z) \cdot \frac{p_n^{(j)}(z)}{p_n(z)} = \lambda_n \tag{3}$$

where $\lambda_n = \sum_{j=1}^{j_0} q_{j,j} \cdot \frac{n!}{(n-j)!}$. We will now use the result in Lemma 2 to estimate each term in (3).

* Denote by j_m the largest j for which d is attained. Then $d = (j_m - j_0)/(j_m - \deg Q_{j_m}) \Rightarrow d(\deg Q_{j_m} - j_m) + j_m = j_0$, and $j_m - \deg Q_{j_m} = (j_m - j_0)/d$. By Lemma 2 we have:

$$\frac{1}{K_{j_m}} \cdot n^{j_0} \cdot \frac{1}{s^{\frac{j_m - j_0}{d}}} \leq \left\| Q_{j_m}(z) \cdot \frac{p^{(j_m)}}{p} \right\|_{2sn^d} \leq K_{j_m} \cdot n^{j_0} \cdot \frac{1}{s^{\frac{j_m - j_0}{d}}}. \tag{4}$$

Note that the exponent of s is positive since $j_m > j_0$ and $d > 0$. In what follows we will only need the left-hand side of the above inequality.

* Consider the remaining (if there are any) $j_0 < j < j_m$ for which d is attained. For such j we have (using the right-hand side inequality of Lemma 2):

$$\begin{aligned} \left\| Q_j(z) \cdot \frac{p^{(j)}}{p} \right\|_{2sn^d} &\leq K_j n^{j_0} \cdot \frac{1}{s^{\frac{j-j_0}{d}}} = K_j n^{j_0} \cdot \frac{1}{s^{\frac{j_m - j_0}{d}}} \cdot s^{\frac{j_m - j}{d}} \\ &\leq K_j n^{j_0} \cdot \frac{1}{s^{\frac{j_m - j_0}{d}}} \cdot s^{1/d} \end{aligned} \tag{5}$$

where we have used that $(j_m - j) \geq 1$ and $s < 1 \Rightarrow s^{(j_m - j)/d} \leq s^{1/d}$.

* Consider all $j_0 < j \leq k$ for which d is *not* attained. Then $(j - \deg Q_j) > 0$ and $(j - j_0)/(j - \deg Q_j) < d \Rightarrow d(\deg Q_j - j) + j < j_0$ and we can write $d(\deg Q_j - j) + j \leq j_0 - \delta$ where $\delta > 0$. Then we

have:

$$\left\|Q_j(z)\cdot\frac{p^{(j)}}{p}\right\|_{2sn^d} \leq K_j\cdot n^{d(\deg Q_j-j)+j}\cdot\frac{s^{\deg Q_j}}{s^j}\leq K_j\cdot n^{j_0-\delta}\cdot\frac{s^{\deg Q_j}}{s^j}$$
$$\leq K_j\cdot n^{j_0-\delta}\cdot\frac{1}{s^k}, \qquad (6)$$

where the last inequality follows since $\deg Q_j\geq 0\Rightarrow s^{\deg Q_j}\leq s^0=1$ and $j\leq k\Rightarrow s^j\geq s^k$ since $0<s<1$.

* For $j=j_0$ by definition $\deg Q_{j_0}=j_0$ and thus:

$$\left\|Q_{j_0}(z)\cdot\frac{p^{(j_0)}}{p}\right\|_{2sn^d}\leq K_{j_0}\cdot n^{d(\deg Q_{j_0}-j_0)+j_0}\cdot\frac{s^{\deg Q_{j_0}}}{s^{j_0}}=K_{j_0}\cdot n^{j_0}. \quad (7)$$

* Now consider all $1\leq j\leq j_0-1$. Since $n\geq n_0\Rightarrow A=sn^d>>1$ we get $(sn^d)^{j-\deg Q_j}\geq 1$ and thus:

$$\left\|Q_j(z)\cdot\frac{p^{(j)}}{p}\right\|_{2sn^d} \leq K_j\cdot n^{d(\deg Q_j-j)+j}\cdot\frac{s^{\deg Q_j}}{s^j}=K_j\cdot n^j\cdot(sn^d)^{(\deg Q_j-j)}$$
$$= K_j\cdot n^j\cdot\frac{1}{(sn^d)^{j-\deg Q_j}}\leq K_j\cdot n^j\leq K_j\cdot n^{j_0-1}. \qquad (8)$$

* Finally we estimate the eigenvalue $\lambda_n=\sum_{i=1}^{j_0}q_{j,j}\cdot\frac{n!}{(n-j)!}$, which grows as n^{j_0} for large n, since there exists an integer n_{j_0} and some positive constant K'_{j_0} such that for all $n\geq n_{j_0}$ we obtain:

$$\begin{aligned}|\lambda_n| &\leq \sum_{j=1}^{j_0}|q_{j,j}|\cdot\frac{n!}{(n-j)!}\\ &= |q_{j_0,j_0}|\cdot\frac{n!}{(n-j_0)!}\left[1+\sum_{1\leq j<j_0}\left|\frac{q_{j,j}}{q_{j_0,j_0}}\right|\cdot\frac{(n-j_0)!}{(n-j)!}\right]\\ &\leq K'_{j_0}\cdot n^{j_0}. \qquad (9)\end{aligned}$$

Finally we rewrite the eigenvalue equation (3) as follows:

$$Q_{j_m}(z)\cdot\frac{p_n^{(j_m)}(z)}{p_n(z)}=\lambda_n+\sum_{j\neq j_m}Q_j(z)\frac{p_n^{(j)}(z)}{p_n(z)}.$$

Applying inequalities (5)-(9) to this we obtain

$$
\begin{aligned}
& \left\|Q_{j_m} \cdot \frac{p_n^{(j_m)}(z)}{p_n(z)}\right\|_{2sn^d} \leq |\lambda_n| + \sum_{j \neq j_m} \left\|Q_j \frac{p_n^{(j)}(z)}{p_n(z)}\right\|_{2sn^d} \\
\leq \quad & K'_{j_0} n^{j_0} + K_{j_0} n^{j_0} + \sum_{1 \leq j < j_0} K_j n^{j_0 - 1} \\
+ \quad & \sum_{\substack{j_0 < j \leq k \\ (\frac{j-j_0}{j - \deg Q_j}) < d}} K_j \frac{n^{j_0 - \delta}}{s^k} + \sum_{\substack{j_0 < j < j_m \\ (\frac{j-j_0}{j - \deg Q_j}) = d}} K_j n^{j_0} \frac{s^{1/d}}{s^{\frac{j_m - j_0}{d}}} \\
\leq \quad & K \cdot n^{j_0} + K \cdot \frac{n^{j_0 - \delta}}{s^k} + K \cdot n^{j_0} \frac{s^{1/d}}{s^{\frac{j_m - j_0}{d}}} \qquad (10)
\end{aligned}
$$

for all $n \geq \max(n_0, n(j), n_{j_0})$, where K is some positive constant and $0 < s < 1$. For the term on the left-hand side of the rewritten eigenvalue equation above we obtain using (4) the following estimation:

$$
\frac{1}{K} \cdot n^{j_0} \cdot \frac{1}{s^{\frac{j_m - j_0}{d}}} \leq \frac{1}{K_{j_m}} \cdot n^{j_0} \cdot \frac{1}{s^{\frac{j_m - j_0}{d}}} \leq \left\|Q_{j_m} \cdot \frac{p_n^{(j_m)}(z)}{p_n(z)}\right\|_{2sn^d} \qquad (11)
$$

for some constant $K \geq K_{j_m}$ which also satisfies (10). Now combining (10) and (11) we get

$$
\frac{1}{K} \cdot n^{j_0} \cdot \frac{1}{s^{\frac{j_m - j_0}{d}}} \leq K \cdot n^{j_0} + K \cdot \frac{n^{j_0 - \delta}}{s^k} + K \cdot n^{j_0} \frac{s^{1/d}}{s^{\frac{j_m - j_0}{d}}}.
$$

Dividing this inequality by n^{j_0} and multiplying by K we have

$$
\begin{gathered}
\frac{1}{s^{\frac{j_m - j_0}{d}}} \leq K^2 + K^2 \cdot \frac{1}{n^\delta} \cdot \frac{1}{s^k} + K^2 \cdot \frac{s^{1/d}}{s^{\frac{j_m - j_0}{d}}}. \\
\Leftrightarrow \\
\frac{1}{s^w} \leq K^2 + \frac{K^2}{s^k} \cdot \frac{1}{n^\delta} + K^2 \cdot \frac{s^{1/d}}{s^w} \\
\Leftrightarrow \\
\frac{1}{s^w} [1 - K^2 \cdot s^{1/d}] \leq K^2 + \frac{K^2}{s^k} \cdot \frac{1}{n^\delta}. \qquad (12)
\end{gathered}
$$

where $w = (j_m - j_0)/d > 0$.

In what follows we will obtain a contradiction to this inequality for some small properly chosen $0 < s < 1$ and all sufficiently large n. Since $j_m \in [j_0 + 1, k]$ we have $w = (j_m - j_0)/d \geq 1/d$, and since $s < 1$ we get $s^w \leq s^{1/d} \Rightarrow 1/s^w \geq 1/s^{1/d}$. **Now choose $s^{1/d} = \frac{1}{4K^2}$, where K is the constant in (12)**. Then estimating the left-hand side of (12) we get

$$\frac{1}{s^w}[1 - K^2 \cdot s^{1/d}] \geq \frac{1}{s^{1/d}}[1 - K^2 \cdot s^{1/d}] = 4K^2 - K^2 = 3K^2$$

and thus from (12) we have

$$3K^2 \leq \frac{1}{s^w}[1 - K^2 \cdot s^{1/d}] \leq K^2 + \frac{K^2}{s^k} \cdot \frac{1}{n^\delta}$$

$$\Leftrightarrow$$

$$2K^2 \leq \frac{K^2}{s^k} \cdot \frac{1}{n^\delta}$$

$$\Leftrightarrow$$

$$n^\delta \leq \frac{1}{2} \cdot \frac{1}{s^k} = \frac{1}{2}(2K)^{2dk}.$$

We therefore obtain a contradiction to this inequality, and hence to inequality (12) and thus to the eigenvalue equation (3), if $n^\delta > \frac{1}{2}(2K)^{2dk}$ and $s = 1/(2K)^{2d}$, and consequently all roots of p_n cannot be contained in a disc of radius $s \cdot n^d$ for such choices on s and n. Hence there exists an integer n_0 such that for all $n > n_0$ we have $r_n > s \cdot n^d$. $\square$

3 Open Problems and Conjectures

3.1 The upper bound

Based upon numerical evidence from computer experiments (some of which is presented in [1]) we are led to assert that there exists a constant C_0, which depends on T only, such that

$$r_n \leq C_0 \cdot n^d \tag{13}$$

holds for all sufficiently large integers n. We refer to this as the **upper-bound conjecture**. Computer experiments confirm that the zeros of the scaled eigenpolynomial $q_n(z) = p_n(n^d z)$ are contained in a compact set when $n \to \infty$.

3.2 The measures $\{\mu_n\}$

Consider the sequence of discrete probability measures

$$\mu_n = \frac{1}{n}\sum_{\nu=1}^{\nu=n} \delta(\frac{\alpha_\nu}{n^d})$$

where $\alpha_1, \ldots, \alpha_n$ are the roots of the eigenpolynomial p_n and d is as in Definition 1. Assuming (13) the supports of $\{\mu_n\}$ stay in a compact set in $\mathbb{C}$. Next, by a **tree** we mean a connected compact subset Γ of $\mathbb{C}$ which consists of a finite union of real-analytic curves and where $\hat{\mathbb{C}} \setminus \Gamma$ is simply connected (here $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ is the extended complex plane). Computer experiments from [1] lead us to the following

Conjecture 1. For each operator T the sequence $\{\mu_n\}$ converges weakly to a probability measure μ_T which is supported on a certain tree Γ_T.

3.3 The determination of μ_T

Given $T = \sum_{j=1}^{k} Q_j(z)D^j$ and $Q_j(z) = \sum_{i=0}^{\deg Q_j} q_{j,i}z^i$ we obtain an algebraic function $C_T(z)$ which satisfies the following algebraic equation (also see [1]):

$$q_{j_0,j_0} \cdot z^{j_0} \cdot C_T^{j_0}(z) + \sum_{j \in J} q_{j,\deg Q_j} \cdot z^{\deg Q_j} \cdot C_T^{j}(z) = q_{j_0,j_0},$$

where $J = \{j : (j - j_0)/(j - \deg Q_j) = d\}$, i.e. the sum is taken over all j for which d is attained. In addition C_T is chosen to be the unique single-valued branch which has an expansion

$$C_T(z) = \frac{1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \ldots$$

at $\infty \in \hat{\mathbb{C}}$. Let $\mathbb{D}_T$ be the discriminant locus of C_T, i.e. this is a finite set in $\mathbb{C}$ such that the single-valued branch of C_T in an exterior disc $|z| > R$ can be continued to an (in general multi-valued) analytic function in $\hat{\mathbb{C}} \setminus \mathbb{D}_T$. If Γ_T is a tree which contains $\mathbb{D}_T$, we obtain a single-valued branch of C_T in the simply connected set $\Omega_{\Gamma_T} = \hat{\mathbb{C}} \setminus \Gamma_T$. It is easily seen that this holomorphic function in Ω_{Γ_T} defines a locally integrable function in the sense of Lebesgue outside the nullset Γ_T. A tree Γ_T which contains $\mathbb{D}_T$ is called T-positive if the distribution defined by

$$\nu_{\Gamma_T} = \frac{1}{\pi} \cdot \bar{\partial} C_T / \bar{\partial} \bar{z}$$

is a probability measure.

3.4 Main conjecture

Now we announce the following which is experimentally confirmed in [1]:

For each operator T, the limiting measure μ_T in Conjecture 1 exists. Moreover, its support is a T-positive tree Γ_T and one has the equality $\mu_T = \nu_{\Gamma_T}$ which means that when $z \in \hat{\mathbb{C}} \setminus \Gamma_T$ the following holds:

$$C_T(z) = \int_{\Gamma_T} \frac{d\mu_T(\zeta)}{z - \zeta}.$$

Remark. For *non-degenerate exactly-solvable operators* (i.e. when $\deg Q_k = k$) it was proved in [2] that the roots of all eigenpolynomials stay in a compact set of $\mathbb{C}$, and the unscaled sequence of probability measures $\{\mu_n\}$ converge to a measure supported on a tree, i.e. the analogue of the main conjecture above holds in the non-degenerate case.

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